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Multiple positive solutions for a class of Kirchhoff type equations in $\mathbb{R}^N$

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Abstract

In this paper, we study the following nonlinear Kirchhoff type equation:

$$-\left(a+b\int_{\mathbb{R}^N}|\nabla u|^2\,dx\right)\Delta u+Vu=f(u)+h(x),\quad x\in\mathbb{R}^N,$$

where a, b, V are positive constants, $N=2$ or 3. Under appropriate assumptions on f and h, we get that the equation has two positive solutions by using variational methods.

MSC: 35J60; 35B09; 35A15

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1 Introduction and main results

We consider the following nonlinear Kirchhoff type equation:

$$-\left(a+b\int_{\mathbb{R}^N}|\nabla u|^2\,dx\right)\Delta u+Vu=f(u)+h(x),\quad x\in\mathbb{R}^N,\tag{1.1}$$

where a, b, V are positive constants, $N=2$ or 3.

In recent years, the existence or multiplicity of solutions for the following Kirchhoff type equation

$$-\left(a+b\int_{\mathbb{R}^N}|\nabla u|^2\,dx\right)\Delta u+V(x)u=f(x,u),\quad x\in\mathbb{R}^N,$$

where a, b are positive constants, $N=1,2,3$, has been widely investigated by many authors, for example [1–6], etc. But in those papers, the nonlinearity f satisfies 3-superlinear growth at infinity, which assures the boundedness of any Palais-Smale sequence or Cerami sequence.

Very recently, Guo [7], Li and Ye [8], Liu and Guo [9], Tang and Chen [10] studied respectively the following equation:

$$-\left(a+b\int_{\mathbb{R}^3}|\nabla u|^2\,dx\right)\Delta u+V(x)u=f(u),\quad x\in\mathbb{R}^3,$$

where a, b are positive constants, f only needs to satisfy superlinear growth at infinity. By using the Pohozaev equality, it is easy to obtain a bounded Palais-Smale sequence. Thus they obtained the existence of positive solution.

Inspired by [7–10], we study equation (1.1); in here, very weak conditions are assumed on f. Exactly, $f \in C(\mathbb{R}^+, \mathbb{R})$ satisfies

(f_1) when $N = 2$, there exists $p \in (2, +\infty)$ such that $\lim_{t\to+\infty} \frac{f(t)}{t^{p-1}} = 0$; when $N = 3$, $\lim_{t\to+\infty} \frac{f(t)}{t^5} = 0$;
(f_2) $\lim_{t\to0^+} \frac{f(t)}{t} = m \in (-\infty, V)$;
(f_3) $\lim_{t\to+\infty} \frac{f(t)}{t} = +\infty$.

On h, we make the following hypotheses:

(h_1) $h \in L^2(\mathbb{R}^N) \cap C^1(\mathbb{R}^N)$ is nonnegative and $h \not\equiv 0$;
(h_2) when $N = 2$, $0 \le (\nabla h(x), x) \in L^2(\mathbb{R}^2)$; when $N = 3$, $(\nabla h(x), x) \in L^2(\mathbb{R}^3)$;
(h_3) h is radially symmetric.

By using Ekeland's variational principle [11] and Struwe's monotonicity trick [12], we get the following.

Theorem 1.1 *Suppose that* (f_1)-(f_3) *and* (h_1)-(h_3) *hold. Then there exists* $m_0 > 0$ *such that, when* $(\int_{\mathbb{R}^N} h^2\,dx)^{\frac{1}{2}} < m_0$, *equation* (1.1) *has two positive solutions.*

When $f(t) < 0$, by (f_2) and (f_3), there exists $l > 0$ such that $f(t) + lt \ge 0$ for all $t \ge 0$. Thus equation (1.1) is equivalent to the following equation:

$$-\left(a + b\int_{\mathbb{R}^N} |\nabla u|^2\,dx\right)\triangle u + Wu = k(u) + h(x), \quad x \in \mathbb{R}^N, \tag{1.2}$$

where $W = V + l > 0$ and $k(t) = f(t) + lt \in C(\mathbb{R}^+, \mathbb{R}^+)$ satisfies

(k_1) when $N = 2$, there exists $p \in (2, +\infty)$ such that $\lim_{t\to+\infty} \frac{k(t)}{t^{p-1}} = 0$; when $N = 3$, $\lim_{t\to+\infty} \frac{k(t)}{t^5} = 0$;
(k_2) $\lim_{t\to0^+} \frac{k(t)}{t} = m + l := d \in [0, W)$;
(k_3) $\lim_{t\to+\infty} \frac{k(t)}{t} = +\infty$.

Hence in order to prove Theorem 1.1, we only need to prove the following.

Theorem 1.2 *Suppose that* (k_1)-(k_3) *and* (h_1)-(h_3) *hold. Then there exists* $m_0 > 0$ *such that when* $(\int_{\mathbb{R}^N} h^2\,dx)^{\frac{1}{2}} < m_0$, *equation* (1.2) *has two positive solutions.*

Remark 1.3 Under hypotheses on k, we are not able to obtain directly the boundedness of the Palais-Smale sequences. Inspired by Jeanjean's idea in [13] and [14], we will use an indirect approach, i.e., Struwe's monotonicity trick developed by Jeanjean. It is worth pointing out that comparing with $N = 3$, when $N = 2$, it is more complex to prove the boundedness of the Palais-Smale sequences, which will be seen in Lemma 3.8.

