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The truncation regularization method for identifying the initial value of heat equation on a spherical symmetric domain

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Abstract

In this paper, identifying the initial value for high dimension heat equation with inhomogeneous source on a spherical symmetric domain is investigated. The truncation regularization method is a powerful technique for solving this inverse problem. We prove the convergence estimates between the regularization solution and the exact solution under the prior and the posterior regularization parameter choice rulers. A numerical example is presented to validate the effectiveness of this method.

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Keywords: inverse problem; identifying initial value; ill-posed; truncation method; spherical symmetric domain

1 Introduction

The initial value problem is one of the backward heat conduction problems (BHCPs). These problems have been studied over several decades due to their significance in many engineering problems and practical application problems, such as in welding of iron and steel, quenching of solids in liquids and testing of new thermal protective material.

In this paper, we consider an inhomogeneous heat equation on a symmetric domain as follows:

$$\begin{cases} u_t - \frac{2}{r}u_r - u_{rr} = f(r,t), & 0 < t < T, 0 < r < r_0, \\ u(r,0) = \varphi(r), & 0 \le r \le r_0, \\ u(r_0,t) = 0, & 0 \le t \le T, \\ \lim_{r \to 0} u(r,t) \text{ is bounded}, & 0 < t < T, 0 < r < r_0, \\ u(r,T) = g(r), & 0 \le r \le r_0, \end{cases} \quad (1.1)$$

where r_0 is the radius, $\varphi(r)$ is the initial value. We use the additional condition $u(r,T) = g(r)$ and $f(r,t)$ to determine the initial value $\varphi(r)$. The measured data of $g(r)$ and $f(r,t)$ are $g^\delta(r)$ and $f^\delta(r,t)$, which satisfy

$$\left\| g^\delta(\cdot) - g(\cdot) \right\|_{L^2[0,r_0;r^2]} \le \delta; \qquad \left\| f^\delta(\cdot,t) - f(\cdot,t) \right\|_{L^\infty(0,T;L^2[0,r_0;r^2])} \le \delta. \quad (1.2)$$

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The initial value problem is one of the backward heat conduction problems (BHCPs). A BHCP is severely ill-posed problem [1]. To overcome this difficulty, many scholars proposed some regularization techniques for the BHCP, such as the kernel-based method [2], the mollification method [3], the Fourier regularization method [4], optimal filtering method [5], the iterative method [6], the quasi-reversibility method [7–9], the central difference method [10], the filter regularization method [11], the method of fundamental solutions [12, 13], the boundary element method [14, 15], the group preserving scheme [16], modified Tikhonov regularization method [17], Quasi-boundary value method [18] and so on. But these references about BHCP, there are some drawbacks as follows: firstly, the regularization parameter is a prior choice rule, according to this choice rule, the parameter depends on the prior bound of the exact solution. But in practice we cannot obtain the exact solution, and the inaccurate prior bound may lead to the bad regularized solution. Secondly, they only considered the one dimensional BHCP; however, about high dimensional BHCP, there is little research results. In [19–21], the authors ever considered the high dimensional BHCP, but the regularization parameter is a prior choice. Thirdly, the equation is homogeneous and the measurement data is only one.

The truncation regularization method has been used to solve several inverse problems. In [22, 23], the authors used the truncation method to solve BHCP. In [24–26], the authors used the truncation method to solve a cauchy problem for the Helmholtz equation and the modified Helmholtz equation. In [27–30], the authors used the truncation method to identify the unknown source. In this paper, we mainly use the truncation regularization method to identify the initial value under two parameter choice rules. Moreover, we give an example to show the effectiveness of this method. We also compare the effectiveness between the posterior choice rule and the prior choice rule.

Using the separation of variables, we obtain the solution of the problem (1.1) as follows:

$$u(r, t) = \sum_{n=1}^{\infty} \left(e^{-(\frac{n\pi}{r_0})^2 t} \varphi_n + \int_0^t e^{-(\frac{n\pi}{r_0})^2 (t-\tau)} f_n(\tau) \, d\tau \right) \omega_n(r), \quad (1.3)$$

where

$$\omega_n(r) := \frac{\sqrt{2} n\pi}{\sqrt{r_0^3}} \frac{\sin(n\pi r/r_0)}{(n\pi r/r_0)} \quad (1.4)$$

