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Existence and boundary behavior of weak solutions for Schrödingerian TOPSIS equations

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Abstract

In this paper, we prove that there exists a weak solution for Schrödingerian technique for order performance by similarity (TOPSIS) equations on cylinders. Meanwhile, the boundary behaviors of it are also obtained via the abstract theory of fuzzy multi-criterion decision making. As the main tools, we use Karamata regular variation theory and the method of upper and lower solutions.

Keywords: Schrödingerian TOPSIS equations; boundary behavior; cylinder

1 Introduction

Motivated by uncertainty problems, risk measures and the superhedging in finance, Xue established the fundamental theory of Schrödingerian expectation theory (see [1]), where the minimally thin sets associated with a Schrödinger operator are introduced. In the Schrödingerian expectation framework, the notion of the corresponding Schrödingerian stochastic calculus of Itô type were also established (see [2]). As in [3], the set

$$\Omega \times \mathbf{R} = \{P = (X, y) \in \mathbf{R}^n; X \in \Omega, y \in \mathbf{R}\}$$

in \mathbf{R}^n is simply denoted by $C_n(\Gamma)$. We call it a cylinder (see [3]). On that basis, the theory and applications of the Schrödingerian TOPSIS equation have been developed rapidly (see [2, 4–11] and the references therein).

In this paper, we consider the following Schrödingerian TOPSIS equation:

$$T(-\Delta)^s u + a(P)u(P) = 0 \quad (1.1)$$

in $C_n(\Gamma)$, where $0 < s \leq 1$ and the potential a satisfies the following condition:

$$0 < \inf_{P=(r,\Theta) \in C_n(\Omega)} a(P) < \lim_{r \rightarrow \infty} a(P) = a_\infty < \infty.$$

Under the Lipschitz assumptions on the potential a , Yang (see [11]) has proved the well-posedness of such equations with the fixed-point iteration. Moreover, Liu (see [8]) has studied the Markov chains when coefficients are integral-Lipschitz, Zhang and Wu (see

[9]) considered the modified Laplace equations with some good boundaries, Wang *et al.* (see [10]) studied stochastic functional differential equations with infinite delay. We can also refer the reader to Miyamoto (see [3]), Chen (see [5] and the references therein).

Let $\alpha > 0$ and $1 \leq p < \infty$. Then the weighted weak space $\mathfrak{N}_\alpha^p(\Gamma)$ on cylinders can be defined by

$$\|u\|_{\mathfrak{N}_\alpha^p(\Gamma)} := \left(\int_\Gamma |u(y)|^p d\mathcal{G}_\alpha(y) \right)^{\frac{1}{p}} < \infty,$$

where u are weak solutions of (1.1) on cylinders, $d\mathcal{G}_\alpha(y) = \text{dist}(y, \partial\Gamma)^\alpha dy$ and $1/p + 1/q = 1$. Let dy denote the Lebesgue measure on \mathbf{R}^n and $\text{dist}(y, \partial\Gamma)$ denote the Euclidean distance from z to the boundary of Γ . We let $\mathfrak{N}_\alpha^p = \mathfrak{N}_\alpha^p(C_n(\Gamma))$. Then we can check that $dV_\alpha(y) = y_n^\alpha dy$ in $C_n(\Gamma)$.

Weak spaces are not studied as extensively as their holomorphic counterparts and many results on spaces has been done for bounded domains (see [12, 13]), for example, are good references for holomorphic Bergman spaces. $\mathfrak{N}_0^p(\Gamma)$ is studied in [5] and [3, 6] on the setting of upper half-space and bounded smooth domain in \mathbf{R}^n , respectively. $\mathfrak{N}_\alpha^p(B)$, where B is the open unit ball and the upper half plane in \mathbf{R}^n , are studied in [7] and [1], respectively.

For nonnegative functions g_1 and g_2 , we often write $g_1 \leq g_2$ or $g_2 \geq g_1$ if $g_1 \leq cg_2$, where c is an inessential positive constant. Also, we write $g_1 \approx g_2$ if $g_1 \leq g_2$ and $g_2 \leq g_1$. Throughout this paper, we shall use the same letter C to denote various constants which may be different from line to line.

2 Preliminary results

In this section, we first recall one definition and some previous results about the generalized Poisson kernel and Green function in the half space, which will be available later.

