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Pullback attractor for N-dimensional thermoelastic coupled structure equations

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Abstract

In this paper, proving the pullback asymptotic compactness of processes by the aid of a contractive function in space X_0, we prove the existence of a pullback attractor for N-dimensional nonautonomous thermoelastic coupled structure equations

$$u_{tt}+\alpha\Delta^2u-\left[\beta+\sigma\left(\int_\Omega(\nabla u)^2\,dx\right)\right]\Delta u+\gamma\Delta\theta+g(u)+\eta u_t=h(x,t),$$

$$\text{in }\Omega\times[\tau,\infty),$$

$$\theta_t-\Delta\theta-\gamma\Delta u_t=q(x,t)\quad\text{in }\Omega\times[\tau,\infty),$$

with the lateral load distribution function $h(x,t)$ and the external heat supply function $q(x,t)$ unnecessarily bounded. The nonlinear source term $g(u)$ is essentially $k_1(u+\frac{|u|^{\rho-1}u}{\rho+1})$ $(k_1>0)$ with $1<\rho\le\frac{N}{N-2}$ if $N\ge3$ and $1<\rho<\infty$ if $N=1,2$.

Keywords: pullback attractor; pullback asymptotically compact; contractive function; thermoelastic coupled structure equations

1 Introduction

In this paper, we consider the pullback asymptotic behavior of the following nonautonomous thermoelastic coupled structure equations:

$$u_{tt}+\alpha\Delta^2u-\left[\beta+\sigma\left(\int_\Omega(\nabla u)^2\,dx\right)\right]\Delta u+\gamma\Delta\theta+g(u)+\eta u_t=h(x,t)\quad\text{in }\Omega\times[\tau,\infty),\tag{1.1}$$

$$\theta_t-\Delta\theta-\gamma\Delta u_t=q(x,t)\quad\text{in }\Omega\times[\tau,\infty),\tag{1.2}$$

in a bounded domain $\Omega\subset R^N$ with smooth boundary. Here α,β,γ,η are all positive constants, which arise from a model of the nonlinear thermoelastic coupled vibration structure with clamped ends for simultaneously considering the medium damping, the viscous effect, and the nonlinear constitutive relation and thermoelasticity based on a theory of non-Fourier heat flux. The system is supplemented with the boundary conditions

$$u(x,t)|_{\partial\Omega}=\frac{\partial u}{\partial\nu}(x,t)|_{\partial\Omega}=0,\qquad\theta(x,t)|_{\partial\Omega}=0,\quad t\ge\tau,\tag{1.3}$$

for every $t > 0$, and the initial conditions

$$u(x,\tau) = u_0(x), \qquad u_t(x,\tau) = v_0(x), \qquad \theta(x,\tau) = \theta_0(x), \quad x \in \Omega, \tag{1.4}$$

where $u_0(x)$, $v_0(x)$ and $\theta_0(x)$ are assigned initial value functions.

Here the unknown variables $u(x,t)$ and $\theta(x,t)$ represent the vertical deflection of the structure and vertical component of the temperature gradient, respectively. The subscript t denotes the derivative with respect to t, $\sigma(\cdot)$ is the nonlinearity of the material and continuous nonnegative nonlinear real function, $g(u)$ is the source term, $h(x,t)$ is the lateral load distribution, and $q(x,t)$ is the external heat supply. Moreover, the source term $g(u)$ is essentially $k_1(u + \frac{|u|^{\rho-1}u}{\rho+1})$ ($k_1 > 0$) with $1 < \rho \le \frac{N}{N-2}$ if $N \ge 3$ and $1 < \rho < \infty$ if $N = 1,2$. Assumptions on nonlinear functions $\sigma(\cdot), g(\cdot)$ and the external force function $h(x,t), q(x,t)$ will be specified later.

It is well known the global attractor on autonomous thermoelastic coupled structure equations has been considered in many papers. We refer the reader to [1–4] and the references therein.

However, in the actual life, the real systems are mostly nonautonomous. Recently, the nonautonomous infinite-dimensional dynamical system attracted attention of many people. For example, Chepyzhov and Vishik [5] firstly extended the notion of global attractor in the autonomous case to the nonautonomous case, which led to the concept of a uniform attractor. But the uniform attractor [6] was not applicable to nonautonomous systems with possibly unbounded trajectories as time increases to infinity (see [7–12]). To handle such problems, some new concepts and theories were brought up for nonautonomous case, and thus the pullback attractors were developed in [13–17], and they are a useful tool in understanding the dynamics of nonautonomous dynamical systems.

In this paper, we use the concept of pullback asymptotic compactness given in [7], and we prove the pullback asymptotic compactness by the method in [14] for nonautonomous system (1.1)-(1.4). Our fundamental assumptions on $\sigma(\cdot)$, $g(\cdot)$, $h(x,t)$, and $q(x,t)$ are given as follows.

Assumption 1 *We assume that $\sigma(\cdot) \in C^1(R)$ satisfy*

$$\sigma(z)z \ge \hat{\sigma}(z) \ge 0, \quad \forall z \ge 0, \tag{1.5}$$

where $\hat{\sigma}(z) = \int_0^z \sigma(z)\,dz$. This condition is promptly satisfied if $\sigma(\cdot)$ is nondecreasing with $\sigma(0) = 0$.

Assumption 2 *The nonlinear term $g(\cdot)$ is a $C^1(R,R)$ function satisfying the following assumptions*:

(H_1) *There exists a constant k_2 such that*

$$\bigl|g(u) - g(v)\bigr| \le k_2|u-v|\bigl(1 + |u|^{\rho-1} + |v|^{\rho-1}\bigr). \tag{1.6}$$

(H_2) *If $\hat{g}(s)$ is the primitive of $g(s)$, that is, $\hat{g}(s) = \int_0^s g(\tau)\,d\tau$, then*

$$\liminf_{|s|\to\infty} \frac{\hat{g}(s)}{s^2} \ge 0, \tag{1.7}$$

and there exists a constant k_3 such that

$$\left|\hat{g}(u) - \hat{g}(v)\right| \leq k_3\left(u + v + |u|^\rho + |v|^\rho\right)|u - v|, \quad \forall u, v \in R. \tag{1.8}$$

(H_3) *There exists a constant $C_0 \geq 1$ such that*

$$\liminf_{|s|\to\infty} \frac{sg(s) - C_0\hat{g}(s)}{s^2} \geq 0. \tag{1.9}$$

Assumption 3 *Functions $h(x,t)$ and $q(x,t)$ for $t \in R$, $x \in \Omega$ are locally square integrable in time, that is, $h(x,t), q(x,t) \in L^2_{\mathrm{loc}}(R, L^2(\Omega))$, and for any $t \in R$,*

$$\int_{-\infty}^t e^{\delta s}\left\|h(x,s)\right\|^2 ds < \infty \tag{1.10}$$

and

$$\int_{-\infty}^t e^{\delta s}\left\|q(x,s)\right\|^2 ds < \infty, \tag{1.11}$$

where $\delta > 0$ is a small real number, which will be characterized later.

Under these assumptions, we prove the existence of a pullback attractor for nonautonomous thermoelastic coupled structure equation system (1.1)-(1.4).

2 Preliminaries

We first introduce the following abbreviations:

$$H = L^2(\Omega), \qquad \|\cdot\| = \|\cdot\|_{L^2(\Omega)}.$$

Let $(\cdot,\cdot)$ denote the H-inner product, and let $\|\nabla\cdot\|$ and $\|\triangle\cdot\|$ be the norms of $H_0^1(\Omega)$ and $H_0^2(\Omega)$, respectively.

We denote the space

$$X_0 = H_0^2(\Omega) \times L^2(\Omega) \times L^2(\Omega)$$

equipped with the norm

$$\|\cdot\|_{X_0}^2 = \|\triangle u\|^2 + \|v\|^2 + \|\theta\|^2.$$

The sign $H_1 \hookrightarrow\hookrightarrow H_2$ denotes compact embedding of H_1 into H_2. For brevity, we use the same letter C to denote different positive constants.

3 Abstract results

In this section, we recall some definitions and results concerning the pullback attractor for nonautonomous dynamical systems. These definitions and results can be found in [11–15] and the references therein.

Let (X_0, d) be a complete metric space, and let (Q, ρ) be a metric space which is called the parameter space. We define a nonautonomous dynamical system by a cocycle mapping

$\Phi: R_+ \times Q \times X_0 \to X_0$, which is driven by an autonomous dynamical system θ acting on a parameter space Q. Specifically, $\theta = \{\theta_t\}_{t\in R}$ is a dynamical system on Q, that is, it is a group of homeomorphisms under composition on Q with the properties that:

(1) $\theta_0(q) = q$ for all $q \in Q$;
(2) $\theta_{t+\tau}(q) = \theta_t(\theta_\tau(q))$ for all $t, \tau \in R$;
(3) The mapping $(t,q) \to \theta_t(q)$ is continuous.

Definition 3.1 ([13–18]) A mapping Φ is said to be a cocycle on X_0 with respect to group θ if

(1) $\Phi(0,q,x) = x$ for all $(q,x) \in Q \times X_0$;
(2) $\Phi(t+s,q,x) = \Phi(s,\theta_t(q),\Phi(t,q,x))$ for all $s,t \in R_+$ and all $(q,x) \in Q \times X_0$;
(3) the mapping $\Phi(t,q,\cdot): X_0 \to X_0$ is continuous for all $(t,q) \in R^+ \times Q$.

Definition 3.2 ([13–18]) A family of nonempty compact sets $A = \{A_q\}_{q\in Q}$ is said to be a pullback (or cocycle) attractor if, for each $q \in Q$, it satisfies

(1) $\Phi(t,q,A_q) = A_{\theta_t(q)}$ for all $t \in R^+$ (Φ-invariance);
(2) $\lim_{t\to\infty} \operatorname{dist}(\Phi(t,\theta_{-t}(q),B),A_q) = 0$ for any bounded subset $B \subset X$ (pullback attracting).

Definition 3.3 ([13–18]) A family $D = \{D_q\}_{q\in Q} \in K$ is said to be pullback absorbing if for each $q \in Q$ and any bounded subset B of X_0, there exists $t_0(q,B) \geq 0$ such that

$$\Phi\bigl(t,\theta_{-t}(q),B_{\theta_{-t}(q)}\bigr) \subset D_q \quad \text{for all } t \geq t_0. \tag{3.1}$$

Definition 3.4 ([13]) Let (θ,Φ) be a nonautonomous dynamical system on $Q \times X_0$, and let $D = \{D_q\}_{q\in Q}$ be a family of bounded subsets of X_0. The cocycle Φ is said to be pullback D-asymptotically compact if for any sequences $t_n \to \infty$ and $x_n \in D_{\theta_{-t_n}(q)}$, the sequence $\Phi(t_n,\theta_{-t_n}(q),x_n)$ is precompact in X_0.

Lemma 3.1 ([17]) *Let (θ,Φ) be a nonautonomous dynamical system on $Q \times X_0$. Assume that the family $D = \{D_q\}_{q\in Q}$ is pullback absorbing for Φ and Φ is pullback D-asymptotically compact. Then Φ possesses attractor $A = \{A_q\}_{q\in Q}$, and*

$$A_q = \bigcap_{t\geq 0}\overline{\bigcup_{s\geq t}\Phi\bigl(s,\theta{-}s(q),D_{\theta_{-s}(q)}\bigr)}, \quad q \in Q.$$

For this matter, first we give the following concept and lemma.

