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Subelliptic equations with singular nonlinearities on the Heisenberg group

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Abstract

In this paper, we consider the Dirichlet boundary value problem to singular semilinear subelliptic equation on the Heisenberg group

$$-\Delta_{\mathbb{H}} u = \frac{1}{u^\gamma} + f(u), \quad \gamma > 0.$$

We prove the positivity and continuity up to the boundary for the weak solutions. We also conclude monotonicity of cylindrical solutions to the problem based on a study of the equation $-\Delta_{\mathbb{H}} u_0 = \frac{1}{u_0^\gamma}$. The main technique is a generalization of the moving plane method to the Heisenberg group.

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1 Introduction

Crandall, Rabinowitz, and Tartar [1] considered the Dirichlet problem to elliptic equation with singular nonlinearity and deduced the existence and continuity of solutions. The Dirichlet problem on the sufficiently regular bounded domain to the elliptic equation

$$\Delta u + p(x) \frac{1}{u^\gamma} = 0, \quad \gamma > 0$$

was treated by Lazer and McKenna [2], and the existence of solutions with smoothing in the domain and continuity up to the boundary was given under the assumption $p(x) > 0$. Boccardo and Orsina [3] derived the existence, regularity, and nonexistence for the Dirichlet problem to the model

$$-\Delta u = \frac{f(x)}{u^\gamma},$$

where the results depend on γ and the summability of $f(x)$ in some Lebesgue spaces.

Canino and Degiovanni [4] applied a variational approach to the singular semilinear elliptic equation

$$-\Delta u = u^{-\beta} + g(x, u), \quad \beta > 0,$$

and obtained some properties of solutions u and u_0 , where u_0 is the solution of the Dirichlet problem to the equation

$$-\Delta u_0 = u_0^{-\beta}.$$

Based on the approach in [4] and the moving plane method, the monotonicity properties of weak solutions of the Dirichlet problem to the singular elliptic equation

$$-\Delta u = \frac{1}{u^\gamma} + f(u)$$

were proved by Canino, Grandinetti, and Sciunzi [5].

Along with the development of theory of degenerate elliptic partial differential equations, there were many attempts to extend various results of elliptic equations to subelliptic equations. Birindelli and Prajapat [6] derived a Liouville theorem for the nonnegative cylindrical solutions of the equation

$$\Delta_{\mathbb{H}^n} u + u^p = 0$$

in the Heisenberg group $\mathbb{H}^n = (\mathbb{R}^{2n+1}, \circ)$, where $\Delta_{\mathbb{H}^n}$ is the sub-Laplacian on \mathbb{H}^n , $0 < p < \frac{Q+2}{Q-2}$, by using the CR inversion to a regular function u . Xu [7] determined that there exist $C^\infty(\overline{\Omega})$ solutions of the Dirichlet problem

$$\begin{cases} \sum_{j=1}^m X_j^* X_j + cu = f(x, u) & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega, \end{cases}$$

where f and φ satisfy proper smooth conditions, $\{X_j\}$ ($j = 1, \dots, m$) is a system of vector fields satisfying Hörmander’s condition, Ω is a bounded domain, and its boundary $\partial\Omega$ is smooth and noncharacteristic for $\{X_j\}$. By extending the Perron method in the Euclidean space to the Carnot group and constructing barrier functions, the existence and uniqueness of solutions for the Dirichlet problem

$$\begin{cases} Lu + f(x, u) = 0 & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega, \end{cases}$$

where L is the sub-Laplacian on the Carnot group, were established by Yuan and Yuan [8]. We also refer the readers to [9] and [10] for the related results.

The purpose of this paper is to consider the Dirichlet problem in \mathbb{H}^n to the subelliptic equation with singular nonlinearities

$$\begin{cases} -\Delta_{\mathbb{H}^n} u = \frac{1}{u^\gamma} + f(u), & \xi \in \Omega, \\ u = 0, & \xi \in \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded convex domain (see [11, 12]) in \mathbb{H}^n with boundary $\partial\Omega$ satisfying the Wiener criterion (for any $a \in (0, 1)$, let the C_j be the capacity of the set $B_{\mathbb{H}^n}(\xi_0, a^j) \cap \Omega^C$; then $\xi_0 \in \partial\Omega$ is a regular point if and only if $\sum_{j=0}^\infty C_j/a^{j(Q-2)}$ diverges; see [13, 14] etc.), $\gamma > 0$, and f satisfies the following condition:

(F) f is a nonnegative nondecreasing locally Lipschitz function, $f(s) > 0$ for any $s > 0$, and $f(0) \geq 0$.

We say that $u \in H^1(\Omega)$ is a weak solution to (1.1) if for any $\varphi \in C_0^\infty(\Omega)$,

$$\int_{\Omega} \nabla_{\mathbb{H}} u \cdot \nabla_{\mathbb{H}} \varphi = \int_{\Omega} \left(\frac{1}{u^\gamma} + f \right) \varphi.$$

We first consider the Dirichlet problem

$$\begin{cases} -\Delta_{\mathbb{H}} u_0 = \frac{1}{u_0^\gamma}, & \xi \in \Omega, \\ u_0 = 0, & \xi \in \partial\Omega. \end{cases} \tag{1.2}$$

By borrowing the method in [3] to our setting we prove that existence, uniqueness, and boundedness of weak positive solutions u_n for the approximation problem

$$\begin{cases} -\Delta_{\mathbb{H}} u_n = \frac{1}{(u_n + \frac{1}{n})^\gamma}, & \xi \in \Omega, \\ u_n = 0, & \xi \in \partial\Omega, \end{cases} \tag{1.3}$$

for each fixed n (the bound of u_n in $H_0^1(\Omega)$ is local when $\gamma > 1$), and then we obtain the existence and uniqueness of a weak solution u of (1.2) as the limit of an increasing sequence of positive solutions $\{u_n\}$. Since $\partial\Omega$ satisfies the Wiener criterion, we use the sub-super solution method similarly to [4] and get that the positive solution u_0 of (1.2) is continuous up to $\partial\Omega$. The monotonicity of u_0 is shown via the moving plane method.

Similarly to the method of energy functional in [4], we define the functional $F_f : L^2(\Omega) \rightarrow (-\infty, +\infty]$ to problem (1.1) by

$$F_f(u) = \begin{cases} \frac{1}{2} \int_{\Omega} |\nabla_{\mathbb{H}}(u - u_0)|^2 + \int_{\Omega} G_0(\xi, u - u_0) - \int_{\Omega} f(\xi, u - u_0) & \text{if } u \in u_0 + H_0^1(\Omega), \\ +\infty & \text{otherwise,} \end{cases}$$

where $G_0 : \Omega \times \mathbb{R} \rightarrow [0, +\infty]$ is the Borel function

$$G_0(\xi, s) = \Phi(u_0(\xi) + s) - \Phi(u_0(\xi)) + s u_0^{-\beta}(\xi)$$

with

$$\Phi(s) = \begin{cases} -\int_1^s t^{-\beta} & \text{if } s \geq 0, \\ +\infty & \text{if } s < 0. \end{cases}$$

It follows that $G_0(\xi, 0) = 0$ and $G_0(\xi, \cdot)$ is convex and lower semicontinuous for any $\xi \in \Omega$ and that F_f is convex, lower semicontinuous, and coercive with $F_f(u_0) = 0$. The effective domain of F_f is $\{u \in L^2(\Omega) : F_f(u) < +\infty\}$, which is independent of f . Moreover, if u_0 is the minimum of F_0 , then the effective domain of F_f is

$$\{u \in u_0 + H_0^1(\Omega)\} \subseteq H_{loc}^1(\Omega),$$

and we see that $u \in H^1_{\text{loc}}(\Omega)$ is a weak solution to (1.1). Motivated by the idea in [5], we use the decomposition

$$u = u_0 + w$$

and demonstrate two maximum principles for w satisfying

$$-\Delta_{\mathbb{H}} w = -\Delta_{\mathbb{H}} u + \Delta_{\mathbb{H}} u_0. \tag{1.4}$$

Then the monotonicity of w is derived. Based on this, we arrive at the monotonicity of u .

To obtain the monotonicity of u , we apply the moving plane method on \mathbb{H}^n in [6] (see also [15–18]) by adopting the simpler way of [15] in the proof. Since $\Delta_{\mathbb{H}}$ is not invariant with respect to the general reflection, Birindelli and Prajapat [6] introduced the so-called H-reflection and pointed out that $\Delta_{\mathbb{H}}$ is invariant with respect to the H-reflection. Applying the H-reflection and the moving plane method, we prove that the cylindrical solution of (1.1) is monotone.