is the orthonormal eigenfunction system with weight r^2 on $[0, r_0]$. It is also a complete system in $L^2[0, r_0; r^2]$. Now let $\varphi_n = (\varphi(r), \omega_n(r)), f_n(\tau) = (f(r, t), \omega_n(r))$ and $g_n = (g(r), \omega_n(r))$, $h_n = \varphi_n + \int_0^T e^{(\frac{n\pi}{r_0})^2 \tau} f_n(\tau) \, d\tau$. Using $u(r, T) = g(r)$, we have

$$g(r) = \sum_{n=1}^{\infty} \left(e^{-(\frac{n\pi}{r_0})^2 T} \varphi_n + \int_0^T e^{-(\frac{n\pi}{r_0})^2 (T-\tau)} f_n(\tau) \, d\tau \right) \omega_n(r), \quad (1.5)$$

$$g_n = e^{-(\frac{n\pi}{r_0})^2 T} \varphi_n + \int_0^T e^{-(\frac{n\pi}{r_0})^2 (T-\tau)} f_n(\tau) \, d\tau. \quad (1.6)$$

Define operator $K : h(r) \to g(r)$, then

$$g(r) = Kh(r) = \sum_{n=1}^{\infty} e^{-(\frac{n\pi}{r_0})^2 T} \left(\varphi_n + \int_0^T e^{(\frac{n\pi}{r_0})^2 \tau} f_n(\tau) \, d\tau \right) \omega_n(r). \quad (1.7)$$

The operator K is a linear self-adjoint compact operator, and

$$k_n = e^{-(\frac{n\pi}{r_0})^2 T} \tag{1.8}$$

is for the singular values of K. Using (1.4), (1.6) and equation (1.7) can be rewritten as

$$\big(g(r), \omega_n(r)\big) = \big(h(r), \omega_n(r)\big) e^{-(\frac{n\pi}{r_0})^2 T}. \tag{1.9}$$

So

$$\varphi(r) = \sum_{n=1}^{\infty} \frac{g_n - \int_0^T e^{-(\frac{n\pi}{r_0})^2 (T-\tau)} f_n(\tau)\, d\tau}{e^{-(\frac{n\pi}{r_0})^2 T}} \omega_n(r). \tag{1.10}$$

We give a prior bound on the initial value, *i.e.,*

$$\big\|\varphi(\cdot)\big\|_p \le E, \quad p > 0, \tag{1.11}$$

where $E > 0$ is a constant and $\|\cdot\|_p$ denotes the norm in Sobolev space which is defined as follows:

$$\big\|\varphi(\cdot)\big\|_p := \left(\sum_{n=1}^{\infty} \big(1 + n^2\big)^p \big|\big(\varphi(\cdot), \omega_n(\cdot)\big)\big|^2 \right)^{\frac{1}{2}}. \tag{1.12}$$

This article is organized as follows. Section 2 presents some preliminaries results. Section 3 presents the convergence estimates under two parameter choice rules. In Section 4, a numerical example is proposed to show the effectiveness of this method. In Section 5, a brief conclusion is given.

2 Some auxiliary results

Throughout this paper, $L^2[0, r_0; r^2]$ denotes the Hilbert space of Lebesgue measurable function φ with weight r^2 on $[0, r_0]$. $(\cdot, \cdot)$ and $\|\cdot, \cdot\|$ denote the inner and norm on $L^2[0, r_0; r^2]$, respectively, with the norm

$$\|\varphi\| = \left(\int_0^{r_0} r^2 \big|\varphi(r)\big|^2 dr \right)^{\frac{1}{2}}. \tag{2.1}$$

Lemma 2.1 ([31, 32]) *For any $n \ge 1$, we have*

$$\frac{C_1}{n\pi} \le e^{-(\frac{n\pi}{r_0})^2 T} \le \frac{C_2}{n\pi}, \tag{2.2}$$

where C_1, C_2 are constants.

Lemma 2.2 *Suppose $f \in L^{\infty}(0, T; L^2[0, r_0; r^2])$, then there exists a positive M such that*

$$\left\| g_n - \int_0^T e^{-(\frac{n\pi}{r_0})^2} f_n(\tau)\, d\tau \right\| \le \sqrt{2\big(\|g\|_{L^2[0,r_0;r^2]}^2 + M\|F\|_{L^{\infty}(0,T;L^2[0,r_0;r^2])}^2\big)}, \tag{2.3}$$

where $M := \frac{T}{2}(\frac{r_0}{\pi})^2(1 - e^{(\frac{\pi}{r_0})^2})$.