2 Preliminaries

From now on, we will use the following notations.

- $E := \{u \in H^1(\mathbb{R}^N) : u(x) = u(|x|)\}$ is the usual Sobolev space endowed with the norm

$$\|u\| = \left(\int_{\mathbb{R}^N} |\nabla u|^2 + u^2\,dx\right)^{\frac{1}{2}}.$$

- $D^{1,2}(\mathbb{R}^N)$ is completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm

$$\|u\|_{D^{1,2}(\mathbb{R}^N)} = \left(\int_{\mathbb{R}^N} |\nabla u|^2\,dx\right)^{\frac{1}{2}}.$$

- For any $1 \le p < \infty$, $L^p(\mathbb{R}^N)$ denotes the Lebesgue space and its norm is denoted by

$$|u|_p = \left(\int_{\mathbb{R}^N} |u|^p\,dx\right)^{\frac{1}{p}}.$$

- $\langle\cdot,\cdot\rangle$ denotes the action of dual, $(\cdot,\cdot)$ denotes the inner product in $\mathbb{R}^N$.
- C, C_i denote various positive constants.

Since we are looking for positive solution, we may assume that $k(t) = 0$ for all $t < 0$. Under the assumptions on k and h, it is obvious that the functional $I : E \to \mathbb{R}$ defined by

$$I(u) = \frac{a}{2}\int_{\mathbb{R}^N} |\nabla u|^2\,dx + \frac{b}{4}\left(\int_{\mathbb{R}^N} |\nabla u|^2\,dx\right)^2 + \frac{W}{2}\int_{\mathbb{R}^N} u^2\,dx - \int_{\mathbb{R}^N} K(u)\,dx - \int_{\mathbb{R}^N} hu\,dx$$

is of class C^1, where $K(t) = \int_0^t k(s)\,ds$ and

$$\begin{aligned}\langle I'(u), v\rangle = {}& \left(a + b\int_{\mathbb{R}^N} |\nabla u|^2\,dx\right)\int_{\mathbb{R}^N} (\nabla u, \nabla v)\,dx + W\int_{\mathbb{R}^N} uv\,dx - \int_{\mathbb{R}^N} k(u)v\,dx \\ & - \int_{\mathbb{R}^N} hv\,dx,\end{aligned}$$

for all $u, v \in E$. As is well known, the weak solution of equation (1.2) is the critical point of I in E.

3 Proof of the main results

Next lemma can be viewed as a generalization of Struwe's monotonicity trick [12] and is the main tool for obtaining a bounded Palais-Smale sequence.

Lemma 3.1 (see [13] or [14]) *Let X be a Banach space equipped with a norm $\|\cdot\|_X$, and let $J \subset \mathbb{R}^+$ be an interval. We consider a family $\{\Phi_\mu\}_{\mu\in J}$ of C^1-functionals on X of the form*

$$\Phi_\mu(u) = A(u) - \mu B(u), \quad \forall \mu \in J,$$

where $B(u) \ge 0$ for all $u \in X$ and such that either $A(u) \to +\infty$ or $B(u) \to +\infty$ as $\|u\|_X \to +\infty$. We assume that there are two points v_1, v_2 in X such that

$$c_\mu = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} \Phi_\mu\bigl(\gamma(t)\bigr) > \max\bigl\{\Phi_\mu(v_1), \Phi_\mu(v_2)\bigr\},$$

where

$$\Gamma = \{\gamma \in C([0,1],X) : \gamma(0) = v_1, \gamma(1) = v_2\}.$$

Then, for almost every $\mu \in J$, there is a bounded $(PS)_{c_\mu}$ sequence for Φ_μ, that is, there exists a sequence $\{u_n\} \subset X$ such that
(1) *$\{u_n\}$ is bounded in X,*
(2) *$\Phi_\mu(u_n) \to c_\mu$,*
(3) *$\Phi'_\mu(u_n) \to 0$ in X^*, where X^* is the dual of X.*

Remark 3.2 In [13], it is also proved that, under the assumptions of Lemma 3.1, the map $\mu \mapsto c_\mu$ is left-continuous.

In the paper, we set $X := E$, $\|\cdot\|_X := \|\cdot\|$ and $J := [\frac{1}{2}, 1]$. Let us define $I_\mu : E \to \mathbb{R}$ by $I_\mu(u) = A(u) - \mu B(u)$, where

$$A(u) = \frac{a}{2}\int_{\mathbb{R}^N} |\nabla u|^2\,dx + \frac{b}{4}\left(\int_{\mathbb{R}^N} |\nabla u|^2\,dx\right)^2 + \frac{W}{2}\int_{\mathbb{R}^N} u^2\,dx - \int_{\mathbb{R}^N} hu\,dx,$$
$$B(u) = \int_{\mathbb{R}^N} K(u)\,dx.$$

Then $I_1(u) = I(u)$. By (k_1)-(k_3) and (h_1), it is obvious that $I_\mu \in C^1(E,\mathbb{R})$, $B(u) \geq 0$ for all $u \in E$ and $A(u) \geq \frac{\min\{a,W\}}{2}\|u\|^2 - C|h|_2\|u\| \to +\infty$ as $\|u\| \to +\infty$.

