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Solutions of fractional differential equations with p-Laplacian operator in Banach spaces

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Abstract

In this paper, we study the solutions for nonlinear fractional differential equations with p-Laplacian operator nonlocal boundary value problem in a Banach space. By means of the technique of the properties of the Kuratowski noncompactness measure and the Sadovskii fixed point theorem, we establish some new existence criteria for the boundary value problem. As application, an interesting example is provided to illustrate the main results.

Keywords: fractional differential equation; boundary value problem; p-Laplacian operator; noncompactness measure

1 Introduction

A p-Laplacian differential equation was first introduced by Leibenson [1] when he studied the turbulent flow in a porous medium. Converting this fundamental mechanics problem into the existence of solutions to the following p-Laplacian differential equation:

$$\left(\varphi_p\left(u'(t)\right)\right)' = f\left(t, u(t)\right), \quad t \in (0, 1), \tag{1}$$

where $\varphi_p(s) = |s|^{p-2}s$ ($p > 1$) is the p-Laplacian operator, its inverse function is denoted by $\varphi_q(s)$ with $\varphi_q(s) = |s|^{q-2}s$, and p, q satisfy $\frac{1}{p} + \frac{1}{q} = 1$, he solved the practical and significant theoretical problem. Then many important results relative to differential equation (1) with different initial conditions and boundary conditions have been obtained (e.g. [2–14]). Scholars now find that fractional-order models are more adequate than integer-order models for problems in various fields of science such as physics, fluid flows, electrical networks, and many other (e.g. [15–25]). Consequently, the research of fractional differential equations with p-Laplacian operator BVP has already become a focus in recent years, and it has developed very rapidly (e.g. [26, 27]).

The authors in [28] studied the existence of positive solutions for the nonlinear fractional equation with p-Laplacian operator

$$\mathcal{D}_{0_+}^\alpha\left(\phi_p\left(\mathcal{D}_{0_+}^\alpha u(t)\right)\right) = f\left(t, u(t)\right),$$

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with the boundary conditions

$$u(0) = 0, \qquad u(1) = au(\xi), \qquad \mathcal{D}_{0_+}^{\alpha} u(0) = 0,$$

where $0 < \alpha \leq 2, 0 < \eta \leq 1, 0 < \xi < 1, 0 \leq a \leq 1$. By using the Guo-Krasnosel'skii fixed point theorem and the Leggett-Williams theorem, some sufficient conditions for the existence positive solutions have been obtained. The authors in [29] considered the above equation with the boundary conditions

$$u(0) = 0, \qquad \mathcal{D}_{0_+}^{\alpha} u(0) = 0, \qquad u(1) = a\mathcal{D}_{0_+}^{\alpha} u(1).$$

Some existence and multiplicity results of positive solutions have been obtained. In [30], the authors also considered the same equation with the boundary conditions

$$u(0) = 0, \qquad u(1) = au(\xi), \qquad \mathcal{D}_{0_+}^{\alpha} u(0) = 0, \qquad \mathcal{D}_{0_+}^{\alpha} u(1) = b\mathcal{D}_{0_+}^{\alpha} u(\eta),$$

where $0 \leq a, b \leq 1$. They obtained the existence of at least one positive solution by means of the upper and lower solutions method.

As far as we know, few results have been obtained to the solutions of the fractional order differential equations with p-Laplacian operator nonlocal boundary value problem (BVP):

$$\begin{cases} -\mathcal{D}_{0_+}^{\beta}(\varphi_p(\mathcal{D}_{0_+}^{\alpha}x))(t) = f(t, x(t)), & 0 < t < 1, \\ x(0) = \theta, \qquad \mathcal{D}_{0_+}^{\alpha}x(0) = \theta, \\ \mathcal{D}_{0_+}^{\gamma}x(1) = \sum_{i=1}^{m-2} \alpha_i \mathcal{D}_{0_+}^{\gamma}x(\xi_i), \end{cases} \tag{2}$$

in a Banach space E, where $\mathcal{D}_{0_+}^{\alpha}$, $\mathcal{D}_{0_+}^{\beta}$ and $\mathcal{D}_{0_+}^{\gamma}$ are the standard Riemann-Liouville fractional derivatives, θ is the zero element of E, $1 < \alpha \leq 2, 0 < \beta, \gamma \leq 1, \alpha - \gamma - 1 \geq 0$, $I = [0, 1], f : I \times E \to E$ is continuous, $\alpha_i \geq 0$ $(i = 1, 2, \ldots, m - 2), 0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1$, $\sum_{i=1}^{m-2} \alpha_i \xi_i^{\alpha-\gamma-1} < 1$. We establish some existence of solutions to BVP (2). The technique relies on the properties of the Kuratowski noncompactness measure and the Sadovskii fixed point theorem. Obviously, BVP (2) is more general than the problems discussed in some recent literature (such as [28–30]). Firstly, the boundary conditions are nonlocal, which can cover the well-known Sturm-Liouville boundary conditions as a special case, so we generalize the results of [28]. Secondly, as we generalized the space from the scalar space to the abstract space, our work includes the results of [28–30].

The rest of this paper is organized as follows. In Section 2, we introduce some definitions and lemmas to prove our main results. In Section 3, the existence results of solutions to the BVP are discussed by using the properties of the Kuratowski measure of noncompactness and the Sadovskii fixed point theorem. Finally, one example is provided to illustrate our main results in Section 4.