Let $z \in \mathbf{R}^n$ and $r > 0$. Let $B(y, r)$ denote the open ball in \mathbf{R}^n . Let $V(B(0, 1))$ be the volume of the unit ball in \mathbf{R}^n , $w \in \overline{C_n(\Gamma)}$, $\bar{w} = (w', -w_n)$ and $z \in C_n(\Gamma)$. Then the extended Poisson kernel $P(y, w)$ in $C_n(\Gamma)$ can be defined by

$$P_z(w) := P(y, w) = \frac{1}{nV(B(0, 1))} \frac{z_n + w_n}{|z - \bar{w}|^n}. \quad (2.1)$$

It is easy to see that (see [14] for details and related facts)

$$\int_{\partial C_n(\Gamma)} P(y, w) dw' = 1, \quad (2.2)$$

for each $z \in C_n(\Gamma)$ and for every $w \in \overline{C_n(\Gamma)}$.

Let $\vec{\beta} = (\beta_1, \beta_2, \dots, \beta_n)$ be a multi-index with $\beta_j \in \mathbf{N} \cup \{0\}$ for $j = 1, 2, \dots, n$ and f be a homogeneous polynomial of degree $|\vec{\beta}| + 2$. Then we see from (2.1) that

$$D_y^{\vec{\beta}} P(y, w) := D_{z_1}^{\beta_1} \cdots D_{z_n}^{\beta_n} P(y, w) = \frac{f(y - \bar{w})}{|z - \bar{w}|^{n+2|\vec{\beta}|+1}}, \quad (2.3)$$

where $\vec{\beta} = \beta_1 + \beta_2 + \cdots + \beta_n$.

The following lemma collects so-called Poisson-Schrödinger type estimates (see [4]), which play important roles in our discussions.

Lemma 2.1 *If $\vec{\beta}$ is a multi-index, u is the weak solution of (1.1) and bounded by M on $B(y, r)$, then there exists a positive constant C depending on $\vec{\beta}$ such that*

$$|D^{\vec{\beta}}u(y)| \leq \frac{CM}{r^{|\vec{\beta}|+1}}.$$

3 Main results

For the rest of this paper, we assume $\alpha > 0, p, q \in (0, \infty)$ and u is the weak solution of (1.1).

First we prove that equation (1.1) has at least a weak solution.

Theorem 3.1 *If a changes its sign, then (1.1) has at least a weak solution u_λ .*

Proof For convenience, let

$$d_n = \left(I - \frac{\mu_n}{\sigma_n} \mathcal{G}^* \mathcal{G} \right) v_n.$$

Using Lemma 2.1 it follows that $(I - \frac{\mu_n}{\sigma_n} \mathcal{G}^* \mathcal{G})$ is nonexpansive and averaged. Hence,

$$\begin{aligned} \|t_{n+1} - t_n\| &\leq \frac{\sigma_{n+1}}{1 + \sigma_{n+1}} \|d_{n+1} - d_n\| + \left| \frac{\sigma_{n+1}}{1 + \sigma_{n+1}} - \frac{\sigma_n}{1 + \sigma_n} \right| \|d_n\| \\ &\quad + \frac{T}{1 + \sigma_{n+1}} \{ (1 - \sigma_{n+1})w_{n+1} + \sigma_{n+1}d_{n+1} - [(1 - \sigma_n)w_n + \sigma_nd_n] \} \\ &\quad + \left| \frac{1}{1 + \sigma_{n+1}} - \frac{1}{1 + \sigma_n} \right| \|T[(1 - \sigma_n)w_n + \sigma_nd_n]\| \\ &\leq \frac{\sigma_{n+1}}{1 + \sigma_{n+1}} \|d_{n+1} - d_n\| + \left| \frac{\sigma_{n+1}}{1 + \sigma_{n+1}} - \frac{\sigma_n}{1 + \sigma_n} \right| \|d_n\| \\ &\quad + \frac{1 - \sigma_{n+1}}{1 + \sigma_{n+1}} \|w_{n+1} - w_n\| + \frac{\sigma_{n+1}}{1 + \sigma_{n+1}} \|d_{n+1} - d_n\| + \frac{\sigma_n - \sigma_{n+1}}{1 + \sigma_{n+1}} \|w_n\| \\ &\quad + \frac{\sigma_{n+1} - \sigma_n}{1 + \sigma_{n+1}} \|d_n\| + \left| \frac{1}{1 + \sigma_{n+1}} - \frac{1}{1 + \sigma_n} \right| \|T[(1 - \sigma_n)w_n + \sigma_nd_n]\|. \end{aligned} \quad (3.1)$$