Our main result is the following:

Theorem 1.1 *Let Ω be a bounded convex domain symmetric with respect to T_{λ_1} about H-reflection in \mathbb{H}^n such that $\partial\Omega$ satisfies the Wiener criterion. If $u \in H^1_0(\Omega)$ ($\gamma \leq 1$) or $u \in H^1_{\text{loc}}(\Omega)$ ($\gamma > 1$) is a weak solution to (1.1), then u is positive, and $u \in C(\overline{\Omega})$. If u is a positive cylindrical solution to (1.1), then for any $\lambda_0 < \lambda < \lambda_1$ (see Section 2),*

$$u(\xi) < u_{\lambda}(\xi) \quad \text{for any } \xi \in \Sigma_{\lambda} \tag{1.5}$$

with respect to t (the definitions of T_{λ_1} , $u_{\lambda}(\xi)$, and Σ_{λ} will be given in Section 2).

Since $\Delta_{\mathbb{H}}$ on \mathbb{H}^n is essentially different from Δ on \mathbb{R}^n , the proof of Theorem 1.1 is also essentially different from those for Δ on \mathbb{R}^n . Unlike the Euclidean space case, there is no Poincaré’s inequality for the narrow region in \mathbb{H}^n , so we utilize the fact that the volume of the narrow region is small enough to prove the maximum principle on the narrow region. Because the invariance of $\Delta_{\mathbb{H}}$ does not hold for the reflection $u_{\lambda}(x, y, t) = u(y, 2\lambda - x, t)$ or $u_{\lambda}(x, y, t) = u(2\lambda - y, x, t)$, the monotonicity of solutions to x or y cannot be established. The assumption that the boundary satisfies the Wiener criterion is weaker than the outer sphere or cone condition.

The paper is organized as follows. In Section 2, we collect some well-known facts about \mathbb{H}^n and the subelliptic operator $\Delta_{\mathbb{H}}$. In Section 3, we give the existence, regularity, and monotonicity of u_n to (1.3). Section 4 is devoted to the regularity and monotonicity of u_0 to (1.2) and to the interior gradient estimate for $u_0 \in C^3(\Omega)$. Finally, in Section 5, we derive the maximum principles of w and prove Theorem 1.1.

2 Preliminaries

In this section, we list some facts related to the Heisenberg group and sub-Laplacian $\Delta_{\mathbb{H}}$. For proofs and more details, we refer, for example, to [19–22].

The Heisenberg group \mathbb{H}^n is the Euclidean space \mathbb{R}^{2n+1} ($n \geq 1$) endowed with the group action \circ defined by

$$\xi_0 \circ \xi = \left(x + x_0, y + y_0, t + t_0 + 2 \sum_{i=1}^n (x_i y_{i_0} - y_i x_{i_0}) \right), \tag{2.1}$$

where $\xi = (x_1, \dots, x_n, y_1, \dots, y_n, t) := (x, y, t) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$, $\xi_0 = (x_0, y_0, t_0)$. We denote by δ_λ the dilations on \mathbb{R}^{2n+1} , that is,

$$\delta_\lambda(\xi) = (\lambda x, \lambda y, \lambda^2 t), \tag{2.2}$$

which satisfies $\delta_\lambda(\xi_0 \circ \xi) = \delta_\lambda(\xi_0) \circ \delta_\lambda(\xi)$.

The left-invariant vector fields corresponding to \mathbb{H}^n are of the form

$$\begin{aligned} X_i &= \frac{\partial}{\partial x_i} + 2y_i \frac{\partial}{\partial t}, & i = 1, \dots, n, \\ Y_i &= \frac{\partial}{\partial y_i} - 2x_i \frac{\partial}{\partial t}, & i = 1, \dots, n, \\ T &= \frac{\partial}{\partial t}. \end{aligned}$$

The Heisenberg gradient of a function u is defined as

$$\nabla_{\mathbb{H}} u = (X_1 u, \dots, X_n u, Y_1 u, \dots, Y_n u). \tag{2.3}$$

The sub-Laplacian $\Delta_{\mathbb{H}}$ on \mathbb{H}^n is

$$\Delta_{\mathbb{H}} = \sum_{i=1}^n X_i^2 + Y_i^2 \tag{2.4}$$

with the expansion

$$\Delta_{\mathbb{H}} = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2} + \frac{\partial^2}{\partial y_i^2} + 4y_i \frac{\partial^2}{\partial x_i \partial t} - 4x_i \frac{\partial^2}{\partial y_i \partial t} + 4(x_i^2 + y_i^2) \frac{\partial^2}{\partial t^2}.$$

It is easy to check that

$$[X_i, Y_j] = -4T\delta_{ij}, \quad [X_i, X_j] = [Y_i, Y_j] = 0, \quad i, j = 1, \dots, n,$$

and $\{X_1, \dots, X_n, Y_1, \dots, Y_n\}$ satisfies Hörmander’s rank condition (see [22]). In particular, this implies that $\Delta_{\mathbb{H}}$ is hypoelliptic (see [22]) and the solution of equation including $\Delta_{\mathbb{H}}$ satisfies the maximum principle (see [20, 23]).

Denote by $Q = 2n + 2$ the homogeneous dimension of \mathbb{H}^n . The norm $|\xi|_{\mathbb{H}}$ is the distance of ξ to the origin (see [21]):

$$|\xi|_{\mathbb{H}} = \left(\sum_{i=1}^n (x_i^2 + y_i^2) + t^2 \right)^{\frac{1}{4}}. \tag{2.5}$$

Afterward, we also denote $r = (|x|^2 + |y|^2)^{\frac{1}{2}}$. Using this norm, we can define the distance between two points in \mathbb{H}^n in the natural way:

$$d_{\mathbb{H}}(\xi, \eta) = |\eta^{-1} \circ \xi|_{\mathbb{H}},$$

where η^{-1} denotes the inverse of η with respect to the group action \circ , that is, $\eta^{-1} = -\eta$.

The open ball of radius $R > 0$ centered at ξ_0 is the set

$$B_{\mathbb{H}}(\xi_0, R) = \{\eta \in \mathbb{H}^n \mid d_{\mathbb{H}}(\eta, \xi_0) < R\}.$$

By the dilation of the group, $\xi \rightarrow |\xi|_{\mathbb{H}}$ is homogeneous of degree one with respect to δ_λ , and

$$|B_{\mathbb{H}}(\xi_0, R)| = |B_{\mathbb{H}}(0, R)| = |B_{\mathbb{H}}(0, 1)|R^Q,$$

where $|\cdot|$ denotes the Lebesgue measure. Noting that X_i and Y_i are homogeneous of degree minus one with respect to δ_λ , that is,

$$X_i(\delta_\lambda) = \lambda \delta_\lambda(X_i), \quad Y_i(\delta_\lambda) = \lambda \delta_\lambda(Y_i),$$

we see that $\Delta_{\mathbb{H}}$ is homogeneous of degree minus two and left invariant.

Define the Sobolev space

$$H^1(\Omega) = \{u \mid u, |\nabla_{\mathbb{H}} u| \in L^2(\Omega)\},$$

which is a Banach space with the norm

$$\|u\|_{H^1(\Omega)} = \|u\|_{L^2(\Omega)} + \|\nabla_{\mathbb{H}} u\|_{L^2(\Omega)}.$$

Denote by $H_0^1(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $H^1(\Omega)$.

If $u(x, y, t) = u(r, t)$, then we say that u is cylindrical. It is easy to see that

$$\Delta_{\mathbb{H}} u(r, t) = \frac{\partial^2 u}{\partial r^2} + \frac{2n-1}{r} \frac{\partial u}{\partial r} + 4r^2 \frac{\partial^2 u}{\partial t^2}. \tag{2.6}$$

If the weak solution $u \in H_0^1(\Omega)$ to (1.1) is cylindrical, then u is said to be the cylindrical solution to (1.1).

For any $\xi = (x, y, t) \in \mathbb{H}^n$ and $\lambda \in \mathbb{R}$, we set

$$T_\lambda = \{\xi \in \mathbb{H}^n \mid t = \lambda\}$$

and

$$\Sigma_\lambda = \{\xi \in \Omega \mid t < \lambda\}.$$

Let $\xi_\lambda = (y, x, 2\lambda - t)$ be the H-reflection of ξ about T_λ , and let

$$\Sigma'_\lambda = \{\xi_\lambda \in \Omega \mid \xi \in \Sigma_\lambda\}.$$

Denote

$$\lambda_0 = \min\{\lambda : T_\lambda \cap \overline{\Omega} \neq \emptyset\}$$

and

$$\lambda_1 = \max\{\lambda : \Sigma_\lambda \cup T_\lambda \cup \Sigma'_\lambda = \Omega\}.$$

Similarly to [6], we denote by u_λ the H-reflection of u about T_λ :

$$u_\lambda(x, y, t) = u_\lambda(r, t) = u(r, 2\lambda - t) = u(y, x, 2\lambda - t).$$

The following result is from [24].

