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Nehari-type ground state solutions for asymptotically periodic fractional Kirchhoff-type problems in $\mathbb{R}^N$

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Abstract
In this paper, we studied the following fractional Kirchhoff-type equation:

$$\left(a+b\int_{\mathbb{R}^N}\left|(-\triangle)^{\frac{\alpha}{2}}u\right|^2\,\mathrm{d}x\right)(-\triangle)^{\alpha}u+V(x)u=f(x,u),\quad x\in\mathbb{R}^N,$$

where a, b are positive constants, $\alpha\in(0,1)$, $N\in(2\alpha,4\alpha)$, $(-\triangle)^{\alpha}$ is the fractional Laplacian operator, $V(x)$ and $f(x,u)$ are periodic or asymptotically periodic in x. Under some weaker conditions on the nonlinearity, we obtain the existence of ground state solutions for the above problem in periodic case and asymptotically periodic case, respectively. In particular, our results unify both asymptotically cubic and super-cubic nonlinearities, which are new even for $\alpha=1$.

MSC: 35J20; 35J25; 35J60

Keywords: fractional Kirchhoff-type problem; Nehari-type ground state solution; asymptotically periodic; asymptotically cubic and super-cubic growth

1 Introduction

In this paper, we studied the following fractional Kirchhoff-type problem:

$$\left(a+b\int_{\mathbb{R}^N}\left|(-\triangle)^{\frac{\alpha}{2}}u\right|^2\,\mathrm{d}x\right)(-\triangle)^{\alpha}u+V(x)u=f(x,u),\quad x\in\mathbb{R}^N,\tag{1.1}$$

where $\alpha\in(0,1]$, a, b are positive constants. The fractional Laplacian operator $(-\triangle)^{\alpha}$ is defined as:

$$(-\triangle)^{\alpha}u(x)=\frac{1}{C(\alpha)}\int_{\mathbb{R}^N}\frac{u(x)-u(y)}{|x-y|^{N+2\alpha}}\,\mathrm{d}x,\quad x\in\mathbb{R}^N,$$

which can be viewed as the infinitesimal generators of a Lévy stable diffusion processes [1]. In order to reduce our statements, we first assume that the potential V and nonlinearity f satisfy the following basic assumptions:

(V) $V\in\mathcal{C}(\mathbb{R}^N)\cap L^{\infty}(\mathbb{R}^N)$ and $\inf_{\mathbb{R}^N}V(x)>0$;

(F1) $f \in \mathcal{C}(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$ and there exist constants $C > 0$ and $2 < p < 2_\alpha^* := \frac{2N}{N-2\alpha}$ such that

$$\left|f(x,t)\right| \le C\left(1 + |t|^{p-1}\right), \quad \forall (x,t) \in \mathbb{R}^N \times \mathbb{R};$$

(F2) $f(x,t) = o(|t|)$ as $t \to 0$ uniformly in $x \in \mathbb{R}^N$;
(F3) $\lim_{|t|\to\infty} \frac{\int_0^t f(x,s)\,\mathrm{d}s}{|t|^{4+N-4\alpha}} = \infty$ uniformly in $x \in \mathbb{R}^N$;
(F4) there exists a constant $\theta_0 \in (0,1)$ such that for any $x \in \mathbb{R}^N$, $t > 0$ and $\tau \ne 0$

$$\left[\frac{f(x,\tau)}{\tau^3} - \frac{f(x,t\tau)}{(t\tau)^3}\right] \operatorname{sign}(1-t) + \theta_0 V(x) \frac{|1-t^2|}{(t\tau)^2} \ge 0.$$

In the recent years, fractional and nonlocal operators arise in the description of various phenomena in the pure mathematical research and concrete real-world applications, such as fractional quantum mechanics [2, 3], physics and chemistry [4], obstacle problems [5], optimization and finance [6], conformal geometry and minimal surfaces [7] and so on.

If $\alpha = 1$, Problem (1.1) formally reduces to the well-known Kirchhoff Dirichlet equation:

$$-\left(a + b\int_\Omega |\nabla u|^2\,\mathrm{d}x\right)\triangle u + V(x)u = f(x,u), \quad x \in \mathbb{R}^N, \tag{1.2}$$

which is related to the stationary analog of the equations

$$u_{tt} - \left(a + b\int_\Omega |\nabla u|^2\,\mathrm{d}x\right)\triangle u = f(x,u). \tag{1.3}$$

Equations of this type were first proposed by Kirchhoff [8] to describe the transversal oscillations of a stretched string. For more details in physical aspects, we refer the reader to [7, 9, 10]. Particularly, after Lions [11] introduced an abstract functional analysis framework to (1.2), the Kirchhoff-type problems received increasingly more attention by various authors. There are many existence and multiplicity results for (1.2), for example, Zhang and Zhang [12] proved the existence of Nehari-type ground state solutions for asymptotically periodic Kirchhoff-type problem when f satisfies (F1), (F2) and the following conditions:

(F3′) $\lim_{|t|\to\infty} \frac{\int_0^t f(x,s)\,\mathrm{d}s}{|t|^4} = \infty$ uniformly in $x \in \Omega$;
(F4′) $\frac{f(x,t)}{|t|^3}$ is nondecreasing in t on $\mathbb{R} \setminus \{0\}$ for every $x \in \mathbb{R}^N$.

For more recent results concerning Kirchhoff-type problems, see *e.g.* [13–22].

When $a = 1$ and $b = 0$, then (1.1) reduces to the following fractional Schrödinger problem:

$$(-\triangle)^\alpha u + V(x)u = f(x,u), \quad x \in \mathbb{R}^N. \tag{1.4}$$

Since $(-\triangle)^\alpha$ is a nonlocal operator, the usual analysis tools for elliptic PDEs are invalid for (1.4). This causes some mathematical difficulties which make the study of such a problem particularly interesting. After Caffarelli and Silvestrein transformed the nonlocal problem into a local problem by the extension theorem in [23], there have been a large number of works focused on the study of fractional Schrödinger equations. Recently, Secchi [24] obtained the existence of positive solutions by using the Nehari manifold method. Chang [25] proved the existence of a positive ground state solution of (1.4) when $f(x,t)$ is asymptotically linear with respect to t at infinity. Zhang, Zhang and Mi [26] established the existence of solutions for (1.4) in periodic case and asymptotically periodic case via variational methods. We refer the reader to [27–32] and the references therein.

Although the fractional Schrödinger equations have been widely studied, to the best of our knowledge, there are few papers concerning on the fractional Kirchhoff-type problems like (1.1) in the literature. Recently, under decay assumptions on V, Liu, Marco and Zhang [33] considered (1.1) where $N = 2$ with $\alpha \in (\frac{1}{2},1)$ or $N = 3$ with $\alpha \in (\frac{3}{4},1)$ and $f(x,t) = f(t)$ satisfies:

(H1) $f \in \mathcal{C}^1(\mathbb{R}^+,\mathbb{R})$ and $\lim_{t\to 0}\frac{f(t)}{t} = 0$;
(H2) there are $D > 0$ and $2 < q < 2^*_\alpha$ such that $f(t) \geq t^{t^{2^*_\alpha}-1} + Dt^{q-1}$ for any $t \geq 0$.

We point out that they obtained the existence of ground state solutions for (1.1) when D large enough in (H2). Note that $2^*_\alpha > 4$ when $N = 2$ with $\alpha \in (\frac{1}{2},1)$ or $N = 3$ with $\alpha \in (\frac{3}{4},1)$, which means f satisfies super-cubic condition (F3′). In fact, under (F1) and (F3′), it is easy to verify the Mountain Pass geometry for energy functional. Different from their work, we assume that f is non-autonomous and satisfies (F3) which implies (F3′). Evidently, (F1) and (F3) suggest that $N \in (2\alpha, 4\alpha)$ with $\alpha \in (0,1)$ and that is what we are concerned about in this paper. Our results can be regarded as the complementary work of [33]. We also cite [34–41] for related results.