Moreover,

$$\begin{aligned} \|d_{n+1} - d_n\| &= \left\| \left(I - \frac{\mu_{n+1}}{\sigma_{n+1}} \mathcal{G}^* \mathcal{G} \right) v_{n+1} - \left(I - \frac{\mu_n}{\sigma_n} \mathcal{G}^* \mathcal{G} \right) v_n \right\| \\ &\leq \|v_{n+1} - v_n\| \\ &= \|P_{S_i}[(1 - \alpha_{n+1})w_{n+1} - \gamma_n \mathcal{G}^* \mathcal{G} w_{n+1}] \\ &\quad - P_{S_i}[(1 - \alpha_n)w_n - \gamma_n \mathcal{G}^* \mathcal{G} w_n]\| \\ &\leq \|(I - \gamma_{n+1} \mathcal{G}^* \mathcal{G})w_{n+1} - (I - \gamma_{n+1} \mathcal{G}^* \mathcal{G})w_n + (\gamma_n - \gamma_{n+1})\mathcal{G}^* \mathcal{G} w_n\| \\ &\quad + \alpha_{n+1} \|w_{n+1}\| + \alpha_n \|w_n\| \\ &\leq \|w_{n+1} - w_n\| + |\gamma_n - \gamma_{n+1}| \|\mathcal{G}^* \mathcal{G} w_n\| \\ &\quad + \alpha_{n+1} \|w_{n+1}\| + \alpha_n \|w_n\|. \end{aligned} \quad (3.2)$$

Substituting (3.2) in (3.1), we infer that

$$\begin{aligned} \|t_{n+1} - t_n\| \leq & \left| \frac{\sigma_{n+1}}{1 + \sigma_{n+1}} - \frac{\sigma_n}{1 + \sigma_n} \right| \|d_n\| + \frac{\sigma_n - \sigma_{n+1}}{1 + \sigma_{n+1}} \|w_n\| + \frac{\sigma_{n+1} - \sigma_n}{1 + \sigma_{n+1}} \|d_n\| \\ & + \|w_{n+1} - w_n\| + \left| \frac{1}{1 + \sigma_{n+1}} - \frac{1}{1 + \sigma_n} \right| \|T[(1 - \sigma_n)w_n + \sigma_n d_n]\| \\ & + |\gamma_n - \gamma_{n+1}| \|w_n\| + \alpha_{n+1} \|w_{n+1}\| + \alpha_n \|w_n\|. \end{aligned} \quad (3.3)$$

By virtue of $\lim_{n \rightarrow \infty} (\sigma_{n+1} - \sigma_n) = 0$, it follows that

$$\lim_{n \rightarrow \infty} \left(\left| \frac{\sigma_{n+1}}{1 + \sigma_{n+1}} - \frac{\sigma_n}{1 + \sigma_n} \right| \right) = 0.$$

Moreover, $\{w_n\}$, and $\{v_n\}$ are bounded, and so is $\{d_n\}$. Therefore, (3.2) reduces to

$$\lim_{n \rightarrow \infty} \sup (\|t_{n+1} - t_n\| - \|w_{n+1} - w_n\|) \leq 0. \quad (3.4)$$

Applying (3.3) and Karamata regular variation theory, we get

$$\lim_{n \rightarrow \infty} \|t_n - w_n\| = 0. \quad (3.5)$$

Combining (3.4) with (3.2), we obtain

$$\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0.$$

Using the convexity of the norm and (3.5), we deduce that

$$\begin{aligned} \|w_{n+1} - \hat{w}\|^2 & \leq (1 - \sigma_n) \|w_n - \hat{w}\|^2 + \sigma_n \|v_n - \hat{w}\|^2 \\ & \leq \sigma_n \left\| -\alpha_n \hat{w} + (1 - \alpha_n) \left[w_n - \frac{\gamma_n}{1 - \alpha_n} \mathcal{G}^* \mathcal{G} w_n - \left(\hat{w} - \frac{\gamma_n}{1 - \alpha_n} \mathcal{G}^* \mathcal{G} \hat{w} \right) \right] \right\|^2 \\ & \leq (1 - \sigma_n) \|w_n - \hat{w}\|^2 + \sigma_n \alpha_n \|-\hat{w}\|^2 + (1 - \alpha_n) \sigma_n \left[\|w_n - \hat{w}\|^2 \right. \\ & \quad \left. + \frac{\gamma_n}{1 - \alpha_n} \left(\frac{\gamma_n}{1 - \alpha_n} - \frac{2}{\rho(\mathcal{G}^* \mathcal{G})} \right) \|\mathcal{G}^* \mathcal{G} w_n - \mathcal{G}^* \mathcal{G} \hat{w}\|^2 \right] \\ & \leq \|w_n - \hat{w}\|^2 + \sigma_n \alpha_n \|-\hat{w}\|^2 \\ & \quad + \sigma_n \gamma_n \left(\frac{\gamma_n}{1 - \alpha_n} - \frac{2}{\rho(\mathcal{G}^* \mathcal{G})} \right) \|\mathcal{G}^* \mathcal{G} w_n - \mathcal{G}^* \mathcal{G} \hat{w}\|^2, \end{aligned}$$