Lemma 2.1 *If $f \in L^\infty(\Omega)$ and $u \in H_0^1(\Omega)$ is a solution to the subelliptic equation*

$$-\Delta_{\mathbb{H}}u = f,$$

then

$$\sup_{\Omega} u \leq C \|f\|_{L^\infty(\Omega)},$$

where C is a positive constant depending only on n and Ω .

Let us state Sobolev’s and Poincaré’s inequalities in \mathbb{H}^n ; see [25, 26].

Lemma 2.2 *Let U be a bounded domain in \mathbb{H}^n , and let $\Omega \Subset U$. If $1 < p < Q$ and $u \in W_0^{1,p}(\Omega)$, then there exists $C > 0$ depending on n, p , and Ω , such that for any $1 \leq q \leq \frac{pQ}{Q-p}$,*

$$\left(\int_{\Omega} |u|^q\right)^{\frac{1}{q}} \leq C \left(\int_{\Omega} |\nabla_{\mathbb{H}}u|^p\right)^{\frac{1}{p}}. \tag{2.7}$$

If $1 \leq p < \infty$ and $u \in W_0^{1,p}(\Omega)$, then

$$\int_{\Omega} |u|^p \leq C \int_{\Omega} |\nabla_{\mathbb{H}}u|^p. \tag{2.8}$$

Lemma 2.3 ([20, 23]) *Let Ω be a bounded domain, and let $K(\xi) > 0$. If u satisfies*

$$-\Delta_{\mathbb{H}}u + Ku \geq 0 \quad \text{on } \Omega; \quad u = 0 \quad \text{in } \partial\Omega,$$

then $u \geq 0$ on Ω . Furthermore, $u > 0$ on Ω unless $u \equiv 0$.

3 Solutions u_n to (1.3)

Let Ω be a bounded domain of \mathbb{H}^n , and let $\gamma > 0$. In this section, we prove the existence, uniqueness, and boundedness of solutions to the Dirichlet problem (1.3). We will work by approximation similarly to [3].

Lemma 3.1 *Problem (1.3) has a nonnegative solution $u_n \in H_0^1(\Omega) \cap L^\infty(\Omega)$ for any given n .*

Proof Let $v \in L^2(\Omega)$. It is well known that the Dirichlet problem to the subelliptic equation

$$\begin{cases} -\Delta_{\mathbb{H}} v^* = \frac{1}{(|v| + \frac{1}{n})^\gamma}, & \xi \in \Omega, \\ v^* = 0, & \xi \in \partial\Omega, \end{cases} \tag{3.1}$$

has a unique solution $v^* = S(v)$. Multiplying by v^* the equation and integrating over Ω , we have

$$\int_{\Omega} |\nabla_{\mathbb{H}} v^*|^2 = \int_{\Omega} (-\Delta_{\mathbb{H}} v^*) v^* = \int_{\Omega} \frac{v^*}{(|v| + \frac{1}{n})^\gamma} \leq n^\gamma \int_{\Omega} |v^*|.$$

By (2.8) in Lemma 2.2 and Hölder’s inequality we obtain

$$\int_{\Omega} (v^*)^2 \leq C \int_{\Omega} |\nabla_{\mathbb{H}} v^*|^2 \leq C n^\gamma \int_{\Omega} |v^*| \leq C n^\gamma \left(\int_{\Omega} |v^*|^2 \right)^{\frac{1}{2}}$$

and

$$\|v^*\|_{L^2(\Omega)} \leq C n^\gamma.$$

Then the ball in $L^2(\Omega)$ of radius $C n^\gamma$ is invariant for $S(v) = |v|$. Using the Sobolev embedding [25], we see that S is continuous and compact on $L^2(\Omega)$, and by Schauder’s fixed point theorem there exists $u_n \in H_0^1(\Omega)$ such that $u_n = S(u_n)$, that is, u_n is a solution to the problem

$$\begin{cases} -\Delta_{\mathbb{H}} u_n = \frac{1}{(|u_n| + \frac{1}{n})^\gamma}, & \xi \in \Omega, \\ u_n = 0, & \xi \in \partial\Omega. \end{cases}$$

Noting that $\frac{1}{(|u_n| + \frac{1}{n})^\gamma} \geq 0$, Lemma 2.3 gives $u_n \geq 0$, and then u_n solves (1.3). Since $\frac{1}{(u_n + \frac{1}{n})^\gamma}$ in (1.3) belongs to $L^\infty(\Omega)$, Lemma 2.1 implies that $u_n \in L^\infty(\Omega)$. \square

Lemma 3.2 *A solution of (1.3) is unique.*

Proof Since v_n satisfies

$$-\Delta_{\mathbb{H}} v_n = \frac{1}{(v_n + \frac{1}{n})^\gamma}$$

in the distribution sense, it follows that

$$-\Delta_{\mathbb{H}}(u_n - v_n) = \frac{1}{(u_n + \frac{1}{n})^\gamma} - \frac{1}{(v_n + \frac{1}{n})^\gamma} = \frac{(v_n + \frac{1}{n})^\gamma - (u_n + \frac{1}{n})^\gamma}{(u_n + \frac{1}{n})^\gamma (v_n + \frac{1}{n})^\gamma}. \tag{3.2}$$

Taking $(u_n - v_n)^+$ as the test function and observing that

$$\left(\left(v_n + \frac{1}{n} \right)^\gamma - \left(u_n + \frac{1}{n} \right)^\gamma \right) (u_n - v_n)^+ \leq 0, \tag{3.3}$$

it follows by (3.2) that

$$0 \leq \int_{\Omega} |\nabla_{\mathbb{H}}(u_n - v_n)^+|^2 \leq 0,$$

and therefore $(u_n - v_n)^+ = 0$ a.e. in Ω , which implies $u_n \leq v_n$. In the same way, taking $(v_n - u_n)^+$ as the test function, we can prove that $u_n \geq v_n$. This proves the uniqueness. \square

Lemma 3.3 *The nonnegative solution sequence $\{u_n\}$ from (1.3) is nondecreasing with respect to n , and for every $\omega \Subset \Omega$, there exists a constant $c_{\omega} > 0$, independent on n , such that, for any $\xi \in \omega$,*

$$u_n(\xi) \geq c_{\omega} \quad \text{for every } n. \tag{3.4}$$

Proof For $\gamma > 0$, we have that

$$-\Delta_{\mathbb{H}}u_n = \frac{1}{(u_n + \frac{1}{n})^{\gamma}} \leq \frac{1}{(u_n + \frac{1}{n+1})^{\gamma}}$$

in the distribution sense, and then

$$\begin{aligned} -\Delta_{\mathbb{H}}(u_n - u_{n+1}) &\leq \frac{1}{(u_n + \frac{1}{n+1})^{\gamma}} - \frac{1}{(u_{n+1} + \frac{1}{n+1})^{\gamma}} \\ &= \frac{(u_{n+1} + \frac{1}{n+1})^{\gamma} - (u_n + \frac{1}{n+1})^{\gamma}}{(u_n + \frac{1}{n+1})^{\gamma}(u_{n+1} + \frac{1}{n+1})^{\gamma}}. \end{aligned} \tag{3.5}$$

Taking $(u_n - u_{n+1})^+$ as the test function in (3.5) and using

$$\left(\left(u_{n+1} + \frac{1}{n+1} \right)^{\gamma} - \left(u_n + \frac{1}{n+1} \right)^{\gamma} \right) (u_n - u_{n+1})^+ \leq 0, \tag{3.6}$$

we have

$$0 \leq \int_{\Omega} |\nabla_{\mathbb{H}}(u_n - u_{n+1})^+|^2 \leq 0,$$

which implies $(u_n - u_{n+1})^+ = 0$ a.e. in Ω , that is, $u_n \leq u_{n+1}$, so $\{u_n\}$ is nondecreasing.