In this paper, we are concerned with the existence of ground state solutions for the asymptotically periodic fractional Kirchhoff-type problems (1.1) involving asymptotically cubic or super-cubic nonlinearities. To this end, we must overcome three main difficulties: (I) when f is asymptotically cubic (*i.e.* $\lim_{|t|\to\infty}\frac{|f(x,t)|}{|t|^3} = V_\infty \geq (\not\equiv) 0$), there is no Mountain-Pass structure for (1.1) and the standard variational methods cannot be used on Nehari manifold; (II) when $V(x)$ and $f(x,u)$ are asymptotically periodic in x, many effective methods for periodic problems are invalid for asymptotically periodic ones; (III) when f is not differentiable, Nehari manifold may not be a $\mathcal{C}^1$ manifold, so it is difficult to prove the minimizer of the variational functional over Nehari manifold is a critical point. To overcome these difficulties, we will introduce some new methods and analytical techniques in this paper.

In order to precisely state our result we denote by $\mathcal{H}$ the class of functions $h \in \mathcal{C}(\mathbb{R}^N,\mathbb{R}) \cap L^\infty(\mathbb{R}^N,\mathbb{R})$ such that, for every $\epsilon > 0$, the set $\{x \in \mathbb{R}^N : |h(x)| \geq \epsilon\}$ has finite Lebesgue measure. Moreover, for the potential V and the nonlinear term f, we suppose that:

(V0) $V \in \mathcal{C}(\mathbb{R}^N,(0,\infty))$ and $V(x)$ is 1-periodic in x;
(F0) $f(x,t)$ is 1-periodic in x;
(V0′) $V(x) = V_0(x) + V_1(x)$, $V_0, V_1 \in \mathcal{C}(\mathbb{R}^N,\mathbb{R}^N)$, and $V_0(x)$ is 1-periodic in x, $V_1(x) \leq 0$ for $x \in \mathbb{R}^N$, and $V_1 \in \mathcal{H}$;
(F0′) $f(x,t) = f_0(x,t) + f_1(x,t)$, $f_0 \in \mathcal{C}(\mathbb{R}^N \times \mathbb{R}^N,\mathbb{R})$, $f_0(x,t)$ is 1-periodic in x, and for any $x \in \mathbb{R}^N$, $t > 0$ and $\tau \neq 0$,

$$\left[\frac{f_0(x,\tau)}{\tau^3} - \frac{f_0(x,t\tau)}{(t\tau)^3}\right]\operatorname{sign}(1-t) + V_0(x)\frac{|1-t^2|}{(t\tau)^2} \geq 0; \tag{1.5}$$

$f_1 \in \mathcal{C}(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$, satisfies

$$\left|f_1(x,t)\right| \le h(x)\left(|t| + |t|^{q-1}\right), \quad f_1(x,t)t \ge 0, \tag{1.6}$$

where $F_1(x,t) = \int_0^t f_1(x,s)\,\mathrm{d}s$, $q \in (2, 2_\alpha^*)$ and $h \in \mathcal{H}$.

Inspired by the previously mentioned work, especially [21, 42, 43], we seek definite answers to overcome the above three difficulties. Firstly, we use a new trick to show the following set:

$$\mathcal{E} = \left\{u \in E : b\left\|(-\triangle)^{\frac{\alpha}{2}}u\right\|_2^4 + \int_{\mathbb{R}^N}\left[V(x)u^2 - f(x,u)u\right]\mathrm{d}x < 0\right\},$$

is not empty (see Lemma 2.4), and construct a new minimax characterization (see Lemma 2.6) under (F3). Secondly, for the asymptotically periodic case, we adopt a technique introducing in [21] by combining the quantitative deformation lemma and degree theory, to overcome the difficulties caused by the dropping of periodicity of $V(x)$ and $f(x,u)$ in x (see Section 4). Thirdly, because the argument based on Nehari manifold approach (see Szulkin and Weth [44]) becomes invalid under condition (F4) instead of (F4′), we will use the non-Nehari manifold approach developed in [42, 43]. It relies on finding a minimizing Cerami sequence for variational functional related to (1.1) outside the Nehari manifold by using the diagonal method (see Lemma 2.8). Note that (F3) implies (F3′), which covers the asymptotically cubic and super-cubic nonlinearities due to $N \in (2\alpha, 4\alpha)$.

Now, we state the main results of this paper. In the periodic case, we establish the following theorem.

Theorem 1.1 *Assume that V and f satisfy* (V0) *and* (F0)-(F4). *Then Problem* (1.1) *has a ground state solution $u_0 \in E$ such that $\Phi(u_0) = \inf_{\mathcal{N}} \Phi > 0$. Moreover,*

$$b\left\|(-\triangle)^{\frac{\alpha}{2}}u_0\right\|_2^4 + \int_{\mathbb{R}^N}\left[V(x)u_0^2 - f(x,u_0)u_0\right]\mathrm{d}x < 0.$$

In the asymptotically periodic case, we establish the following theorem.

Theorem 1.2 *Assume that V and f satisfy* (V0′), (F0′) *and* (F1)-(F4). *Then Problem* (1.1) *has a ground state solution $u_0 \in E$ such that $\Phi(u_0) = \inf_{\mathcal{N}} \Phi > 0$. Moreover,*

$$b\left\|(-\triangle)^{\frac{\alpha}{2}}u_0\right\|_2^4 + \int_{\mathbb{R}^N}\left[V(x)u_0^2 - f(x,u_0)u_0\right]\mathrm{d}x < 0.$$

Remark 1.3 Since the term $\|(-\triangle)^{\frac{\alpha}{2}}u\|_2^4$ is homogeneous of degree 4, the greatest part of the literature focuses on the study of (1.1) with f satisfying the super-cubic condition (F3′). However, there are few papers consider f satisfies asymptotically cubic. In order to treat asymptotically cubic or super-cubic nonlinearities in a unified way, we use a weaker condition (F3) instead of (F3′) where $N \in (2\alpha, 4\alpha)$ with $\alpha \in (0,1)$. Our results strongly improve the previous results on the existence of ground state solutions for (1.1), which are new even for $\alpha = 1$. Furthermore, we reduce the usual Nehari-type monotonic condition to a weaker condition (F4). In fact, there are many functions satisfying (F1)-(F4), but not satisfying (F3′) and (F4′), and some typical examples are introduced in [21].

Throughout this paper, we denote the norm of $L^s(\mathbb{R}^N)$ by $\|u\|_s$ for $s \geq 2$, $B_r(x) = \{y \in \mathbb{R}^N : |y-x| < r\}$, and C are various positive constants.

2 Preliminaries

First, we give some notations. A complete introduction to fractional Sobolev spaces can be found in [45]; we offer in the succeeding discussion a short review. We define the homogeneous fractional Sobolev space $\mathcal{D}^{\alpha,2}(\mathbb{R}^N)$ as follows:

$$\mathcal{D}^{\alpha,2}(\mathbb{R}^N) = \left\{u \in L^{2_\alpha^*}(\mathbb{R}^N) : \frac{|u(x)-u(y)|}{|x-y|^{\frac{N}{2}+\alpha}} \in L^2(\mathbb{R}^N \times \mathbb{R}^N)\right\},$$

which is the completion of $C_0^\infty(\mathbb{R}^N)$ under the norm

$$\|u\|_{\mathcal{D}^{\alpha,2}(\mathbb{R}^N)} = \left(\int_{\mathbb{R}^N} \left|(-\triangle)^{\frac{\alpha}{2}} u\right|^2 \,\mathrm{d}x\right)^{1/2} = \left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2\alpha}} \,\mathrm{d}x \,\mathrm{d}y\right)^{1/2}.$$