Lemma 3.3 *Assume that (k_1)-(k_3) and (h_1) hold. Then there exist $\rho > 0$, $\alpha > 0$ and $m_0 > 0$ such that $I_\mu(u)|_{\|u\|=\rho} \geq \alpha$ for all h satisfying $|h|_2 < m_0$ and for all $\mu \in J$.*

Proof First, we consider $N = 2$. It follows from (k_1) and (k_2) that, for all $t \in \mathbb{R}$, we have

$$|K(t)| \leq \frac{W+d}{4}|t|^2 + C|t|^p. \tag{3.1}$$

By (3.1), the Hölder inequality and the Sobolev inequality, for all $\mu \in J$ and $u \in E$, one has

$$\begin{aligned}
I_\mu(u) &\geq \frac{a}{2}\int_{\mathbb{R}^2} |\nabla u|^2\,dx + \frac{W}{2}\int_{\mathbb{R}^2} u^2\,dx - \int_{\mathbb{R}^2} K(u)\,dx - \int_{\mathbb{R}^2} hu\,dx \\
&\geq \frac{a}{2}\int_{\mathbb{R}^2} |\nabla u|^2\,dx + \frac{W}{2}\int_{\mathbb{R}^2} u^2\,dx - \frac{W+d}{4}\int_{\mathbb{R}^2} u^2\,dx - C\int_{\mathbb{R}^2} |u|^p\,dx - |h|_2|u|_2 \\
&\geq \frac{\min\{2a, W-d\}}{4}\|u\|^2 - C_1\|u\|^p - C_2|h|_2\|u\| \\
&= \|u\|\left(\frac{\min\{2a, W-d\}}{4}\|u\| - C_1\|u\|^{p-1} - C_2|h|_2\right).
\end{aligned}$$

Let $g_1(t) = \frac{\min\{2a,W-d\}}{4}t - C_1 t^{p-1}$ for $t \geq 0$. Since $p > 2$, we know that there exists a constant $\rho > 0$ such that $\max_{t\geq 0} g_1(t) = g_1(\rho) > 0$. Choose $m_0 = \frac{1}{2C_2}g_1(\rho)$, then there exists $\alpha > 0$ such that $I_\mu(u)|_{\|u\|=\rho} \geq \alpha$ for all h satisfying $|h|_2 < m_0$.

Next when $N = 3$, it follows from (k_1) and (k_2) that, for all $t \in \mathbb{R}$, we have

$$|K(t)| \leq \frac{W+d}{4}|t|^2 + C|t|^6. \tag{3.2}$$

By (3.2), the Hölder inequality and the Sobolev inequality, for all $\mu \in J$ and $u \in E$, one has

$$\begin{aligned} I_\mu(u) &\geq \frac{a}{2}\int_{\mathbb{R}^3}|\nabla u|^2\,dx + \frac{W}{2}\int_{\mathbb{R}^3}u^2\,dx - \int_{\mathbb{R}^3}K(u)\,dx - \int_{\mathbb{R}^3}hu\,dx \\ &\geq \frac{a}{2}\int_{\mathbb{R}^3}|\nabla u|^2\,dx + \frac{W}{2}\int_{\mathbb{R}^3}u^2\,dx - \frac{W+d}{4}\int_{\mathbb{R}^3}u^2\,dx - C\int_{\mathbb{R}^3}|u|^6\,dx - |h|_2|u|_2 \\ &\geq \frac{\min\{2a, W-d\}}{4}\|u\|^2 - C_3\|u\|^6 - C_4|h|_2\|u\| \\ &= \|u\|\left(\frac{\min\{2a, W-d\}}{4}\|u\| - C_3\|u\|^5 - C_4|h|_2\right). \end{aligned}$$

Let $g_2(t) = \frac{\min\{2a,W-d\}}{4}t - C_3t^5$ for $t \geq 0$, we know that there exists a constant $\rho > 0$ such that $\max_{t\geq 0} g_2(t) = g_2(\rho) > 0$. Choose $m_0 = \frac{1}{2C_4}g_2(\rho)$, then there exists $\alpha > 0$ such that $I_\mu(u)|_{\|u\|=\rho} \geq \alpha$ for all h satisfying $|h|_2 < m_0$. □

Lemma 3.4 *Assume that* (k_1)-(k_3) *and* (h_1) *hold. Then* $-\infty < c := \inf\{I(u) : \|u\| \leq \rho\} < 0$, *where* ρ *is given by Lemma* 3.3.

Proof Since $h \in L^2(\mathbb{R}^N)$ and $h \not\equiv 0$, then for $\varepsilon = \frac{|h|_2}{2}$, there exists $\phi \in C_0^\infty(\mathbb{R}^N)$ such that $|h-\phi|_2 < \varepsilon$. Thus

$$\int_{\mathbb{R}^N}\left(h^2 - h\phi\right)dx \leq \int_{\mathbb{R}^N}\left|h^2 - h\phi\right|dx \leq |h-\phi|_2|h|_2 < \varepsilon|h|_2,$$

and then

$$\int_{\mathbb{R}^N}h\phi\,dx \geq |h|_2^2 - \varepsilon|h|_2 = \frac{|h|_2^2}{2} > 0.$$