2 Preliminaries

For convenience, we present here the necessary definitions and preliminary facts which are used throughout this paper.

Definition 2.1 (see [31]) Let $\alpha > 0$, the fractional integral of order $\alpha > 0$ of a function $x : (0, \infty) \to R$ is given by

$$\mathit{I}_{0+}^{\alpha} x(t) = \frac{1}{\Gamma(\alpha)} \int_0^x \frac{x(t)}{(t-s)^{1-\alpha}} \, dt,$$

provided that the right-hand side is pointwise defined on $(0, \infty)$.

Definition 2.2 (see [31]) The Riemann-Liouville standard fractional derivative of order $\alpha > 0$ of a continuous function $x : [0, \infty) \to R$ is given by

$$\mathcal{D}_{0+}^{\alpha} x(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt} \right)^n \int_0^t \frac{x(s)}{(t-s)^{\alpha-n+1}} \, ds,$$

where $n = [\alpha] + 1$, $[\alpha]$ denotes the integer part of the real number α, provided that the right-hand side integral is pointwise defined on $[0, \infty)$.

Proposition 2.1 (see [32, 33]) *Let x be integrable,*
(1) *if $\alpha > 0$, then*

$$\mathit{I}_{0+}^{\alpha} \mathcal{D}_{0+}^{\alpha} x(t) = x(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n},$$

where $c_i \in R$, $i = 1, 2, 3, \ldots, N$, N is the smallest integer greater than or equal to α.
(2) *if $\beta > \alpha > 0$, then*

$$\mathcal{D}_{0+}^{\alpha} \mathit{I}_{0+}^{\beta} x(t) = \mathit{I}_{0+}^{\beta-\alpha} x(t),$$
$$\mathcal{D}_{0+}^{\alpha} \mathit{I}_{0+}^{\alpha} x(t) = x(t).$$

Definition 2.3 (Kuratowski measure of noncompactness, see [34]) Let E be a real Banach space, S be a bounded set in E, the Kuratowski measure of noncompactness of S is given by

$$\alpha(S) = \inf \left\{ \delta > 0 : S = \bigcup_{i=1}^{m} S_i, \operatorname{diam}(S_i) < \delta, i = 1, 2, \ldots, m \right\},$$

where $\operatorname{diam}(S_i)$ denotes the diameters of S_i.

Remark 2.1 From the definition, it is obvious that $0 \le \alpha(S) < \infty$.

Definition 2.4 (k-set contraction operator, see [34]) Let E_1 and E_2 be real Banach spaces, $D \subset E_1$, $A : D \to E_2$ is a continuous and bounded operator. If there exists a constant $k \ge 0$ such that $\alpha(A(S)) \le k\alpha(S)$ for any bounded set S in D, then A is called a k-set contraction operator.

Remark 2.2 When $k < 1$, A is called a strict set contraction operator. It is easy to prove that a strict set contraction operator is a condensing operator.

Now, we denote

$$Q(I) = \left\{ x \in C[I,E] : \sup_{t \in I} \frac{\|x(t)\|}{1+t} < +\infty \right\},$$

where $C[I,E]$ is the Banach space of a continuous function $x : I \to E$ with the norm $\|x\|_C = \max_{t \in I} \|x(t)\|$. It is easy to see that $Q(I)$ is a Banach space with the norm $\|x\|_Q = \sup_{t \in I} \frac{\|x(t)\|}{1+t}$. The basic space used in this paper is $Q(I)$. The Kuratowski measure of noncompactness in E, $C[I,E]$ and $Q(I)$ are denoted by $\alpha_E(\cdot)$, $\alpha_C(\cdot)$ and $\alpha_Q(\cdot)$, respectively.

The following properties of the Kuratowski noncompactness measure and the Sadovskii fixed point theorem are needed for our discussion.

Lemma 2.1 (see [35]) *If $H \subset C[I,E]$ is bounded and equicontinuous, then $\alpha_E(H(t))$ is continuous on I and $\alpha_C(H) = \max_{t \in I} \alpha_E(H(t))$, $\alpha_E(\int_I x(t)\,dt : x \in H) \leq \int_I \alpha_E(H(t))\,dt$, where $H(t) = \{x(t) : x \in H\}$ for each $t \in I$.*

Lemma 2.2 (Sadovskii, see [34]) *Let D be a bounded, closed and convex subset of the Banach space E. If the operator $A : D \to D$ is condensing, then A has a fixed point in D.*

3 Main results

For simplicity of presentation, we give some notations and list some conditions as follows:

$$M = \sigma \left[1 + \frac{m-1}{1 - \sum_{i=1}^{m-2} \alpha_i \xi_i^{\alpha-\gamma-1}} \right], \qquad \bar{M} = \max_{t \in I} \{a(t), b(t)\},$$

$$\sigma = \frac{(\Gamma(\beta))^{1-q}}{\Gamma(\alpha)}, \qquad K_r = \{x \in E : \|x\| \leq r\},$$

$$K_R = \{x \in Q(I) : \|x\|_Q \leq R\}, \qquad K_\rho = \{x \in Q(I) : \|x\|_Q \leq \rho\}.$$

(H_1) There exist nonnegative functions $a, b \in C[0,1]$ such that

$$\int_0^t \|f(s,x)\|\,ds \leq \varphi_p[a(t)\|x\| + b(t)], \quad \forall t \in I, x \in E,$$

and

$$\int_0^1 (1+t)a(t)\,dt < M, \qquad \int_0^1 b(t)\,dt < +\infty.$$

(H_2) For any $r > 0$, $[\alpha, \beta] \subset I$, $f(t,x)$ is uniformly continuous on $[\alpha, \beta] \times K_r$ and $f(t,x) \geq 0$.