Moreover, the embedding $\mathcal{D}^{\alpha,2}(\mathbb{R}^N) \hookrightarrow L^{2_\alpha^*}(\mathbb{R}^N)$ is continuous and for any $\alpha \in (0,1)$, there exists a best constant S_α such that

$$S_\alpha = \inf_{u \in \mathcal{D}^{\alpha,2}(\mathbb{R}^N)} \frac{\int_{\mathbb{R}^N} |(-\triangle)^{\frac{\alpha}{2}} u|^2 \,\mathrm{d}x}{(\int_{\mathbb{R}^N} |u^{2_\alpha^*}| \,\mathrm{d}x)^{\frac{2}{2_\alpha^*}}}.$$

The fractional Sobolev space $H^\alpha(\mathbb{R}^N)$ can be described by

$$H^\alpha(\mathbb{R}^N) = \left\{u \in L^2(\mathbb{R}^N) : \frac{|u(x)-u(y)|}{|x-y|^{\frac{N}{2}+\alpha}} \in L^2(\mathbb{R}^N \times \mathbb{R}^N)\right\},$$

endowed with the natural norm

$$\|u\|_{H^\alpha(\mathbb{R}^N)} = \left(\int_{\mathbb{R}^N} |u|^2 \,\mathrm{d}x + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2\alpha}} \,\mathrm{d}x \,\mathrm{d}y\right)^{1/2}.$$

It is well known that $H^\alpha(\mathbb{R}^N)$ is continuously embedded into $L^q(\mathbb{R}^N)$, and compactly embedded into $L^q_{\mathrm{loc}}(\mathbb{R}^N)$ for $2 \leq q \leq 2_\alpha^* := \frac{2N}{N-2\alpha}$.

Under assumption (V), we see that

$$E = \left\{u \in H^\alpha(\mathbb{R}^N) : \int_{\mathbb{R}^N} \left(a\left|(-\triangle)^{\frac{\alpha}{2}} u\right|^2 + V(x)u^2\right) \mathrm{d}x < +\infty\right\}$$

is a Hilbert space equipped with the norm

$$\|u\| = \left(\int_{\mathbb{R}^N} \left(a\left|(-\triangle)^{\frac{\alpha}{2}} u\right|^2 + V(x)u^2\right) \mathrm{d}x\right)^{1/2}.$$

Let

$$\begin{aligned}\Phi(u) = {}& \frac{1}{2}\int_{\mathbb{R}^N} \left(a\left|(-\triangle)^{\frac{\alpha}{2}} u\right|^2 + V(x)u^2\right) \mathrm{d}x \\ &+ \frac{b}{4}\left(\int_{\mathbb{R}^N} \left|(-\triangle)^{\frac{\alpha}{2}} u\right|^2 \mathrm{d}x\right)^2 - \int_{\mathbb{R}^N} F(x,u)\,\mathrm{d}x, \quad \forall u \in E. \qquad (2.1)\end{aligned}$$

From (F1) and (F2), it is easy to see that $\Phi \in \mathcal{C}^1(E,\mathbb{R})$ as a functional, and that

$$\begin{aligned}\langle \Phi'(u), v\rangle &= \int_{\mathbb{R}^N} \big(a(-\triangle)^{\frac{\alpha}{2}}u(-\triangle)^{\frac{\alpha}{2}}v + V(x)uv\big)\,\mathrm{d}x \\ &\quad + b\int_{\mathbb{R}^N} \big|(-\triangle)^{\frac{\alpha}{2}}u\big|^2\,\mathrm{d}x \int_{\mathbb{R}^N} (-\triangle)^{\frac{\alpha}{2}}u(-\triangle)^{\frac{\alpha}{2}}v\,\mathrm{d}x \\ &\quad - \int_{\mathbb{R}^N} f(x,u)v\,\mathrm{d}x, \quad \forall u,v\in E. \end{aligned} \tag{2.2}$$

Clearly, any critical point of Φ is a weak solution of (1.1). Set

$$\mathcal{N} := \big\{u\in E : \langle \Phi'(u), u\rangle = 0, u\neq 0\big\}. \tag{2.3}$$

Lemma 2.1 *Assume that* (V), (F1), (F2) *and* (F4) *hold. Then*

$$\Phi(u) \geq \Phi(tu) + \frac{1-t^4}{4}\langle \Phi'(u), u\rangle + \frac{(1-\theta_0)(1-t^2)^2}{4}\|u\|^2, \quad \forall u\in E, t\geq 0. \tag{2.4}$$

Proof For any $x\in\mathbb{R}^N$, $t\geq 0$, $\tau\in\mathbb{R}\setminus\{0\}$, (F4) yields

$$\begin{aligned}&\frac{1-t^4}{4}\tau f(x,\tau) + F(x,t\tau) - F(x,\tau) + \frac{\theta_0 V(x)}{4}\big(1-t^2\big)^2\tau^2 \\ &\quad = \int_t^1 \left[\frac{f(x,\tau)}{\tau^3} - \frac{f(x,\xi\tau)}{(\xi\tau)^3} + \theta_0 V(x)\frac{1-\xi^2}{(\xi\tau)^2}\right]\xi^3\tau^4\,\mathrm{d}\xi \geq 0. \end{aligned} \tag{2.5}$$

By (2.1), (2.2) and (2.5), we have

$$\begin{aligned}\Phi(u) - \Phi(tu) &= \frac{1-t^2}{2}\|u\|^2 + \frac{1-t^4}{4}b\big\|(-\triangle)^{\frac{\alpha}{2}}\big\|_2^4 + \int_{\mathbb{R}^N}\big[F(x,tu) - F(x,u)\big]\,\mathrm{d}x \\ &= \frac{1-t^4}{4}\langle \Phi'(u), u\rangle + \frac{(1-t^2)^2}{2}\|u\|^2 \\ &\quad + \int_{\mathbb{R}^N}\left[\frac{1-t^4}{4}f(x,u)u + F(x,tu) - F(x,u)\right]\mathrm{d}x \\ &\geq \frac{1-t^4}{4}\langle \Phi'(u), u\rangle + \frac{(1-\theta_0)(1-t^2)^2}{4}\|u\|^2 \\ &\quad + \int_{\mathbb{R}^N}\left[\frac{1-t^4}{4}f(x,u)u + F(x,tu) - F(x,u) + \frac{\theta_0 V(x)}{4}\big(1-t^2\big)^2u^2\right]\mathrm{d}x \\ &\geq \frac{1-t^4}{4}\langle \Phi'(u), u\rangle + \frac{(1-\theta_0)(1-t^2)^2}{4}\|u\|^2. \end{aligned}$$

This shows that (2.4) holds. □

Corollary 2.2 *Assume that* (V), (F1), (F2) *and* (F4) *hold. Then, for an* $u\in\mathcal{N}$,

$$\Phi(u) \geq \Phi(tu) + \frac{(1-\theta_0)(1-t^2)^2}{4}\|u\|^2, \quad \forall t\geq 0. \tag{2.6}$$

Corollary 2.3 *Assume that* (V), (F1), (F2) *and* (F4) *hold. Then, for any* $u\in\mathcal{N}$,

$$\Phi(u) = \max_{t\geq 0}\Phi(tu). \tag{2.7}$$

Under (F3), to show $\mathcal{N} \neq \emptyset$ in our situation, we have to overcome the competing effect of $\int_{\mathbb{R}^N} |(-\triangle)^{\frac{\alpha}{2}} u|^2 \,\mathrm{d}x$. To this end, we define a set $\mathcal{E}$ as follows:

$$\mathcal{E} = \left\{u \in E : b\left\|(-\triangle)^{\frac{\alpha}{2}} u\right\|_2^4 + \int_{\mathbb{R}^N} \left[V(x)u^2 - f(x,u)u\right] \mathrm{d}x < 0\right\}. \tag{2.8}$$

Lemma 2.4 *Assume that* (V) *and* (F1)-(F4) *hold. Then* $\mathcal{E} \neq \emptyset$. *Moreover,* $\mathcal{N} \subset \mathcal{E}$.