Proof For $t \in [0, T]$,

$$\left| f_n(t) \right|^2 \leq \sum_{n=1}^{\infty} \left| \left(f(\cdot, t), \omega_n \right) \right|^2 \leq \| f \|^2_{L^{\infty}(0,T;L^2[0,r_0;r^2])}.$$

Thus

$$\left\| g_n - \int_0^T e^{-(\frac{n\pi}{r_0})^2(T-\tau)} f_n(\tau) \, d\tau \right\|^2 = \sum_{n=1}^{\infty} \left(g_n - \int_0^T e^{-(\frac{n\pi}{r_0})^2(T-\tau)} f_n(\tau) \, d\tau \right)^2$$

$$\leq 2 \sum_{n=1}^{\infty} g_n^2 + 2 \sum_{n=1}^{\infty} \left| \int_0^T e^{-(\frac{n\pi}{r_0})^2(T-\tau)} f_n(\tau) \, d\tau \right|^2$$

$$\leq 2 \sum_{n=1}^{\infty} g_n^2 + 2 \sum_{n=1}^{\infty} \int_0^T e^{-2(\frac{n\pi}{r_0})^2(T-\tau)} \, d\tau \int_0^T f_n^2(\tau) \, d\tau$$

$$\leq 2 \left(\| g \|^2_{L^2[0,r_0;r^2]} + M \| f \|^2_{L^{\infty}(0,T;L^2[0,r_0;r^2])} \right),$$

where $M := T \sup_{n \in \mathbb{N}} \left(\int_0^T e^{-2(\frac{n\pi}{r_0})^2(T-\tau)} \, d\tau \right) = \frac{T}{2} (\frac{r_0}{\pi})^2 (1 - e^{(\frac{\pi}{r_0})^2})$. $\square$

3 Regularization method and convergence estimate

It is obvious that the instability arises in the components of large n in the solution. It is natural to imagine that we should replace $e^{(\frac{n\pi}{r_0})^2 T}$ by a bounded approximation or eliminate the noise in the input data. In this paper, we eliminate all the components of large n from the solution and define the truncation regularized solution as follows:

$$\varphi^{N,\delta}(r) = \sum_{n=1}^{N} \frac{g_n^{\delta} - \int_0^T e^{-(\frac{n\pi}{r_0})^2(T-\tau)} f_n^{\delta}(\tau) \, d\tau}{e^{-(\frac{n\pi}{r_0})^2 T}} \omega_n(r). \tag{3.1}$$

3.1 Error estimate under a prior parameter choice rule

Theorem 3.1 *Let $\varphi(r)$ given by* (1.10) *be the exact solution of problem* (1.1). *Let $\varphi^{N,\delta}(r)$ given by* (3.1) *be the regularization solution. Choosing the regularization parameter $N = [\gamma]$, where $\gamma = (\frac{E}{\delta})^{\frac{1}{p+1}}$, then we obtain the following estimate*:

$$\left\| \varphi^{N,\delta}(\cdot) - \varphi(\cdot) \right\| \leq \left(\frac{\sqrt{2(M+1)}\pi}{C_1} + 1 \right) E^{\frac{1}{p+1}} \delta^{\frac{p}{p+1}}, \tag{3.2}$$

where $[\gamma]$ denotes the largest integer less than or equal to γ.

Proof By the triangle inequality, we have

$$\left\| \varphi^{N,\delta}(\cdot) - \varphi(\cdot) \right\| \leq \left\| \varphi^{N,\delta}(\cdot) - \varphi^N(\cdot) \right\| + \left\| \varphi^N(\cdot) - \varphi(\cdot) \right\|. \tag{3.3}$$