Hence

$$I(t\phi) \leq \frac{at^2}{2}\int_{\mathbb{R}^N}|\nabla\phi|^2\,dx + \frac{bt^4}{4}\left(\int_{\mathbb{R}^N}|\nabla\phi|^2\,dx\right)^2 + \frac{Wt^2}{2}\int_{\mathbb{R}^N}\phi^2\,dx - t\int_{\mathbb{R}^N}h\phi\,dx < 0$$

for $t > 0$ small enough. Then we get $c = \inf\{I(u) : \|u\| \leq \rho\} < 0$. $c > -\infty$ is obvious. □

In order to prove the compactness, we define $g(t) = k(t) - dt$, $\forall t \in \mathbb{R}$. Then, by (k_1) and (k_2), we get that

$$\lim_{t\to 0^+}\frac{g(t)}{t} = 0, \tag{3.3}$$

and when $N = 2$,

$$\lim_{t\to +\infty}\frac{g(t)}{t^{p-1}} = 0, \tag{3.4}$$

when $N = 3$,

$$\lim_{t\to +\infty}\frac{g(t)}{t^5} = 0. \tag{3.5}$$

Lemma 3.5 *Suppose that* (k_1)-(k_3), (h_1) *and* (h_3) *hold. Assume that* $\{u_n\} \subset E$ *is a bounded Palais-Smale sequence of* I_μ *for each* $\mu \in J$. *Then* $\{u_n\}$ *has a convergent subsequence in* E.

Proof Since $\{u_n\}$ is bounded in E and $E \hookrightarrow L^s(\mathbb{R}^3)$, $\forall s \in (2,6)$, $E \hookrightarrow L^s(\mathbb{R}^2)$, $\forall s \in (2,+\infty)$ are compact (see [15]), up to a subsequence, we can assume that there exists $u \in E$ such that $u_n \rightharpoonup u$ in E, $u_n \to u$ in $L^s(\mathbb{R}^3)$, $\forall s \in (2,6)$, $u_n \to u$ in $L^s(\mathbb{R}^2)$, $\forall s \in (2,+\infty)$, $u_n(x) \to u(x)$ a.e. in $\mathbb{R}^N$.

By (3.3) and (3.4), for any $\varepsilon > 0$, we have

$$\bigl|g(t)\bigr| \le \varepsilon|t| + C_\varepsilon|t|^{p-1}, \quad \forall t \ge 0. \tag{3.6}$$

Then, by (3.6) and the Hölder inequality, one has

$$\begin{aligned}
&\left|\int_{\mathbb{R}^2} g(u_n)(u_n - u)\,dx\right| \\
&\quad \le \varepsilon \int_{\mathbb{R}^2} |u_n||u_n - u|\,dx + C_\varepsilon \int_{\mathbb{R}^2} |u_n|^{p-1}|u_n - u|\,dx \\
&\quad \le \varepsilon|u_n|_2|u_n - u|_2 + C_\varepsilon\left(\int_{\mathbb{R}^2} |u_n|^p\,dx\right)^{\frac{p-1}{p}} |u_n - u|_p \\
&\quad \le C\varepsilon + o_n(1).
\end{aligned}$$

Similarly, we can obtain that

$$\left|\int_{\mathbb{R}^2} g(u)(u_n - u)\,dx\right| = o_n(1).$$

By (3.3) and (3.5), for any $\varepsilon > 0$, we have

$$\bigl|g(t)\bigr| \le \varepsilon\bigl(|t| + |t|^5\bigr) + C_\varepsilon|t|^3, \quad \forall t \ge 0. \tag{3.7}$$

Hence, by (3.7) and the Hölder inequality, one has

$$\begin{aligned}
&\left|\int_{\mathbb{R}^3} g(u_n)(u_n - u)\,dx\right| \\
&\quad \le \varepsilon \int_{\mathbb{R}^3} |u_n||u_n - u|\,dx + \varepsilon \int_{\mathbb{R}^3} |u_n|^5|u_n - u|\,dx + C_\varepsilon \int_{\mathbb{R}^3} |u_n|^3|u_n - u|\,dx \\
&\quad \le \varepsilon|u_n|_2|u_n - u|_2 + \varepsilon\left(\int_{\mathbb{R}^3} |u_n|^6\,dx\right)^{\frac{5}{6}} |u_n - u|_6 + C_\varepsilon\left(\int_{\mathbb{R}^3} |u_n|^{\frac{9}{2}}\,dx\right)^{\frac{2}{3}} |u_n - u|_3 \\
&\quad \le C\varepsilon + o_n(1).
\end{aligned}$$

Similarly, we can obtain that

$$\left|\int_{\mathbb{R}^3} g(u)(u_n - u)\,dx\right| = o_n(1).$$

Hence when $N = 2$ or 3, one has

$$\left|\int_{\mathbb{R}^N} \big(g(u_n) - g(u)\big)(u_n - u)\,dx\right| = o_n(1).$$

It is clear that

$$\big\langle I'_\mu(u_n) - I'_\mu(u), u_n - u\big\rangle = o_n(1)$$

and

$$b\left(\int_{\mathbb{R}^N} \big(|\nabla u|^2 - |\nabla u_n|^2\big)\,dx\right)\int_{\mathbb{R}^N} \big(\nabla u, \nabla(u_n - u)\big)\,dx = o_n(1).$$