Proof For any fixed $u \in E$ with $u \neq 0$, set $u_t(x) = tu(t^{-1}x)$ for $t > 0$. By (V), one has

$$\begin{aligned} & b\left\|(-\triangle)^{\frac{\alpha}{2}} u_t\right\|_2^4 + \int_{\mathbb{R}^N} \left[V(x)u_t^2 - f(x,u_t)u_t\right] \mathrm{d}x \\ &= t^{4+2N-4\alpha} b\left\|(-\triangle)^{\frac{\alpha}{2}} u\right\|_2^4 + t^{2+N} \int_{\mathbb{R}^N} V(tx)u^2 \,\mathrm{d}x - t^N \int_{\mathbb{R}^N} f(tx,tu)tu \,\mathrm{d}x \\ &\leq t^{2+N}\left[t^{2+N-4\alpha} b\left\|(-\triangle)^{\frac{\alpha}{2}} u\right\|_2^4 + \|V\|_\infty \|u\|_2^2 - \int_{\mathbb{R}^N} \frac{f(tx,tu)tu}{t^2} \,\mathrm{d}x\right] \\ &= t^{2+N}\left\{t^{2+N-4\alpha}\left[b\left\|(-\triangle)^{\frac{\alpha}{2}} u\right\|_2^4 - \int_{\mathbb{R}^N} \frac{f(tx,tu)tu}{t^{4+N-4\alpha}} \,\mathrm{d}x\right] + \|V\|_\infty \|u\|_2^2\right\}. \end{aligned} \tag{2.9}$$

Note that, for $u \neq 0$, $F(tx,tu)/|tu|^{4+N-4\alpha} \to +\infty$ as $t \to +\infty$ by (F3), where $F(x,t) = \int_0^t f(x,s)\,\mathrm{d}s$. From (2.5) with $t = 0$, one has

$$\frac{1}{4}\tau f(x,\tau) - F(x,\tau) + \frac{\theta_0 V(x)}{4}\tau^2 \geq 0, \quad \forall x \in \mathbb{R}^N, \tau \in \mathbb{R}, \tag{2.10}$$

then we have

$$\frac{f(tx,tu)tu}{|tu|^{4+N-4\alpha}} \to +\infty, \quad \text{as } t \to +\infty \text{ uniformly in } x \in \mathbb{R}^N. \tag{2.11}$$

For $N \in (2\alpha, 4\alpha)$, thus from (2.9) and (2.11), one has

$$b\left\|(-\triangle)^{\frac{\alpha}{2}} u_t\right\|_2^4 + \int_{\mathbb{R}^3} \left[V(x)u_t^2 - f(x,u_t)u_t\right] \mathrm{d}x \to -\infty, \quad \text{as } t \to +\infty.$$

Thus, taking $v = u_T$ for T large, we have $v \in \mathcal{E}$. From (2.2), it is easy to see that and $\mathcal{N} \subset \mathcal{E}$. □

Lemma 2.5 *Assume that* (V) *and* (F1)-(F4) *hold. If* $u \in \mathcal{E}$, *then there exists a unique* $t(u) > 0$ *such that* $t(u) \in \mathcal{N}$.

Proof First, we prove the existence of $t(u)$. In view of Lemma 2.4, let $u \in \mathcal{E}$ be fixed and define a function $g(t) = \langle \Phi'(tu), tu\rangle$ on $[0, +\infty)$. By (F4), one has

$$\theta_0 V(x)(t\tau)^2 - f(x,t\tau)t\tau \leq \left[\theta_0 V(x)\tau^2 - f(x,\tau)\tau\right]t^4, \quad \forall x \in \mathbb{R}^N, t \geq 1, \tau \in \mathbb{R}, \tag{2.12}$$

Since $u \in \mathcal{E}$, (2.12) yields

$$\begin{aligned} & bt^4\left\|(-\triangle)^{\frac{\alpha}{2}} u\right\|_2^4 + \int_{\mathbb{R}^N} \left[\theta_0 V(x)(tu)^2 - f(x,tu)tu\right] \mathrm{d}x \\ &\leq t^4\left(b\left\|(-\triangle)^{\frac{\alpha}{2}} u\right\|_2^4 + \int_{\mathbb{R}^N} \left[\theta_0 V(x)u^2 - f(x,u)u\right] \mathrm{d}x\right) < 0, \quad \forall t \geq 1. \end{aligned} \tag{2.13}$$

It follows from (2.1) and (2.13) that

$$\begin{aligned} g(t) &= t^2\left[a\left\|(-\triangle)^{\frac{\alpha}{2}}u\right\|_2^2+(1-\theta_0)\int_{\mathbb{R}^N}V(x)u^2\,\mathrm{d}x\right]+bt^4\left\|(-\triangle)^{\frac{\alpha}{2}}u\right\|_2^4 \\ &\quad+\int_{\mathbb{R}^N}\left[\theta_0V(x)(tu)^2-f(x,tu)tu\right]\mathrm{d}x \\ &\le t^2\left[a\left\|(-\triangle)^{\frac{\alpha}{2}}u\right\|_2^2+(1-\theta_0)\int_{\mathbb{R}^N}V(x)u^2\,\mathrm{d}x\right] \\ &\quad+t^4\left(b\left\|(-\triangle)^{\frac{\alpha}{2}}u\right\|_2^4+\int_{\mathbb{R}^N}\left[\theta_0V(x)u^2-f(x,u)u\right]\mathrm{d}x\right). \end{aligned} \tag{2.14}$$

For (F1), (F2) and (2.14), it is easy to verify that $g(0)=0$, $g(t)>0$ for $t>0$ small and $g(t)<0$ for t large because of $u\in\mathcal{E}$. Therefore, there exists $t_0=t(u)>0$ so that $g(t_0)=0$ and $t(u)u\in\mathcal{N}$.

Next, we prove the uniqueness. For any given $u\in\mathcal{E}$, let $t_1,t_2>0$ such that $g(t_1)=g(t_2)=0$. Jointly with (2.6), one has

$$\Phi(t_1u)\ge\Phi(t_2u)+\frac{(1-\theta_0)(t_1^2-t_2^2)^2}{4t_1^2}\|u\|^2 \tag{2.15}$$

and

$$\Phi(t_2u)\ge\Phi(t_1u)+\frac{(1-\theta_0)(t_2^2-t_1^2)^2}{4t_2^2}\|u\|^2. \tag{2.16}$$

Both (2.15) and (2.16) imply $t_1=t_2$. Hence, $t(u)>0$ is unique for any $u\in\mathcal{E}$. □

Lemma 2.6 *Assume that* (V) *and* (F1)-(F4) *hold. Then*

$$\inf_{u\in\mathcal{N}}\Phi(u)=c=\inf_{u\in\mathcal{E},u\neq0}\max_{t\ge0}\Phi(tu)>0.$$

Proof Corollary 2.3 and Lemma 2.5 imply that $c=\inf_{u\in\mathcal{E},u\neq0}\max_{t\ge0}\Phi(tu)$. From Lemma 2.1, it is easy to see that $c>0$. □

Lemma 2.7 *Assume that* (V) *and* (F1)-(F4) *hold. Then there exist a constant* $c_*\in(0,c]$ *and a sequence* $u_n\subset E$ *satisfying*

$$\Phi(u_n)\to c_*,\qquad \left\|\Phi'(u_n)\right\|\left(1+\|u_n\|\right)\to0. \tag{2.17}$$