(H_3) For all $t \in [0,1]$, bounded subsets $W \subset E$, there exists a positive constant $l < \frac{1}{4M}$ such that

$$\alpha_E(f(s,W)) \leq \varphi_p(l\alpha_E(W)).$$

In order to discuss the BVP, the preliminary lemmas are given in this section.

Lemma 3.1 *Given* $y \in C[0, 1]$ *and* $y \geq 0$. *Then the following BVP*

$$
\begin{cases}
-\mathcal{D}_{0+}^{\beta}(\varphi_p(\mathcal{D}_{0+}^{\alpha}x))(t) = y(t), & 0 < t < 1, \\
x(0) = 0, \qquad \mathcal{D}_{0+}^{\alpha}x(0) = 0, \\
\mathcal{D}_{0+}^{\gamma}x(1) = \sum_{i=1}^{m-2} \alpha_i \mathcal{D}_{0+}^{\gamma}x(\xi_i),
\end{cases}
\tag{3}
$$

has a unique solution satisfying

$$
x(t) = -\sigma \int_0^t (t-s)^{\alpha-1} \varphi_q\left(\int_0^s (s-\tau)^{\beta-1} y(\tau)\,d\tau\right) ds
$$

$$
+ \sigma \frac{t^{\alpha-\gamma-1}}{1 - \sum_{i=1}^{m-2} \alpha_i \xi_i^{\alpha-\gamma-1}} \int_0^1 (1-s)^{\alpha-\gamma-1} \varphi_q\left(\int_0^s (s-\tau)^{\beta-1} y(\tau)\,d\tau\right) ds
$$

$$
- \sigma \frac{t^{\alpha-\gamma-1}}{1 - \sum_{i=1}^{m-2} \alpha_i \xi_i^{\alpha-\gamma-1}}
$$

$$
\times \sum_{i=1}^{m-2} \alpha_i \int_0^{\xi_i} (\xi_i-s)^{\alpha-\gamma-1} \varphi_q\left(\int_0^s (s-\tau)^{\beta-1} y(\tau)\,d\tau\right) ds.
\tag{4}
$$

Proof Step 1. From [36, Lemma 2.3], we know the following BVP

$$
\begin{cases}
-\mathcal{D}_{0+}^{\alpha}x(t) = y(t), & 0 < t < 1, \\
x(0) = 0, \qquad \mathcal{D}_{0+}^{\gamma}x(1) = \sum_{i=1}^{m-2} \alpha_i \mathcal{D}_{0+}^{\gamma}x(\xi_i)
\end{cases}
$$

has a unique solution satisfying

$$
x(t) = -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s)\,ds
$$

$$
+ \frac{1}{\Gamma(\alpha)} \frac{t^{\alpha-\gamma-1}}{1 - \sum_{i=1}^{m-2} \alpha_i \xi_i^{\alpha-\gamma-1}} \int_0^1 (1-s)^{\alpha-\gamma-1} y(s)\,ds
$$

$$
- \frac{1}{\Gamma(\alpha)} \frac{t^{\alpha-\gamma-1}}{1 - \sum_{i=1}^{m-2} \alpha_i \xi_i^{\alpha-\gamma-1}} \sum_{i=1}^{m-2} \alpha_i \int_0^{\xi_i} (\xi_i-s)^{\alpha-\gamma-1} y(s)\,ds.
\tag{5}
$$

Step 2. Let $u = \mathcal{D}_{0+}^{\alpha}x(t)$ and $v = \varphi_p(u)$. It is easy to know that $u = \varphi_q(v)$. By Proposition 2.1, the solution of the following initial value problem

$$
\begin{cases}
-\mathcal{D}_{0+}^{\beta}v(t) = y(t), & 0 < t < 1, \\
v(0) = 0,
\end{cases}
$$

can be written as $v(t) = -\mathcal{I}_{0+}^{\beta}y(t)$, $t \in [0, 1]$. Combining with the expression of u, we know that the solution of (3) satisfies

$$
\begin{cases}
-\mathcal{D}_{0+}^{\alpha}x(t) = \varphi_p^{-1}(-\mathcal{I}_{0+}^{\beta}y(t)), & 0 < t < 1, \\
x(0) = 0, \qquad \mathcal{D}_{0+}^{\gamma}x(1) = \sum_{i=1}^{m-2} \alpha_i \mathcal{D}_{0+}^{\gamma}x(\xi_i).
\end{cases}
\tag{6}
$$

As we have stated in Step 1, we can easily get the solution of BVP (6) as follows:

$$
\begin{aligned}
x(t) = {} & \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \varphi_q\left(-\mathcal{I}_{0+}^\beta y(t)\right) ds \\
& - \frac{1}{\Gamma(\alpha)} \frac{t^{\alpha-\gamma-1}}{1 - \sum_{i=1}^{m-2} \alpha_i \xi_i^{\alpha-\gamma-1}} \int_0^1 (1-s)^{\alpha-\gamma-1} \varphi_q\left(-\mathcal{I}_{0+}^\beta y(t)\right) ds \\
& + \frac{1}{\Gamma(\alpha)} \frac{t^{\alpha-\gamma-1}}{1 - \sum_{i=1}^{m-2} \alpha_i \xi_i^{\alpha-\gamma-1}} \sum_{i=1}^{m-2} \alpha_i \int_0^{\xi_i} (\xi_i-s)^{\alpha-\gamma-1} \varphi_q\left(-\mathcal{I}_{0+}^\beta y(t)\right) ds.
\end{aligned}
$$

Since $y(t) \geq 0, t \in [0,1]$, we have $\varphi_p^{-1}(-\mathcal{I}_{0+}^\beta y(t)) = -(\mathcal{I}_{0+}^\beta y(t))^{q-1}$, which implies that the solution of (3) is given by (4).