Note that

$$\begin{aligned}\big\langle I'_\mu(u_n) - I'_\mu(u), u_n - u\big\rangle &= \left(a + b\int_{\mathbb{R}^N} |\nabla u_n|^2\,dx\right)\int_{\mathbb{R}^N} \big|\nabla(u_n - u)\big|^2\,dx \\ &\quad + (W - \mu d)\int_{\mathbb{R}^N} |u_n - u|^2\,dx \\ &\quad - b\left(\int_{\mathbb{R}^N} \big(|\nabla u|^2 - |\nabla u_n|^2\big)\,dx\right)\int_{\mathbb{R}^N} \big(\nabla u, \nabla(u_n - u)\big)\,dx \\ &\quad - \mu\int_{\mathbb{R}^N} \big(g(u_n) - g(u)\big)(u_n - u)\,dx \\ &\geq \min\{a, W - \mu d\}\|u_n - u\|^2 \\ &\quad - b\left(\int_{\mathbb{R}^N} \big(|\nabla u|^2 - |\nabla u_n|^2\big)\,dx\right)\int_{\mathbb{R}^N} \big(\nabla u, \nabla(u_n - u)\big)\,dx \\ &\quad - \mu\int_{\mathbb{R}^N} \big(g(u_n) - g(u)\big)(u_n - u)\,dx.\end{aligned}$$

Therefore we get that $\|u_n - u\| \to 0$ as $n \to \infty$. □

Proof of the first solution of Theorem 1.2 By Lemma 3.4 and Ekeland's variational principle [11], there exists a sequence $\{u_n\} \subset E$ such that $\|u_n\| \leq \rho$, $I(u_n) \to c$ and $I'(u_n) \to 0$ as $n \to \infty$. From Lemma 3.5 with $\mu = 1$, there exists $u_0 \in E$ such that $u_n \to u_0$ in E and then $I'(u_0) = 0$ and $I(u_0) = c < 0$. Put $u_0^- := \max\{-u_0, 0\}$, one has

$$\begin{aligned}0 &= \big\langle I'(u_0), u_0^-\big\rangle \\ &= -a\int_{\mathbb{R}^N} \big|\nabla u_0^-\big|^2\,dx - b\int_{\mathbb{R}^N} |\nabla u_0|^2\,dx\int_{\mathbb{R}^N} \big|\nabla u_0^-\big|^2\,dx - W\int_{\mathbb{R}^N} \big|u_0^-\big|^2\,dx \\ &\quad - \int_{\mathbb{R}^N} hu_0^- dx,\end{aligned} \tag{3.8}$$

which implies that $u_0^- = 0$ and then $u_0 \geq 0$. By the strong maximum principle, we get $u_0 > 0$. □

For ρ and α in Lemma 3.3, we have following result.

Lemma 3.6 *Assume that* (k_1)-(k_3) *and* (h_1) *hold. Then*

$(*)$ $\exists v_2 \in E$ *with* $\|v_2\| > \rho$ *such that* $I_\mu(v_2) < 0$, $\forall \mu \in J$.
$(**)$ $c_\mu = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I_\mu(\gamma(t)) > \max\{I_\mu(0), I_\mu(v_2)\}$, $\forall \mu \in J$, *where*

$$\Gamma = \bigl\{\gamma \in C\bigl([0,1], E\bigr) : \gamma(0) = 0, \gamma(1) = v_2\bigr\}.$$

Proof It follows from (k_3) that, for any $L > 0$, there exists $C_L > 0$ such that, for all $t \geq 0$, one has

$$K(t) \geq Lt^2 - C_L. \tag{3.9}$$

Fix $0 \leq w \in C_0^\infty(\mathbb{R}^N)$ with $\operatorname{supp} w \subset B_1 := \{x \in \mathbb{R}^N : |x| < 1\}$ and $w \not\equiv 0$. Define $w_t(x) = tw(\frac{x}{t^2})$ for $t > 0$, then

$$\operatorname{supp} w_t = \bigl\{t^2 y : y \in \operatorname{supp} w\bigr\}.$$

By direct computation, we have

$$\begin{aligned}
&\int_{\mathbb{R}^N} |\nabla w_t|^2\,dx = t^{2N-2} \int_{\mathbb{R}^N} |\nabla w|^2\,dx,\\
&\int_{\mathbb{R}^N} w_t^2\,dx = t^{2N+2} \int_{\mathbb{R}^N} w^2\,dx
\end{aligned}$$

and, by (3.9),

$$\begin{aligned}
\int_{\mathbb{R}^N} K(w_t)\,dx &= \int_{\operatorname{supp} w_t} K(w_t)\,dx\\
&\geq L \int_{\operatorname{supp} w_t} w_t^2\,dx - C_L \int_{\operatorname{supp} w_t} dx\\
&\geq Lt^{2N+2} \int_{\operatorname{supp} w} w^2\,dx - C_L \int_{\{t^2 y: y\in B_1\}} dx\\
&= Lt^{2N+2} \int_{\mathbb{R}^N} w^2\,dx - C_L C t^{2N}.
\end{aligned}$$