Proof We use the non-Nehari manifold approach developed in [42, 43] to show (2.17). From (F1), (F2) and (2.1), we know that there exist $\delta_0>0$ and $\rho_0>0$ such that

$$\Phi(u)\ge\rho_0,\qquad \|u\|=\delta_0. \tag{2.18}$$

Choose $v_k\in\mathcal{N}\subset\mathcal{E}$ such that

$$c\le\Phi(v_k)<c+\frac{1}{k},\qquad k\in\mathbb{N}. \tag{2.19}$$

For Lemma 2.1, it is easy to see that $\Phi(tv_k) < 0$ for large $t > 0$. In fact, if $\Phi(tv_k) \geq 0$ for large $t > 0$. By (2.4) and $v_k \in \mathcal{N}$, we have

$$\Phi(v_k) \geq \Phi(tv_k) + \frac{(1-\theta_0)(1-t^2)^2}{4}\|v_k\|^2, \tag{2.20}$$

which contradicts (2.19). From the mountain pass lemma, there exists a sequence $\{u_{k,n}\}_{n\in\mathbb{N}} \subset E$ satisfying

$$\Phi(u_{k,n}) \to c_k, \qquad \|\Phi'(u_{k,n})\|\bigl(1+\|u_{k,n}\|\bigr) \to 0, \quad k \in \mathbb{N}, \tag{2.21}$$

where $c_k \in [\rho_0, \sup_{t\geq 0}\Phi(tv_k)]$. By (2.7) and (2.19), one has

$$\rho_0 \leq c_k \leq \sup_{t\geq 0}\Phi(tv_k) = \Phi(v_k) < c + \frac{1}{k}, \quad k \in \mathbb{N}. \tag{2.22}$$

Therefore, by (2.21) and (2.22), for $k \in \mathbb{N}$, one has

$$\Phi(u_{k,n}) \to c_k \in \left[\rho_0, c + \frac{1}{k}\right), \qquad \|\Phi'(u_{k,n})\|\bigl(1+\|u_{k,n}\|\bigr) \to 0, \quad \text{as } n \to \infty. \tag{2.23}$$

In view of (2.23), for $k = 1$, there exists $n_1 > 0$ large enough such that

$$\rho_0 \leq \Phi(u_{1,n_1}) < c_1 + 1, \qquad \|\Phi'(u_{1,n_1})\|\bigl(1+\|u_{1,n_1}\|\bigr) < 1; \tag{2.24}$$

for $k = 2$, there exists $n_2 > n_1 > 0$ large enough such that

$$\rho_0 \leq \Phi(u_{2,n_2}) < c_2 + 1/2, \qquad \|\Phi'(u_{2,n_2})\|\bigl(1+\|u_{2,n_2}\|\bigr) < 1/2. \tag{2.25}$$

In this way, we can choose a sequence $\{n_k\} \subset \mathbb{N}$ with $n_k \to \infty$ as $k \to \infty$ such that

$$\Phi(u_{k,n_k}) \in \left[\rho_0, c + \frac{1}{k}\right), \qquad \|\Phi'(u_{k,n_k})\|\bigl(1+\|u_{k,n_k}\|\bigr) < \frac{1}{k}, \quad k \in \mathbb{N}. \tag{2.26}$$

Let $u_k = u_{k,n_k}$, $k \in \mathbb{N}$. Then, going if necessary to a subsequence, we conclude from (2.26) that

$$\Phi(u_n) \to c_*, \qquad \|\Phi'(u_n)\|\bigl(1+\|u_n\|\bigr) \to 0. \tag{2.27}$$

□

Lemma 2.8 *Assume that* (V) *and* (F1)-(F4) *hold. Then any sequence* $\{u_n\} \subset E$ *satisfying* (2.17) *is bounded in* E.

Proof By (2.2), (2.10) and (2.17), one has

$$\begin{aligned} c_* + o(1) &= \Phi(u_n) - \frac{1}{4}\langle\Phi'(u_n), u_n\rangle \\ &\geq \frac{1-\theta_0}{4}\|u_n\|^2 + \int_{\mathbb{R}^N}\left(\frac{1}{4}f(x,u)u - F(x,u) + \frac{\theta_0 V(x)}{4}u^2\right)\mathrm{d}x \\ &\geq \frac{1-\theta_0}{4}\|u_n\|^2. \end{aligned} \tag{2.28}$$

This shows that the sequence $\{u_n\}$ is bounded in E. □

Next, we prove the minimizer of the constrained problem is a critical point, which plays a crucial role in the asymptotically periodic case.

Lemma 2.9 *Assume that* (V) *and* (F1)-(F4) *hold. If* $u_0 \in \mathcal{N}$ *and* $\Phi(u_0) = c$, *then* u_0 *is a critical point of* Φ.

Proof Analogous to the proof of [21], it is easy to show this lemma by combining the quantitative deformation lemma and degree theory. □

3 The periodic case

Proof of Theorem 1.1 Lemma 2.7 implies the existence of a sequence $\{u_n\} \subset E$ satisfying (2.17), then

$$\Phi(u_n) \to c_*, \qquad \|\Phi'(u_n)\|\big(1 + \|u_n\|\big) \to 0. \tag{3.1}$$

By Lemma 2.8, $\{u_n\}$ is bounded in E. Thus, there exists $C > 0$ such that $\|u_n\|_2 \le C$. If

$$\delta := \limsup_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_1(y)} |u_n|^2 \,\mathrm{d}x = 0, \tag{3.2}$$

then by Lion's concentration compactness principle [46], we have $u_n \to 0$ in $L^q(\mathbb{R}^N)$ for $2 < q < 2^*_\alpha$. By (F1) and (F2), for $\varepsilon = c_*/2C^2$, there exists $C_\varepsilon > 0$ such that

$$\limsup_{n\to\infty} \int_{\mathbb{R}^N} \left|\frac{1}{2} f(x,u_n)u_n - F(x,u_n)\right| \mathrm{d}x \le \frac{3}{2}\varepsilon C^2 + C_\varepsilon \lim_{n\to\infty} \|u_n\|_p^p = \frac{3c_*}{4}. \tag{3.3}$$

From (2.1), (2.2), (3.1) and (3.3), one has

$$\begin{aligned} c_* &= \Phi(u_n) - \frac{1}{2}\big\langle \Phi'(u_n), u_n\big\rangle + o(1) \\ &= -\frac{b}{4}\big\|(-\triangle)^{\frac{\alpha}{2}} u_n\big\| + \int_{\mathbb{R}^N} \left[\frac{1}{2} f(x,u_n)u_n - F(x,u_n)\right] \mathrm{d}x \\ &\le \frac{3c_*}{4}. \end{aligned}$$

This contradiction shows $\delta > 0$.

Going if necessary to a subsequence, we may assume the existence of $k_n \in \mathbb{Z}^N$ such that

$$\int_{B_2(k_n)} |u_n|^2 \,\mathrm{d}x > \frac{\delta}{2}. \tag{3.4}$$

Let $v_n(x) = u_n(x + k_n)$, then

$$\int_{B_2(0)} |v_n|^2 \,\mathrm{d}x > \frac{\delta}{2}. \tag{3.5}$$

Since $V(x)$ and $f(x,u)$ are periodic on x, we have

$$\Phi(v_n) \to c_*, \qquad \|\Phi'(v_n)\|\big(1 + \|v_n\|\big) \to 0. \tag{3.6}$$

Passing to a subsequence, we have $v_n \rightharpoonup \bar{v}$ in E, $v_n \to \bar{v}$ in $L^q_{\mathrm{loc}}(\mathbb{R}^N)$ for $2 < q < 2^*_\alpha$ and $v_n \to \bar{v}$ a.e. on $\mathbb{R}^N$. Thus, (3.5) implies that $\bar{v} \neq 0$. Let $l := \lim_{n\to\infty} \|(-\triangle)^{\frac{\alpha}{2}}\bar{v}_n\|_2$, then the weakly lower semi-continuous of the norm implies that $\|(-\triangle)^{\frac{\alpha}{2}}\bar{v}\|_2 \le l$. It is easy to check that

$$\begin{aligned}&\langle \Phi'(\bar{v}), \varphi\rangle + \left(l^2 - \left\|(-\triangle)^{\frac{\alpha}{2}}\bar{v}\right\|_2^2\right)\int_{\mathbb{R}^N}(-\triangle)^{\frac{\alpha}{2}}\bar{v}(-\triangle)^{\frac{\alpha}{2}}\varphi\,\mathrm{d}x\\&\quad = \lim_{n\to\infty}\langle \Phi'(\bar{v}_n), \varphi\rangle = 0, \quad \forall \varphi \in \mathcal{C}_0^\infty(\mathbb{R}^N).\end{aligned} \tag{3.7}$$