Noting that $u_1 \in L^{\infty}(\Omega)$ by Lemma 3.1 and using Lemma 2.1 to the equation $-\Delta_{\mathbb{H}}u_1 = \frac{1}{(u_1+1)^{\gamma}} \leq 1$, we get that there exists a constant $C > 0$, only depending on Ω and Q , such that $\|u_1\|_{L^{\infty}(\Omega)} \leq C$ and

$$-\Delta_{\mathbb{H}}u_1 = \frac{1}{(u_1 + 1)^{\gamma}} \geq \frac{1}{(\|u_1\|_{L^{\infty}(\Omega)} + 1)^{\gamma}} \geq \frac{1}{(C + 1)^{\gamma}}.$$

Lemma 2.3 implies that $u_1 > 0$ in Ω , and then (3.4) also holds for u_1 . By the monotonicity of u_n , (3.4) holds for u_n . \square

Remark 3.1 Using (3.4), it follows that

$$-\Delta_{\mathbb{H}}u_n = \frac{1}{(u_n + \frac{1}{n})^{\gamma}} \leq \frac{1}{(u_n)^{\gamma}} \leq \frac{1^{\gamma}}{c_{\omega}},$$

and Lemma 2.1 yields

$$\sup_{\omega} u_n \leq C.$$

Lemma 3.4 *Let $u_n \in H_0^1(\Omega)$ be the solution to (1.3). Then u_n is bounded if $0 < \gamma \leq 1$ and locally bounded if $\gamma > 1$.*

Proof If $\gamma > 1$, then by taking u_n as the test function in (1.3) and using Lemma 3.3 we have

$$\int_{\Omega} |\nabla_{\mathbb{H}} u_n|^2 = \int_{\Omega} (-\Delta_{\mathbb{H}} u_n) u_n = \int_{\Omega} \frac{u_n}{(u_n + \frac{1}{n})^{\gamma}} \leq \int_{\Omega} \frac{1}{(u_n + \frac{1}{n})^{\gamma-1}} \leq \frac{|\Omega|}{c_{\omega}^{\gamma-1}}.$$

This shows that u_n is locally bounded. For $0 < \gamma \leq 1$, choosing u_n as the test function in (1.3) and using Hölder’s inequality, we get that

$$\begin{aligned} \int_{\Omega} |\nabla_{\mathbb{H}} u_n|^2 &= \int_{\Omega} (-\Delta_{\mathbb{H}} u_n) u_n = \int_{\Omega} \frac{u_n}{(u_n + \frac{1}{n})^{\gamma}} \leq \int_{\Omega} u_n^{1-\gamma} \\ &\leq |\Omega|^{1-\frac{1}{m}} \left(\int_{\Omega} u_n^{(1-\gamma)m} \right)^{\frac{1}{m}}, \quad m > 1. \end{aligned} \tag{3.7}$$

If $\gamma = 1$, then

$$\int_{\Omega} |\nabla_{\mathbb{H}} u_n|^2 = \int_{\Omega} (-\Delta_{\mathbb{H}} u_n) u_n = \int_{\Omega} \frac{u_n}{u_n + \frac{1}{n}} \leq |\Omega|.$$

If $0 < \gamma < 1$, the taking m in (3.7) satisfying $2^* = (1 - \gamma)m$ and applying (2.7), we arrive at

$$\left(\int_{\Omega} u_n^{2^*} \right)^{\frac{1}{2^*}} \leq C \left(\int_{\Omega} |\nabla_{\mathbb{H}} u_n|^2 \right)^{\frac{1}{2}} \leq C |\Omega|^{1-\frac{1}{m}} \left(\int_{\Omega} u_n^{2^*} \right)^{\frac{1}{2m}}.$$

Since $\frac{1}{2^*} > \frac{1}{2m}$, this inequality implies $u_n \in L^{2^*}(\Omega)$. Using this estimate again together with (3.7), we have

$$\|\nabla_{\mathbb{H}} u_n\|_{L^2(\Omega)} \leq C,$$

so that u_n is bounded in $H_0^1(\Omega)$. □

Remark 3.2 Note that $u_n \in C(\overline{\Omega})$ since the right-hand side in (1.3) is nonsingular.

Lemma 3.5 *Let $u_n \in C(\overline{\Omega}) \cap H_0^1(\Omega)$ ($\gamma \leq 1$) and $u_n \in C(\overline{\Omega}) \cap H_{loc}^1(\Omega)$ ($\gamma > 1$) be solutions to (1.3). If $u_n \leq (u_n)_{\lambda}$ on $\partial\Omega$, then $u_n \leq (u_n)_{\lambda}$ in Ω with respect to t , and then $u_n < (u_n)_{\lambda}$ in Ω unless $u_n \equiv (u_n)_{\lambda}$.*

Proof Since u_n satisfies (1.3) and $\Delta_{\mathbb{H}}$ is invariant under the H-reflection, we have

$$-\Delta_{\mathbb{H}}(u_n)_{\lambda} = \frac{1}{((u_n)_{\lambda} + \frac{1}{n})^{\gamma}}. \tag{3.8}$$

Take positive functions $\phi_i \in C_0^\infty(\Omega)$, $\text{supp } \phi_i \subseteq \text{supp } (u_n - (u_n)_\lambda)^+$, $i = 1, 2, 3, \dots$, such that ϕ_i converges to $(u_n - (u_n)_\lambda)^+$ in $C(\bar{\Omega}) \cap H_0^1(\Omega)$. Using ϕ_i as the test function in (3.8), we have by combining (1.3) that

$$\begin{aligned} 0 &\leq \lim_{i \rightarrow \infty} \int_{\Omega} \nabla_{\mathbb{H}}(u_n - (u_n)_\lambda) \cdot \nabla_{\mathbb{H}}\phi_i \\ &= \lim_{i \rightarrow \infty} \int_{\Omega} (-\Delta_{\mathbb{H}}(u_n - (u_n)_\lambda))\phi_i \\ &= \lim_{i \rightarrow \infty} \int_{\Omega} \left(\frac{1}{(u_n + \frac{1}{n})^\gamma} - \frac{1}{((u_n)_\lambda + \frac{1}{n})^\gamma} \right) \phi_i \\ &= \int_{\Omega} \frac{(((u_n)_\lambda + \frac{1}{n})^\gamma - (u_n + \frac{1}{n})^\gamma)(u_n - (u_n)_\lambda)^+}{(u_n + \frac{1}{n})^\gamma ((u_n)_\lambda + \frac{1}{n})^\gamma} \\ &\leq 0. \end{aligned}$$

Then $(u_n - (u_n)_\lambda)^+ = 0$ a.e. in Ω , which implies by Lemma 2.3 that $u_n \leq (u_n)_\lambda$ in Ω .

Assume that there exists $\xi_0 \in \Omega$ such that $u_n(\xi_0) = (u_n)_\lambda(\xi_0)$ and let $R = R(\xi_0) > 0$ be such that $B(\xi_0, R) \Subset \Omega$. We have that

$$-\Delta_{\mathbb{H}}(u_n - (u_n)_\lambda) = \frac{1}{(u_n + \frac{1}{n})^\gamma} - \frac{1}{((u_n)_\lambda + \frac{1}{n})^\gamma} \quad \text{in } B(\xi_0, R) \tag{3.9}$$

in the distribution sense. Since $u_n(\xi_0) > 0$, we can assume that $u_n > 0$ in $B(\xi_0, R)$ for small R . By Lemma 3.3, $u_n \geq c_{B(\xi_0, R)} > 0$. Therefore,

$$\frac{1}{(u_n + \frac{1}{n})^\gamma} - \frac{1}{((u_n)_\lambda + \frac{1}{n})^\gamma} = c(\zeta)(u_n - (u_n)_\lambda),$$

where $c(\zeta) = \frac{-\gamma}{\zeta^{\gamma+1}}$ with ζ between $u_n + \frac{1}{n}$ and $(u_n)_\lambda + \frac{1}{n}$. We see that $c(\zeta)$ is bounded by using $u_n \geq c_{B(\xi_0, R)}$. Thus there exists a positive constant $\Lambda \geq |c(\zeta)|$ such that

$$\frac{1}{(u_n + \frac{1}{n})^\gamma} - \frac{1}{((u_n)_\lambda + \frac{1}{n})^\gamma} + \Lambda w \geq 0 \quad \text{in } B(\xi_0, R).$$

Putting this into (3.9), we have

$$-\Delta_{\mathbb{H}}(u_n - (u_n)_\lambda) + \Lambda(u_n - (u_n)_\lambda) \geq 0 \quad \text{in } B(\xi_0, R).$$

By Lemma 2.3 this implies that $u_n \equiv (u_n)_\lambda$ in $B(\xi_0, R)$. The usual covering argument shows that $u_n \equiv (u_n)_\lambda$ in Ω . □

4 Solutions to (1.2)

In this section, we get the regularity of solutions to (1.2) by using results to (1.3). Also, we obtain the monotonicity of cylindrical solutions to (1.2). The gradient estimate and Harnack inequality for classical solutions to (1.2) are provided by calculating the Harnack quantity, which is similar to Li [27] and Ma and Wei [28].