The following lemma is a straightforward conclusion of Lemma 3.1. □

Lemma 3.2 *Suppose that condition* (H_1) *is satisfied. Then BVP* (2) *has a unique solution satisfying*

$$
\begin{aligned}
x(t) = {} & \sigma \int_0^t (t-s)^{\alpha-1} \varphi_q\left(\int_0^s (s-\tau)^{\beta-1} f(\tau, x(\tau)) d\tau\right) ds \\
& + \frac{\sigma t^{\alpha-\gamma-1}}{1 - \sum_{i=1}^{m-2} \alpha_i \xi_i^{\alpha-\gamma-1}} \int_0^1 (1-s)^{\alpha-\gamma-1} \varphi_q\left(\int_0^s (s-\tau)^{\beta-1} f(\tau, x(\tau)) d\tau\right) ds \\
& - \frac{\sigma t^{\alpha-\gamma-1}}{1 - \sum_{i=1}^{m-2} \alpha_i \xi_i^{\alpha-\gamma-1}} \\
& \times \sum_{i=1}^{m-2} \alpha_i \int_0^{\xi_i} (\xi_i-s)^{\alpha-\gamma-1} \varphi_q\left(\int_0^s (s-\tau)^{\beta-1} f(\tau, x(\tau)) d\tau\right) ds.
\end{aligned}
$$

Proof The proof is similar to Lemma 3.1, so we omit.

For any $x \in Q(I)$, we define the operator T by

$$
\begin{aligned}
(Tx)(t) = {} & -\sigma \int_0^t (t-s)^{\alpha-1} \varphi_q\left(\int_0^s (s-\tau)^{\beta-1} f(\tau, x(\tau)) d\tau\right) ds \\
& + \frac{\sigma t^{\alpha-\gamma-1}}{1 - \sum_{i=1}^{m-2} \alpha_i \xi_i^{\alpha-\gamma-1}} \int_0^1 (1-s)^{\alpha-\gamma-1} \varphi_q\left(\int_0^s (s-\tau)^{\beta-1} f(\tau, x(\tau)) d\tau\right) ds \\
& - \frac{\sigma t^{\alpha-\gamma-1}}{1 - \sum_{i=1}^{m-2} \alpha_i \xi_i^{\alpha-\gamma-1}} \\
& \times \sum_{i=1}^{m-2} \alpha_i \int_0^{\xi_i} (\xi_i-s)^{\alpha-\gamma-1} \varphi_q\left(\int_0^s (s-\tau)^{\beta-1} f(\tau, x(\tau)) d\tau\right) ds.
\end{aligned}
\tag{7}
$$

□

Remark 3.1 Lemma 3.2 indicates that the existence of solution to BVP (2) is equivalent to the existence of the fixed point of the operator T.

Lemma 3.3 *Suppose that conditions* (H_1) *and* (H_2) *are satisfied. Then the operator* T : $Q(I) \to Q(I)$ *is continuous and bounded.*

Proof Step 1. For any $x \in Q(I)$, we prove that $(Tx)(t) \in Q(I)$. By condition (H$_1$), together with the definition of operator T, we have

$$
\left\| \frac{(Tx)(t)}{1+t} \right\|
$$

$$
\leq \left\| \frac{\sigma}{1+t} \int_0^t (t-s)^{\alpha-1} \varphi_q \left(\int_0^s (s-\tau)^{\beta-1} f(\tau, x(\tau)) \, d\tau \right) ds \right\|
$$

$$
+ \left\| \frac{\sigma}{1+t} \frac{t^{\alpha-\gamma-1}}{1 - \sum_{i=1}^{m-2} \alpha_i \xi_i^{\alpha-\gamma-1}} \int_0^1 (1-s)^{\alpha-\gamma-1} \varphi_q \left(\int_0^s (s-\tau)^{\beta-1} f(\tau, x(\tau)) \, d\tau \right) ds \right\|
$$

$$
+ \left\| \frac{\sigma}{1+t} \frac{t^{\alpha-\gamma-1}}{1 - \sum_{i=1}^{m-2} \alpha_i \xi_i^{\alpha-\gamma-1}} \right.
$$

$$
\left. \times \sum_{i=1}^{m-2} \alpha_i \int_0^{\xi_i} (\xi_i - s)^{\alpha-\gamma-1} \varphi_q \left(\int_0^s (s-\tau)^{\beta-1} f(\tau, x(\tau)) \, d\tau \right) ds \right\|
$$

$$
\leq \sigma \int_0^1 (1-s)^{\alpha-1} \varphi_q \left(\int_0^s \| f(\tau, x(\tau)) \| \, d\tau \right) ds
$$

$$
+ \frac{\sigma}{1 - \sum_{i=1}^{m-2} \alpha_i \xi_i^{\alpha-\gamma-1}} \int_0^1 \varphi_q \left(\int_0^s \| f(\tau, x(\tau)) \| \, d\tau \right) ds
$$