Definition 3.5 ([17]) Let X_0 be a Banach space, and let B be a bounded subset of X_0. We call a function $\phi(\cdot,\cdot)$ defined on $X_0 \times X_0$ a contractive function on $B \times B$ if for any sequence $\{x_n\}_{n\in N} \subset B$, there exists a subsequence $\{x_{nk}\}_{k\in N} \subset \{x_n\}_{n\in N}$ such that

$$\lim_{k\to\infty}\lim_{l\to\infty}\phi(x_{nk},x_{nl}) = 0. \tag{3.2}$$

Lemma 3.2 ([17]) *Let (θ,Φ) be a nonautonomous dynamical system on $Q \times X_0$. Suppose that bounded families $D = \{D_q\}_{q\in Q}$ and $\tilde{D} = \{\tilde{D}_q\}_{q\in Q}$ are such that, for any $q \in Q$, there* *exists* $t_q = t(q, D, \tilde{D}) \geq 0$ *such that*

$$\Phi\bigl(t, \theta_{-t}(q), D_{\theta_{-t}(q)}\bigr) \subset \tilde{D}_q \quad \text{for all } t \geq t_q. \tag{3.3}$$

Assume that, for any $\varepsilon > 0$ *and* $q \in Q$, *there exist* $t = t(\varepsilon, \tilde{D}, q) \geq 0$ *and a contractive function* $\phi_{t,q}(\cdot,\cdot)$ *defined on* $\tilde{D}_{\theta_{-t}(q)} \times \tilde{D}_{\theta_{-t}(q)}$ *such that*

$$\bigl\| \Phi\bigl(t, \theta_{-t}(q), x\bigr) - \Phi\bigl(t, \theta_{-t}(q), y\bigr) \bigr\|_{X_0} \leq \varepsilon + \phi_{t,q}(x, y) \quad \text{for all } x, y \in \tilde{D}_{\theta_{-t}(q)},$$

where $\phi_{t,q}$ *depends on* t, q. *Then* Φ *is pullback D-asymptotically compact in* X_0.

4 Global solutions and pullback attracting set

Using the classical Galerkin method, we can establish our main theorem of this section on the existence and uniqueness of a global solution to problem (1.1)-(1.4).

Theorem 4.1 *Assume that* $h(x,t), q(x,t) \in L^2_{\mathrm{loc}}(R, L^2(\Omega))$ *and that assumptions* (H_1)-(H_3) *on the function* $g(\cdot)$ *hold. Then for any* $(u_0, v_0, \theta_0) \in X_0$, *problem* (1.1)-(1.4) *has a unique solution* (u, u_t, θ) *satisfying* $(u, u_t, \theta) \in C^0(R_\tau; X_0)$, *where* $R_\tau = [\tau, \infty)$.

For simplicity, we write $y(r) = (u(r), \partial_r u(r), \theta(r)) = (u(r), v(r), \theta(r))$, $y_0 = (u_0, v_0, \theta_0)$. We denote by X_0 the space of vector functions $y(r) = (u(r), v(r), \theta(r))$ with the norm $\|y\|^2_{X_0} = \|\triangle u\|^2 + \|v\|^2 + \|\theta\|^2$.

We can construct the nonautonomous dynamical system generated by problem (1.1)-(1.4) in X_0. We consider $Q = R$ and $\theta_t \tau = \tau + t$. Then we define

$$\Phi(t, \tau, y_0) = y(t+\tau; \tau, y_0) = \bigl(u(t+\tau), v(t+\tau), \theta(t+\tau)\bigr), \quad \tau \in R, t \geq 0, y_0 \in X_0. \tag{4.1}$$

The uniqueness of a solution to problem (1.1)-(1.4) implies that

$$\Phi(t+s, \tau, y_0) = \Phi\bigl(t, s+\tau, \Phi(s, \tau, y_0)\bigr), \quad \tau \in R, t, s \geq 0, y_0 \in X_0,$$

and, for all $\tau \in R, t \geq 0$, the mapping $\Phi(t, \tau, \cdot) : X_0 \to X_0$ defined by (4.1) is continuous. Consequently, for any $(t, \tau) \in R^+ \times R$, the mapping $\Phi(t, \tau, \cdot)$ defined by (4.1) is a continuous cocycle on X_0.

Another main result of this section is as follows.

Theorem 4.2 *Suppose* $\alpha > 3\gamma$, $h(x,t)$ *and* $q(x,t) \in L^2_{\mathrm{loc}}(R; H)$ *satisfy* (1.10) *and* (1.11) *with* δ *satisfying* $0 < \delta < \varepsilon_0$ $(0 < \varepsilon_0 \leq \min\{\frac{\sqrt{1+4\mu^2}-1}{2}, \frac{4\alpha\lambda^2}{6\eta+5}, \frac{\alpha}{3}, \sqrt{9+3\eta}-3, \frac{\alpha\lambda^2}{\eta}\})$. *Then there exist a family of bounded sets* $D = \{D_q\}_{q \in Q}$ *in* X_0 *which is pullback absorbing for* Φ *defined by* (4.1) *and a family of bounded sets* $\tilde{D} = \{\tilde{D}_q\}_{q \in Q}$ *satisfying* (3.3).

Proof Let $t_0 \in R$, $\tau \geq 0$, and $y_0 = (u_0, v_0, \theta_0) \in X_0$ be fixed. Define

$$u(r) = u(r, t_0 - \tau, u_0), \qquad v(r) = u'(r, t_0 - \tau, v_0), \quad \text{and}$$

$$\theta(r) = \theta(r, t_0 - \tau, \theta_0) \quad \text{for } r \geq t_0 - \tau$$

and

$$\big(u(r),v(r),\theta(r)\big)=\Phi(r-t_0+\tau,t_0-\tau,y_0)\quad \text{for } r\ge t_0-\tau.$$

Multiplying equations (1.1) and (1.2) by $p=u_t+\varepsilon_0 u$ and θ, respectively, and then summing, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dr}\big[\|p\|^2+\alpha\|\triangle u\|^2-\eta\varepsilon_0\|u\|^2+\beta\|\nabla u\|^2+\hat{\sigma}\big(\|\nabla u\|^2\big)+\|\theta\|^2\big]\\
&\quad-\varepsilon_0\|p\|^2+\varepsilon_0\alpha\|\triangle u\|^2+\eta\|p\|^2-\eta\varepsilon_0^2\|u\|^2+\varepsilon_0\beta\|\nabla u\|^2+\varepsilon_0\sigma\big(\|\nabla u\|^2\big)\|\nabla u\|^2\\
&\quad+\varepsilon_0\gamma(\triangle u,\theta)+\|\nabla\theta\|^2+\varepsilon_0^2(u,p)+\big(g(u),p\big)\\
&=(h,p)+(q,\theta).
\end{aligned}\tag{4.2}$$

For simplicity, define $\phi(u)=\int_\Omega\hat{g}(u)\,dx$. By assumption (1.7) on $g(\cdot)$ it is obvious that $\phi(u)\ge 0$. By assumption (1.9) on $g(\cdot)$ we have

$$\big(g(u),u\big)-C_0\phi(u)+\frac{\varepsilon_0}{4}\|u\|^2\ge -M,$$

where $C_0\ge 1$. So

$$\begin{aligned}
\big(g(u),p\big)&=\big(g(u),u_t\big)+\varepsilon_0\big(g(u),u\big)\\
&\ge\int_\Omega\frac{d}{dr}\hat{g}(u)\,dx+\varepsilon_0C_0\phi(u)-\frac{\varepsilon_0^2}{4}\|u\|^2-\varepsilon_0M\\
&\ge\frac{d}{dr}\phi(u)+\varepsilon_0C_0\phi(u)-\frac{\varepsilon_0^2}{4}\|u\|^2-\varepsilon_0M.
\end{aligned}\tag{4.3}$$

By Young's inequality we have

$$\big|(h,p)\big|\le\frac{1}{\eta}\|h\|^2+\frac{\eta}{4}\|p\|^2\tag{4.4}$$

and

$$\big|(q,\theta)\big|\le\frac{\lambda_0^2}{2}\|\theta\|^2+\frac{1}{2\lambda_0^2}\|q\|^2\le\frac{1}{2}\|\nabla\theta\|^2+\frac{1}{2\lambda_0^2}\|q\|^2,\tag{4.5}$$

where λ_0 is the first eigenvalue of ∇ in $L^2(\Omega)$.

By (4.3)-(4.5) from (4.2) we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dr}\big[\|p\|^2+\alpha\|\triangle u\|^2-\eta\varepsilon_0\|u\|^2+\beta\|\nabla u\|^2+\hat{\sigma}\big(\|\nabla u\|^2\big)+\|\theta\|^2+2\phi(u)\big]\\
&\quad-\varepsilon_0\|p\|^2+\varepsilon_0\alpha\|\triangle u\|^2+\frac{3\eta}{4}\|p\|^2-\eta\varepsilon_0^2\|u\|^2+\varepsilon_0\beta\|\nabla u\|^2+\varepsilon_0\sigma\big(\|\nabla u\|^2\big)\|\nabla u\|^2\\
&\quad+\varepsilon_0\gamma(\triangle u,\theta)+\frac{1}{2}\|\nabla\theta\|^2+\varepsilon_0^2(u,p)+\varepsilon_0C_0\phi(u)-\frac{\varepsilon_0^2}{4}\|u\|^2\\
&\le\frac{1}{\eta}\|h\|^2+\frac{1}{2\lambda_0^2}\|q\|^2+\varepsilon_0M.
\end{aligned}\tag{4.6}$$

By Young's inequality we have

$$\varepsilon_0\gamma(\triangle u,\theta) \ge -\frac{\varepsilon_0\gamma}{2}\|\triangle u\|^2 - \frac{\varepsilon_0\gamma}{2}\|\theta\|^2 \tag{4.7}$$

and

$$\varepsilon_0^2(u,p) \ge -\varepsilon_0^2\left(\|u\|^2 + \frac{1}{4}\|p\|^2\right). \tag{4.8}$$

So

$$\begin{aligned}
&\left(\frac{3\eta}{4} - \varepsilon_0\right)\|p\|^2 + \varepsilon_0\alpha\|\triangle u\|^2 - \eta\varepsilon_0^2\|u\|^2 + \varepsilon_0\beta\|\nabla u\|^2 + \varepsilon_0\sigma\left(\|\nabla u\|^2\right)\|\nabla u\|^2 \\
&\quad + \varepsilon_0\gamma(\triangle u,\theta) + \frac{1}{2}\|\nabla\theta\|^2 + \varepsilon_0^2(u,p) + \varepsilon_0 C_0\phi(u) - \frac{\varepsilon_0^2}{4}\|u\|^2 \\
&\ge \left(\frac{3\eta}{4} - \varepsilon_0 - \frac{\varepsilon_0^2}{4}\right)\|p\|^2 + \left(\varepsilon_0\alpha - \frac{\varepsilon_0\gamma}{2}\right)\|\triangle u\|^2 + \left(-\eta\varepsilon_0^2 - \frac{5\varepsilon_0^2}{4}\right)\|u\|^2 \\
&\quad + \frac{1}{2}\|\nabla\theta\|^2 - \frac{\varepsilon_0\gamma}{2}\|\theta\|^2 + \varepsilon_0\beta\|\nabla u\|^2 + \varepsilon_0\sigma\left(\|\nabla u\|^2\right)\|\nabla u\|^2 + \varepsilon_0 C_0\phi(u) \\
&\ge \left(\frac{3\eta}{4} - \varepsilon_0 - \frac{\varepsilon_0^2}{4}\right)\|p\|^2 + \left(\frac{2\varepsilon_0\alpha}{3} - \frac{\varepsilon_0\gamma}{2}\right)\|\triangle u\|^2 + \left(\frac{\varepsilon_0\alpha\lambda^2}{3} - \eta\varepsilon_0^2 - \frac{5\varepsilon_0^2}{4}\right)\|u\|^2 \\
&\quad + \left(\frac{\lambda_0^2}{2} - \frac{\varepsilon_0\gamma}{2}\right)\|\theta\|^2 + \varepsilon_0\beta\|\nabla u\|^2 + \varepsilon_0\sigma\left(\|\nabla u\|^2\right)\|\nabla u\|^2 + \varepsilon_0 C_0\phi(u),
\end{aligned}$$

where λ is the first eigenvalue of $\triangle$ in $L^2(\Omega)$. Let

$$L(u,p,\theta) = \frac{1}{2}\left(\|p\|^2 + \alpha\|\triangle u\|^2 - \eta\varepsilon_0\|u\|^2 + \beta\|\nabla u\|^2 + \hat{\sigma}\left(\|\nabla u\|^2\right) + \|\theta\|^2 + 2\phi(u)\right) \ge 0$$

and

$$\begin{aligned}
Y(u,p,\theta) &= \left(\frac{3\eta}{4} - \varepsilon_0 - \frac{\varepsilon_0^2}{4}\right)\|p\|^2 + \left(\frac{2\varepsilon_0\alpha}{3} - \frac{\varepsilon_0\gamma}{2}\right)\|\triangle u\|^2 \\
&\quad + \left(\frac{\varepsilon_0\alpha\lambda^2}{3} - \eta\varepsilon_0^2 - \frac{5\varepsilon_0^2}{4}\right)\|u\|^2 + \left(\frac{\lambda_0^2}{2} - \frac{\varepsilon_0\gamma}{2}\right)\|\theta\|^2 \\
&\quad + \varepsilon_0\beta\|\nabla u\|^2 + \varepsilon_0\sigma\left(\|\nabla u\|^2\right)\|\nabla u\|^2 + \varepsilon_0 C_0\phi(u).
\end{aligned}$$