4.1 Regularity and symmetry of solutions to (1.2)

Lemma 4.1 *Problem (1.2) allows a unique nonnegative solution $u \in H_0^1(\Omega)$ ($\gamma \leq 1$) or $u \in H_{loc}^1(\Omega)$ ($\gamma > 1$), and (3.4) is true for u_0 , that is, $u_0(\xi) \geq c_\omega$ for $\xi \in \omega \subset\subset \Omega$.*

Proof For every n , since the solution $u_n \in H_0^1(\Omega)$ to (1.3) is bounded by Lemma 3.4 and u_n converges pointwise to u_0 in Ω by Remark 3.1, it follows that u_n converges weakly to u_0 in $H_0^1(\omega)$, $\omega \Subset \Omega$, that is, for any $\varphi \in C_0^1(\Omega)$ such that $\text{supp } \varphi = \omega$,

$$\lim_{n \rightarrow \infty} \int_\omega \nabla_{\mathbb{H}} u_n \cdot \nabla_{\mathbb{H}} \varphi = \int_\omega \nabla_{\mathbb{H}} u_0 \cdot \nabla_{\mathbb{H}} \varphi. \tag{4.1}$$

Noting that u_n satisfies (3.4), we have that, for any $\varphi \in C_0^1(\Omega)$,

$$0 \leq \left| \frac{\varphi}{(u_n + \frac{1}{n})^\gamma} \right| \leq \frac{\|\varphi\|_{L^\infty(\Omega)}}{c_\omega^\gamma}.$$

By Lebesgue’s dominated convergence theorem,

$$\lim_{n \rightarrow \infty} \int_\Omega \frac{\varphi}{(u_n + \frac{1}{n})^\gamma} = \int_\Omega \frac{\varphi}{u_0^\gamma}. \tag{4.2}$$

This implies by (4.1) and (4.2) that problem (1.2) has a solution u_0 .

Using $u_n \geq c_\omega$ on $\omega \Subset \Omega$, this yields $u_0 \geq 0$ in Ω . The uniqueness can be proved as in the proof of Lemma 3.2. □

For the equation $\Delta_{\mathbb{H}} z = 0$ in Ω , by [29] we have $z \in C^2(\Omega)$.

Lemma 4.2 *If $\partial\Omega$ satisfies the Wiener criterion and $z_1 \in H_0^1(\Omega) \cap C^2(\Omega)$ solves*

$$-\Delta_{\mathbb{H}} z_1 = 1 \quad \text{in } \Omega,$$

then $z_1 \in C(\overline{\Omega})$ is positive with $z_1|_{\partial\Omega} = 0$.

Proof Let $z_2(\xi) = \frac{x_1^2}{2}$. Then $\Delta_{\mathbb{H}} z_2 = 1$. Since $\partial\Omega$ satisfies the Wiener criterion, it follows by [13] (p. 1158) that there exists $z \in C(\overline{\Omega}) \cap C^2(\Omega)$ such that

$$\Delta_{\mathbb{H}} z = 0 \quad \text{in } \Omega \quad \text{and} \quad z|_{\partial\Omega} = z_2.$$

We easily see that $z - z_2 \in H_0^1(\Omega) \cap C^2(\Omega)$ satisfies

$$-\Delta_{\mathbb{H}}(z - z_2) = 1 \quad \text{in } \Omega,$$

and therefore $z_1 = z - z_2 \in C(\overline{\Omega})$ is the required function with $z_1 > 0$ and $z_1|_{\partial\Omega} = 0$. □

Lemma 4.3 *If $\partial\Omega$ satisfies the Wiener criterion, then there exists a solution u_0 to problem (1.2), and $u_0 \in C(\overline{\Omega})$.*

Proof Taking $u_0^* = ((\gamma + 1)z_1)^{\frac{1}{\gamma+1}}$, we have by Lemma 4.2 that

$$\begin{aligned} &\Delta_{\mathbb{H}} u_0^* + (u_0^*)^{-\gamma} \\ &= -\gamma((\gamma + 1)z_1)^{\frac{-2\gamma-1}{\gamma+1}} |\nabla_{\mathbb{H}} z_1|^2 + ((\gamma + 1)z_1)^{\frac{-\gamma}{\gamma+1}} \Delta_{\mathbb{H}} z_1 + ((\gamma + 1)z_1)^{\frac{-\gamma}{\gamma+1}} \\ &= -\gamma((\gamma + 1)z_1)^{\frac{-2\gamma-1}{\gamma+1}} |\nabla_{\mathbb{H}} z_1|^2 \leq 0. \end{aligned} \tag{4.3}$$

Noting that u_0 is nonnegative, we know by (4.3) that

$$0 \leq u_0 \leq u_0^* = ((\gamma + 1)z_1)^{\frac{1}{\gamma+1}},$$

that is, u_0^* and 0 are the super- and subsolutions of (1.2), respectively. This implies the existence of a solution u_0 to (1.2). Using $z_1 \in C(\overline{\Omega})$ in Lemma 4.2, we have $u_0 \in C(\overline{\Omega})$. \square

Lemma 4.4 *If $\partial\Omega$ satisfies the Wiener criterion, $u_0 \in C(\overline{\Omega}) \cap H_0^1(\Omega)$ ($\gamma \leq 1$) or $u_0 \in C(\overline{\Omega}) \cap H_{loc}^1(\Omega)$ ($\gamma > 1$) is a cylindrical solution to problem (1.2), then for any $\lambda_0 < \lambda < \lambda_1$, we have*

$$u_0(\xi) < (u_0)_\lambda(\xi) \quad \text{for all } \xi \in \Sigma_\lambda \tag{4.4}$$

with respect to t .

Proof Step 1. Suppose that λ is sufficiently close to λ_0 . Then by Lemma 3.3 we have that $u_n > 0$ in Σ_λ , and then $u_n \leq (u_n)_\lambda$ on $\partial\Sigma_\lambda$. Since $u_n = (u_n)_\lambda$ on the hyperplane T_λ , it follows that $u_n \leq (u_n)_\lambda$ in Σ_λ , and then $u_n < (u_n)_\lambda$ on Σ_λ by Lemma 3.5.

Step 2. Since Step 1 provides a starting point, we can move the plane T_λ to the right keeping $u_n \leq (u_n)_\lambda$ on Σ_λ until its limiting position λ_1 . Denoting

$$\bar{\lambda} = \sup\{\lambda \mid u_n \leq (u_n)_\lambda \text{ in } \Sigma_\mu, \mu \leq \lambda\},$$

we claim that $\bar{\lambda} = \lambda_1$. Indeed, suppose $\bar{\lambda} < \lambda_1$; we will show that the plane T_λ can be moved further to the right or, to be more rigorous, there exists $\varepsilon > 0$ such that, for any $\lambda \in (\bar{\lambda}, \bar{\lambda} + \varepsilon)$,

$$u_n \leq (u_n)_\lambda. \tag{4.5}$$

This is a contradiction to the definition of $\bar{\lambda}$. Hence $\bar{\lambda} = \lambda_1$.

To prove (4.5), we separate $\Sigma_{\bar{\lambda}+\varepsilon}$ into two parts, $\Sigma_{\bar{\lambda}-\varepsilon}$ and $\Sigma_{\bar{\lambda}+\varepsilon} \setminus \Sigma_{\bar{\lambda}-\varepsilon}$ for $\varepsilon > 0$.

Clearly, $(u_n)_{\bar{\lambda}} - u_n \geq c_\varepsilon > 0$ in $\Sigma_{\bar{\lambda}-\varepsilon}$ for any $\varepsilon > 0$. Since $(u_n)_\lambda - u_n$ depends continuously on λ , there exists $\varepsilon > 0$ sufficiently small such that, for all $\lambda \in (\bar{\lambda}, \bar{\lambda} + \varepsilon)$, we have

$$(u_n)_\lambda - u_n \geq 0 \quad \text{in } \Sigma_{\bar{\lambda}-\varepsilon}. \tag{4.6}$$

On the other hand, since $(u_n)_\lambda - u_n \geq 0$ on $\partial(\Sigma_{\bar{\lambda}+\varepsilon} \setminus \Sigma_{\bar{\lambda}-\varepsilon})$ for fixed $\varepsilon > 0$, it follows by Lemma 3.5 that

$$(u_n)_\lambda - u_n \geq 0 \quad \text{in } \Sigma_{\bar{\lambda}+\varepsilon} \setminus \Sigma_{\bar{\lambda}-\varepsilon}. \tag{4.7}$$

Combining (4.6) and (4.7), we prove (4.5).

Using the equality $\bar{\lambda} = \lambda_1$ and the convergence of u_n to u_0 a.e., we get that $u_0 \leq (u_0)_\lambda$ on Σ_λ . So (4.4) is proved by Lemma 2.3. \square

4.2 Harnack inequality for a solution to (1.2)

Results in the subsection are not used in the sequel, but they are of independent interest.

Lemma 4.5 *Let $u_0 \in C^3(\Omega)$ be a positive solution to (1.2). Then, for any $R = R(\xi_0) > 0$, $\xi_0 \in \Omega$, and $B(\xi_0, 2R) \subset \Omega$, there exists a positive constant $C = C(R)$ such that*

$$|\nabla_{\mathbb{H}} u_0|^2 \leq C u_0^2 \quad \text{in } B(\xi_0, R). \tag{4.8}$$

Proof Let $w_0 = \log u_0$. Then

$$\nabla_{\mathbb{H}} w_0 = u_0^{-1} \nabla_{\mathbb{H}} u_0 \tag{4.9}$$

and

$$\Delta_{\mathbb{H}} w_0 = -|\nabla_{\mathbb{H}} w_0|^2 - \frac{1}{u_0^{\gamma+1}}. \tag{4.10}$$

Taking a smooth cut-off function ϕ with $\phi = 1$ in $B(\xi_0, R)$ and $\phi = 0$ outside $B(\xi_0, 2R)$, define

$$P = \phi |\nabla_{\mathbb{H}} w_0|^2,$$

which as usual is called the Harnack quantity.