Therefore

$$\begin{aligned}
&I_\mu(w_t)\\
&\quad= \frac{a}{2} \int_{\mathbb{R}^N} |\nabla w_t|^2\,dx + \frac{b}{4}\biggl(\int_{\mathbb{R}^N} |\nabla w_t|^2\,dx\biggr)^2 + \frac{W}{2} \int_{\mathbb{R}^N} w_t^2\,dx\\
&\qquad - \mu \int_{R^N} K(w_t)\,dx - \int_{\mathbb{R}^N} h w_t\,dx\\
&\quad\leq \frac{at^{2N-2}}{2} \int_{\mathbb{R}^N} |\nabla w|^2\,dx + \frac{bt^{4N-4}}{4}\biggl(\int_{\mathbb{R}^N} |\nabla w|^2\,dx\biggr)^2 + \frac{Wt^{2N+2}}{2} \int_{\mathbb{R}^N} w^2\,dx\\
&\qquad - \frac{Lt^{2N+2}}{2} \int_{\mathbb{R}^N} w^2\,dx + C_L C t^{2N}
\end{aligned}$$

for all $\mu \in J$. When $N = 2$, we choose $L = 2W$. When $N = 3$, we choose $L = 2W + b\frac{(\int_{\mathbb{R}^N} |\nabla w|^2\,dx)^2}{\int_{\mathbb{R}^N} w^2\,dx}$. Then $I_\mu(w_t) \to -\infty$ as $t \to +\infty$. Hence there exists $t' > 0$ such that $v_2 := w_{t'}$ with $\|v_2\| > \rho$ and $I_\mu(v_2) < 0$, $\forall \mu \in J$. This completes the proof of $(*)$.

By Lemma 3.3 and the definition of c_μ, for all $\mu \in J$, we have

$$0 < \alpha \le c_1 \le c_\mu \le c_{\frac{1}{2}} \le \max_{t\in[0,1]} I_{\frac{1}{2}}(tv_2) < +\infty.$$

Therefore, by $I_\mu(0) = 0$ and $I_\mu(v_2) < 0$, we obtain the proof of $(**)$. □

So far we have verified all the conditions of Lemma 3.1. Then there exists $\{\mu_j\} \subset J$ such that

(i) $\mu_j \to 1^-$ as $j \to \infty$, $\{u_n^j\}$ is bounded in E;
(ii) $I_{\mu_j}(u_n^j) \to c_{\mu_j}$ as $n \to \infty$;
(iii) $I'_{\mu_j}(u_n^j) \to 0$ as $n \to \infty$.

Using (i)-(iii) and Lemma 3.5, there exists $u_j \in E$ such that $u_n^j \to u_j$ in E as $n \to \infty$ and then $I_{\mu_j}(u_j) = c_{\mu_j}$ and $I'_{\mu_j}(u_j) = 0$. Hence, from $I_{\mu_j}(u_j) = c_{\mu_j}$ and $\langle I'_{\mu_j}(u_j), u_j\rangle = 0$, we get respectively

$$\begin{aligned} &\frac{a}{2}\int_{\mathbb{R}^N} |\nabla u_j|^2\,dx + \frac{b}{4}\left(\int_{\mathbb{R}^N} |\nabla u_j|^2\,dx\right)^2 + \frac{W}{2}\int_{\mathbb{R}^N} u_j^2\,dx \\ &\quad - \mu_j\int_{\mathbb{R}^N} K(u_j)\,dx - \int_{\mathbb{R}^N} hu_j\,dx = c_{\mu_j}, \end{aligned} \tag{3.10}$$

$$\begin{aligned} &a\int_{\mathbb{R}^N} |\nabla u_j|^2\,dx + b\left(\int_{\mathbb{R}^N} |\nabla u_j|^2\,dx\right)^2 + W\int_{\mathbb{R}^N} u_j^2\,dx \\ &\quad - \mu_j\int_{\mathbb{R}^N} k(u_j)u_j\,dx - \int_{\mathbb{R}^N} hu_j\,dx = 0. \end{aligned} \tag{3.11}$$

Next, for obtaining $\{u_j\}$ is bounded in E, we need the following lemma (Pohozaev type identity). The proof is similar to Lemma 2.6 in [16], and we omit its proof in here.

Lemma 3.7 *Suppose that* (h_1) *and* (h_2) *hold. If* $I'_\mu(u) = 0$, *we have*

$$\begin{aligned} &\frac{a(N-2)}{2}\int_{\mathbb{R}^N} |\nabla u|^2\,dx + \frac{b(N-2)}{2}\left(\int_{\mathbb{R}^N} |\nabla u|^2\,dx\right)^2 + \frac{NW}{2}\int_{\mathbb{R}^N} u^2\,dx \\ &\quad - N\mu\int_{\mathbb{R}^N} K(u)\,dx - N\int_{\mathbb{R}^N} hu\,dx - \int_{\mathbb{R}^N} \big(\nabla h(x), x\big)u\,dx = 0. \end{aligned}$$

Since $I'_{\mu_j}(u_j) = 0$, by Lemma 3.7, we get that

$$\begin{aligned} &\frac{a(N-2)}{2}\int_{\mathbb{R}^N} |\nabla u_j|^2\,dx + \frac{b(N-2)}{2}\left(\int_{\mathbb{R}^N} |\nabla u_j|^2\,dx\right)^2 + \frac{NW}{2}\int_{\mathbb{R}^N} u_j^2\,dx \\ &\quad - N\mu_j\int_{\mathbb{R}^N} K(u_j)\,dx - N\int_{\mathbb{R}^N} hu_j\,dx - \int_{\mathbb{R}^N} \big(\nabla h(x), x\big)u_j\,dx = 0. \end{aligned} \tag{3.12}$$

Lemma 3.8 *Assume that* (k_1)-(k_3) *and* (h_1)-(h_3) *hold. Then* $\{u_j\}$ *is bounded in* E.