Then considering $0 < \varepsilon_0 \le \min\{\frac{\lambda_0^2}{\gamma+1}, \frac{4\alpha\lambda^2}{6\eta+15}, \sqrt{9+3\eta}-3, \frac{\alpha\lambda^2}{\eta}\}$, $\alpha > 3\gamma$, and $C_0 \ge 1$, from (1.5) we get

$$\begin{aligned}
&Y(u,p,\theta) - \varepsilon_0 L(u,p,\theta) \\
&\ge \left(\frac{3\eta}{4} - \frac{3\varepsilon_0}{2} - \frac{\varepsilon_0^2}{4}\right)\|p\|^2 + \left(\frac{2\varepsilon_0\alpha}{3} - \frac{\varepsilon_0\gamma}{2} - \frac{\varepsilon_0\alpha}{2}\right)\|\triangle u\|^2 \\
&\quad + \left(\frac{\varepsilon_0\alpha\lambda^2}{3} - \frac{\eta\varepsilon_0^2}{2} - \frac{5\varepsilon_0^2}{4}\right)\|u\|^2 + \left(\frac{\lambda_0^2}{2} - \frac{\varepsilon_0\gamma}{2} - \frac{\varepsilon_0}{2}\right)\|\theta\|^2 \\
&\quad + \varepsilon_0\sigma\left(\|\nabla u\|^2\right)\|\nabla u\|^2 - \varepsilon_0\hat{\sigma}\left(\|\nabla u\|^2\right) + \varepsilon_0(C_0 - 1)\phi(u) \ge 0.
\end{aligned}$$

From (4.6) we have

$$\frac{d}{dr}L(u,p,\theta) + \varepsilon_0 L(u,p,\theta) \le \frac{1}{\eta}\|h\|^2 + \frac{1}{2\lambda_0^2}\|q\|^2 + \varepsilon_0 M. \tag{4.9}$$

Note that

$$\frac{d}{dr}e^{\delta r}L(u,p,\theta) = \delta e^{\delta r}L(u,p,\theta) + e^{\delta r}\frac{d}{dr}L(u,p,\theta),$$

so by (4.9) we have

$$\begin{aligned}\frac{d}{dr}\bigl[e^{\delta r}L(u,p,\theta)\bigr] \le{}& (\delta - \varepsilon_0)e^{\delta r}L(u,p,\theta)\\ &+ e^{\delta r}\left(\frac{1}{\eta}\|h\|^2 + \frac{1}{2\lambda_0^2}\|q\|^2 + \varepsilon_0 M\right).\end{aligned} \tag{4.10}$$

By integrating (4.10) over the interval $[t_0 - \tau, t_0]$, with $L(u,p,\theta) \ge 0$, we obtain

$$\begin{aligned}e^{\delta t_0}L\bigl(u(t_0),p(t_0),\theta(t_0)\bigr) \le{}& e^{\delta(t_0-\tau)}L\bigl(u(t_0-\tau),p(t_0-\tau),\theta(t_0-\tau)\bigr)\\ &+ (\delta-\varepsilon_0)\int_{t_0-\tau}^{t_0} e^{\delta s}L\bigl(u(s),p(s),\theta(s)\bigr)\,ds\\ &+ \int_{t_0-\tau}^{t_0} e^{\delta s}\left(\frac{1}{\eta}\|h\|^2 + \frac{1}{2\lambda_0^2}\|q\|^2\right)ds\\ &+ \frac{\varepsilon_0 M}{\delta}\bigl(e^{\delta t_0} - e^{\delta(t_0-\tau)}\bigr).\end{aligned} \tag{4.11}$$

Since $\delta < \varepsilon_0$, from (4.11) we have

$$\begin{aligned}&\bigl\|p(t_0)\bigr\|^2 + \alpha\bigl\|\triangle u(t_0)\bigr\|^2 - \eta\varepsilon_0\bigl\|u(t_0)\bigr\|^2 + \beta\bigl\|\nabla u(t_0)\bigr\|^2\\ &\quad + \hat{\sigma}\bigl(\bigl\|\nabla u(t_0)\bigr\|^2\bigr) + \bigl\|\theta(t_0)\bigr\|^2 + 2\phi\bigl(u(t_0)\bigr)\\ &\le e^{-\delta\tau}\bigl(\bigl\|p(t_0-\tau)\bigr\|^2 + \alpha\bigl\|\triangle u(t_0-\tau)\bigr\|^2 - \eta\varepsilon_0\bigl\|u(t_0-\tau)\bigr\|^2 + \beta\bigl\|\nabla u(t_0-\tau)\bigr\|^2\\ &\quad + \hat{\sigma}\bigl(\bigl\|\nabla u(t_0-\tau)\bigr\|^2\bigr) + \bigl\|\theta(t_0-\tau)\bigr\|^2 + 2\phi\bigl(u(t_0-\tau)\bigr)\bigr)\\ &\quad + e^{-\delta t_0}\int_{t_0-\tau}^{t_0} e^{\delta s}\left(\frac{2}{\eta}\|h\|^2 + \frac{1}{\lambda_0^2}\|q\|^2\right)ds + \frac{2\varepsilon_0 M}{\delta}\bigl(1 - e^{-\delta\tau}\bigr).\end{aligned} \tag{4.12}$$

If we take $C_1 = \max\{2, 1 + \frac{2\varepsilon_0^2}{\lambda^2}\}$, then since $\|u\|^2 \le \frac{1}{\lambda^2}\|\triangle u\|^2$, we have

$$\begin{aligned}&\bigl\|\triangle u(t_0)\bigr\|^2 + \bigl\|v(t_0)\bigr\|^2 + \bigl\|\theta(t_0)\bigr\|^2\\ &\le \bigl\|\triangle u(t_0)\bigr\|^2 + \bigl\|v(t_0) + \varepsilon_0 u(t_0) - \varepsilon_0 u(t_0)\bigr\|^2 + \bigl\|\theta(t_0)\bigr\|^2\\ &\le \bigl\|\triangle u(t_0)\bigr\|^2 + 2\bigl\|p(t_0)\bigr\|^2 + 2\varepsilon_0^2\bigl\|u(t_0)\bigr\|^2 + \bigl\|\theta(t_0)\bigr\|^2\\ &\le \left(1 + \frac{2\varepsilon_0^2}{\lambda^2}\right)\bigl\|\triangle u(t_0)\bigr\|^2 + 2\bigl\|p(t_0)\bigr\|^2 + \bigl\|\theta(t_0)\bigr\|^2\\ &= C_1\bigl(\bigl\|\triangle u(t_0)\bigr\|^2 + \bigl\|p(t_0)\bigr\|^2 + \bigl\|\theta(t_0)\bigr\|^2\bigr).\end{aligned} \tag{4.13}$$

On the other hand, setting $C_2 = \min\{1, \alpha - \frac{\eta\varepsilon_0}{\lambda^2}\}$, we obtain

$$\begin{aligned} &L\big(u(t_0), p(t_0), \theta(t_0)\big) \\ &\quad \geq \left(\big\|p(t_0)\big\|^2 + \left(\alpha - \frac{\eta\varepsilon_0}{\lambda^2}\right) \big\|\triangle u(t_0)\big\|^2 + \beta \big\|\nabla u(t_0)\big\|^2 \right. \\ &\qquad \left. + \hat{\sigma}\big(\big\|\nabla u(t_0)\big\|^2\big) + \big\|\theta(t_0)\big\|^2 + 2\phi\big(u(t_0)\big) \right) \\ &\quad \geq C_2\big(\big\|p(t_0)\big\|^2 + \big\|\triangle u(t_0)\big\|^2 + \big\|\theta(t_0)\big\|^2\big). \end{aligned} \tag{4.14}$$

So from (4.13)-(4.14) we get

$$\begin{aligned} &\big\|\triangle u(t_0)\big\|^2 + \big\|v(t_0)\big\|^2 + \big\|\theta(t_0)\big\|^2 \\ &\quad \leq C_1\big(\big\|\triangle u(t_0)\big\|^2 + \big\|p(t_0)\big\|^2 + \big\|\theta(t_0)\big\|^2\big) \\ &\quad \leq \frac{C_1}{C_2} L\big(u(t_0), p(t_0), \theta(t_0)\big) \\ &\quad \leq \frac{C_1}{C_2}\bigg\{ e^{-\delta\tau}\big(\big\|p(t_0-\tau)\big\|^2 + \alpha\big\|\triangle u(t_0-\tau)\big\|^2 - \eta\varepsilon_0\big\|u(t_0-\tau)\big\|^2 + \beta\big\|\nabla u(t_0-\tau)\big\|^2 \\ &\qquad + \hat{\sigma}\big(\big\|\nabla u(t_0-\tau)\big\|^2\big) + \big\|\theta(t_0-\tau)\big\|^2 + 2\phi\big(u(t_0-\tau)\big)\big) \\ &\qquad + e^{-\delta t_0}\int_{t_0-\tau}^{t_0} e^{\delta s}\left(\frac{2}{\eta}\|h\|^2 + \frac{1}{\lambda_0^2}\|q\|^2\right) ds + \frac{2\varepsilon_0 M}{\delta}\big(1 - e^{-\delta\tau}\big)\bigg\}. \end{aligned} \tag{4.15}$$

Let $\hat{D}_{\delta,X_0}$ ($\hat{D}_{\delta,X_0}$ denotes the class of all families $D = \{D_t\}_{t\in R}$) be given. For all $y(t_0 - \tau) = y_0 \in D(t_0 - \tau)$, $t \in R$, and $\tau \geq 0$, from assumption (1.8) on $\hat{g}(\cdot)$ we know that $\phi(u(t_0 - \tau))$ is bounded. Using the midvalue theorem of integration, from the assumption that $\sigma(\cdot) \in C^1(R)$ we have that $\hat{\sigma}(\|\nabla u(t_0 - \tau)\|^2)$ is bounded, too. So from (4.15) we easily obtain

$$\begin{aligned} &\big\|\Phi(\tau, t_0 - \tau, y_0)\big\|_{X_0}^2 \\ &\quad \leq \frac{C_1}{C_2}\bigg\{ e^{-\delta\tau}\big(\big\|p(t_0-\tau)\big\|^2 + \alpha\big\|\triangle u(t_0-\tau)\big\|^2 - \eta\varepsilon_0\big\|u(t_0-\tau)\big\|^2 + \beta\big\|\nabla u(t_0-\tau)\big\|^2 \\ &\qquad + \hat{\sigma}\big(\big\|\nabla u(t_0-\tau)\big\|^2\big) + \big\|\theta(t_0-\tau)\big\|^2 + 2\phi\big(u(t_0-\tau)\big)\big) \\ &\qquad + e^{-\delta t_0}\int_{-\infty}^{t_0} e^{\delta s}\left(\frac{2}{\eta}\|h\|^2 + \frac{1}{\lambda_0^2}\|q\|^2\right) ds + \frac{2\varepsilon_0 M}{\delta}\big(1 - e^{-\delta\tau}\big)\bigg\} \end{aligned} \tag{4.16}$$

for all $y_0 \in D(t_0 - \tau)$, $t_0 \in R$, and $\tau \geq 0$. Set

$$(R_t)^2 = 2\frac{C_1}{C_2}e^{-\delta t_0}\int_{-\infty}^{t} e^{\delta s}\left(\frac{2}{\eta}\big\|h(s)\big\|^2 + \frac{1}{\lambda_0^2}\big\|q(s)\big\|^2\right) ds + \frac{4C_1\varepsilon_0 M}{C_2\delta} \tag{4.17}$$

and consider the family D of closed balls in X_0 defined by $D_t = \{y \in X_0, \|y\|_{X_0} \leq R_t\}$. It is easy to check the family $D = \{D_t\}_{t\in R}$ is a bounded family of pullback absorbing sets in X_0.