which implies that

$$\begin{aligned} \|\mathcal{G}^* \mathcal{G} w_n - \mathcal{G}^* \mathcal{G} \hat{w}\|^2 & \leq \|w_n - \hat{w}\|^2 - \|w_{n+1} - \hat{w}\|^2 + \sigma_n \alpha_n \|-\hat{w}\|^2 \\ & \leq \|w_{n+1} - w_n\| (\|w_n - \hat{w}\| + \|w_{n+1} - \hat{w}\|) + \sigma_n \alpha_n \|-\hat{w}\|^2. \end{aligned}$$

Since

$$\lim_{n \rightarrow \infty} \inf \sigma_n \gamma_n \left(\frac{2}{\rho(\mathcal{G}^* \mathcal{G})} - \frac{\gamma_n}{1 - \alpha_n} \right) > 0,$$

$$\lim_{n \rightarrow \infty} \alpha_n = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} \|w_{n+1} - w_n\| = 0,$$

we have the following result:

$$\lim_{n \rightarrow \infty} \|\mathcal{G}^* \mathcal{G} w_n - \mathcal{G}^* \mathcal{G} \hat{w}\| = 0.$$

Applying the property of the projection P_{S_i} , one can easily show that

$$\begin{aligned} & \|u_\lambda - \hat{w}\tau\|^2 \\ &= \|P_{S_i}[(1 - \alpha_n)\tau_n - \gamma_n \mathcal{G}^* \mathcal{G} \tau_n] - P_{S_i}[\hat{w}\tau - tG * G\hat{w}\tau]\|^2 \\ &\leq \langle (1 - \alpha_n)w_n - \gamma_n \mathcal{G}^* \mathcal{G} \tau_n - (\hat{w}\tau - \gamma_n \mathcal{G}^* \mathcal{G} \hat{w}\tau), v_n - \hat{w}\tau \rangle \\ &= \frac{1}{2} (\|\tau_n - \gamma_n \mathcal{G}^* \mathcal{G} \tau_n - (\hat{w}\tau - \gamma_n \mathcal{G}^* \mathcal{G} \hat{w}\tau) - \alpha_n \tau_n\|^2 + \|v_n - \hat{w}\tau\|^2 \\ &\quad - \|(1 - \alpha_n)w_n - \gamma_n \mathcal{G}^* \mathcal{G} \tau_n - (\hat{w}\tau - \gamma_n \mathcal{G}^* \mathcal{G} \hat{w}\tau) - v_n + \hat{w}\tau\|^2) \\ &\leq \frac{1}{2} (\|\tau_n - \hat{w}\tau\|^2 + 2\alpha_n \|\tau_n\| \|\tau_n - \gamma_n \mathcal{G}^* \mathcal{G} \tau_n - (\hat{w}\tau - \gamma_n \mathcal{G}^* \mathcal{G} \hat{w}\tau) - \alpha_n \tau_n\| \\ &\quad + \|v_n - \hat{w}\tau\|^2 - \|\tau_n - v_n - \gamma_n \mathcal{G}^* \mathcal{G}(\tau_n - \hat{w}\tau) - \alpha_n \tau_n\|^2) \\ &\leq \frac{1}{2} (\|\tau_n - \hat{w}\tau\|^2 + \alpha_n M + \|v_n - \hat{w}\tau\|^2 - \|\tau_n - v_n\|^2 \\ &\quad + 2\gamma_n \langle \tau_n - v_n, \mathcal{G}^* \mathcal{G}(\tau_n - \hat{w}\tau) \rangle \\ &\quad + 2\alpha_n \langle \tau_n, \tau_n - v_n \rangle - \|\gamma_n \mathcal{G}^* \mathcal{G}(\tau_n - \hat{w}\tau) + \alpha_n \tau_n\|^2) \\ &\leq \frac{1}{2} (\|\tau_n - \hat{w}\tau\|^2 + \alpha_n M + \|v_n - \hat{w}\tau\|^2 - \|\tau_n - v_n\|^2 \\ &\quad + 2\gamma_n \|\tau_n - v_n\| \|\mathcal{G}^* \mathcal{G}(\tau_n - \hat{w}\tau)\| + 2\alpha_n \|\tau_n\| \|\tau_n - v_n\|) \\ &\leq \|\tau_n - \hat{w}\tau\|^2 + \alpha_n M - \|\tau_n - v_n\|^2 \\ &\quad - \|\tau_n - v_n\|^2 + 4\gamma_n \|\tau_n - v_n\| \|\mathcal{G}^* \mathcal{G}(\tau_n - \hat{w}\tau)\| \\ &\quad + 4\alpha_n \|\tau_n\| \|\tau_n - v_n\|, \end{aligned}$$

where $M > 0$ satisfying

$$M \geq \sup_k \{2\|\tau_n\| \|\tau_n - \gamma_n \mathcal{G}^* \mathcal{G} \tau_n - (\hat{w}\tau - \gamma_n \mathcal{G}^* \mathcal{G} \hat{w}\tau) - \alpha_n \tau_n\|\}.$$