Since $\mathcal{C}_0^\infty(\mathbb{R}^N)$ is dense in E, (3.7) implies $\langle \Phi'(\bar{v}), \varphi\rangle \le 0$. We claim that $\langle \Phi'(\bar{v}), \bar{v}\rangle = 0$. In fact, if $\langle \Phi'(\bar{v}), \bar{v}\rangle < 0$, together with $\langle \Phi'(t\bar{v}), t\bar{v}\rangle > 0$ for small $t > 0$, then there exists $t_1 \in (0,1)$ such that $\langle \Phi'(t_1\bar{v}), t_1\bar{v}\rangle = 0$ and $\Phi(t_1\bar{v}) \ge c$. Using (F4), we have

$$f(x,t\tau)t\tau \le f(x,\tau)\tau t^4 + \theta_0 V(x)\left(1-t^2\right)(t\tau)^2, \quad \forall x \in \mathbb{R}^N, 0 \le t \le 1, \tau \in \mathbb{R}. \tag{3.8}$$

Note that (2.5) implies

$$\begin{aligned}F(x,t\tau) \ge{}& \frac{t^4-1}{4}f(x,\tau)\tau + F(x,\tau)\\&- \frac{1-2t+t^4}{4}\theta_0 V(x)\tau^2, \quad \forall x \in \mathbb{R}^N, 0 \le t \le 1, \tau \in \mathbb{R}.\end{aligned} \tag{3.9}$$

Combining (3.8) and (3.9), we have

$$\begin{aligned}&\frac{1}{4}f(x,t\tau)t\tau - F(x,t\tau) + \frac{\theta_0 V(x)}{4}(t\tau)^2\\&\quad \le \frac{1}{4}f(x,\tau)\tau - F(x,\tau) + \frac{\theta_0 V(x)}{4}\tau^2, \quad \forall x \in \mathbb{R}^N, 0 \le t \le 1, \tau \in \mathbb{R}.\end{aligned} \tag{3.10}$$

Then by (2.1), (2.2), (3.6) and (3.10), the weakly lower semi-continuity of the norm and Fatou's lemma, we have

$$\begin{aligned}c &\le \Phi(t_1\bar{v}) - \frac{1}{4}\langle \Phi'(t_1\bar{v}), t_1\bar{v}\rangle\\&= \frac{a\theta_0}{4}t_1^2\left\|(-\triangle)^{\frac{\alpha}{2}}\bar{v}\right\|_2^2 + \frac{1-\theta_0}{4}t_1^2\|\bar{v}\|^2\\&\quad + \int_{\mathbb{R}^N}\left[\frac{1}{4}f(x,t_1\bar{v})t_1\bar{v} - F(x,t_1\bar{v}) + \frac{\theta_0 V(x)}{4}(t_1\bar{v})^2\right]\mathrm{d}x\\&< \frac{a\theta_0}{4}\left\|(-\triangle)^{\frac{\alpha}{2}}\bar{v}\right\|_2^2 + \frac{1-\theta_0}{4}\|\bar{v}\|^2 + \int_{\mathbb{R}^N}\left[\frac{1}{4}f(x,\bar{v})\bar{v} - F(x,\bar{v}) + \frac{\theta_0 V(x)}{4}(\bar{v})^2\right]\mathrm{d}x\\&\le \lim_{n\to\infty}\left\{\frac{a\theta_0}{4}\left\|(-\triangle)^{\frac{\alpha}{2}}v_n\right\|_2^2 + \frac{1-\theta_0}{4}\|v_n\|^2\right.\\&\quad \left. + \int_{\mathbb{R}^N}\left[\frac{1}{4}f(x,v_n)v_n - F(x,v_n) + \frac{\theta_0 V(x)}{4}v_n^2\right]\mathrm{d}x\right\}\\&= \lim_{n\to\infty}\left\{\Phi(v_n) - \frac{1}{4}\langle \Phi'(v_n), v_n\rangle\right\} \le c,\end{aligned} \tag{3.11}$$

which is impossible. Thus, we get $\bar{v} \in \mathcal{N}$ and $\Phi(\bar{v}) \ge c$. Jointly with (3.7), we have $\lim_{n\to\infty}\|(-\triangle)^{\frac{\alpha}{2}}v_n\|_2 = \|(-\triangle)^{\frac{\alpha}{2}}\bar{v}\|_2$ and $\Phi'(\bar{v}) = 0$. On the other hand, from (2.10), (3.6), the weakly lower semi-continuous of norm and Fatou's lemma, one has

$$\begin{aligned}
c \geq c_* &= \Phi(v_n) - \frac{1}{4}\langle \Phi'(v_n), v_n \rangle \\
&= \lim_{n\to\infty}\left\{ \frac{a\theta_0}{4}\left\|(-\triangle)^{\frac{\alpha}{2}} v_n\right\|_2^2 + \frac{1-\theta_0}{4}\|v_n\|^2 \right. \\
&\quad \left. + \int_{\mathbb{R}^N}\left[\frac{1}{4}f(x,v_n)v_n - F(x,v_n) + \frac{\theta_0 V(x)}{4}v_n^2\right]\mathrm{d}x \right\} \\
&\geq \liminf_{n\to\infty}\left[\frac{a\theta_0}{4}\left\|(-\triangle)^{\frac{\alpha}{2}} v_n\right\|_2^2 + \frac{1-\theta_0}{4}\|v_n\|^2\right] \\
&\quad + \liminf_{n\to\infty}\int_{\mathbb{R}^N}\left[\frac{1}{4}f(x,v_n)v_n - F(x,v_n) + \frac{\theta_0 V(x)}{4}v_n^2\right]\mathrm{d}x \\
&\geq \frac{1}{4}\|\bar{v}\|^2 + \int_{\mathbb{R}^N}\left[\frac{1}{4}f(x,\bar{v})v - F(x,\bar{v})\right]\mathrm{d}x \\
&= \Phi(\bar{v}) - \frac{1}{4}\langle \Phi'(\bar{v}), \bar{v}\rangle = \Phi(\bar{v}).
\end{aligned} \tag{3.12}$$

This shows that $\Phi(\bar{v}) \leq c$ and so $\Phi(\bar{v}) = c = \inf_{\mathcal{N}} \Phi > 0$. □

4 The asymptotically periodic case

In this section, we have $V(x) = V_0(x) + V_1(x)$ and $f(x,u) = f_0(x,u) + f_1(x,u)$. Define the functional Φ_0 as follows:

$$\begin{aligned}
\Phi_0(u) &= \frac{1}{2}\int_{\mathbb{R}^N}\left(a\left|(-\triangle)^{\frac{\alpha}{2}}u\right|^2 + V_0(x)u^2\right)\mathrm{d}x \\
&\quad + \frac{b}{4}\left(\int_{\mathbb{R}^N}\left|(-\triangle)^{\frac{\alpha}{2}}u\right|^2\mathrm{d}x\right)^2 - \int_{\mathbb{R}^N}F_0(x,u)\,\mathrm{d}x, \quad \forall u \in E,
\end{aligned} \tag{4.1}$$

where $F_0(x,u) := \int_{\mathbb{R}^N} f_0(x,s)\,ds$. By (V0′), (F0′), (F1) and (F2), we have $\Phi_0 \in \mathcal{C}^1(E,\mathbb{R})$ and

$$\begin{aligned}
\langle \Phi_0'(u), v\rangle &= \int_{\mathbb{R}^N}\left(a(-\triangle)^{\frac{\alpha}{2}}u(-\triangle)^{\frac{\alpha}{2}}v + V_0(x)uv\right)\mathrm{d}x \\
&\quad + b\int_{\mathbb{R}^N}\left|(-\triangle)^{\frac{\alpha}{2}}u\right|^2\mathrm{d}x\int_{\mathbb{R}^N}(-\triangle)^{\frac{\alpha}{2}}u(-\triangle)^{\frac{\alpha}{2}}v\,\mathrm{d}x \\
&\quad - \int_{\mathbb{R}^N} f_0(x,u)v\,\mathrm{d}x, \quad \forall u,v \in E.
\end{aligned} \tag{4.2}$$

By a standard argument, we have the following lemma.