Choose a number $\tilde{\delta}$ such that

$$\tilde{\delta} < \min\left\{\frac{\eta}{2}, 2\lambda_0^2, \frac{\eta - \varepsilon'}{2 + 5\varepsilon'}, \delta, \right\}, \tag{4.18}$$

where $0 < \varepsilon' < \min\{\frac{\sqrt{(9+\lambda_0^2)^2+40\eta\lambda_0^2}-(9+\lambda_0^2)}{20}, \eta\}$. Then, reasoning as before, (4.16) is also true if we replace δ by $\tilde{\delta}$.

Now, we letting $y_0 \in D(t_0 - \tau)$, we deduce that

$$
\begin{aligned}
&\left\|\Phi(\tau, t_0 - \tau, y_0)\right\|_{X_0}^2 \\
&\quad \le \frac{C_1}{C_2}\left\{e^{-\tilde{\delta}\tau} C\left(R_{t_0-\tau}^2 + R_{t_0-\tau}\right) + e^{-\tilde{\delta}t_0}\int_{-\infty}^{t_0} e^{\tilde{\delta}s}\left(\frac{2}{\eta}\|h\|^2 + \frac{1}{\lambda_0^2}\|q\|^2\right) ds \right. \\
&\qquad \left. + \frac{2\varepsilon_0 M}{\delta}\left(1 - e^{-\tilde{\delta}\tau}\right)\right\}.
\end{aligned}
\tag{4.19}
$$

If we set

$$
(\tilde{R}_t)^2 = 2\frac{C_1}{C_2}e^{-\tilde{\delta}t}\int_{-\infty}^{t} e^{\tilde{\delta}s}\left(\frac{2}{\eta}\left\|h(s)\right\|^2 + \frac{1}{\lambda_0^2}\left\|q(s)\right\|^2\right) ds + \frac{4C_1\varepsilon_0 M}{C_2\delta}
$$

and

$$
\tilde{D}_t = \left\{y \in X_0, \|y\|_{X_0} \le \tilde{R}_t\right\},
$$

then the family $\tilde{D} = \{\tilde{D}_t\}_{t\in R}$ satisfies (3.3). The proof is finished. □

5 The pullback attractor in X_0

In this section, we prove the pullback attractor in X_0.

Theorem 5.1 *Assume that assumptions* (H_1)-(H_3) *of* $g(\cdot)$ *hold and that* $h(x,t), q(x,t) \in L_{\mathrm{loc}}^2(R,H)$ *satisfy* (1.10) *and* (1.11) *with some* $\tilde{\delta}$ *satisfying* $0 < \tilde{\delta} < \min\{\frac{\eta}{2}, 2\lambda_0^2, \frac{\eta-\varepsilon'}{2+5\varepsilon'}, \delta\}$ $(0 < \varepsilon' < \min\{\frac{\sqrt{(9+\lambda_0^2)^2+40\eta\lambda_0^2}-(9+\lambda_0^2)}{20}, \eta\})$. *Then there exists a pullback attractor* $A = \{A_t\}_{t\in R}$ *in* X_0 *for the nonautonomous dynamical system* (θ, Φ) *defined by* (4.1).

Proof Fix $t_0 \in R$. Let $y_i(t) = (u_i(t), v_i(t), \theta_i(t))$ $(i = 1, 2)$ be the corresponding weak solution to $y_0^i = (u_0^i, v_0^i, \theta_0^i) \in \tilde{D}_{t_0-\tau}$, where $\tau \ge 0$, and let $w(t) = u_1(t) - u_2(t)$, $\tilde{\theta}(t) = \theta_1(t) - \theta_2(t)$. Then $(w, \tilde{\theta})$ satisfy

$$
\begin{aligned}
&w_{tt} + \alpha\triangle^2 w - \beta\triangle w - \left(\sigma\left(\int_\Omega (\nabla u_1)^2\,dx\right)\triangle u_1 - \sigma\left(\int_\Omega (\nabla u_2)^2\,dx\right)\triangle u_2\right) \\
&\qquad + \gamma\triangle\tilde{\theta} + \triangle g + \eta w_t = 0,
\end{aligned}
\tag{5.1}
$$

$$
\tilde{\theta}_t - \triangle\tilde{\theta} - \gamma\triangle w_t = 0 \tag{5.2}
$$

with the initial condition $(w(0), w_t(0), \tilde{\theta}(0)) = (u_0^1, v_0^1, \theta_0^1) - (u_0^2, v_0^2, \theta_0^2)$, where $\triangle g = g(u_1) - g(u_2)$.

Define

$$
E_u(t) = \frac{1}{2}\left(\|u_t\|^2 + \|\triangle u\|^2 + \|\theta\|^2\right) = \frac{1}{2}\left\|\phi(t - t_0 + \tau, t_0 - \tau, y_0)\right\|_{X_0}^2
$$

and

$$
F(t) = \frac{1}{2}\left(\|w_t\|^2 + \alpha\|\triangle w\|^2 + \beta\|\nabla w\|^2 + \sigma\left(\|\nabla u_1\|^2\right)\|\nabla w\|^2 + \|\tilde{\theta}\|^2\right).
$$

First, we have

$$\begin{aligned}
F(t) &\geq \frac{1}{2}\left(\|w_t\|^2 + \alpha\|\triangle w\|^2 + \|\tilde{\theta}\|^2\right) \\
&\geq \frac{1}{2}C^1\left(\|w_t\|^2 + \|\triangle w\|^2 + \|\tilde{\theta}\|^2\right) \\
&= C^1 E_w(t),
\end{aligned} \tag{5.3}$$

where $C^1 = \min\{1, \alpha\}$.

Multiplying (5.1) by $e^{\tilde{\delta}t}w_t$ and (5.2) by $e^{\tilde{\delta}t}\tilde{\theta}$ and then summing, we obtain

$$\begin{aligned}
&\frac{d}{dt}\left[e^{\tilde{\delta}t}F(t)\right] + e^{\tilde{\delta}t}\left(\eta\|w_t\|^2 + \|\nabla\tilde{\theta}\|^2\right) \\
&\quad = \tilde{\delta}e^{\tilde{\delta}t}F(t) \\
&\qquad - e^{\tilde{\delta}t}\left(\int_\Omega \triangle g w_t\,dx - \sigma'\left(\|\nabla u_1\|^2\right)\int_\Omega \nabla u_1\nabla u_{1t}\,dx\|\nabla w\|^2 - \int_\Omega \triangle\sigma\, w_t\,dx\right),
\end{aligned} \tag{5.4}$$

where $\triangle\sigma = \sigma(\|\nabla u_1\|^2) - \sigma(\|\nabla u_2\|^2)$, and $\tilde{\delta}$ satisfies (4.18).

Integrating (5.4) from s to t_0, we have

$$\begin{aligned}
&e^{\tilde{\delta}t_0}F(t_0) - e^{\tilde{\delta}s}F(s) + \int_s^{t_0} e^{\tilde{\delta}\xi}\left(\eta\|w_t\|^2 + \|\nabla\tilde{\theta}\|^2\right)d\xi \\
&\quad = \tilde{\delta}\int_s^{t_0} e^{\tilde{\delta}\xi}F(\xi)\,d\xi - \int_s^{t_0} e^{\tilde{\delta}\xi}\left(\int_\Omega \triangle g w_t\,dx\right)d\xi \\
&\qquad + \int_s^{t_0} e^{\tilde{\delta}\xi}\left(\sigma'\left(\|\nabla u_1\|^2\right)\int_\Omega \nabla u_1\nabla u_{1t}\,dx\|\nabla w\|^2 + \int_\Omega \triangle\sigma\, w_t\,dx\right)d\xi.
\end{aligned} \tag{5.5}$$

Integrating (5.5) from $t_0 - \tau$ to t_0 with respect to s, we obtain

$$\begin{aligned}
&\tau e^{\tilde{\delta}t_0}F(t_0) - \int_{t_0-\tau}^{t_0} e^{\tilde{\delta}s}F(s)\,ds + \int_{t_0-\tau}^{t_0}\int_s^{t_0} e^{\tilde{\delta}\xi}\left(\eta\|w_t\|^2 + \|\nabla\tilde{\theta}\|^2\right)d\xi\,ds \\
&\quad = \tilde{\delta}\int_{t_0-\tau}^{t_0}\int_s^{t_0} e^{\tilde{\delta}\xi}F(\xi)\,d\xi\,ds - \int_{t_0-\tau}^{t_0}\int_s^{t_0} e^{\tilde{\delta}\xi}\left(\int_\Omega \triangle g w_t\,dx\right)d\xi\,ds \\
&\qquad + \int_{t_0-\tau}^{t_0}\int_s^{t_0} e^{\tilde{\delta}\xi}\left(\sigma'\left(\|\nabla u_1\|^2\right)\int_\Omega \nabla u_1\nabla u_{1t}\,dx\|\nabla w\|^2 + \int_\Omega \triangle\sigma\, w_t\,dx\right)d\xi\,ds.
\end{aligned} \tag{5.6}$$

Similarly, multiplying (5.1) by $e^{\tilde{\delta}t}w$, we have

$$\begin{aligned}
&\frac{d}{dt}\left[e^{\tilde{\delta}t}(w_t, w)\right] + e^{\tilde{\delta}t}\left(\alpha\|\triangle w\|^2 + \beta\|\nabla w\|^2 + \sigma\left(\|\nabla u_1\|^2\right)\|\nabla w\|^2\right) \\
&\quad = (\tilde{\delta} - \eta)e^{\tilde{\delta}t}(w_t, w) + e^{\tilde{\delta}t}\|w_t\|^2 \\
&\qquad - e^{\tilde{\delta}t}\left(\int_\Omega \triangle g w\,dx - \gamma\int_\Omega \triangle\tilde{\theta} w\,dx - \int_\Omega \triangle\sigma\triangle u_2 w\,dx\right).
\end{aligned} \tag{5.7}$$

First, integrating (5.7) over $[s, t_0]$, we get that

$$\begin{aligned}
&\int_s^{t_0} e^{\tilde{\delta}\xi}\left(\alpha\|\triangle w\|^2+\beta\|\nabla w\|^2+\sigma\left(\|\nabla u_1\|^2\right)\|\nabla w\|^2\right)d\xi+e^{\tilde{\delta}t_0}\left(w_t(t_0),w(t_0)\right)\\
&\quad=e^{\tilde{\delta}s}\left(w_t(s),w(s)\right)+(\tilde{\delta}-\eta)\int_s^{t_0}e^{\tilde{\delta}\xi}(w_t,w)\,d\xi+\int_s^{t_0}e^{\tilde{\delta}\xi}\|w_t\|^2\,d\xi\\
&\qquad-\int_s^{t_0}\int_\Omega e^{\tilde{\delta}\xi}\triangle gw\,dx\,d\xi-\gamma\int_s^{t_0}\int_\Omega e^{\tilde{\delta}\xi}\triangle\tilde{\theta}w\,dx\,d\xi\\
&\qquad-\int_s^{t_0}\int_\Omega e^{\tilde{\delta}\xi}\triangle\sigma\triangle u_2 w\,dx\,d\xi.
\end{aligned}\tag{5.8}$$