At the maximum point of P , we have $\nabla_{\mathbb{H}} P = 0$ and $\Delta_{\mathbb{H}} P \leq 0$. Then

$$\nabla_{\mathbb{H}} (|\nabla_{\mathbb{H}} w_0|^2) = -\phi^{-1} \nabla_{\mathbb{H}} \phi |\nabla_{\mathbb{H}} w_0|^2 = -\phi^{-2} P \nabla_{\mathbb{H}} \phi, \tag{4.11}$$

$$|\nabla_{\mathbb{H}} w_0|^2 \Delta_{\mathbb{H}} \phi + 2 \nabla_{\mathbb{H}} \phi \cdot \nabla_{\mathbb{H}} (|\nabla_{\mathbb{H}} w_0|^2) + \phi \Delta_{\mathbb{H}} (|\nabla_{\mathbb{H}} w_0|^2) \leq 0. \tag{4.12}$$

Using the basic formula

$$\Delta_{\mathbb{H}} |\nabla_{\mathbb{H}} v|^2 = 2((\nabla_{\mathbb{H}} \Delta_{\mathbb{H}} v, \nabla_{\mathbb{H}} v) + |D_{\mathbb{H}}^2 v|^2), \tag{4.13}$$

for any smooth function v in \mathbb{H}^n , it follows

$$\phi \Delta_{\mathbb{H}} (|\nabla_{\mathbb{H}} w_0|^2) = 2\phi (\nabla_{\mathbb{H}} \Delta_{\mathbb{H}} w_0, \nabla_{\mathbb{H}} w_0) + 2\phi |D_{\mathbb{H}}^2 w_0|^2. \tag{4.14}$$

Let us estimate the right-hand side in (4.14). For the first term, we have

$$\begin{aligned} 2\phi (\nabla_{\mathbb{H}} \Delta_{\mathbb{H}} w_0, \nabla_{\mathbb{H}} w_0) &= 2\phi \left(\nabla_{\mathbb{H}} \left(-|\nabla_{\mathbb{H}} w_0|^2 - \frac{1}{u_0^{\gamma+1}} \right), \nabla_{\mathbb{H}} w_0 \right) \\ &= -2\phi (\nabla_{\mathbb{H}} (|\nabla_{\mathbb{H}} w_0|^2), \nabla_{\mathbb{H}} w_0) + 2(\gamma + 1)\phi \left(\frac{\nabla_{\mathbb{H}} u_0}{u_0^{\gamma+2}}, \nabla_{\mathbb{H}} w_0 \right) \\ &= -2\phi (-\phi^{-2} P \nabla_{\mathbb{H}} \phi, \nabla_{\mathbb{H}} w_0) + 2(\gamma + 1)\phi \frac{|\nabla_{\mathbb{H}} w_0|^2}{u_0^{\gamma+1}} \end{aligned}$$

$$\begin{aligned}
 &= 2\phi^{-1}(\nabla_{\mathbb{H}}\phi, \nabla_{\mathbb{H}}w_0)P + \frac{2(\gamma + 1)}{u_0^{\gamma+1}}P \\
 &\geq -\phi^{-1}|\nabla_{\mathbb{H}}\phi|^2P - \phi^{-1}|\nabla_{\mathbb{H}}w_0|^2P + \frac{2(\gamma + 1)}{u_0^{\gamma+1}}P \\
 &= -\phi^{-1}|\nabla_{\mathbb{H}}\phi|^2P - \phi^{-2}P^2 + \frac{2(\gamma + 1)}{u_0^{\gamma+1}}P.
 \end{aligned} \tag{4.15}$$

Due to (4.10), the second term of (4.14) satisfies

$$\begin{aligned}
 2\phi|D_{\mathbb{H}}^2w_0|^2 &\geq 2\phi|\Delta_{\mathbb{H}}w_0|^2 \\
 &= 2\phi\left| -|\nabla_{\mathbb{H}}w_0|^2 - \frac{1}{u_0^{\gamma+1}} \right|^2 \\
 &= 2\phi|\nabla_{\mathbb{H}}w_0|^4 + \frac{4\phi|\nabla_{\mathbb{H}}w_0|^2}{u_0^{\gamma+1}} + \frac{2\phi}{u_0^{2(\gamma+1)}} \\
 &\geq \frac{2}{\phi}P^2 + \frac{4}{u_0^{\gamma+1}}P.
 \end{aligned} \tag{4.16}$$

Substituting (4.15) and (4.16) into (4.14), we obtain

$$\phi\Delta_{\mathbb{H}}(|\nabla_{\mathbb{H}}w_0|^2) \geq -\frac{|\nabla_{\mathbb{H}}\phi|^2}{\phi}P - \frac{P^2}{\phi^2} + \frac{2\gamma + 6}{u_0^{\gamma+1}}P + \frac{2}{\phi}P^2. \tag{4.17}$$

From (4.12), $|\nabla_{\mathbb{H}}w_0|^2 = \frac{P}{\phi}$, (4.11), and (4.17) we have that

$$\begin{aligned}
 0 &\geq \phi\Delta_{\mathbb{H}}(|\nabla_{\mathbb{H}}w_0|^2) + \frac{\Delta_{\mathbb{H}}\phi}{\phi}P - \frac{2|\nabla_{\mathbb{H}}\phi|^2}{\phi}P \\
 &\geq \frac{\Delta_{\mathbb{H}}\phi}{\phi}P - \frac{2|\nabla_{\mathbb{H}}\phi|^2}{\phi^2}P - \frac{|\nabla_{\mathbb{H}}\phi|^2}{\phi}P - \frac{P^2}{\phi^2} + \frac{2\gamma + 6}{u_0^{\gamma+1}}P + \frac{2}{\phi}P^2,
 \end{aligned} \tag{4.18}$$

and so

$$-\frac{\Delta_{\mathbb{H}}\phi}{\phi} + \frac{2|\nabla_{\mathbb{H}}\phi|^2}{\phi^2} + \frac{|\nabla_{\mathbb{H}}\phi|^2}{\phi} - \frac{2\gamma + 6}{u_0^{\gamma+1}} \geq \frac{2\phi - 1}{\phi^2}P.$$

This gives

$$\frac{2\phi - 1}{\phi^2}P \leq -\frac{\Delta_{\mathbb{H}}\phi}{\phi} + \frac{2|\nabla_{\mathbb{H}}\phi|^2}{\phi^2} + \frac{|\nabla_{\mathbb{H}}\phi|^2}{\phi} \quad \text{on } B(\xi_0, 2R).$$

Using the bounds $|\nabla_{\mathbb{H}}\phi| \leq CR^{-1}$ and $|\Delta_{\mathbb{H}}\phi| \leq CR^{-2}$, we get

$$(2\phi - 1)P \leq -\phi\Delta_{\mathbb{H}}\phi + 2|\nabla_{\mathbb{H}}\phi|^2 + \phi|\nabla_{\mathbb{H}}\phi|^2 \leq \frac{C}{R^2}. \tag{4.19}$$

Since $\phi = 1$ on $B(\xi_0, R)$, by (4.19) we have that

$$|\nabla_{\mathbb{H}}w_0|^2 \leq C(R) \quad \text{on } B(\xi_0, R), \tag{4.20}$$

which implies (4.8). □

Theorem 4.6 *Let $u_0 \in C^3(\Omega)$ be a positive solution to (1.2). Then, for any $R = R(\xi_0) > 0$, $\xi_0 \in \Omega$, and $B(\xi_0, 2R) \subset \Omega$, there exists $C = C(R) > 0$ such that, for all $\xi, \eta \in B(\xi_0, \frac{R}{2})$,*

$$u_0(\xi) \leq C u_0(\eta). \tag{4.21}$$

Proof Let I be the shortest curve in $B(\xi_0, R)$ joining ξ to η . Then the length of I is clearly at most $2R$. Integrating the quantity $|\nabla_{\mathbb{H}} \log u_0|$ along I yields

$$\log u_0(\xi) - \log u_0(\eta) \leq \int_I |\nabla_{\mathbb{H}} \log u_0|. \tag{4.22}$$

Applying (4.20), we get

$$\int_I |\nabla_{\mathbb{H}} \log u_0| \leq \int_I C \leq C. \tag{4.23}$$

Now (4.21) is obtained from (4.22) and (4.23). □

5 Properties of w and proof of Theorem 1.1

In this section, we prove several properties of $w = u - u_0$. Similarly to [5], we provide two maximum principles; see Lemmas 5.2 and 5.4. The idea to check monotonicity is inspired by [5, 15].