$$
+ \frac{\sigma(m-2)}{1 - \sum_{i=1}^{m-2} \alpha_i \xi_i^{\alpha-\gamma-1}} \int_0^1 \varphi_q \left(\int_0^s \| f(\tau, x(\tau)) \| \, d\tau \right) ds
$$

$$
\leq M \int_0^1 \varphi_q \left(\int_0^s \| f(\tau, x(\tau)) \| \, d\tau \right) ds
$$

$$
\leq M \int_0^1 \left[a(s) \| x \| + b(s) \right] ds
$$

$$
\leq M \left[\int_0^1 (1+s) a(s) \, ds \| x \|_Q + \int_0^1 b(s) \, ds \right]
$$

$$
< +\infty. \tag{8}
$$

This means that $(Tx)(t)$ is well defined and $(Tx)(t) \in Q(I)$ for any $x \in Q(I)$.

Step 2. It is time to show that T is a bounded operator. For any $x \in B_\rho$, from (8), we get

$$
\left\| \frac{(Tx)(t)}{1+t} \right\| \leq M \bar{M} (2\rho + 1). \tag{9}
$$

So T maps bounded sets into bounded sets in $Q(I)$, it follows that T is a bounded operator.

Step 3. It remains to show that T is continuous on $Q(I)$. Let $x_n, x \in Q(I)$ with $\lim_{n \to +\infty} \| x_n - x \|_Q \to 0$. It is trivial to see that $\{x_n\}$ is a bounded subset of $Q(I)$. As a result, there exists a constant $\eta > 0$ such that $\| x_n \|_Q \leq \eta$ for all $n \geq 1$. Taking limit, we see $\| x \|_Q \leq \eta$. Taking (H$_2$) into consideration, we know that for any $\varepsilon > 0$, there exists $N > 0$ such that

$$
\left\| f(s, x_n(s)) - f(s, x(s)) \right\| \leq M^{\frac{1}{1-q}} \varepsilon^{\frac{1}{q-1}}, \quad \forall n \geq N, s \in I. \tag{10}
$$

According to (8), a routine computation gives rise to the inequality as follows:

$$
\left\| \frac{(Tx_n)(t)}{1+t} - \frac{(Tx)(t)}{1+t} \right\|
$$

$$
\leq \frac{M}{1+t} \left\{ \int_0^1 \varphi_q \left(\int_0^s \| f(\tau, x_n(\tau)) - f(\tau, x(\tau)) \| \, d\tau \right) ds \right\}
$$

$$
\leq \frac{\varepsilon}{1+t}
$$

$$
< \varepsilon. \tag{11}
$$

It follows that $\| \frac{(Tx_n)(t)}{1+t} - \frac{(Tx)(t)}{1+t} \|_Q < \varepsilon$. Thus $T : Q(I) \to Q(I)$ is continuous. This completes the proof of Lemma 3.3. $\qquad\square$

Lemma 3.4 *Let condition* (H_1) *be satisfied and V be a bounded subset of $Q(I)$. Then $\frac{(TV)(t)}{1+t}$ is equicontinuous on* $[0,1]$.

Proof In fact, in the light of the boundedness of V, namely, for any $x \in V$, there exists $\tilde{\eta} > 0$ such that $\|x\|_Q \leq \tilde{\eta}$. Without loss of generality, suppose that $t_1, t_2 \in I$ with $t_1 < t_2$ by means of the monotonicity of $\frac{(t-s)^{\alpha-1}}{1+t}$ in t for $s < t$ and the mean value theorem. Combining with the definition of operator T, we have

$$
\left\| \frac{(Tx)(t_2)}{1+t_2} - \frac{(Tx)(t_1)}{1+t_1} \right\|
$$

$$
\leq \sigma \left\| \frac{1}{1+t_2} \int_0^{t_2} (t_2-s)^{\alpha-1} \varphi_q \left(\int_0^s (s-\tau)^{\beta-1} f(\tau, x(\tau)) \, d\tau \right) ds \right.
$$

$$
\left. - \frac{1}{1+t_1} \int_0^{t_1} (t_1-s)^{\alpha-1} \varphi_q \left(\int_0^s (s-\tau)^{\beta-1} f(\tau, x(\tau)) \, d\tau \right) ds \right\|
$$

$$
+ \frac{\sigma}{1 - \sum_{i=1}^{m-2} \alpha_i \xi_i^{\alpha-\gamma-1}} \left\| \frac{t_2^{\alpha-\gamma-1}}{1+t_2} \int_0^1 (1-s)^{\alpha-\gamma-1} \varphi_q \left(\int_0^s (s-\tau)^{\beta-1} f(\tau, x(\tau)) \, d\tau \right) ds \right.
$$

$$
\left. - \frac{t_1^{\alpha-\gamma-1}}{1+t_1} \int_0^1 (1-s)^{\alpha-\gamma-1} \varphi_q \left(\int_0^s (s-\tau)^{\beta-1} f(\tau, x(\tau)) \, d\tau \right) ds \right\|
$$

$$
+ \frac{\sigma}{1 - \sum_{i=1}^{m-2} \alpha_i \xi_i^{\alpha-\gamma-1}}
$$

$$
\times \left\| \frac{t_2^{\alpha-\gamma-1}}{1+t_2} \sum_{i=1}^{m-2} \alpha_i \int_0^{\xi_i} (\xi_i-s)^{\alpha-\gamma-1} \varphi_q \left(\int_0^s (s-\tau)^{\beta-1} f(\tau, x(\tau)) \, d\tau \right) ds \right.
$$