Then integrating (5.8) over $[t_0-\tau, t_0]$ with respect to s, we obtain

$$\begin{aligned}
&\tilde{\delta}\int_{t_0-\tau}^{t_0}\int_s^{t_0} e^{\tilde{\delta}\xi}\left(\alpha\|\triangle w\|^2+\beta\|\nabla w\|^2+\sigma\left(\|\nabla u_1\|^2\right)\|\nabla w\|^2\right)d\xi\,ds\\
&\quad=-\tilde{\delta}\tau e^{\tilde{\delta}t_0}\left(w_t(t_0),w(t_0)\right)+\tilde{\delta}\int_{t_0-\tau}^{t_0}e^{\tilde{\delta}s}\left(w_t(s),w(s)\right)ds\\
&\qquad+\tilde{\delta}(\tilde{\delta}-\eta)\int_{t_0-\tau}^{t_0}\int_s^{t_0}e^{\tilde{\delta}\xi}(w_t,w)\,d\xi\,ds+\tilde{\delta}\int_{t_0-\tau}^{t_0}\int_s^{t_0}e^{\tilde{\delta}\xi}\|w_t\|^2\,d\xi\,ds\\
&\qquad-\tilde{\delta}\int_{t_0-\tau}^{t_0}\int_s^{t_0}\int_\Omega e^{\tilde{\delta}\xi}\triangle gw\,dx\,d\xi\,ds-\gamma\tilde{\delta}\int_{t_0-\tau}^{t_0}\int_s^{t_0}\int_\Omega e^{\tilde{\delta}\xi}\triangle\tilde{\theta}w\,dx\,d\xi\,ds\\
&\qquad-\tilde{\delta}\int_{t_0-\tau}^{t_0}\int_s^{t_0}\int_\Omega e^{\tilde{\delta}\xi}\triangle\sigma\triangle u_2 w\,dx\,d\xi\,ds.
\end{aligned}\tag{5.9}$$

Substituting (5.9) into (5.6), we deduce that

$$\begin{aligned}
&\tau e^{\tilde{\delta}t_0}F(t_0)-\int_{t_0-\tau}^{t_0}e^{\tilde{\delta}s}F(s)\,ds+\int_{t_0-\tau}^{t_0}\int_s^{t_0}e^{\tilde{\delta}\xi}\left(\eta\|w_t\|^2+\|\nabla\tilde{\theta}\|^2\right)d\xi\,ds\\
&\quad=\tilde{\delta}\int_{t_0-\tau}^{t_0}\int_s^{t_0}e^{\tilde{\delta}\xi}\frac{1}{2}\left(\|w_t\|^2+\alpha\|\triangle w\|^2+\beta\|\nabla w\|^2+\sigma\left(\|\nabla u_1\|^2\right)\|\nabla w\|^2+\|\tilde{\theta}\|^2\right)d\xi\,ds\\
&\qquad-\int_{t_0-\tau}^{t_0}\int_s^{t_0}e^{\tilde{\delta}\xi}\left(\int_\Omega\triangle gw_t\,dx-\sigma'\left(\|\nabla u_1\|^2\right)\int_\Omega\nabla u_1\nabla u_{1t}\,dx\|\nabla w\|^2\right.\\
&\qquad\left.-\int_\Omega\triangle\sigma w_t\,dx\right)d\xi\,ds\\
&\quad=\tilde{\delta}\int_{t_0-\tau}^{t_0}\int_s^{t_0}e^{\tilde{\delta}\xi}\frac{1}{2}\left(\|w_t\|^2+\|\tilde{\theta}\|^2\right)d\xi\,ds-\frac{1}{2}\tilde{\delta}\tau e^{\tilde{\delta}t_0}\left(w_t(t_0),w(t_0)\right)\\
&\qquad+\frac{1}{2}\tilde{\delta}\int_{t_0-\tau}^{t_0}e^{\tilde{\delta}s}\left(w_t(s),w(s)\right)ds+\frac{1}{2}\tilde{\delta}(\tilde{\delta}-\eta)\int_{t_0-\tau}^{t_0}\int_s^{t_0}e^{\tilde{\delta}\xi}(w_t,w)\,d\xi\,ds\\
&\qquad+\frac{1}{2}\tilde{\delta}\int_{t_0-\tau}^{t_0}\int_s^{t_0}e^{\tilde{\delta}\xi}\|w_t\|^2\,d\xi\,ds-\frac{1}{2}\tilde{\delta}\int_{t_0-\tau}^{t_0}\int_s^{t_0}\int_\Omega e^{\tilde{\delta}\xi}\triangle gw\,dx\,d\xi\,ds\\
&\qquad-\frac{1}{2}\gamma\tilde{\delta}\int_{t_0-\tau}^{t_0}\int_s^{t_0}\int_\Omega e^{\tilde{\delta}\xi}\triangle\tilde{\theta}w\,dx\,d\xi\,ds-\frac{1}{2}\tilde{\delta}\int_{t_0-\tau}^{t_0}\int_s^{t_0}\int_\Omega e^{\tilde{\delta}\xi}\triangle\sigma\triangle u_2 w\,dx\,d\xi\,ds\\
&\qquad-\int_{t_0-\tau}^{t_0}\int_s^{t_0}e^{\tilde{\delta}\xi}\left(\int_\Omega\triangle gw_t\,dx-\sigma'\left(\|\nabla u_1\|^2\right)\int_\Omega\nabla u_1\nabla u_{1t}\,dx\|\nabla w\|^2\right.\\
&\qquad\left.-\int_\Omega\triangle\sigma w_t\,dx\right)d\xi\,ds.
\end{aligned}\tag{5.10}$$

Second, integrating (5.7) from $t_0 - \tau$ to t_0, we have

$$\begin{aligned}
&\int_{t_0-\tau}^{t_0} e^{\tilde{\delta}\xi}\left(\|w_t\|^2 + \alpha\|\triangle w\|^2 + \beta\|\nabla w\|^2 + \sigma\left(\|\nabla u_1\|^2\right)\|\nabla w\|^2 + \|\tilde{\theta}\|^2\right) d\xi \\
&\quad = e^{\tilde{\delta}(t_0-\tau)}\big(w_t(t_0-\tau), w(t_0-\tau)\big) - e^{\tilde{\delta}t_0}\big(w_t(t_0), w(t_0)\big) \\
&\qquad + (\tilde{\delta}-\eta)\int_{t_0-\tau}^{t_0} e^{\tilde{\delta}\xi}(w_t, w)\, d\xi + \int_{t_0-\tau}^{t_0} e^{\tilde{\delta}\xi}\|w_t\|^2\, d\xi \\
&\qquad - \int_{t_0-\tau}^{t_0}\int_\Omega e^{\tilde{\delta}\xi}\triangle g w\, dx\, d\xi - \gamma\int_{t_0-\tau}^{t_0}\int_\Omega e^{\tilde{\delta}\xi}\triangle\tilde{\theta} w\, dx\, d\xi \\
&\qquad - \int_{t_0-\tau}^{t_0}\int_\Omega e^{\tilde{\delta}\xi}\triangle\sigma\triangle u_2 w\, dx\, d\xi + \int_{t_0-\tau}^{t_0} e^{\tilde{\delta}\xi}\left(\|w_t\|^2 + \|\tilde{\theta}\|^2\right) d\xi.
\end{aligned} \tag{5.11}$$

Substituting (5.11) into (5.10) and noting that $\tilde{\delta} < \frac{\eta}{2}$, we have

$$\begin{aligned}
&\tau e^{\tilde{\delta}t_0}F(t_0) + \int_{t_0-\tau}^{t_0} e^{\tilde{\delta}s}F(s)\, ds + \int_{t_0-\tau}^{t_0}\int_s^{t_0} e^{\tilde{\delta}\xi}\|\nabla\tilde{\theta}\|^2\, d\xi\, ds \\
&\quad \le \frac{\tilde{\delta}}{2}\int_{t_0-\tau}^{t_0}\int_s^{t_0} e^{\tilde{\delta}\xi}\|\tilde{\theta}\|^2\, d\xi\, ds - \left(\frac{1}{2}\tilde{\delta}\tau + 1\right)e^{\tilde{\delta}t_0}\big(w_t(t_0), w(t_0)\big) \\
&\qquad + \frac{1}{2}\tilde{\delta}\int_{t_0-\tau}^{t_0} e^{\tilde{\delta}s}\big(w_t(s), w(s)\big)\, ds + \frac{1}{2}\tilde{\delta}(\tilde{\delta}-\eta)\int_{t_0-\tau}^{t_0}\int_s^{t_0} e^{\tilde{\delta}\xi}(w_t, w)\, d\xi\, ds \\
&\qquad - \frac{1}{2}\tilde{\delta}\int_{t_0-\tau}^{t_0}\int_s^{t_0}\int_\Omega e^{\tilde{\delta}\xi}\triangle g w\, dx\, d\xi\, ds - \frac{1}{2}\gamma\tilde{\delta}\int_{t_0-\tau}^{t_0}\int_s^{t_0}\int_\Omega e^{\tilde{\delta}\xi}\triangle\tilde{\theta} w\, dx\, d\xi\, ds \\
&\qquad - \frac{1}{2}\tilde{\delta}\int_{t_0-\tau}^{t_0}\int_s^{t_0}\int_\Omega e^{\tilde{\delta}\xi}\triangle\sigma\triangle u_2 w\, dx\, d\xi\, ds + \int_{t_0-\tau}^{t_0}\int_s^{t_0} e^{\tilde{\delta}\xi}\left(\int_\Omega \triangle g w_t\, dx\right) d\xi\, ds \\
&\qquad - \int_{t_0-\tau}^{t_0}\int_s^{t_0} e^{\tilde{\delta}\xi}\left(\sigma'\left(\|\nabla u_1\|^2\right)\int_\Omega \nabla u_1\nabla u_{1t}\, dx\|\nabla w\|^2 + \int_\Omega \triangle\sigma w_t\, dx\right) d\xi\, ds \\
&\qquad + e^{\tilde{\delta}(t_0-\tau)}\big(w_t(t_0-\tau), w(t_0-\tau)\big) + (\tilde{\delta}-\eta)\int_{t_0-\tau}^{t_0} e^{\tilde{\delta}\xi}(w_t, w)\, d\xi + \int_{t_0-\tau}^{t_0} e^{\tilde{\delta}\xi}\|w_t\|^2\, d\xi \\
&\qquad - \int_{t_0-\tau}^{t_0}\int_\Omega e^{\tilde{\delta}\xi}\triangle g w\, dx\, d\xi - \gamma\int_{t_0-\tau}^{t_0}\int_\Omega e^{\tilde{\delta}\xi}\triangle\tilde{\theta} w\, dx\, d\xi \\
&\qquad - \int_{t_0-\tau}^{t_0}\int_\Omega e^{\tilde{\delta}\xi}\triangle\sigma\triangle u_2 w\, dx\, d\xi + \int_{t_0-\tau}^{t_0} e^{\tilde{\delta}\xi}\left(\|w_t\|^2 + \|\tilde{\theta}\|^2\right) d\xi.
\end{aligned} \tag{5.12}$$