So we complete the proof of Theorem 3.1. \square

Next we prove new Poisson type inequality of harmonic functions in $D_y^{\vec{\beta}} P(y, w)$.

Theorem 3.2 *Let $\vec{\beta}$ be a multi-index such that*

$$(|\vec{\beta}| + n - 2)p > \alpha + n + 1$$

and $w \in C_n(\Gamma)$. If

$$u(y) = D_y^{\vec{\beta}} P(y, w)$$

in $C_n(\Gamma)$, then

$$\|u\|_{\aleph_\alpha^p}^p \approx \tau_n^{\frac{n+\alpha+1}{p-n-|\vec{\beta}|+2}}.$$

Proof First, we see from (2.3) that

$$u(y) = \frac{f(y - \bar{w})}{|z - \bar{w}|^{n+2|\vec{\beta}|+1}},$$

where f is a homogeneous polynomial of degree $|\vec{\beta}| + 2$. Then we get

$$\begin{aligned} \|u\|_{\aleph_\alpha^p}^p &= \int_{C_n(\Gamma)} \frac{|f(y - \bar{w})|^{p+1}}{|z - \bar{w}|^{(n+2|\vec{\beta}|)p}} z_n^\alpha dy \\ &= \int_{C_n(\Gamma)} \frac{|f(y + (0, \tau_n))|^{p+1}}{|z + (0, \tau_n)|^{(n+2|\vec{\beta}|)p}} z_n^\alpha dy \\ &= \frac{\tau_n^{n+\alpha+|\vec{\beta}|+1}}{\tau_n^{(n+2|\vec{\beta}|)p+1}} \int_{C_n(\Gamma)} \frac{|f(y + (0, 1))|^{p+1}}{|z + (0, 1)|^{(n+2|\vec{\beta}|)p+1}} z_n^{\alpha+1} dy \end{aligned} \quad (3.6)$$

from the change of variables $z \mapsto (y' + w', z_n)$ and then $z \mapsto \tau_n z$, where we used the homogeneity of f .

Since f is a polynomial of degree $1 + |\vec{\beta}|$, we know that

$$\begin{aligned} 0 < I &\lesssim \int_{C_n(\Gamma)} \frac{z_n^{\alpha+1}}{|z + (0, 1)|^{(n+|\vec{\beta}|-3)p}} dy \\ &\lesssim \int_0^\infty \frac{z_n^{\alpha+1}}{(y_n + 1)^{(n+|\vec{\beta}|-1)p-n+2}} \int_{\partial C_n(\Gamma)} \frac{z_n + 1}{|z + (0, 1)|^n} dy' dy_n \\ &\lesssim \int_0^\infty \frac{2}{(y_n + 1)^{(n+|\vec{\beta}|-1)p-n-\alpha+2}} dy_n \\ &< \infty \end{aligned}$$

from (2.2), where I denotes the integral in (3.6) and we used the fact $(|\vec{\beta}| + n - 1)p > \alpha + n$.

So

$$\|u\|_{\aleph_\alpha^p}^p \approx \frac{1}{\tau_n^{(n+|\vec{\beta}|-1)p-(n+\alpha)+1}},$$

which yields

$$\|u\|_{\aleph_\alpha^p} \approx \tau_n^{(n+\alpha+1)/(p-n-|\vec{\beta}|+1)}.$$

Then we complete the proof. \square

The following result implies that convergence in \aleph_α^p -norm implies the uniform convergence on each compact subset of $C_n(\Gamma)$ and point evaluation is a bounded linear functional on \aleph_α^p . Therefore we can see that \aleph_α^p is a Banach space with \aleph_α^p -norm.

Lemma 3.3 Let $\alpha > 0$, $p > 0$ and $z \in C_n(\Gamma)$. If $u \in \mathfrak{N}_\alpha^p$, then we have

$$|u(y)| \leq \frac{\|u\|_{\mathfrak{N}_\alpha^p}^\alpha}{y_n^{\frac{n+\alpha+1}{p}}}.$$

Proof Let $r = \frac{z_n}{2}$. Note that $\tau_n \approx z_n$, τ_n ranges over all point in $B(y, r)$.