Lemma 5.1 *If $\partial\Omega$ satisfies the Wiener criterion, then the weak solution $u \in H_0^1(\Omega)$ ($\gamma \leq 1$) or $u \in H_{loc}^1(\Omega)$ ($\gamma > 1$) to problem (1.1) is positive, and $u \in C(\overline{\Omega})$.*

Proof By the hypothesis $f \geq 0$ we see that u is a supersolution to the equation

$$-\Delta_{\mathbb{H}} u_0 = \frac{1}{u_0^\gamma}.$$

Therefore, $u \geq u_0 > 0$ in Ω .

Since $u = u_0 + w$, it follows that $w \geq 0$ in Ω . Due to (1.4), we have that w satisfies

$$\Delta_{\mathbb{H}} w + \frac{1}{(u_0 + w)^\gamma} + f(u_0 + w) - \frac{1}{u_0^\gamma} = 0 \quad \text{in } \Omega; \quad w|_{\partial\Omega} = 0. \tag{5.1}$$

Since the equation in (5.1) has no any singularity with respect to w , the continuity in Ω of w is clearly obtained; refer to [7, 8, 10] etc. Now we only need to check the continuity of w near the boundary $\partial\Omega$.

Indeed, by the nonnegativity of f we easily get that $w_* = 0$ is a subsolution to (5.1). Also, note that $w^* = ((\gamma + 1)z_1)^{\frac{1}{\gamma+1}}$ satisfies

$$\Delta_{\mathbb{H}} w^* + f(u_0 + w^*) \leq 0,$$

that is,

$$-\gamma((\gamma + 1)z_1)^{\frac{-2\gamma-1}{\gamma+1}} |\nabla_{\mathbb{H}} z_1|^2 - ((\gamma + 1)z_1)^{\frac{-\gamma}{\gamma+1}} + f(u_0 + ((\gamma + 1)z_1)^{\frac{1}{\gamma+1}}) \leq 0 \tag{5.2}$$

near the boundary ($z_1 \rightarrow 0$ at this moment), which yields

$$0 \leq w \leq ((\gamma + 1)z_1)^{\frac{1}{\gamma+1}}$$

near the boundary, and this shows that w is continuous near $\partial\Omega$ and then $w \in C(\overline{\Omega})$. Hence $u = u_0 + w \in C(\overline{\Omega})$. □

Lemma 5.2 *Suppose that $\partial\Omega$ satisfies the Wiener criterion, $u \in C(\overline{\Omega}) \cap H_0^1(\Omega)$ ($\gamma \leq 1$) or $u \in C(\overline{\Omega}) \cap H_{loc}^1(\Omega)$ ($\gamma > 1$) is a solution to problem (1.1), f is locally Lipschitz continuous, $f(s) > 0$ for any $s > 0$, and $f(0) \geq 0$. Then $u > u_0$ in Ω , and $w > 0$ in Ω .*

Proof By contradiction assume that there exists $\xi_0 \in \Omega$ such that

$$w(\xi_0) = u(\xi_0) - u_0(\xi_0) = 0.$$

Take $R = R(\xi_0) > 0$ such that $B(\xi_0, R) \subset\subset \Omega$. Then

$$-\Delta_{\mathbb{H}^n} w = -\Delta_{\mathbb{H}^n} u + \Delta_{\mathbb{H}^n} u_0 = \frac{1}{(u_0 + w)^\gamma} + f(u) - \frac{1}{u_0^\gamma} \geq \frac{1}{(u_0 + w)^\gamma} - \frac{1}{u_0^\gamma} \quad \text{in } B(\xi_0, R). \tag{5.3}$$

Since $u_0(\xi_0) > 0$ by Lemma 4.1, we can assume that $u_0 > 0$ in $B(\xi_0, R)$. Since

$$\frac{1}{(u_0 + w)^\gamma} - \frac{1}{u_0^\gamma} = c(\xi)(u_0 + w - u_0) = c(\xi)w$$

and $|c(\xi)| \leq \Lambda$ for a positive constant Λ , it follows that

$$\frac{1}{(u_0 + w)^\gamma} - \frac{1}{u_0^\gamma} + \Lambda w \geq 0 \quad \text{in } B(\xi_0, R).$$

Combining this with (5.3), we get

$$-\Delta_{\mathbb{H}^n} w + \Lambda w \geq 0 \quad \text{in } B(\xi_0, R).$$

By Lemma 2.3 we have $w \equiv 0$ in $B(\xi_0, R)$. The covering argument shows that $w \equiv 0$ in Ω . Therefore $f \equiv 0$ by (5.3), which is impossible. □

Lemma 5.3 ([5]) *For $\gamma > 0$, consider the function*

$$g_\gamma(a, b, c, d) := a^\gamma (a + b)^\gamma (c + d)^\gamma + a^\gamma c^\gamma (c + d)^\gamma - c^\gamma (a + b)^\gamma (c + d)^\gamma - a^\gamma c^\gamma (a + b)^\gamma,$$

where $(a, b, c, d) \in D := \{(a, b, c, d) | 0 \leq a \leq c; 0 \leq d \leq b\} \subset \mathbb{R}^4$. Then $g_\gamma \leq 0$ in D .

Lemma 5.4 (Narrow region maximum principle) *Assume that $\partial\Omega$ satisfies the Wiener criterion and T is a hyperplane in \mathbb{H}^n . For $\lambda \in \mathbb{R}$, denote*

$$T_\lambda = \{ \xi = (x, y, t) \in \mathbb{H}^n | t = \lambda \},$$

$$H = \{ \xi = (x, y, t) \in \mathbb{H}^n | t < \lambda \}.$$

Let Ω' be a bounded narrow region in H contained in $\{\xi \in \mathbb{H}^n | \lambda - l < t < \lambda\}$ with small $l > 0$. Suppose that $u \in C(\overline{\Omega}) \cap H_0^1(\Omega)$ ($\gamma \leq 1$) or $u \in C(\overline{\Omega}) \cap H_{loc}^1(\Omega)$ ($\gamma > 1$) is a cylindrical solution to problem (1.1) and that f satisfies (F). If $w \leq w_\lambda$ on $\partial\Omega'$, then for a sufficiently small l , we have $w \leq w_\lambda$ in Ω' with respect to t . Furthermore, $w < w_\lambda$ in Ω' , unless $w \equiv w_\lambda$.

Proof Since u satisfies problem (1.1), we have in the distribution sense that

$$-\Delta_{\mathbb{H}}(u_0 + w) = \frac{1}{(u_0 + w)^\gamma} + f(u_0 + w) \quad \text{in } \Omega, \tag{5.4}$$

and

$$-\Delta_{\mathbb{H}}((u_0)_\lambda + w_\lambda) = \frac{1}{((u_0)_\lambda + w_\lambda)^\gamma} + f((u_0)_\lambda + w_\lambda) \quad \text{in } \Omega. \tag{5.5}$$

Also,

$$-\Delta_{\mathbb{H}}(u_0)_\lambda = \frac{1}{(u_0)_\lambda^\gamma} \quad \text{in } \Omega, \tag{5.6}$$

and

$$\begin{aligned} -\Delta_{\mathbb{H}}w_\lambda &= -\Delta_{\mathbb{H}}((u_0)_\lambda + w_\lambda) + \Delta_{\mathbb{H}}(u_0)_\lambda \\ &= \frac{1}{((u_0)_\lambda + w_\lambda)^\gamma} + f((u_0)_\lambda + w_\lambda) - \frac{1}{(u_0)_\lambda^\gamma} \quad \text{in } \Omega. \end{aligned} \tag{5.7}$$

Consider positive functions $\psi_i \in C_0^\infty(\Omega')$ with $\text{supp } \psi_i \subseteq \text{supp}(w - w_\lambda)^+$ and $\psi_i \rightarrow (w - w_\lambda)^+ \in H_0^1(\Omega')$ in $H_0^1(\Omega')$. Taking ψ_i as the test function, by (5.1) and (5.7) we get that

$$\begin{aligned} &\int_{\Omega'} \nabla_{\mathbb{H}}(w - w_\lambda) \cdot \nabla_{\mathbb{H}}\psi_i \\ &= \int_{\Omega'} (-\Delta_{\mathbb{H}}(w - w_\lambda))\psi_i \\ &= \int_{\Omega'} \left(\frac{1}{(u_0 + w)^\gamma} + f(u_0 + w) - \frac{1}{u_0^\gamma} - \frac{1}{((u_0)_\lambda + w_\lambda)^\gamma} - f((u_0)_\lambda + w_\lambda) + \frac{1}{(u_0)_\lambda^\gamma} \right) \psi_i \\ &= \int_{\Omega'} \left(\frac{1}{(u_0)_\lambda^\gamma} - \frac{1}{u_0^\gamma} + \frac{1}{(u_0 + w)^\gamma} - \frac{1}{((u_0)_\lambda + w_\lambda)^\gamma} \right) \psi_i \\ &\quad + \int_{\Omega'} (f(u_0 + w) - f((u_0)_\lambda + w_\lambda))\psi_i. \end{aligned} \tag{5.8}$$