(I) Using the Schwarz and Young inequalities, for $t_0 - \tau \le s < t_0$, we have

$$\frac{1}{2}\tilde{\delta}\int_{t_0-\tau}^{t_0} e^{\tilde{\delta}s}\big(w_t(s), w(s)\big)\, ds \le \frac{\tilde{\delta}^2}{8\varepsilon'}\int_{t_0-\tau}^{t_0} e^{\tilde{\delta}s}\big\|w(s)\big\|^2\, ds + \varepsilon'\int_{t_0-\tau}^{t_0} e^{\tilde{\delta}s}\big\|w_t(s)\big\|^2\, ds, \tag{5.13}$$

$$\begin{aligned}
&\frac{1}{2}\tilde{\delta}(\tilde{\delta}-\eta)\int_{t_0-\tau}^{t_0}\int_s^{t_0} e^{\tilde{\delta}\xi}(w_t, w)\, d\xi\, ds \\
&\quad \le \frac{1}{2}\tau\tilde{\delta}(\tilde{\delta}-\eta)\int_{t_0-\tau}^{t_0} e^{\tilde{\delta}\xi}\|w_t\|\|w\|\, d\xi \\
&\quad \le \frac{[\frac{1}{2}\tau\tilde{\delta}(\tilde{\delta}-\eta)]^2}{4\varepsilon'}\int_{t_0-\tau}^{t_0} e^{\tilde{\delta}\xi}\|w\|^2\, d\xi + \varepsilon'\int_{t_0-\tau}^{t_0} e^{\tilde{\delta}\xi}\|w_t\|^2\, d\xi,
\end{aligned} \tag{5.14}$$

$$
\begin{aligned}
&-\frac{1}{2}\gamma\tilde{\delta}\int_{t_0-\tau}^{t_0}\int_s^{t_0}\int_\Omega e^{\tilde{\delta}\xi}\triangle\tilde{\theta}w\,dx\,d\xi\,ds\\
&\quad\le\frac{\varepsilon'}{\lambda_0^2}\int_{t_0-\tau}^{t_0}e^{\tilde{\delta}\xi}\|\nabla\tilde{\theta}\|^2\,d\xi+\frac{(\tau\frac{1}{2}\gamma\tilde{\delta})^2}{4\varepsilon'}\int_{t_0-\tau}^{t_0}e^{\tilde{\delta}\xi}\|\nabla w\|^2\,d\xi,
\end{aligned}
\tag{5.15}
$$

$$
(\tilde{\delta}-\eta)\int_{t_0-\tau}^{t_0}e^{\tilde{\delta}\xi}(w_t,w)\,d\xi\le\frac{(\tilde{\delta}-\eta)^2}{8\varepsilon'}\int_{t_0-\tau}^{t_0}e^{\tilde{\sigma}\xi}\|w\|^2\,d\xi+2\varepsilon'\int_{t_0-\tau}^{t_0}e^{\tilde{\sigma}\xi}\|w_t\|^2\,d\xi,
\tag{5.16}
$$

and

$$
-\gamma\int_{t_0-\tau}^{t_0}\int_\Omega e^{\tilde{\delta}\xi}\triangle\tilde{\theta}w\,dx\,d\xi\le\frac{\varepsilon'}{\lambda_0^2}\int_{t_0-\tau}^{t_0}e^{\tilde{\delta}\xi}\|\nabla\tilde{\theta}\|^2\,d\xi+\frac{\lambda_0^2\gamma^2}{4\varepsilon'}\int_{t_0-\tau}^{t_0}e^{\tilde{\delta}\xi}\|\nabla w\|^2\,d\xi,
\tag{5.17}
$$

where $0<\varepsilon'<\min\{\frac{\sqrt{(9+\lambda_0^2)^2+40\eta\lambda_0^2}-(9+\lambda_0^2)}{20},\eta\}$.

(II) By assumption (1.6) on $g(\cdot)$, the Hölder inequality, and the embedding theorem combined with (4.19), for $t_0-\tau\le s<t_0$, we have

$$
\begin{aligned}
&\frac{1}{2}\tilde{\delta}\int_{t_0-\tau}^{t_0}\int_s^{t_0}\int_\Omega e^{\tilde{\delta}\xi}\triangle gw\,dx\,d\xi\,ds\\
&\quad\le\frac{1}{2}\tilde{\delta}\tau\left(\int_{t_0-\tau}^{t_0}e^{\tilde{\delta}\xi}\int_\Omega\big|g(u_2)-g(u_1)\big|^2\,dx\,d\xi\right)^{\frac{1}{2}}\left(\int_{t_0-\tau}^{t_0}e^{\tilde{\delta}\xi}\|w\|^2\,d\xi\right)^{\frac{1}{2}}\\
&\quad\le\frac{1}{2}\tilde{\delta}C\tau\left(\int_{t_0-\tau}^{t_0}e^{\tilde{\delta}\xi}\int_\Omega\big(1+|u_1|^{2\rho-2}+|u_2|^{2\rho-2}\big)|w|^2\,dx\,d\xi\right)^{\frac{1}{2}}\\
&\qquad\times\left(\int_{t_0-\tau}^{t_0}e^{\tilde{\delta}\xi}\|w\|^2\,d\xi\right)^{\frac{1}{2}}\\
&\quad\le\frac{1}{2}\tilde{\delta}C\tau\left(\int_{t_0-\tau}^{t_0}e^{\tilde{\delta}\xi}\int_\Omega\big(|u_1|^2+|u_2|^2+|u_1|^{2\rho}+|u_2|^{2\rho}\big)\,dx\,d\xi\right)^{\frac{1}{2}}\\
&\qquad\times\left(\int_{t_0-\tau}^{t_0}e^{\tilde{\delta}\xi}\|w\|^2\,d\xi\right)^{\frac{1}{2}}\\
&\quad\le\frac{1}{2}\tilde{\delta}C\tau\left(\int_{t_0-\tau}^{t_0}e^{\tilde{\delta}\xi}\big(\|\nabla u_1\|^2+\|\nabla u_2\|^2+\|\nabla u_1\|^{2\rho}+\|\nabla u_2\|^{2\rho}\big)\,d\xi\right)^{\frac{1}{2}}\\
&\qquad\times\left(\int_{t_0-\tau}^{t_0}e^{\tilde{\delta}\xi}\|w\|^2\,d\xi\right)^{\frac{1}{2}}\\
&\quad\le C_{t_0,\tau}\left(\int_{t_0-\tau}^{t_0}e^{\tilde{\delta}\xi}\|w\|^2\,d\xi\right)^{\frac{1}{2}},
\end{aligned}
\tag{5.18}
$$

and similarly we also have

$$
-\int_{t_0-\tau}^{t_0}\int_\Omega e^{\tilde{\delta}\xi}\triangle gw\,dx\,d\xi\le C_{t_0,\tau}\left(\int_{t_0-\tau}^{t_0}e^{\tilde{\sigma}\xi}\|w\|^2\,d\xi\right)^{\frac{1}{2}}.
\tag{5.19}
$$

(III) By the value theorem, (4.18), the continuity of $\sigma(\cdot)$, and the Schwarz inequality, for $t_0-\tau \le s < t_0$, we obtain that

$$\begin{aligned}
&-\frac{1}{2}\tilde{\delta}\int_{t_0-\tau}^{t_0}\int_s^{t_0}\int_\Omega e^{\tilde{\delta}\xi}\triangle\sigma\triangle u_2 w\,dx\,d\xi\,ds \\
&\le \frac{1}{2}\tilde{\delta}\tau\int_{t_0-\tau}^{t_0} e^{\tilde{\delta}\xi}\sigma'(\xi_1)\big(\|\triangle u_1\|^2-\|\triangle u_2\|^2\big)\|\triangle u_2\|\|w\|\,d\xi \\
&\le C_{t_0,\tau}\int_{t_0-\tau}^{t_0} e^{\tilde{\delta}\xi}\|w\|^2\,d\xi, \qquad (5.20)
\end{aligned}$$

where ξ_1 is between $\|\nabla u_1\|^2$ and $\|\nabla u_2\|^2$. Similarly, we have

$$\int_{t_0-\tau}^{t_0}\int_\Omega e^{\tilde{\delta}\xi}\triangle\sigma\triangle u_2 w\,dx\,d\xi \le C_{t_0,\tau}\int_{t_0-\tau}^{t_0} e^{\tilde{\delta}\xi}\|w\|^2\,d\xi, \qquad (5.21)$$

$$\begin{aligned}
&-\int_{t_0-\tau}^{t_0}\int_s^{t_0} e^{\tilde{\delta}\xi}\sigma'\big(\|\nabla u_1\|^2\big)\int_\Omega \nabla u_1\nabla u_{1t}\,dx\|\nabla w\|^2\,d\xi\,ds \\
&\le C\tau C_{t_0,\tau}\int_{t_0-\tau}^{t_0} e^{\tilde{\delta}\xi}\|\nabla w\|^2\,d\xi, \qquad (5.22)
\end{aligned}$$

and

$$\begin{aligned}
&-\int_{t_0-\tau}^{t_0}\int_s^{t_0} e^{\tilde{\delta}\xi}\int_\Omega \triangle\sigma w_t\,dx\,d\xi\,ds \\
&\le \tau C_{t_0,\tau}\int_{t_0-\tau}^{t_0} e^{\tilde{\delta}\xi}\|w\|\|w_t\|\,d\xi \\
&\le \frac{(\tau C_{t_0,\tau})^2}{4\varepsilon'}\int_{t_0-\tau}^{t_0} e^{\tilde{\delta}\xi}\|w\|^2\,d\xi + \varepsilon'\int_{t_0-\tau}^{t_0} e^{\tilde{\delta}\xi}\|w_t\|^2\,d\xi. \qquad (5.23)
\end{aligned}$$

(IV) Since $\tilde{\delta} \le 2\lambda_0^2$, we have

$$\begin{aligned}
\int_{t_0-\tau}^{t_0}\int_s^{t_0} e^{\tilde{\delta}\xi}\|\nabla\tilde{\theta}\|^2\,d\xi\,ds &\ge \lambda_0^2\int_{t_0-\tau}^{t_0}\int_s^{t_0} e^{\tilde{\delta}\xi}\|\tilde{\theta}\|^2\,d\xi \\
&\ge \frac{\tilde{\delta}}{2}\int_{t_0-\tau}^{t_0}\int_s^{t_0} e^{\tilde{\delta}\xi}\|\tilde{\theta}\|^2\,d\xi\,ds. \qquad (5.24)
\end{aligned}$$

So, substituting (5.13)-(5.23) into (5.12), by (5.24) we obtain

$$\begin{aligned}
&\tau e^{\tilde{\delta}t_0}F(t_0)+\int_{t_0-\tau}^{t_0} e^{\tilde{\delta}s}F(s)\,ds \\
&\le \int_{t_0-\tau}^{t_0} e^{\tilde{\delta}\xi}\|\tilde{\theta}\|^2\,d\xi - \left(\frac{1}{2}\tilde{\delta}\tau+1\right)e^{\tilde{\delta}t_0}\big(w_t(t_0),w(t_0)\big) \\
&\quad + \left(\frac{\tilde{\delta}^2}{8\varepsilon'}+\frac{[\frac{1}{2}\tau\tilde{\delta}(\tilde{\delta}-\eta)]^2}{4\varepsilon'}+\frac{(\tau C_{t_0,\tau})^2}{4\varepsilon'}+\frac{(\tilde{\delta}-\eta)^2}{8\varepsilon'}+2C_{t_0,\tau}\right)\int_{t_0-\tau}^{t_0} e^{\tilde{\delta}\xi}\|w\|^2\,d\xi \\
&\quad + \big(2+5\varepsilon'\big)\int_{t_0-\tau}^{t_0} e^{\tilde{\delta}\xi}\|w_t\|^2\,d\xi + 2C_{t_0,\tau}\left(\int_{t_0-\tau}^{t_0} e^{\tilde{\delta}\xi}\|w\|^2\,d\xi\right)^{\frac{1}{2}}
\end{aligned}$$

$$
\begin{aligned}
&+ \frac{2\varepsilon'}{\lambda_0^2} \int_{t_0-\tau}^{t_0} e^{\tilde{\delta}\xi} \|\nabla\tilde{\theta}\|^2 \, d\xi \\
&+ \left(\frac{(\tau\frac{1}{2}\gamma\tilde{\delta})^2}{4\varepsilon'} + \frac{\lambda_0^2\gamma^2}{4\varepsilon'} + C\tau C_{t_0,\tau} + \tau C_{t_0,\tau} \right) \int_{t_0-\tau}^{t_0} e^{\tilde{\delta}\xi} \|\nabla w\|^2 \, d\xi \\
&+ \int_{t_0-\tau}^{t_0} \int_s^{t_0} e^{\tilde{\delta}\xi} \int_\Omega \triangle g w_t \, dx \, d\xi \, ds + e^{\tilde{\delta}(t_0-\tau)} \big(w_t(t_0-\tau), w(t_0-\tau) \big).
\end{aligned} \tag{5.25}
$$