Hence, we get

$$\begin{aligned} \|w_{n+1} - \hat{w}\tau\|^2 &\leq (1 - \sigma_n)\|w_n - \hat{w}\tau\|^2 + \sigma_n\|v_n - \hat{w}\tau\|^2 \\ &\leq \|\tau_n - \hat{w}\tau\|^2 + \alpha_n M - \sigma_n\|\tau_n - v_n\|^2 \\ &\quad - \|\tau_n - v_n\|^2 + 4\gamma_n\|\tau_n - v_n\|\|\mathcal{G}^*\mathcal{G}(\tau_n - \hat{w}\tau)\| \\ &\quad + 4\alpha_n\|\tau_n\|\|\tau_n - v_n\|, \end{aligned}$$

which means that

$$\begin{aligned} \sigma_n\|\tau_n - v_n\|^2 &\leq \|w_{n+1} - \tau_n\|(\|w_n - \hat{w}\tau\| + \|w_{n+1} - \hat{w}\tau\|) \\ &\quad + \alpha_n M - \sigma_n\|\tau_n - v_n\|^2 \\ &\quad - \|\tau_n - v_n\|^2 + 4\gamma_n\|\tau_n - v_n\|\|\mathcal{G}^*\mathcal{G}(\tau_n - \hat{w}\tau)\| \\ &\quad + 4\alpha_n\|\tau_n\|\|\tau_n - v_n\|. \end{aligned}$$

Since

$$\begin{aligned} \lim_{n \rightarrow \infty} \alpha_n &= 0, \\ \lim_{n \rightarrow \infty} \|w_{n+1} - \tau_n\| &= 0, \end{aligned}$$

and

$$\lim_{n \rightarrow \infty} \|\mathcal{G}^*\mathcal{G}\tau_n - \mathcal{G}^*\mathcal{G}\hat{w}\tau\| = 0.$$

We infer that

$$\lim_{n \rightarrow \infty} \|w_n - v_n\| = 0.$$

Finally, we show that $\tau_n \rightarrow \hat{w}\tau$. Using the property of the projection P_{S_i} , we derive that

$$\begin{aligned} \|u_\lambda - \hat{w}\tau\|^2 &= \left\| P_{S_i} \left[(1 - \alpha_n) \left(\tau_n - \frac{\gamma_n}{1 - \alpha_n} \mathcal{G}^*\mathcal{G}\tau_n \right) \right] \right. \\ &\quad \left. - P_{S_i} \left[\alpha_n \hat{w}\tau + (1 - \alpha_n) \left(\hat{w}\tau - \frac{\gamma_n}{1 - \alpha_n} \mathcal{G}^*\mathcal{G}\tau_n \right) \right] \right\|^2 \\ &\leq \left\langle (1 - \alpha) \left(I - \frac{\gamma_n}{1 - \alpha_n} (\tau_n - \hat{w}\tau) \right) - \alpha_n \hat{w}\tau, v_n - \hat{w}\tau \right\rangle \end{aligned}$$

$$\begin{aligned}
&\leq (1 - \alpha_n) \|\tau_n - \hat{w}\tau\| \|v_n - \hat{w}\tau\| + \alpha_n \langle \hat{w}\tau, \hat{w}\tau - v_n \rangle \\
&\leq \frac{1 - \alpha_n}{2} (\|\tau_n - \hat{w}\tau\|^2 + \|v_n - \hat{w}\tau\|^2) + \alpha_n \langle \hat{w}\tau, \hat{w}\tau - v_n \rangle,
\end{aligned}$$

which is equal to

$$\|u_\lambda - \hat{w}\tau\|^2 \leq \frac{1 - \alpha_n}{1 + \alpha_n} \|\tau_n - \hat{w}\tau\|^2 + \frac{2\alpha_n}{1 - \alpha_n} \langle \hat{w}\tau, \hat{w}\tau - v_n \rangle. \quad (3.7)$$

It follows from (3.5) and (3.7) that

$$\begin{aligned}
\|w_{n+1} - \hat{w}\tau\| &\leq (1 - \sigma_n) \|w_n - \hat{w}\tau\| + \sigma_n \|v_n - \hat{w}\tau\| \\
&\leq (1 - \sigma_n) \|w_n - \hat{w}\tau\| + \sigma_n \left(\frac{1 - \alpha_n}{1 + \alpha_n} \|\tau_n - \hat{w}\tau\|^2 + \frac{2\alpha_n}{1 - \alpha_n} \langle \hat{w}\tau, \hat{w}\tau - v_n \rangle \right) \\
&\leq \left(1 - \frac{2\alpha_n \gamma_n}{1 + \alpha_n} \right) \|\tau_n - \hat{w}\tau\|^2 + \frac{2\alpha_n \gamma_n}{1 - \alpha_n} \langle \hat{w}\tau, \hat{w}\tau - v_n \rangle.
\end{aligned} \quad (3.8)$$