Proof It follows from (3.10) and (3.12) that

$$a\int_{\mathbb{R}^N}|\nabla u_j|^2\,dx+\frac{b(4-N)}{4}\left(\int_{\mathbb{R}^N}|\nabla u_j|^2\,dx\right)^2+\int_{\mathbb{R}^N}\bigl(\nabla h(x),x\bigr)u_j\,dx=Nc_{\mu_j}. \tag{3.13}$$

Be similar to (3.8), by $I'_{\mu_j}(u_j)=0$, we obtain $u_j\geq 0$.

Firstly, we consider $N=2$. From (3.13) and $c_{\mu_j}\leq c_{\frac{1}{2}}$, we get

$$\begin{aligned}a\int_{\mathbb{R}^2}|\nabla u_j|^2\,dx &\leq a\int_{\mathbb{R}^2}|\nabla u_j|^2\,dx+\frac{b}{2}\left(\int_{\mathbb{R}^2}|\nabla u_j|^2\,dx\right)^2\\ &\quad-2c_{\mu_j}+2c_{\mu_j}\\ &=-\int_{\mathbb{R}^2}\bigl(\nabla h(x),x\bigr)u_j\,dx+2c_{\mu_j}. \end{aligned}\tag{3.14}$$

Since $(\nabla h(x),x)\geq 0$, by (3.14) and $u_j\geq 0$, one has $\{\int_{\mathbb{R}^2}|\nabla u_j|^2\,dx\}$ is bounded. Next we prove $\{\int_{\mathbb{R}^2}u_j^2\,dx\}$ is bounded. Inspired by [14], we suppose by contradiction that $\lambda_j:=|u_j|_2\to+\infty$. Define $w_j:=u_j(\lambda_j x)$, then

$$\int_{\mathbb{R}^2}|\nabla w_j|^2\,dx=\int_{\mathbb{R}^2}|\nabla u_j|^2\,dx\leq C$$

and

$$\int_{\mathbb{R}^2}|w_j|^2\,dx=\frac{1}{\lambda_j^2}\int_{\mathbb{R}^2}|u_j|^2\,dx=1. \tag{3.15}$$

Hence $\{w_j\}$ is bounded in E. Up to a subsequence, we may assume that $w_j\rightharpoonup w$ in E, $w_j\to w$ in $L^s(\mathbb{R}^2)$, $\forall s\in(2,+\infty)$, $w_j\to w$ in $L^s_{\mathrm{loc}}(\mathbb{R}^2)$, $\forall s\in[1,+\infty)$, $w_j(x)\to w(x)$ a.e. in $\mathbb{R}^2$. By $I'_{\mu_j}(u_j)=0$, one has

$$-\left(a+b\int_{\mathbb{R}^2}|\nabla w_j|^2\,dx\right)\frac{1}{\lambda_j^2}\triangle w_j+(W-d\mu_j)w_j=\mu_j g(w_j)+h(\lambda_j x). \tag{3.16}$$

For any $v\in C_0^\infty(\mathbb{R}^2)$, one has

$$\left|\int_{\mathbb{R}^2}h(\lambda_j x)v\,dx\right|\leq|v|_2\left(\int_{\mathbb{R}^2}\bigl|h(\lambda_j x)\bigr|^2\,dx\right)^{\frac{1}{2}}=\frac{1}{\lambda_j}|v|_2|h|_2\to 0 \tag{3.17}$$

and by the Lebesgue dominated convergence theorem, we have

$$\left|\int_{\mathbb{R}^2}g(w_j)v\,dx-\int_{\mathbb{R}^2}g(w)v\,dx\right|\leq C\int_{\operatorname{supp}v}\bigl|g(w_j)-g(w)\bigr|\,dx\to 0. \tag{3.18}$$

Hence by (3.16)-(3.18), we have $(W-d)w=g(w)$ in $\mathbb{R}^2$, from which we get that $w=0$. Indeed, since 0 is an isolated solution of $(W-d)z=g(z)$, $w=0$. Therefore by (3.6), (3.15) and (3.16), one has

$$\begin{aligned}
W-d &= (W-d)\int_{\mathbb{R}^2}|w_j|^2\,dx \\
&\le \left(a+b\int_{\mathbb{R}^2}|\nabla w_j|^2\,dx\right)\frac{1}{\lambda_j^2}\int_{\mathbb{R}^2}|\nabla w_j|^2\,dx+(W-d\mu_j)\int_{\mathbb{R}^2}|w_j|^2\,dx \\
&= \mu_j\int_{\mathbb{R}^2}g(w_j)w_j\,dx+\int_{\mathbb{R}^2}h(\lambda_j x)w_j\,dx \\
&\le \varepsilon\int_{\mathbb{R}^2}|w_j|^2\,dx+C_\varepsilon\int_{\mathbb{R}^2}|w_j|^p\,dx+\frac{1}{\lambda_j}|h|_2|w_j|_2 \\
&\le C\varepsilon+o_n(1),
\end{aligned}$$

which implies a contradiction. Hence $\{\int_{\mathbb{R}^2}|u_j|^2\,dx\}$ is bounded and then $\{u_j\}$ is bounded in E.