On the other hand, integrating (5.4) over $[t_0 - \tau, t_0]$, we get that

$$
\begin{aligned}
&e^{\tilde{\delta}t_0} F(t_0) - e^{\tilde{\delta}(t_0-\tau)} F(t_0-\tau) + \int_{t_0-\tau}^{t_0} e^{\tilde{\delta}\xi} \big(\eta\|w_t\|^2 + \|\nabla\tilde{\theta}\|^2 \big) \, d\xi \\
&= \tilde{\delta} \int_{t_0-\tau}^{t_0} e^{\tilde{\delta}\xi} F(\xi) \, d\xi - \int_{t_0-\tau}^{t_0} e^{\tilde{\delta}\xi} \int_\Omega \triangle g w_t \, dx \, d\xi \\
&\quad - \int_{t_0-\tau}^{t_0} e^{\tilde{\delta}\xi} \left(\sigma'\big(\|\nabla u\|^2\big) \int_\Omega \nabla u_1 \nabla u_{1t} \, dx \|\nabla w\|^2 - \int_\Omega \triangle\sigma w_t \, dx \right) d\xi.
\end{aligned} \tag{5.26}
$$

By the continuity of $\sigma'(\cdot)$ combined with (4.19) we get

$$
- \int_{t_0-\tau}^{t_0} e^{\tilde{\delta}\xi} \sigma'\big(\|\nabla u_1\|^2\big) \int_\Omega \nabla u_1 \nabla u_{1t} \, dx \|\nabla w\|^2 \, d\xi \le C_{t_0,\tau} \int_{t_0-\tau}^{t_0} e^{\tilde{\delta}\xi} \|\nabla w\|^2 \, d\xi. \tag{5.27}
$$

By the value theorem and (4.19) we have

$$
\begin{aligned}
- \int_{t_0-\tau}^{t_0} e^{\tilde{\delta}\xi} \int_\Omega \triangle\sigma w_t \, dx \, d\xi &\le C_{t_0,\tau} \int_{t_0-\tau}^{t_0} e^{\tilde{\delta}\xi} \|w\| \|w_t\| \, d\xi \\
&\le \frac{C_{t_0,\tau}^2}{4\varepsilon'} \int_{t_0-\tau}^{t_0} e^{\tilde{\delta}\xi} \|w\|^2 \, d\xi + \varepsilon' \int_{t_0-\tau}^{t_0} e^{\tilde{\delta}\xi} \|w_t\|^2 \, d\xi.
\end{aligned} \tag{5.28}
$$

From (5.26) combined with (5.27)-(5.28) we have

$$
\begin{aligned}
&\big(2 + 5\varepsilon'\big) \int_{t_0-\tau}^{t_0} e^{\tilde{\delta}\xi} \|w_t\|^2 \, d\xi \\
&\le \frac{(2+5\varepsilon')}{\eta-\varepsilon'} e^{\tilde{\delta}(t_0-\tau)} F(t_0-\tau) + \frac{(2+5\varepsilon')\tilde{\delta}}{\eta-\varepsilon'} \int_{t_0-\tau}^{t_0} e^{\tilde{\delta}\xi} F(\xi) \, d\xi \\
&\quad - \frac{(2+5\varepsilon')}{\eta-\varepsilon'} \int_{t_0-\tau}^{t_0} e^{\tilde{\delta}\xi} \int_\Omega \triangle g w_t \, dx \, d\xi + \frac{(2+5\varepsilon')}{\eta-\varepsilon'} C_{t_0,\tau} \int_{t_0-\tau}^{t_0} e^{\tilde{\delta}\xi} \|\nabla w\|^2 \, d\xi \\
&\quad + \frac{(2+5\varepsilon')}{\eta-\varepsilon'} \frac{C_{t_0,\tau}^2}{4\varepsilon'} \int_{t_0-\tau}^{t_0} e^{\tilde{\delta}\xi} \|w\|^2 \, d\xi - \frac{(2+5\varepsilon')}{\eta-\varepsilon'} \int_{t_0-\tau}^{t_0} e^{\tilde{\delta}\xi} \|\nabla\tilde{\theta}\|^2 \, d\xi.
\end{aligned} \tag{5.29}
$$

Substituting (5.29) into (5.25), we obtain

$$
\begin{aligned}
&\tau e^{\tilde{\delta}t_0} F(t_0) + \int_{t_0-\tau}^{t_0} e^{\tilde{\delta}s} F(s) \, ds \\
&\le \int_{t_0-\tau}^{t_0} e^{\tilde{\delta}\xi} \|\tilde{\theta}\|^2 \, d\xi - \left(\frac{1}{2}\tilde{\delta}\tau + 1 \right) e^{\tilde{\delta}t_0} \big(w_t(t_0), w(t_0) \big) + C_{t_0,\tau}^2 \int_{t_0-\tau}^{t_0} e^{\tilde{\delta}\xi} \|w\|^2 \, d\xi \\
&\quad + \frac{(2+5\varepsilon')}{\eta-\varepsilon'} e^{\tilde{\delta}(t_0-\tau)} F(t_0-\tau) + \frac{(2+5\varepsilon')\tilde{\delta}}{\eta-\varepsilon'} \int_{t_0-\tau}^{t_0} e^{\tilde{\delta}\xi} F(\xi) \, d\xi
\end{aligned}
$$

$$\begin{aligned}
&-\frac{(2+5\varepsilon')}{\eta-\varepsilon'}\int_{t_0-\tau}^{t_0}e^{\tilde{\delta}\xi}\int_\Omega \triangle g w_t\,dx\,d\xi-\frac{(2+5\varepsilon')}{\eta-\varepsilon'}\int_{t_0-\tau}^{t_0}e^{\tilde{\delta}\xi}\|\nabla\tilde{\theta}\|^2\,d\xi\\
&+2C_{t_0,\tau}\left(\int_{t_0-\tau}^{t_0}e^{\tilde{\delta}\xi}\|w\|^2\,d\xi\right)^{\frac{1}{2}}+\frac{2\varepsilon'}{\lambda_0^2}\int_{t_0-\tau}^{t_0}e^{\tilde{\delta}\xi}\|\nabla\tilde{\theta}\|^2\,d\xi+C_{t_0,\tau}^3\int_{t_0-\tau}^{t_0}e^{\tilde{\delta}\xi}\|\nabla w\|^2\,d\xi\\
&+\int_{t_0-\tau}^{t_0}\int_s^{t_0}e^{\tilde{\delta}\xi}\int_\Omega\triangle g w_t\,dx\,d\xi\,ds+e^{\tilde{\delta}(t_0-\tau)}\big(w_t(t_0-\tau),w(t_0-\tau)\big),
\end{aligned}\tag{5.30}$$

where $C_{t_0,\tau}^2=\frac{\tilde{\delta}^2}{8\varepsilon'}+\frac{[\frac{1}{2}\tau\tilde{\delta}(\tilde{\delta}-\eta)]^2}{4\varepsilon'}+\frac{(\tau C_{t_0,\tau})^2}{4\varepsilon'}+\frac{(\tilde{\delta}-\eta)^2}{8\varepsilon'}+2C_{t_0,\tau}+\frac{(2+5\varepsilon')}{\eta-\varepsilon'}\frac{C_{t_0,\tau}^2}{4\varepsilon'}$ and $C_{t_0,\tau}^3=\frac{(\tau\frac{1}{2}\gamma\tilde{\delta})^2}{4\varepsilon'}+\frac{\lambda_0^2\gamma^2}{4\varepsilon'}+C\tau C_{t_0,\tau}+\tau C_{t_0,\tau}+\frac{(2+5\varepsilon')}{\eta-\varepsilon'}C_{t_0,\tau}$.

Since $0<\varepsilon'<\min\{\frac{\sqrt{(9+\lambda_0^2)^2+40\eta\lambda_0^2}-(9+\lambda_0^2)}{20},\eta\}$, we have

$$-\frac{(2+5\varepsilon')}{\eta-\varepsilon'}\int_{t_0-\tau}^{t_0}e^{\tilde{\delta}\xi}\|\nabla\tilde{\theta}\|^2\,d\xi+\frac{2\varepsilon'}{\lambda_0^2}\int_{t_0-\tau}^{t_0}e^{\tilde{\delta}\xi}\|\nabla\tilde{\theta}\|^2\,d\xi+\int_{t_0-\tau}^{t_0}e^{\tilde{\delta}\xi}\|\tilde{\theta}\|^2\,d\xi\leq 0.$$

So from (5.30) we obtain

$$\begin{aligned}
&\tau e^{\tilde{\delta}t_0}F(t_0)+\int_{t_0-\tau}^{t_0}e^{\tilde{\delta}s}F(s)\,ds\\
&\quad\leq-\left(\frac{1}{2}\tilde{\delta}\tau+1\right)e^{\tilde{\delta}t_0}\big(w_t(t_0),w(t_0)\big)+C_{t_0,\tau}^2\int_{t_0-\tau}^{t_0}e^{\tilde{\delta}\xi}\|w\|^2\,d\xi\\
&\quad\quad+\frac{(2+5\varepsilon')}{\eta-\varepsilon'}e^{\tilde{\delta}(t_0-\tau)}F(t_0-\tau)+\frac{(2+5\varepsilon')\tilde{\delta}}{\eta-\varepsilon'}\int_{t_0-\tau}^{t_0}e^{\tilde{\delta}\xi}F(\xi)\,d\xi\\
&\quad\quad-\frac{2(2+5\varepsilon')}{\eta-\varepsilon'}\int_{t_0-\tau}^{t_0}e^{\tilde{\delta}\xi}\int_\Omega\triangle g w_t\,dx\,d\xi\\
&\quad\quad+2C_{t_0,\tau}\left(\int_{t_0-\tau}^{t_0}e^{\tilde{\delta}\xi}\|w\|^2\,d\xi\right)^{\frac{1}{2}}+C_{t_0,\tau}^3\int_{t_0-\tau}^{t_0}e^{\tilde{\delta}\xi}\|\nabla w\|^2\,d\xi\\
&\quad\quad+\int_{t_0-\tau}^{t_0}\int_s^{t_0}e^{\tilde{\delta}\xi}\int_\Omega\triangle g w_t\,dx\,d\xi\,ds+e^{\tilde{\delta}(t_0-\tau)}\big(w_t(t_0-\tau),w(t_0-\tau)\big).
\end{aligned}\tag{5.31}$$

Since $\tilde{\delta}\leq\frac{\eta-\varepsilon'}{2+5\varepsilon'}$, we have

$$\begin{aligned}
\tau e^{\tilde{\delta}t_0}F(t_0)\leq{}&-\left(\frac{1}{2}\tilde{\delta}\tau+1\right)e^{\tilde{\delta}t_0}\big(w_t(t_0),w(t_0)\big)+C_{t_0,\tau}^2\int_{t_0-\tau}^{t_0}e^{\tilde{\delta}\xi}\|w\|^2\,d\xi\\
&+\frac{(2+5\varepsilon')}{\eta-\varepsilon'}e^{\tilde{\delta}(t_0-\tau)}F(t_0-\tau)-\frac{(2+5\varepsilon')}{\eta-\varepsilon'}\int_{t_0-\tau}^{t_0}e^{\tilde{\delta}\xi}\int_\Omega\triangle g w_t\,dx\,d\xi\\
&+2C_{t_0,\tau}\left(\int_{t_0-\tau}^{t_0}e^{\tilde{\delta}\xi}\|w\|^2\,d\xi\right)^{\frac{1}{2}}+C_{t_0,\tau}^3\int_{t_0-\tau}^{t_0}e^{\tilde{\delta}\xi}\|\nabla w\|^2\,d\xi\\
&+\int_{t_0-\tau}^{t_0}\int_s^{t_0}e^{\tilde{\delta}\xi}\int_\Omega\triangle g w_t\,dx\,d\xi\,ds+e^{\tilde{\delta}(t_0-\tau)}\big(w_t(t_0-\tau),w(t_0-\tau)\big).
\end{aligned}\tag{5.32}$$