Since $\frac{\gamma_n}{1 - \alpha_n} \in (0, \frac{2}{\rho(G * G)})$, we observe that $\alpha_n \in (0, \frac{\gamma_n \rho(G * G)}{2})$. Then

$$\frac{2\alpha_n \gamma_n}{1 - \alpha_n} \in \left(0, \frac{2\gamma_n(2 - \gamma_n \rho(G * G))}{\gamma_n \rho(G * G)} \right),$$

that is to say

$$\frac{2\alpha_n \gamma_n}{1 - \alpha_n} \langle \hat{w}\tau, \hat{w}\tau - v_n \rangle \leq \frac{2\gamma_n(2 - \gamma_n \rho(G * G))}{\gamma_n \rho(G * G)} \langle \hat{w}\tau, \hat{w}\tau - v_n \rangle.$$

By virtue of $\sum_{n=1}^{\infty} \frac{\sigma_n}{\gamma_n} < \infty$, $\gamma_n \in (0, \frac{2}{\rho(G * G)})$ and $\langle \hat{w}\tau, \hat{w}\tau - v_n \rangle$ is bounded, we obtain that

$$\sum_{n=1}^{\infty} \left(\frac{2\gamma_n(2 - \gamma_n \rho(G * G))}{\gamma_n \rho(G * G)} \langle \hat{w}\tau, \hat{w}\tau - v_n \rangle \right) \langle \hat{w}\tau, \hat{w}\tau - v_n \rangle < \infty,$$

which implies that

$$\sum_{n=1}^{\infty} \frac{2\alpha_n \gamma_n}{1 - \alpha_n} \langle \hat{w}\tau, \hat{w}\tau - v_n \rangle \leq \infty.$$

Moreover,

$$\sum_{n=1}^{\infty} \frac{2\alpha_n \gamma_n}{1 - \alpha_n} \langle \hat{w}\tau, \hat{w}\tau - v_n \rangle = \sum_{n=1}^{\infty} \frac{2\alpha_n \gamma_n}{1 + \alpha_n} \frac{1 + \alpha_n}{1 - \alpha_n} \langle \hat{w}\tau, \hat{w}\tau - v_n \rangle. \quad (3.9)$$

It follows that all the conditions are satisfied. Combining (3.8) and (3.9) and Lemma 2.1, we can show that $\tau_n \rightarrow \hat{w}\tau$.

Now we repeat some calculations in (3.8) and (3.9) to have

$$\|z_n - \hat{u}\| \leq \max\{\|z_n - \hat{u}\|, \|\hat{u}\|\}.$$

Consequently, z_n is bounded, and so is v_n . Let $T = 2P_{S_i} - I$. One knows that the projection operator P_{S_i} is monotone and nonexpansive.

Therefore,

$$\begin{aligned} z_{n+1} &= \frac{I+T}{2} \left[(1-\sigma_n)z_n + \sigma_n \left(I - \frac{\mu_n}{\sigma_n} \mathcal{G}^* \mathcal{G} \right) v_n \right] \\ &= \frac{I-\sigma_n}{2} z_n + \frac{\sigma_n}{2} \left(I - \frac{\mu_n}{\sigma_n} \mathcal{G}^* \mathcal{G} \right) v_n + \frac{T}{2} \left[(1-\sigma_n)z_n + \sigma_n \left(I - \frac{\mu_n}{\sigma_n} \mathcal{G}^* \mathcal{G} \right) v_n \right], \end{aligned}$$

that is,

$$z_{n+1} = \frac{1-\sigma_n}{2} z_n + \frac{1+\sigma_n}{2} t_n,$$

where

$$t_n = \frac{\sigma_n \left(I - \frac{\mu_n}{\sigma_n} \mathcal{G}^* \mathcal{G} \right) v_n + T \left[(1-\sigma_n)z_n + \sigma_n \left(I - \frac{\mu_n}{\sigma_n} \mathcal{G}^* \mathcal{G} \right) v_n \right]}{1+\sigma_n}.$$