Secondly, for $N=3$, we have a simple proof. From (3.13), (h_2) and $c_{\mu_j}\le c_{\frac{1}{2}}$, we get

$$\begin{aligned}
a\int_{\mathbb{R}^3}|\nabla u_j|^2\,dx &\le a\int_{\mathbb{R}^3}|\nabla u_j|^2\,dx+\frac{b}{4}\left(\int_{\mathbb{R}^3}|\nabla u_j|^2\,dx\right)^2-3c_{\mu_j}+3c_{\mu_j} \\
&= -\int_{\mathbb{R}^3}\big(\nabla h(x),x\big)u_j\,dx+3c_{\mu_j} \\
&\le \big|\big(\nabla h(x),x\big)\big|_2|u_j|_2+3c_{\frac{1}{2}} \\
&\le C|u_j|_2+3c_{\frac{1}{2}}.
\end{aligned}\tag{3.19}$$

We prove directly $\{\int_{\mathbb{R}^3}u_j^2\,dx\}$ is bounded. Similar to (3.19), we obtain

$$\begin{aligned}
\frac{b}{4}\left(\int_{\mathbb{R}^3}|\nabla u_j|^2\,dx\right)^2 &\le a\int_{\mathbb{R}^3}|\nabla u_j|^2\,dx+\frac{b}{4}\left(\int_{\mathbb{R}^3}|\nabla u_j|^2\,dx\right)^2-3c_{\mu_j}+3c_{\mu_j} \\
&\le C|u_j|_2+3c_{\frac{1}{2}}.
\end{aligned}\tag{3.20}$$

By the Hölder inequality, we have

$$\left(\int_{\mathbb{R}^3}|\nabla u_j|^2\,dx\right)^3\le C\left(\int_{\mathbb{R}^3}u_j^2\,dx\right)^{\frac{3}{4}}+C.\tag{3.21}$$

By (3.3) and (3.5), for all $t\in\mathbb{R}$, one has

$$\big|g(t)t\big|\le\frac{W-d}{2}|t|^2+C|t|^6.\tag{3.22}$$

From (3.11), (3.21), (3.22), $\mu_j\le 1$ and $D^{1,2}(\mathbb{R}^3)\hookrightarrow L^6(\mathbb{R}^3)$, it follows that

$$\begin{aligned}
(W-d)\int_{\mathbb{R}^3}u_j^2\,dx &\le a\int_{\mathbb{R}^3}|\nabla u_j|^2\,dx+b\left(\int_{\mathbb{R}^3}|\nabla u_j|^2\,dx\right)^2+(W-\mu_j d)\int_{\mathbb{R}^3}u_j^2\,dx \\
&= \mu_j\int_{\mathbb{R}^3}g(u_j)u_j\,dx+\int_{\mathbb{R}^3}hu_j\,dx
\end{aligned}$$

$$\leq \frac{W-d}{2}\int_{\mathbb{R}^3} u_j^2\,dx + C\int_{\mathbb{R}^3} u_j^6\,dx + |h|_2\left(\int_{\mathbb{R}^3} u_j^2\,dx\right)^{\frac{1}{2}}$$
$$\leq \frac{W-d}{2}\int_{\mathbb{R}^3} u_j^2\,dx + C\left(\int_{\mathbb{R}^3} |\nabla u_j|^2\,dx\right)^3 + |h|_2\left(\int_{\mathbb{R}^3} u_j^2\,dx\right)^{\frac{1}{2}}$$
$$\leq \frac{W-d}{2}\int_{\mathbb{R}^3} u_j^2\,dx + C\left(\int_{\mathbb{R}^3} u_j^2\,dx\right)^{\frac{3}{4}} + C + |h|_2\left(\int_{\mathbb{R}^3} u_j^2\,dx\right)^{\frac{1}{2}},$$

which implies that $\{\int_{\mathbb{R}^3} u_j^2\,dx\}$ is bounded. Combining with (3.19), we get that $\{u_j\}$ is bounded in E. □

Proof of the second solution of Theorem 1.2 By $I_{\mu_j}(u_j) = c_{\mu_j}$, $I'_{\mu_j}(u_j) = 0$, $\mu_j \to 1^-$ and Remark 3.2, we get $I(u_j) \to c_1$ and $I'(u_j) \to 0$ as $n \to +\infty$. By Lemmas 3.5 and 3.8, there exists $v_0 \in E$ such that $u_j \to v_0$ in E as $n \to +\infty$ and then $I(v_0) = c_1 > 0$, $I'(v_0) = 0$. Be similar to (3.8), we get $v_0 \geq 0$. By the strong maximum principle, one has $v_0 > 0$. □

4 Conclusions

The goal of this paper is to study the multiplicity of positive solutions for the following nonlinear Kirchhoff type equation:

$$-\left(a + b\int_{\mathbb{R}^N} |\nabla u|^2\,dx\right)\triangle u + Vu = f(u) + h(x), \quad x \in \mathbb{R}^N,$$

where a, b, V are positive constants, $N = 2$ or 3. Under very weak conditions on f, we get that the equation has two positive solutions by using variational methods.

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