We firstly give an estimate for the first term. From (1.2) and (2.2), we can get

$$\left\| \varphi^{N,\delta}(\cdot) - \varphi^N(\cdot) \right\|^2$$

$$= \left\| \sum_{n=1}^{N} \frac{g_n^{\delta} - \int_0^T e^{-(\frac{n\pi}{r_0})^2(T-\tau)} f_n^{\delta}(\tau) \, d\tau}{e^{-(\frac{n\pi}{r_0})^2 T}} \omega_n(r) - \sum_{n=1}^{N} \frac{g_n - \int_0^T e^{-(\frac{n\pi}{r_0})^2(T-\tau)} f_n(\tau) \, d\tau}{e^{-(\frac{n\pi}{r_0})^2 T}} \omega_n(r) \right\|^2$$

$$= \left\| \sum_{n=1}^{N} \frac{(g_n^\delta - g_n) - \int_0^T e^{-(\frac{n\pi}{r_0})^2(T-\tau)}(f_n^\delta(\tau) - f_n(\tau))\,d\tau}{e^{-(\frac{n\pi}{r_0})^2 T}} \omega_n(r) \right\|^2$$

$$= \sum_{n=1}^{N} \left(\frac{(g_n^\delta - g_n) - |\int_0^T e^{-(\frac{n\pi}{r_0})^2(T-\tau)}(f_n^\delta(\tau) - f_n(\tau))\,d\tau|}{e^{-(\frac{n\pi}{r_0})^2 T}} \right)^2$$

$$\leq 2 \sum_{n=1}^{N} \left(\frac{g_n^\delta - g_n}{e^{-(\frac{n\pi}{r_0})^2 T}} \right)^2 + 2 \sum_{n=1}^{N} \frac{|\int_0^T e^{-(\frac{n\pi}{r_0})^2(T-\tau)}(f_n^\delta(\tau) - f_n(\tau))\,d\tau|^2}{(e^{-(\frac{n\pi}{r_0})^2 T})^2}$$

$$\leq 2 \max_{n \leq N} \left(\frac{n\pi}{C_1} \right)^2 \sum_{n=1}^{\infty} (g_n^\delta - g_n)^2 + 2M \max_{n \leq N} \left(\frac{n\pi}{C_1} \right)^2 \sum_{n=1}^{\infty} (f_n^\delta(\tau) - f_n(\tau))^2$$

$$\leq 2 \frac{N^2 \pi^2 \delta^2}{C_1^2} + 2M \frac{N^2 \pi^2 \delta^2}{C_1^2}$$

$$= 2(M+1) \frac{N^2 \pi^2 \delta^2}{C_1^2}.$$

Thus

$$\left\| \varphi^{N,\delta}(\cdot) - \varphi^N(\cdot) \right\| \leq \sqrt{2(M+1)} \frac{N\pi\delta}{C_1}. \tag{3.4}$$

Applying conditions (1.12) and (2.2), we obtain

$$\left\| \varphi(\cdot) - \varphi^N(\cdot) \right\|^2$$

$$= \left\| \sum_{n=1}^{\infty} \frac{g_n - \int_0^T e^{-(\frac{n\pi}{r_0})^2(T-\tau)} f_n(\tau)\,d\tau}{e^{-(\frac{n\pi}{r_0})^2 T}} \omega_n(r) \right.$$

$$\left. - \sum_{n=1}^{N} \frac{g_n - \int_0^T e^{-(\frac{n\pi}{r_0})^2(T-\tau)} f_n(\tau)\,d\tau}{e^{-(\frac{n\pi}{r_0})^2 T}} \omega_n(r) \right\|^2$$

$$= \left\| \sum_{n=N+1}^{\infty} \frac{g_n - \int_0^T e^{-(\frac{n\pi}{r_0})^2(T-\tau)} f_n(\tau)\,d\tau}{e^{-(\frac{n\pi}{r_0})^2 T}} \omega_n(r) \right\|^2$$

$$= \sum_{n=N+1}^{\infty} \left(\frac{g_n - \int_0^T e^{-(\frac{n\pi}{r_0})^2(T-\tau)} f_n(\tau)\,d\tau}{e^{-(\frac{n\pi}{r_0})^2 T}} \right)^2$$

$$= \sum_{n=N+1}^{\infty} \left((1+n^2)^{\frac{-p}{2}} (1+n^2)^{\frac{p}{2}} \left(\frac{g_n - \int_0^T e^{-(\frac{n\pi}{r_0})^2(T-\tau)} f_n(\tau)\,d\tau}{e^{-(\frac{n\pi}{r_0})^2 T}} \right) \right)^2$$

$$\leq \max_{n > N} \left((n+1)^{-2p} \right) \sum_{n=1}^{\infty} \left((1+n^2)^{\frac{p}{2}} \left(\frac{g_n - \int_0^T e^{-(\frac{n\pi}{r_0})^2(T-\tau)} f_n(\tau)\,d\tau}{e^{-(\frac{n\pi}{r_0})^2 T}} \right) \right)^2$$

$$\leq (N+1)^{-2p} E^2.$$

Hence

$$\left\| \varphi^{N,\delta}(\cdot) - \varphi(\cdot) \right\| \leq \sqrt{2(M+1)} \frac{N\pi\delta}{C_1} + (N+1)^{-p} E. \tag{3.5}$$