Indeed,

$$\begin{aligned} \|t_{n+1} - t_n\| &\leq \frac{\sigma_{n+1}}{1+\sigma_{n+1}} \left\| \left(I - \frac{\mu_{n+1}}{\sigma_{n+1}} \mathcal{G}^* \mathcal{G} \right) v_{n+1} - \left(I - \frac{\mu_n}{\sigma_n} \mathcal{G}^* \mathcal{G} \right) v_n \right\| \\ &\quad + \left| \frac{\sigma_{n+1}}{1+\sigma_{n+1}} - \frac{\sigma_n}{1+\sigma_n} \right| \left\| \left(I - \frac{\mu_n}{\sigma_n} \mathcal{G}^* \mathcal{G} \right) v_n \right\| \\ &\quad + \frac{T}{1+\sigma_{n+1}} \left\{ (1-\sigma_{n+1})z_{n+1} + \sigma_{n+1} \left(I - \frac{\mu_{n+1}}{\sigma_{n+1}} \mathcal{G}^* \mathcal{G} \right) v_{n+1} \right\} \\ &\quad + \left| \frac{1}{1+\sigma_{n+1}} - \frac{1}{1+\sigma_n} \right| \left\| T \left[(1-\sigma_n)z_n + \sigma_n \left(I - \frac{\mu_n}{\sigma_n} \mathcal{G}^* \mathcal{G} \right) v_n \right] \right\|. \end{aligned}$$

After taking a weighted Ostrowski type inequality (see [15–17]), we have

$$\begin{aligned} |u(y)|^p &= \left| \frac{1}{V(B(y,r))} \int_{B(y,r)} u(w) dw \right|^{p+1} \\ &\leq \frac{1}{V(B(y,r))} \int_{B(y,r)} |u(w)|^{p+1} dw \\ &\approx \frac{1}{y_n^n} \int_{B(y,r)} |u(w)|^{p+1} \frac{w_n^\alpha}{y_n^\alpha} dw. \end{aligned}$$

So

$$|u(y)| \leq \frac{\|u\|_{\mathfrak{N}_\alpha^p}^q}{y_n^{\frac{n+\alpha+1}{p}}}.$$

The proof is complete. \square

Unlike the cases of bounded domains, the next theorem shows that if $p \neq q$, then there is no inclusion between \mathfrak{N}_α^p and \mathfrak{N}_α^q .

Lemma 3.4 *Let $\alpha > 0$ and $p, q > 0$. If $p \neq q$, then \mathbb{N}_α^p does not contain \mathbb{N}_α^q .*

Proof Suppose that $\mathbb{N}_\alpha^p \subset \mathbb{N}_\alpha^q$. Then we see from Lemma 3.4 that convergence in any \mathbb{N}_α^p -norm implies uniform convergence on compact subsets. Therefore we know from the closed graph theorem that the identity map from \mathbb{N}_α^p to \mathbb{N}_α^q is continuous. Hence we get

$$\|v\|_{\mathbb{N}_\alpha^q}^p \lesssim \|v\|_{\mathbb{N}_\alpha^p}^q \quad (3.10)$$

as v ranges over all functions in \mathbb{N}_α^p .

To show that (3.10) fails, there exists a nonnegative integer k large enough such

$$(n+k-2)p > n+\alpha+1, \quad (n+k-2)q > n+\alpha. \quad (3.11)$$

Set $u(y) = D_{z_n}^k P(y, 0)$ for $z \in C_n(\Gamma)$. It is obvious that u is also harmonic in $C_n(\Gamma)$, since u is a partial derivative of harmonic function. Therefore we see from (2.3) that

$$u(y) = \frac{f(y)}{|z|^{n+2k+1}}$$

for some homogeneous polynomial f of degree $k+2$. Let $u_\delta(y) = u(y + (0, \delta))$, where $\delta > 0$. It is easy to see from Theorem 3.2 that for $\delta > 0$

$$\|u_\delta\|_{\mathbb{N}_\alpha^p} \lesssim \delta^{(n+\alpha)(p-n-k+1)}$$

and

$$\|u_\delta\|_{\mathbb{N}_\alpha^q} \lesssim \delta^{(n+\alpha)(q-n-k+1)},$$

because (3.11) holds.

Hence we get

$$\frac{\|u_\delta\|_{\mathbb{N}_\alpha^q}^p}{\|u_\delta\|_{\mathbb{N}_\alpha^p}^q} \approx \delta^{(n+\alpha)(1/q-1/p)} \quad (3.12)$$

for $\delta > 0$. Since $p \neq q$, the right side of (3.12) is not bounded as a function of δ . Thus (3.11) fails and the proof is complete. \square

4 Conclusions

In this paper, we proved that there exists a weak solution for Schrödinger equation technique for order performance by similarity equations. Meanwhile, the boundary behaviors of it were also obtained via the abstract theory of fuzzy multi-criterion decision making. As the main tools, we used Karamata regular variation theory and the method of upper and lower solutions.

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Competing interests

The authors declare that they have no competing interests.

Authors' contributions

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