By (5.3) we have

$$\begin{aligned}
E_w(t_0) &\le -\frac{1}{C^1}\left(\frac{1}{2}\tilde{\delta}+\frac{1}{\tau}\right)\big(w_t(t_0),w(t_0)\big)+\frac{1}{C^1\tau}C_{t_0,\tau}^2 e^{-\tilde{\delta}t_0}\int_{t_0-\tau}^{t_0} e^{\tilde{\delta}\xi}\|w\|^2\,d\xi \\
&\quad+\frac{(2+5\varepsilon')}{C^1(\eta-\varepsilon')\tau}e^{\tilde{\delta}(-\tau)}F(t_0-\tau)-\frac{(2+5\varepsilon')}{C^1(\eta-\varepsilon')\tau}e^{-\tilde{\delta}t_0}\int_{t_0-\tau}^{t_0} e^{\tilde{\delta}\xi}\int_\Omega \triangle g w_t\,dx\,d\xi \\
&\quad+2C_{t_0,\tau}\left(\int_{t_0-\tau}^{t_0} e^{\tilde{\delta}\xi}\|w\|^2\,d\xi\right)^{\frac{1}{2}}+\frac{1}{C^1\tau}C_{t_0,\tau}^3 e^{-\tilde{\delta}t_0}\int_{t_0-\tau}^{t_0} e^{\tilde{\delta}\xi}\|\nabla w\|^2\,d\xi \\
&\quad+\frac{1}{C^1\tau}e^{-\tilde{\delta}t_0}\int_{t_0-\tau}^{t_0}\int_s^{t_0} e^{\tilde{\delta}\xi}\int_\Omega \triangle g w_t\,dx\,d\xi\,ds+\frac{1}{C^1\tau}e^{-\tilde{\delta}\tau}\big(w_t(t_0-\tau),w(t_0-\tau)\big).
\end{aligned} \tag{5.33}$$

We set

$$\begin{aligned}
&\phi_{t_0,\tau}\big((u_0^1,v_0^1,\theta_0^1),(u_0^2,v_0^2,\theta_0^2)\big) \\
&\quad=-\frac{1}{C^1}\left(\frac{1}{2}\tilde{\delta}+\frac{1}{\tau}\right)\big(w_t(t_0),w(t_0)\big)+\frac{1}{C^1\tau}e^{-\tilde{\delta}t_0}C_{t_0,\tau}^2\int_{t_0-\tau}^{t_0} e^{\tilde{\delta}\xi}\|w\|^2\,d\xi \\
&\quad\quad-\frac{(2+5\varepsilon')}{C^1\lambda^2\tau}e^{-\tilde{\delta}t_0}\int_{t_0-\tau}^{t_0} e^{\tilde{\delta}\xi}\int_\Omega \triangle g w_t\,dx\,d\xi+2C_{t_0,\tau}\left(\int_{t_0-\tau}^{t_0} e^{\tilde{\delta}\xi}\|w\|^2\,d\xi\right)^{\frac{1}{2}} \\
&\quad\quad+\frac{1}{C^1\tau}C_{t_0,\tau}^3 e^{-\tilde{\delta}t_0}\int_{t_0-\tau}^{t_0} e^{\tilde{\delta}\xi}\|\nabla w\|^2\,d\xi+\frac{1}{C^1\tau}e^{-\tilde{\delta}t_0}\int_{t_0-\tau}^{t_0}\int_s^{t_0} e^{\tilde{\delta}\xi}\int_\Omega \triangle g w_t\,dx\,d\xi\,ds.
\end{aligned} \tag{5.34}$$

Since $\lim_{\tau\to\infty} e^{-\hat{\sigma}\tau}\tilde{R}_{t_0-\tau}^2=0$, for any $\varepsilon>0$, we can find $\tau_0=\tau_0(\varepsilon,\tilde{D},t_0)\ge 0$ such that

$$\frac{(2+5\varepsilon')}{C^1\lambda^2\tau}e^{\tilde{\delta}(-\tau)}F(t_0-\tau)+\frac{1}{C^1\tau}e^{-\tilde{\delta}\tau}\big(w_t(t_0-\tau),w(t_0-\tau)\big)\le\varepsilon.$$

Thus we have

$$E_w(t_0)\le\varepsilon+\phi_{t_0,\tau}\big((u_0^1,v_0^1,\theta_0^1),(u_0^2,v_0^2,\theta_0^2)\big)\quad\text{for all }\big(u_0^i,v_0^i,\theta_0^i\big)\in\tilde{D}_{t_0-\tau_0}.$$

By Lemma 3.1 we only need to show that $\phi_{t_0,\tau_0}(\cdot,\cdot)$ defined by (5.34) is a contractive function on $\tilde{D}_{t_0-\tau_0}\times\tilde{D}_{t_0-\tau_0}$. Let (u_n,u_{nt},θ_n) be the corresponding solutions of $(u_0^n,v_0^n,\theta_0^n)\in\tilde{D}_{t_0-\tau_0}$, $n=1,2,\ldots$. Since $\tilde{D}_{t_0-\tau_0}$ is a bounded subset in X_0, by (4.16) we know that

$$\big\|\big(u_n(s),u_{nt}(s),\theta_n(s)\big)\big\|_{X_0}\le C'_{t_0,\tau_0}<+\infty\quad\text{for all }s\in[t_0-\tau_0,t_0]\text{ and }n\in N, \tag{5.35}$$

where C'_{t_0,τ_0} depends on t_0,τ_0.

Now, we will deal with the right terms in (5.34) one by one.

Without loss of generality, assuming first that

$$u_n\to u\quad\text{weak-star in }L^\infty\big(t_0-\tau_0,t_0;H_0^2(\Omega)\big)$$

and considering that compact embeddings $H_0^2(\Omega)\hookrightarrow\hookrightarrow H_0^1(\Omega)$, we have

$$u_n\to u\quad\text{strongly in }L^2\big(t_0-\tau_0,t_0;H_0^1(\Omega)\big),$$

so we obtain

$$\lim_{n\to\infty}\lim_{m\to\infty}\int_{t_0-\tau_0}^{t_0} e^{\tilde{\delta}\xi}\left\|\nabla u_n(\xi)-\nabla u_m(\xi)\right\|^2\,d\xi=0. \tag{5.36}$$

Second, similarly assuming that

$$u_n\to u\quad\text{weak-star in } L^\infty\bigl(t_0-\tau_0,t_0;H_0^1(\Omega)\bigr)$$

and considering compact embeddings $H_0^1(\Omega)\hookrightarrow\hookrightarrow L^2(\Omega)$, we also get

$$\begin{aligned}
&\lim_{n\to\infty}\lim_{m\to\infty}\int_{t_0-\tau_0}^{t_0} e^{\tilde{\delta}\xi}\left\|u_n(\xi)-u_m(\xi)\right\|^2\,d\xi=0,\\
&\lim_{n\to\infty}\lim_{m\to\infty}\left(\int_{t_0-\tau_0}^{t_0} e^{\tilde{\delta}\xi}\left\|u_n(\xi)-u_m(\xi)\right\|^2\,d\xi\right)^{\frac{1}{2}}=0.
\end{aligned}\tag{5.37}$$

Finally, with

$$\begin{aligned}
&u_n\to u\quad\text{weak-star in } L^\infty\bigl(t_0-\tau_0,t_0;L^2(\Omega)\bigr),\\
&u_{nt}\to u_t\quad\text{weak-star in } L^\infty\bigl(t_0-\tau_0,t_0;L^2(\Omega)\bigr),
\end{aligned}$$

we have

$$u_n\to u\quad\text{strongly in } C\bigl(t_0-\tau_0,t_0;L^2(\Omega)\bigr).$$

So, it is easy to obtain that

$$\begin{aligned}
&\lim_{n\to\infty}\lim_{m\to\infty}\int_\Omega\bigl(u_{nt}(t_0)-u_{mt}(t_0)\bigr)\bigl(u_n(t_0)-u_m(t_0)\bigr)\,dx\\
&\quad=\lim_{n\to\infty}\lim_{m\to\infty}\int_\Omega\bigl(u_{nt}(t_0)-u_{mt}(t_0)\bigr)\bigl(u_n(t_0)-u(t_0)+u(t_0)-u_m(t_0)\bigr)\,dx\\
&\quad=\lim_{n\to\infty}\lim_{m\to\infty}\int_\Omega\bigl(u_{nt}(t_0)-u_{mt}(t_0)\bigr)\bigl(u_n(t_0)-u(t_0)\bigr)\,dx\\
&\qquad+\lim_{n\to\infty}\lim_{m\to\infty}\int_\Omega\bigl(u_{nt}(t_0)-u_{mt}(t_0)\bigr)\bigl(u(t_0)-u_m(t_0)\bigr)\,dx\\
&\quad\le\left\|u_{nt}(t_0)-u_{mt}(t_0)\right\|_{L^\infty(L^2(\Omega))}\left\|u_n(t_0)-u(t_0)\right\|\\
&\qquad+\left\|u_{nt}(t_0)-u_{mt}(t_0)\right\|_{L^\infty(L^2(\Omega))}\left\|u(t_0)-u_m(t_0)\right\|\\
&\quad\le C\bigl[\left\|u_n(t_0)-u(t_0)\right\|+\left\|u(t_0)-u_m(t_0)\right\|\bigr]\\
&\quad\to 0.
\end{aligned}\tag{5.38}$$

By assumptions (1.6) on $g(\cdot)$, using the embedding theorem combined with (5.35), we have

$$\begin{aligned}
&\lim_{n\to\infty}\lim_{m\to\infty}\int_{t_0-\tau_0}^{t_0} e^{\tilde{\delta}\xi}\int_\Omega \big(u_{nt}(\xi)-u_{mt}(\xi)\big)\big(g\big(u_n(\xi)\big)-g\big(u_m(\xi)\big)\big)\,dx\,d\xi \\
&\quad\le \lim_{n\to\infty}\lim_{m\to\infty}\int_{t_0-\tau_0}^{t_0} e^{\tilde{\delta}\xi}\int_\Omega \big(u_{nt}(\xi)-u_{mt}(\xi)\big)k_2\big(u_n(\xi)-u_m(\xi)\big) \\
&\qquad\times \big(1+\big|u_n(\xi)\big|^{\rho-1}+\big|u_m(\xi)\big|^{\rho-1}\big)\,dx\,d\xi \\
&\quad\le C'_{t_0,\tau_0}\lim_{n\to\infty}\lim_{m\to\infty}\left(\int_{t_0-\tau_0}^{t_0} e^{\tilde{\delta}\xi}\big\|\big(u_n(\xi)-u_m(\xi)\big)\big\|^2\,d\xi\right)^{\frac{1}{2}} \\
&\quad= 0.
\end{aligned}\tag{5.39}$$

Similarly, since $\int_s^{t_0} e^{\tilde{\delta}\xi}\int_\Omega (u_{nt}(\xi)-u_{mt}(\xi))(g(u_n(\xi))-g(u_m(\xi)))\,dx\,d\xi$ is bounded for each $s\in[\tau,t_0]$, by (5.39) and the Lebesgue dominated convergence theorem we have

$$\begin{aligned}
&\lim_{n\to\infty}\lim_{m\to\infty}\int_{t_0-\tau_0}^{t_0}\int_s^{t_0} e^{\tilde{\delta}\xi}\int_\Omega \big(u_{nt}(\xi)-u_{mt}(\xi)\big)\big(g\big(u_n(\xi)\big)-g\big(u_m(\xi)\big)\big)\,dx\,d\xi\,ds \\
&\quad= \int_{t_0-\tau_0}^{t_0}\left(\lim_{n\to\infty}\lim_{m\to\infty}\int_s^{t_0} e^{\tilde{\delta}\xi}\int_\Omega \big(u_{nt}(\xi)-u_{mt}(\xi)\big)\big(g\big(u_n(\xi)\big)-g\big(u_m(\xi)\big)\big)\,dx\,d\xi\right)ds \\
&\quad= \int_{t_0-\tau_0}^{t_0} 0\,ds \\
&\quad= 0.
\end{aligned}\tag{5.40}$$

Combining (5.36)-(5.40), we get that $\Phi_{t_0,\tau_0}(\cdot,\cdot)$ is a contractive function on $\tilde{D}_{t_0-\tau_0}\times\tilde{D}_{t_0-\tau_0}$. The proof is finished by Lemma 3.1. □

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Competing interests
The authors declare that they have no competing interests.

Authors' contributions
This paper was mainly completed by DX. YZ gave the exact conditions of the source term *g*(*u*) and dealed with it through the paper. YZ also dealed with the right terms in (5.34). All authors read and approved the final manuscript.

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