$$
\left. - \frac{t_1^{\alpha-\gamma-1}}{1+t_1} \sum_{i=1}^{m-2} \alpha_i \int_0^{\xi_i} (\xi_i-s)^{\alpha-\gamma-1} \varphi_q \left(\int_0^s (s-\tau)^{\beta-1} f(\tau, x(\tau)) \, d\tau \right) ds \right\|
$$

$$
\leq \sigma \left\| \frac{1}{1+t_2} \int_0^{t_1} (t_2-s)^{\alpha-1} \varphi_q \left(\int_0^s (s-\tau)^{\beta-1} f(\tau, x(\tau)) \, d\tau \right) ds \right.
$$

$$
+ \frac{1}{1+t_2} \int_{t_1}^{t_2} (t_2-s)^{\alpha-1} \varphi_q \left(\int_0^s (s-\tau)^{\beta-1} f(\tau, x(\tau)) \, d\tau \right) ds
$$

$$
\left. - \frac{1}{1+t_1} \int_0^{t_1} (t_1-s)^{\alpha-1} \varphi_q \left(\int_0^s (s-\tau)^{\beta-1} f(\tau, x(\tau)) \, d\tau \right) ds \right\|
$$

$$+ \frac{\sigma}{1 - \sum_{i=1}^{m-2} \alpha_i \xi_i^{\alpha-\gamma-1}} \left| \frac{t_2^{\alpha-\gamma-1}}{1 + t_2} - \frac{t_1^{\alpha-\gamma-1}}{1 + t_1} \right|$$

$$\times \left\{ \left\| \int_0^1 (1-s)^{\alpha-\gamma-1} \varphi_q \left(\int_0^s (s-\tau)^{\beta-1} f(\tau, x(\tau)) \, d\tau \right) ds \right\| \right.$$

$$\left. + \left\| \sum_{i=1}^{m-2} \alpha_i \int_0^{\xi_i} (\xi_i - s)^{\alpha-\gamma-1} \varphi_q \left(\int_0^s (s-\tau)^{\beta-1} f(\tau, x(\tau)) \, d\tau \right) ds \right\| \right\}$$

$$\leq \sigma \left| \frac{t_2^\alpha}{1 + t_2} - \frac{t_1^\alpha}{1 + t_1} - \frac{(t_2 - t_1)^\alpha}{1 + t_2} \right| \left\| \int_0^{t_1} \varphi_q \left(\int_0^s (s-\tau)^{\beta-1} f(\tau, x(\tau)) \, d\tau \right) ds \right\|$$

$$+ \sigma \frac{1}{1 + t_2} \left\| \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} \varphi_q \left(\int_0^s (s-\tau)^{\beta-1} f(\tau, x(\tau)) \, d\tau \right) ds \right\|$$

$$+ \frac{\sigma(m-2)}{1 - \sum_{i=1}^{m-2} \alpha_i \xi_i^{\alpha-\gamma-1}} \left| \frac{t_2^{\alpha-\gamma-1}}{1 + t_2} - \frac{t_1^{\alpha-\gamma-1}}{1 + t_1} \right|$$

$$\times \left\| \int_0^1 (1-s)^{\alpha-\gamma-1} \varphi_q \left(\int_0^s (s-\tau)^{\beta-1} f(\tau, x(\tau)) \, d\tau \right) ds \right\|$$

$$\leq \sigma \left| \frac{t_2^\alpha}{1 + t_2} - \frac{t_1^\alpha}{1 + t_1} \right| \int_0^1 \varphi_q \left(\int_0^s \| f(\tau, x(\tau)) \| \, d\tau \right) ds$$

$$+ \sigma \int_{t_1}^{t_2} \varphi_q \left(\int_0^s \| f(\tau, x(\tau)) \| \, d\tau \right) ds$$

$$+ \frac{\sigma(m-2)}{1 - \sum_{i=1}^{m-2} \alpha_i \xi_i^{\alpha-\gamma-1}} \left| \frac{t_2^{\alpha-\gamma-1}}{1 + t_2} - \frac{t_1^{\alpha-\gamma-1}}{1 + t_1} \right| \int_0^1 \varphi_q \left(\int_0^s \| f(\tau, x(\tau)) \| \, d\tau \right) ds$$

$$\leq \sigma \bar{M} \left(2^{\alpha-1} + \frac{m-2}{1 - \sum_{i=1}^{m-2} \alpha_i \xi_i^{\alpha-\gamma-1}} \right) (2\tilde{\eta} + 1)(t_2 - t_1)$$

$$+ \sigma \bar{M} (2\tilde{\eta} + 1)(t_2 - t_1)$$

$$= \sigma \bar{M} \left(1 + 2^{\alpha-1} + \frac{m-2}{1 - \sum_{i=1}^{m-2} \alpha_i \xi_i^{\alpha-\gamma-1}} \right) (2\tilde{\eta} + 1)(t_2 - t_1). \tag{12}$$

Let

$$\delta = \left[\sigma \bar{M} \left(1 + 2^{\alpha-1} + \frac{m-2}{1 - \sum_{i=1}^{m-2} \alpha_i \xi_i^{\alpha-\gamma-1}} \right) (2\tilde{\eta} + 1) \right]^{-1} \cdot \frac{\varepsilon}{2}.$$

It follows from (12) that

$$\left\| \frac{(Tx)(t_2)}{1 + t_2} - \frac{(Tx)(t_1)}{1 + t_1} \right\| < \varepsilon. \tag{13}$$

For the case of $t_1 \geq t_2$, after a tedious computation similar to the one used in the case of $t_1 \leq t_2$, we can also get (13). This ensures that $\frac{(TV)(t)}{1+t}$ is equicontinuous on $[0, 1]$. The proof of Lemma 3.4 is finished. □

The existence of solution to BVP (2) is as follows.