Choosing the regularization parameter $N = [(\frac{E}{\delta})^{\frac{1}{p+1}}]$, we obtain

$$\left\| \varphi^{N,\delta}(\cdot) - \varphi(\cdot) \right\| \leq \left(\frac{\sqrt{2(M+1)}\pi}{C_1} + 1 \right) E^{\frac{1}{p+1}} \delta^{\frac{p}{p+1}}. \tag{3.6}$$

Theorem 3.1 is proved. □

3.2 Error estimate under a posterior parameter choice rule

Choose $|g^\delta(r) - \int_0^T e^{-(\frac{n\pi}{r_0})^2(T-\tau)} f^\delta(r,\tau)\,d\tau| > \tau\delta$. Let $\psi(r) := g^\delta(r) - \int_0^T e^{-(\frac{n\pi}{r_0})^2(T-\tau)} f^\delta(r,\tau)\,d\tau$. Applying a discrepancy principle, we choose a posterior regularization parameter N that satisfies

$$\left\| (I - P_N)\psi(r) \right\| \leq \tau\delta \leq \left\| (I - P_{N-1})\psi(r) \right\|, \tag{3.7}$$

where $P_N : L^2[0, r_0; r^2] \rightarrow \operatorname{span} \omega_n|_{n \leq N}$ is an orthogonal projective operator. I is an identity operator.

Lemma 3.1 *Let* $d(N) = \|(I - P_N)(g^\delta(r) - \int_0^T e^{-(\frac{n\pi}{r_0})^2(T-\tau)} f^\delta(r,\tau)\,d\tau)\|$, *then we have the following conclusions*:

(a) $d(N)$ *is a continuous function*;
(b) $\lim_{N \to \infty} d(N) = 0$;
(c) $\lim_{N \to 0} d(N) = \|g^\delta(r) - \int_0^T e^{-(\frac{n\pi}{r_0})^2(T-\tau)} f^\delta(r,\tau)\,d\tau\|$;
(d) $d(N)$ *is a strictly decreasing function, for any* $N \in [1, \infty)$.

Lemma 3.2 *If conditions* (1.2) *and* (1.11) *hold. Suppose* $\tau > \sqrt{2(M+1)}$. *N is chosen by a posterior choice rule, thus we have*

$$N \leq \left(\frac{C_2}{\pi} \right)^{\frac{1}{p+1}} \left(\frac{E}{(\tau - \sqrt{2(M+1)})\delta} \right)^{\frac{1}{p+1}}. \tag{3.8}$$

Proof Due to (3.5), we obtain

$$\left\| (I - P_{N-1}) \left(g(r) - \int_0^T e^{-(\frac{n\pi}{r_0})^2(T-\tau)} f(r,\tau)\,d\tau \right) \right\|^2$$

$$= \sum_{n=N}^{\infty} \left| g_n - \int_0^T e^{-(\frac{n\pi}{r_0})^2(T-\tau)} f_n(\tau)\,d\tau \right|^2$$

$$= \sum_{n=N}^{\infty} (1 + n^2)^{-p}(1 + n^2)^p e^{-2(\frac{n\pi}{r_0})^2 T} \left(\frac{g_n - \int_0^T e^{-(\frac{n\pi}{r_0})^2(T-\tau)} f_n(\tau)\,d\tau}{e^{-(\frac{n\pi}{r_0})^2 T}} \right)^2$$

$$\leq \max_{n \geq N} \left(n^{-2p} \left(\frac{C_2}{n\pi} \right)^2 \right) \sum_{n=1}^{\infty} \left((1 + n^2)^{\frac{p}{2}} \left(\frac{g_n - \int_0^T e^{-(\frac{n\pi}{r_0})^2(T-\tau)} f_n(\tau)\,d\tau}{e^{-(\frac{n\pi}{r_0})^2 T}} \right) \right)^2$$

$$\leq \frac{C_2^2}{\pi^2} N^{-2(p+1)} E^2.$$

Thus we obtain

$$\left\| (I - P_{N-1}) \left(g(r) - \int_0^T e^{-(\frac{n\pi}{r_0})^2(T-\tau)} f(r,\tau)\,d\tau \right) \right\|^2 \leq \frac{C_2^2}{\pi^2} N^{-2(p+1)} E^2. \tag{3.9}$$