Theorem 4.1 *Assume that* (V0′), (F0′), (F1) *and* (F2) *hold. If* $u_n \rightharpoonup 0$ *in* E, *then*

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} V_1(x)u_n^2\,\mathrm{d}x = 0, \qquad \lim_{n\to\infty}\int_{\mathbb{R}^N} V_1(x)u_n v\,\mathrm{d}x = 0, \quad \forall v \in E; \tag{4.3}$$

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} F_1(x,u_n)\,\mathrm{d}x = 0, \qquad \lim_{n\to\infty}\int_{\mathbb{R}^N} f_1(x,u_n)v\,\mathrm{d}x = 0, \quad \forall v \in E. \tag{4.4}$$

Proof of Theorem 1.2 Lemma 2.7 implies the existence of a sequence $\{u_n\} \subset E$ satisfying (2.17), then

$$\Phi(u_n) \to c_*, \qquad \|\Phi'(u_n)\|(1 + \|u_n\|) \to 0. \tag{4.5}$$

By Lemma 2.8, $\{u_n\}$ is bounded in E. Thus, there exists $C > 0$ such that $\|u_n\|_2 \le C$. Passing to a subsequence, we have $u_n \rightharpoonup \bar{u}$ in E, $u_n \to \bar{u}$ in $L^q_{\mathrm{loc}}(\mathbb{R}^N)$ for $2 \le q < 2^*_\alpha$ and $u_n \to \bar{u}$ a.e. on $\mathbb{R}^N$. There are two possible cases: (i) $\bar{u} = 0$; (ii) $\bar{u} \neq 0$.

Case i). $\bar{u} = 0$. Then $u_n \rightharpoonup 0$ in E, $u_n \to 0$ in $L^q_{\mathrm{loc}}(\mathbb{R}^N)$ for $2 \le q < 2^*_\alpha$ and $u_n \to 0$ a.e. on $\mathbb{R}^N$. Note that

$$\|u\|^2 = \int_{\mathbb{R}^N} \bigl(a\bigl|(-\triangle)^{\frac{\alpha}{2}} u\bigr|^2 + V_0(x)u^2\bigr)\,\mathrm{d}x + \int_{\mathbb{R}^N} V_1(x)u^2\,\mathrm{d}x, \quad \forall u \in E, \tag{4.6}$$

$$\Phi_0(u) = \Phi(u) - \frac{1}{2}\int_{\mathbb{R}^N} V_1(x)u^2\,\mathrm{d}x + \int_{\mathbb{R}^N} F_1(x,u)\,\mathrm{d}x, \quad \forall u \in E, \tag{4.7}$$

and

$$\langle \Phi_0'(u), v\rangle = \langle \Phi'(u), v\rangle - \int_{\mathbb{R}^N} V_1(x)uv\,\mathrm{d}x + \int_{\mathbb{R}^N} f_1(x,u)v\,\mathrm{d}x, \quad \forall u, v \in E. \tag{4.8}$$

By (2.17), (4.3), (4.4), (4.6)-(4.8), one has

$$\Phi_0(u_n) \to c_*, \qquad \|\Phi_0'(u_n)\|(1 + \|u_n\|) \to 0. \tag{4.9}$$

Analogous to the proof of (3.4), there exists $k_n \in \mathbb{Z}^N$, going if necessary to a subsequence, such that

$$\int_{B_2(k_n)} |u_n|^2\,\mathrm{d}x > \frac{\delta}{2}. \tag{4.10}$$

Let us define $v_n(x) = u_n(x + k_n)$. Then

$$\int_{B_2(0)} |v_n|^2\,\mathrm{d}x > \frac{\delta}{2}. \tag{4.11}$$

Since $V(x)$ and $f(x,u)$ are periodic on x, we have

$$\Phi_0(v_n) \to c_* \in (0, c], \qquad \|\Phi_0'(v_n)\|(1 + \|v_n\|) \to 0. \tag{4.12}$$

Passing to a subsequence, we have $v_n \rightharpoonup \bar{v}$ in E, $v_n \to \bar{v}$ in $L^q_{\mathrm{loc}}(\mathbb{R}^N)$ for $2 \le q < 2^*_\alpha$ and $v_n \to \bar{v}$ a.e. on $\mathbb{R}^N$. Thus, (4.10) implies that $\bar{v} \neq 0$. Similar to Corollary 2.3 and Lemma 2.6, by (2.4), (4.7) and (4.8), we can deduce that

$$\Phi_0(u) = \max_{t \ge 0} \Phi_0(tu), \quad \forall u \in \mathcal{N}_0, \qquad \inf_{u \in \mathcal{N}} \Phi_0(u) = c_0 = \inf_{u \in \mathcal{E}_0} \max_{t \ge 0} \Phi_0(tu) > 0, \tag{4.13}$$

where

$$\mathcal{N}_0 := \{u \in E : \langle \Phi_0'(u), u\rangle = 0, u \neq 0\},$$

$$\mathcal{E}_0 = \left\{u \in E : b\bigl\|(-\triangle)^{\frac{\alpha}{2}} u\bigr\|_2^4 + \int_{\mathbb{R}^N} \bigl[V_0(x)u^2 - f_0(x,u)u\bigr]\,\mathrm{d}x < 0\right\}.$$

In view of Theorem 1.1, there exists $v_0 \in \mathcal{N}_0 \subset \mathcal{E}_0$ such that $c_0 = \Phi(v_0)$. Then, from (F0′), (4.7) and (4.13), we get

$$c = \inf_{u\in\mathcal{E}} \max_{t\geq 0} \Phi(tv) \leq \max_{t\geq 0} \Phi(tv_0) \leq \max_{t\geq 0} \Phi_0(tv_0) \leq \Phi_0(v_0) = c_0. \tag{4.14}$$

Similar to (3.7), we have $\langle \Phi_0'(\bar{v}), \bar{v}\rangle \leq 0$. If $\langle \Phi_0'(\bar{v}), \bar{v}\rangle < 0$, together with $\langle \Phi_0'(t\bar{v}), t\bar{v}\rangle > 0$ for small $t > 0$, we see that there exists $t_2 \in (0,1)$ such that $\langle \Phi_0'(t_2\bar{v}), t_2\bar{v}\rangle = 0$ and $\Phi(t_2\bar{v}) \geq c_0$. By (F0′), we have

$$f_0(x, t\tau)t\tau \leq f_0(x,\tau)\tau t^4 + V_0(x)\bigl(1-t^2\bigr)(t\tau)^2, \quad \forall x \in \mathbb{R}^N, 0 \leq t \leq 1, \tau \in \mathbb{R}, \tag{4.15}$$

and

$$\begin{aligned}
&\frac{1-t^4}{4}\tau f_0(x,\tau) + F_0(x,t\tau) - F_0(x,\tau) + \frac{V_0(x)}{4}\bigl(1-t^2\bigr)^2\tau^2 \\
&\quad = \int_t^1 \left[\frac{f_0(x,\tau)}{\tau^3} - \frac{f_0(x,\xi\tau)}{(\xi\tau)^3} + V_0(x)\frac{1-\xi^2}{(\xi\tau)^2}\right]\xi^3\tau^4 \,\mathrm{d}\xi \\
&\quad \geq 0, \quad \forall x \in \mathbb{R}^N, t \geq 0, \tau \in \mathbb{R}\backslash\{0\}.
\end{aligned} \tag{4.16}$$