Theorem 3.1 *Let conditions* (H_1)-(H_3) *be satisfied. Then the BVP has at least one solution belonging to* $Q(I)$.

Proof From Remark 3.1, the main point of our argument is to show that the operator T has a fixed point in $Q(I)$.

Step 1. Take

$$R > \int_0^1 b(s)\,ds \cdot \left[M^{-1} - \int_0^1 (1+s)a(s)\,ds \right]^{-1}.$$

We first prove that $TK_R \subset K_R$. In fact, for any $x \in K_R$ and $t \in I$, by (8), we have

$$
\begin{aligned}
\left\| \frac{(Tx)(t)}{1+t} \right\| &\le M \int_0^1 \varphi_q \left(\int_0^s \| f(\tau, x(\tau)) \| \, d\tau \right) ds \\
&\le M \left[\int_0^1 (1+s)a(s)\,ds \|x\|_D + \int_0^1 b(s)\,ds \right] \\
&\le M \left\{ \int_0^1 (1+s)a(s)\,ds\,R + R \left[M^{-1} - \int_0^1 [(1+s)a(s)]\,ds \right]^{-1} \right\} \\
&< R.
\end{aligned}
\tag{14}
$$

Thus, from Lemma 3.3, $TK_R \subset K_R$ follows.

Step 2. We show that T is a strict set contraction operator. Let $D = \bar{c}o_Q(TK_R)$, i.e., D is the convex closure of TK_R in $Q(I)$. Clearly, D is a nonempty, bounded, convex and closed subset of K_R. By Lemma 3.4, we see $\frac{(TK_R)(t)}{1+t}$ is equicontinuous on I, it follows that $\frac{T(D)(t)}{1+t}$ is equicontinuous on I. By means of the definition of D, it is trivial to see that $D \subset K_R$ and $TK_R \subset D$. According to Lemma 3.3, we know that $T : D \to D$ is bounded and continuous. In addition, it is apparent from (H$_2$) that $\{f(s,x(s)) : x \in D\}$ is equicontinuous on I. Taking (H$_3$) and Lemma 2.1 into consideration, for any $t \in I$ and $U \subset D$, we have

$$
\begin{aligned}
&\alpha_E \left(\frac{(TU)(t)}{1+t} \right) \\
&\le \sigma \left[\int_0^1 (1-s)^{\alpha-1} \varphi_q \left(\int_0^s \alpha_E(\{f(\tau, x(\tau) : x \in U)\}) \, d\tau \right) ds \right. \\
&\quad + \int_0^1 \frac{(1-s)^{\alpha-\gamma-1}}{1 - \sum_{i=1}^{m-2} \alpha_i \xi_i^{\alpha-\gamma-1}} \varphi_q \left(\int_0^s \alpha_E(\{f(\tau, x(\tau) : x \in U)\}) \, d\tau \right) ds \\
&\quad + \left. \frac{m-2}{1 - \sum_{i=1}^{m-2} \alpha_i \xi_i^{\alpha-\gamma-1}} \int_0^1 \varphi_q \left(\int_0^s \alpha_E(\{f(\tau, x(\tau) : x \in U)\}) \, d\tau \right) ds \right] \\
&\le M \int_0^1 \varphi_q \left(\int_0^s \alpha_E(\{f(\tau, x(\tau) : x \in U)\}) \, d\tau \right) ds \\
&= M \int_0^1 \varphi_q \left(\int_0^s \alpha_E(\{f(\tau, V)\}) \, d\tau \right) ds \\
&\le Ml\alpha_E(V),
\end{aligned}
\tag{15}
$$

where $V = \{x(\tau) : \tau \in I, x \in U\}$. For any given $\varepsilon > 0$, we partition U as follows:

$$U = \bigcup_{i=1}^n U_i, \qquad \operatorname{diam}(U_i) < \alpha_D(V) + \frac{\varepsilon}{5}, \quad i = 1, 2, \ldots, n.$$

Moreover, for any $x_i \in U_i$, there exists a partition $0 = t_0 < t_1 < \cdots < t_m = 1$ such that

$$\left\| x_i(s) - x_i(t) \right\| < \frac{\varepsilon}{5}, \quad t, s \in [t_{j-1}, t_j].$$

Let $V_{ij} = \{x(t) : x \subset U_i, t \in [t_{j-1}, t_j]\}$. It is easy to see that V_{ij} is a partition of V, that is, $V = \bigcup_{i=1}^{n} \bigcup_{j=1}^{m} V_{ij}$. Owing to the partition of V, for any $x, y \subset U_i$, $t, s \in [t_{j-1}, t_j]$, we get

$$
\begin{aligned}
\operatorname{diam} V_{ij} &\leq \left\| x(s) - y(t) \right\| \\
&\leq \left\| x(s) - x_i(s) \right\| + \left\| x_i(s) - x_i(t) \right\| + \left\| x_i(t) - y(t) \right\| \\
&\leq (1 + s)\|x - x_i\|_D + \frac{\varepsilon}{5} + (1 + t)\|x_i - y\|_D \\
&\leq 4 \operatorname{diam} U_i + \frac{\varepsilon}{5} \\
&< 4\alpha_D(U) + \varepsilon.
\end{aligned}
\tag{16}
$$