Note that $u_0 \leq (u_0)_\lambda$ in $\Omega' \cap H$ by Lemma 4.4 and that $w \geq w_\lambda$ on $\text{supp } \psi_i$. Applying Lemma 5.3 with $a = u_0$, $c = (u_0)_\lambda$, $b = w$, and $d = w_\lambda$, we have

$$\frac{1}{(u_0)_\lambda^\gamma} - \frac{1}{u_0^\gamma} + \frac{1}{(u_0 + w)^\gamma} - \frac{1}{((u_0)_\lambda + w_\lambda)^\gamma} \leq 0. \tag{5.9}$$

Since f is locally Lipschitz continuous and nondecreasing, it follows that (5.8) and (5.9) imply

$$\begin{aligned} \int_{\Omega'} \nabla_{\mathbb{H}}(w - w_{\lambda}) \cdot \nabla_{\mathbb{H}}\psi_i &\leq \int_{\Omega'} (f(u_0 + w) - f((u_0)_{\lambda} + w_{\lambda}))\psi_i \\ &\leq \int_{\Omega'} (f((u_0)_{\lambda} + w) - f((u_0)_{\lambda} + w_{\lambda}))\psi_i \\ &\leq C \int_{\Omega'} (w - w_{\lambda})\psi_i. \end{aligned} \tag{5.10}$$

Since ψ_i converges to $(w - w_{\lambda})^+$, by Hölder’s inequality and Lemma 2.2 we have

$$\begin{aligned} \int_{\Omega'} |\nabla_{\mathbb{H}}(w - w_{\lambda})^+|^2 &\leq C \int_{\Omega'} |(w - w_{\lambda})^+|^2 \\ &\leq C |\Omega'|^{\frac{Q}{2}} \left(\int_{\Omega'} |(w - w_{\lambda})^+|^{\frac{2Q}{Q-2}} \right)^{\frac{Q-2}{Q}} \\ &\leq C |\Omega'|^{\frac{Q}{2}} \int_{\Omega'} |\nabla_{\mathbb{H}}(w - w_{\lambda})^+|^2. \end{aligned} \tag{5.11}$$

Since $C|\Omega'|^{\frac{Q}{2}} < 1$ for sufficiently small l , it follows that

$$\int_{\Omega'} |\nabla_{\mathbb{H}}(w - w_{\lambda})^+|^2 = 0, \tag{5.12}$$

and then $(w - w_{\lambda})^+ = 0$, that is, $w \leq w_{\lambda}$ in Ω' .

Assume that there exists $\xi_0 \in \Omega'$ such that $w(\xi_0) = w_{\lambda}(\xi_0)$. We have from (5.1) and (5.7) that

$$\begin{aligned} &-\Delta_{\mathbb{H}}(w_{\lambda} - w) \\ &= \left(\frac{1}{u_0^{\gamma}} - \frac{1}{(u_0)_{\lambda}^{\gamma}} + \frac{1}{((u_0)_{\lambda} + w)^{\gamma}} - \frac{1}{(u_0 + w)^{\gamma}} \right) + \frac{1}{((u_0)_{\lambda} + w_{\lambda})^{\gamma}} - \frac{1}{((u_0)_{\lambda} + w)^{\gamma}} \\ &\quad + (f((u_0)_{\lambda} + w_{\lambda}) - f(u_0 + w)). \end{aligned} \tag{5.13}$$

Noting that $u_0 \leq (u_0)_{\lambda}$ in Ω' by Lemma 4.4 and $w \leq w_{\lambda}$ in Ω' from the previous result, by the monotonicity of f we obtain that

$$f((u_0)_{\lambda} + w_{\lambda}) - f(u_0 + w) \geq 0 \quad \text{in } \Omega'. \tag{5.14}$$

Applying Lemma 5.3 yields

$$\frac{1}{(u_0)_{\lambda}^{\gamma}} - \frac{1}{u_0^{\gamma}} + \frac{1}{(u_0 + w)^{\gamma}} - \frac{1}{((u_0)_{\lambda} + w_{\lambda})^{\gamma}} \leq 0 \quad \text{in } \Omega'. \tag{5.15}$$

Putting (5.14) and (5.15) into (5.13) shows that

$$-\Delta_{\mathbb{H}}(w_{\lambda} - w) \geq \frac{1}{((u_0)_{\lambda} + w_{\lambda})^{\gamma}} - \frac{1}{((u_0)_{\lambda} + w)^{\gamma}}. \tag{5.16}$$

Now similarly to the proof of Lemma 5.2, we derive that $w_{\lambda} - w \equiv 0$ in Ω' . □

Remark 5.1 In proving (5.11) and (5.12), although there is no Poincaré’s inequality on the narrow domain in \mathbb{H}^n , Sobolev’s inequality (2.7) and the fact that $|\Omega'|$ is small helps us to derive (5.12).

Lemma 5.5 *If $\partial\Omega$ satisfies the Wiener criterion and $u \in H_0^1(\Omega) \cap C(\overline{\Omega})$ ($\gamma \leq 1$) or $u \in H_{loc}^1(\Omega) \cap C(\overline{\Omega})$ ($\gamma > 1$) is a cylindrical solution of problem (1.1), then, for any $\lambda_0 < \lambda < \lambda_1$, we have*

$$w(\xi) < w_\lambda(\xi) \tag{5.17}$$

for $\xi \in \Sigma_\lambda$ with respect to t .

Proof Step 1. If λ is sufficiently close to λ_0 , we have $w > 0$ in Σ_λ by Lemma 5.2 and $w \leq w_\lambda$ on $\partial\Sigma_\lambda$. Since $w = w_\lambda$ on the hyperplane T_λ , we get that $w \leq w_\lambda$ in Σ_λ , and then $w < w_\lambda$ on Σ_λ by Lemma 5.4.

Step 2. Since Step 1 provides a starting point, we can move the plane T_λ to the right keeping $w \leq w_\lambda$ on Σ_λ to its limiting position λ_1 . We let

$$\bar{\lambda} = \sup\{\lambda \mid w \leq w_\mu \text{ in } \Sigma_\mu, \mu \leq \lambda\}$$

and claim that $\bar{\lambda} = \lambda_1$. Indeed, suppose $\bar{\lambda} < \lambda_1$; we will show that T_λ can further be moved to the right, that is, there exists $\varepsilon > 0$ such that, for any $\lambda \in (\bar{\lambda}, \bar{\lambda} + \varepsilon)$,

$$w \leq w_\lambda. \tag{5.18}$$

This is a contradiction to the definition of $\bar{\lambda}$. Hence $\bar{\lambda} = \lambda_1$.

Now we will prove that (5.18) holds on $\Sigma_{\bar{\lambda}+\varepsilon}$. Let us divide $\Sigma_{\bar{\lambda}+\varepsilon}$ into two parts: $\Sigma_{\bar{\lambda}-\varepsilon}$ and $\Sigma_{\bar{\lambda}+\varepsilon} \setminus \Sigma_{\bar{\lambda}-\varepsilon}$, where $\varepsilon > 0$ is to be chosen.

Since $w_\lambda - w \geq c_\varepsilon > 0$ in $\Sigma_{\bar{\lambda}-\varepsilon}$ and $w_\lambda - w$ depends continuously on λ , there exists $\varepsilon > 0$ sufficiently small such that, for all $\lambda \in (\bar{\lambda}, \bar{\lambda} + \varepsilon)$,

$$w_\lambda - w \geq 0 \text{ in } \Sigma_{\bar{\lambda}-\varepsilon}. \tag{5.19}$$

On the other hand, since $w_\lambda - w \geq 0$ on $\partial(\Sigma_{\bar{\lambda}+\varepsilon} \setminus \Sigma_{\bar{\lambda}-\varepsilon})$ for $\varepsilon > 0$ as before, we have by Lemma 5.4 that

$$w_\lambda - w \geq 0 \text{ in } \Sigma_{\bar{\lambda}+\varepsilon} \setminus \Sigma_{\bar{\lambda}-\varepsilon}. \tag{5.20}$$

Combining (5.19) and (5.20), there follows (5.18).

Then $w \leq w_\lambda$ on Σ_λ . Using Lemma 5.4, we obtain $w < w_\lambda$ on Σ_λ , and now (5.17) is proved. □

Proof of Theorem 1.1 Lemma 4.4 and 5.5 imply (1.5). □

6 Conclusions

In this paper, we obtained the monotonicity of cylindrical solutions to the Dirichlet boundary value problem to singular semilinear subelliptic equation on the Heisenberg group. It is a generalization of the corresponding result in the classical Euclidean setting.

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