Combining (4.15) and (4.16), we have

$$\begin{aligned}
&\frac{1}{4}f_0(x,t\tau)t\tau - F_0(x,t\tau) + \frac{V(x)}{4}(t\tau)^2 \\
&\quad \leq \frac{1}{4}f_0(x,\tau)\tau - F_0(x,\tau) + \frac{V_0(x)}{4}\tau^2, \quad \forall x \in \mathbb{R}^N, 0 \leq t \leq 1, \tau \in \mathbb{R}.
\end{aligned} \tag{4.17}$$

It follows from (4.16) with $t = 0$ that

$$\frac{1}{4}f_0(x,\tau)\tau - F_0(x,\tau) + \frac{V_0(x)}{4}\tau^2 \geq 0, \quad \forall x \in \mathbb{R}^N, \tau \in \mathbb{R}. \tag{4.18}$$

Since $t_2 \in (0,1)$, from (4.1), (4.2), (4.12), (4.14), (4.17) and (4.18), the weakly lower semicontinuity of the norm and Fatou's lemma, we have

$$\begin{aligned}
c_0 &\leq \Phi_0(t_2\bar{v}) - \frac{1}{4}\bigl\langle \Phi_0'(t_2\bar{v}), t_2\bar{v}\bigr\rangle \\
&= \frac{a}{4}t_2^2\bigl\|(-\triangle)^{\frac{\alpha}{2}}\bar{v}\bigr\|_2^2 + \int_{\mathbb{R}^N}\left[\frac{1}{4}f_0(x,t_2\bar{v})t_2\bar{v} - F_0(x,t_2\bar{v}) + \frac{V_0(x)}{4}(t_2\bar{v})^2\right]\mathrm{d}x \\
&< \frac{a}{4}\bigl\|(-\triangle)^{\frac{\alpha}{2}}\bar{v}\bigr\|_2^2 + \int_{\mathbb{R}^N}\left[\frac{1}{4}f_0(x,\bar{v})\bar{v} - F_0(x,\bar{v}) + \frac{V_0(x)}{4}(\bar{v})^2\right]\mathrm{d}x \\
&\leq \lim_{n\to\infty}\left\{\frac{a}{4}\bigl\|(-\triangle)^{\frac{\alpha}{2}}v_n\bigr\|_2^2 + \int_{\mathbb{R}^N}\left[\frac{1}{4}f_0(x,v_n)v_n - F_0(x,v_n) + \frac{V_0(x)}{4}v_n^2\right]\mathrm{d}x\right\} \\
&= \lim_{n\to\infty}\left\{\Phi_0(v_n) - \frac{1}{4}\bigl\langle \Phi_0'(v_n), v_n\bigr\rangle\right\} \leq c \leq c_0,
\end{aligned} \tag{4.19}$$

which is impossible. Thus, we get $\langle \Phi_0'(\bar{v}), \bar{v}\rangle = 0$ and $\Phi_0(\bar{v}) \geq c_0$. By a standard argument, we have $\Phi_0'(\bar{v}) = 0$. On the other hand, from (4.1), (4.2), (4.12) and (4.18), the weakly lower semi-continuity of the norm and Fatou's lemma, we obtain

$$\begin{aligned} c \geq c_* &= \lim_{n\to\infty}\left[\Phi_0(v_n) - \frac{1}{4}\langle\Phi_0'(v_n), v_n\rangle\right] \\ &= \lim_{n\to\infty}\left\{\frac{a}{4}\left\|(-\triangle)^{\frac{\alpha}{2}}v_n\right\|_2^2 + \int_{\mathbb{R}^N}\left[\frac{1}{4}f_0(x,v_n)v_n - F_0(x,v_n) + \frac{V_0(x)}{4}v_n^2\right]\mathrm{d}x\right\} \\ &\geq \liminf_{n\to\infty}\frac{a}{4}\left\|(-\triangle)^{\frac{\alpha}{2}}v_n\right\|_2^2 + \liminf_{n\to\infty}\int_{\mathbb{R}^N}\left[\frac{1}{4}f_0(x,v_n)v_n - F_0(x,v_n) + \frac{V_0(x)}{4}v_n^2\right]\mathrm{d}x \\ &\geq \frac{1}{4}\|\bar{v}\|^2 + \int_{\mathbb{R}^N}\left[\frac{1}{4}f_0(x,\bar{v})v - F_0(x,\bar{v})\right]\mathrm{d}x \\ &= \Phi_0(\bar{v}) - \frac{1}{4}\langle\Phi_0'(\bar{v}),\bar{v}\rangle = \Phi_0(\bar{v}). \end{aligned} \tag{4.20}$$

This implies that $\Phi_0(\bar{v}) \leq c$. By (F0′), we have $\langle\Phi'(\bar{v}),\bar{v}\rangle \leq \langle\Phi_0'(\bar{v}),\bar{v}\rangle = 0$, which implies $\bar{v} \in \mathcal{E}$. Then, by Lemma 2.5, there exists $t_0 = t(\bar{v})$ such that $t_0\bar{v} \in \mathcal{N} \geq c$, and so $\Phi(t_0\bar{v}) \geq c$. Now we prove that $\Phi(t_0\bar{v}) = c$. Arguing indirectly, we assume that $\Phi(t_0\bar{v}) > c$. Then by (V0′), (F0′), (4.1), (4.2) and (4.16), we have

$$\begin{aligned} c &\geq \Phi_0(\bar{v}) \\ &= \Phi_0(t_0\bar{v}) + \frac{1-t_0^4}{4}\langle\Phi_0'(\bar{v}),\bar{v}\rangle + \frac{a(1-t_0)^2}{4}\int_{\mathbb{R}^N}\left|(-\triangle)^{\frac{\alpha}{2}}u\right|^2\mathrm{d}x \\ &\quad + \int_{\mathbb{R}^N}\left[\frac{1-t_0^4}{4}\tau f_0(x,\bar{v}) + F_0(x,t_0\bar{v}) - F_0(x,\bar{v}) + \frac{V_0(x)}{4}\left(1-t_0^2\right)^2(\bar{v})^2\right]\mathrm{d}x \\ &\geq \Phi_0(t_0\bar{v}) = \Phi(t_0\bar{v}) - \frac{1}{2}\int_{\mathbb{R}^N}V_1(x)(t_0\bar{v})^2\,\mathrm{d}x + \int_{\mathbb{R}^N}F_1(x,t_0\bar{v})\,\mathrm{d}x \\ &\geq \Phi(t_0\bar{v}) > c. \end{aligned} \tag{4.21}$$

This contradiction shows that $\Phi(t_0\bar{v}) = c$.

Let $u_0 = t_0\bar{v}$. Then $u_0 \in \mathcal{N}$ and $\Phi(u_0) = c$. In view of Lemma 2.9, we have $\Phi'(u_0) = 0$. This shows that $u_0 \in E$ is a solution for (1.1) with $\Phi(u_0) = \inf_{\mathcal{N}}\Phi > 0$.

Case ii). $\bar{u} \neq 0$. By the same fashion as the last part of the proof of Theorem 1.1, we can prove that $\Phi'(\bar{u}) = 0$ and $\Phi(\bar{u}) = c = \inf_{\mathcal{N}}\Phi$. This shows that $\bar{u} \in E$ is a solution for (1.1) with $\Phi(\bar{u}) = \inf_{\mathcal{N}}\Phi$. □

5 Conclusion

In this paper, by using the variational methods and some weaker conditions, the existence of Nehari-type solutions to equation (1.1) is established. We consider periodic or asymptotically periodic fractional Kirchhoff problems with more general nonlinearity f in $\mathbb{R}^N$, where $2\alpha < N < 4\alpha$ and $\alpha \in (0,1)$, especially f unifies asymptotically cubic and super-cubic nonlinearity, which generalizes and improves the previous results. Meanwhile, (1.1) is a nonlocal problem, so we need to overcome some new difficulties, which involves some new approaches and analytical techniques in our paper.

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