So, $\alpha_E(V) < 4\alpha_D(U) + \varepsilon$, due to ε being arbitrary, we obtain

$$\alpha_E(V) \leq 4\alpha_D(U). \tag{17}$$

By substituting (17) into (15), we have

$$\alpha_E\left(\frac{(TU)(t)}{1 + t} \right) \leq 4Ml\alpha_D(U). \tag{18}$$

Taking the least upper bound of $\alpha_E(\frac{(TU)(t)}{1+t})$ when t is in the set of I, applying [11, Lemma 2.6], we know that

$$\alpha_D(TU) = \sup_{t \in I} \alpha_E\left(\frac{(TU)(t)}{1 + t} \right), \quad \forall U \subset D, \tag{19}$$

where

$$\frac{(TU)(t)}{1 + t} = \left\{ \frac{(Tx)(t)}{1 + t} : x \in U, t \in I \text{ is fixed} \right\} \subset D.$$

Take $L = 4Ml$. From (15), (18) and (19), we get

$$\alpha_D(TU) \leq L\alpha_D(U).$$

Obviously, $0 \leq L < 1$, that is, T is a strict set contraction operator from D to D. Obviously, T is condensing too. It follows from Lemma 2.2 that T has at least one fixed point in D, that is, BVP (2) has at least one solution in $Q(I)$. □

Remark 3.2 If $E = [0, \infty)$, as a special case of Theorem 3.1, we can obtain the following result.

Corollary 3.1 *Let* (H_1)-(H_3) *be satisfied and* $f \in C[[0, 1] \times [0, \infty), [0, \infty)]$. *Then BVP* (2) *has at least one solution in* $Q(I)$.

Proof Letting $E = [0, \infty)$ in Theorem 3.1, we get the desired result. □

4 Illustrative example

Let $E = l_\infty = \{x = (x_1, x_2, \ldots, x_n, \ldots), \sup_n |x_n| < +\infty, t \in I\}$. It is easy to see that E is a Banach space with the norm $\|x\| = \sup_n |x_n|$. We consider the following nonlocal fractional differential equations BVP:

$$
\begin{cases}
-\mathcal{D}_{0+}^{\frac{1}{2}} (\varphi_{\frac{3}{2}} (\mathcal{D}_{0+}^{\frac{3}{2}} x_n))(t) = \left[\frac{x_n(t)}{16(1+t)(1+t^2)} + \frac{\sin(t) + \sqrt{|\sin(x_{n+1}(t))|^2}}{42n^2 e^{\sqrt{t}}} \right]^{\frac{1}{2}}, \\
x_n(0) = 0, \qquad \mathcal{D}_{0+}^{\frac{3}{2}} x_n(0) = 0, \\
\mathcal{D}_{0+}^{\frac{1}{4}} x_n(1) = \frac{1}{4} \mathcal{D}_{0+}^{\frac{1}{4}} x_n(\frac{1}{4}) + \frac{1}{2} \mathcal{D}_{0+}^{\frac{1}{4}} x_n(\frac{3}{4}).
\end{cases}
\tag{20}
$$

BVP (20) can be regarded as a problem with the form of BVP (2), where

$$
f(t, x) = (f_1(t, x), f_2(t, x), \ldots, f_n(t, x), \ldots),
$$

$$
f_n(t, x) = \left[\frac{x_n(t)}{16(1+t)(1+t^2)} + \frac{\sin(t) + \sqrt{|\sin(x_{n+1}(t))|^2}}{42n^2 e^{\sqrt{t}}} \right]^{\frac{1}{2}}.
$$

Clearly, $\alpha = \frac{3}{2}$, $\beta = \frac{1}{2}$, $\gamma = \frac{1}{4}$, $\sum_{i=1}^{m-2} \alpha_i = 0.642 < 1$ and

$$
\int_0^t \|f_n(s, x)\| \, ds \le \varphi_{\frac{3}{2}} \left(\frac{1}{16(1+t)(1+t^2)} + \frac{1}{42 e^{\sqrt{t}}} \right).
$$

Take $l = 0.02$, $a(t) = \frac{1}{16(1+t)(1+t^2)} + \frac{1}{42 e^{\sqrt{t}}}$, $b(t) = \frac{1}{42 e^{\sqrt{t}}}$. In view of $\Gamma(\frac{1}{2}) \approx 1.772$ and $\Gamma(\frac{3}{2}) \approx 0.8862$, by a simple computation, we have

$$
M = \frac{\Gamma(\beta)^{1-q}}{\Gamma(\alpha)} \left(1 + \frac{m-1}{1 - \sum_{i=1}^{m-2} \alpha_i} \right) = 9.589,
$$

$$
\int_0^1 \left[(1+t)a(t) \right] dt = 0.0693 < M^{-1},
$$

$$
\int_0^1 b(t) \, dt = 0.0126 < +\infty.
$$

Therefore, all the conditions of Theorem 3.1 are satisfied. Consequently, we infer that (20) has at least one solution.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
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