On the other hand,

$$
\left\| (I - P_{N-1}) \left(g(r) - \int_0^T e^{-(\frac{n\pi}{r_0})^2(T-\tau)} f(r, \tau) \, d\tau \right) \right\|
$$

$$
\geq \left\| (I - P_{N-1}) \left(g^\delta(r) - \int_0^T e^{-(\frac{n\pi}{r_0})^2(T-\tau)} f^\delta(r, \tau) \, d\tau \right) \right\|
$$

$$
- \left\| (I - P_{N-1}) (g^\delta(r) - g(r)) - \int_0^T e^{-(\frac{n\pi}{r_0})^2(T-\tau)} (f^\delta(r, \tau) - f(r, \tau)) \, d\tau \right\|
$$

$$
\geq \tau \delta - \sqrt{2(M+1)} \delta = (\tau - \sqrt{2(M+1)}) \delta,
$$

i.e.,

$$
\left\| (I - P_{N-1}) \left(g(r) - \int_0^T e^{-(\frac{n\pi}{r_0})^2(T-\tau)} f(r, \tau) \, d\tau \right) \right\| \geq (\tau - \sqrt{2(M+1)}) \delta. \tag{3.10}
$$

Combining (3.9) with (3.10), we obtain

$$
(\tau - \sqrt{2(M+1)}) \delta \leq C_2 (\pi)^{-1} N^{-(p+1)} E.
$$

So

$$
N \leq \left(\frac{C_2}{\pi} \right)^{\frac{1}{p+1}} \left(\frac{E}{(\tau - \sqrt{2(M+1)}) \delta} \right)^{\frac{1}{p+1}}. \tag{3.11}
$$

$\square$

Theorem 3.2 *Let $\varphi(r)$ given by (1.10) be the exact solution of problem (1.1). Let $\varphi^{N,\delta}(r)$ given by (3.1) be the regularization solution. The regularization parameter N is chosen in (3.7). Then we obtain*

$$
\left\| \varphi^{N,\delta}(\cdot) - \varphi(\cdot) \right\| \leq (C_3 + C_4) E^{\frac{1}{p+1}} \delta^{\frac{p}{p+1}}, \tag{3.12}
$$

where $C_3 := \left(\frac{\pi(\tau + \sqrt{2(M+1)})}{C_1} \right)^{\frac{p}{p+1}}$, $C_4 := \frac{\sqrt{2(M+1)}}{C_1} \left(\frac{C_2}{(\tau - \sqrt{2(M+1)})} \right)^{\frac{1}{p+1}} \pi^{\frac{p}{p+1}}$.

Proof According to (3.3), we have

$$
\left\| \varphi^{N,\delta}(\cdot) - \varphi(\cdot) \right\| \leq \left\| \varphi^{N,\delta}(\cdot) - \varphi^N(\cdot) \right\| + \left\| \varphi^N(\cdot) - \varphi(\cdot) \right\|. \tag{3.13}
$$

We first estimate the second term of (3.13). Using (1.12) and Lemma 2.1, we have

$$
\left\| \varphi^N(\cdot) - \varphi(\cdot) \right\|^2
$$

$$
= \left\| \sum_{n=1}^{\infty} \frac{g_n - \int_0^T e^{-(\frac{n\pi}{r_0})^2(T-\tau)} f_n(\tau) \, d\tau}{e^{-(\frac{n\pi}{r_0})^2 T}} \omega_n(r) - \sum_{n=1}^{N} \frac{g_n - \int_0^T e^{-(\frac{n\pi}{r_0})^2(T-\tau)} f_n(\tau) \, d\tau}{e^{-(\frac{n\pi}{r_0})^2 T}} \omega_n(r) \right\|^2
$$

$$
= \left\| \sum_{n=N+1}^{\infty} \frac{g_n - \int_0^T e^{-(\frac{n\pi}{r_0})^2(T-\tau)} f_n(\tau) \, d\tau}{e^{-(\frac{n\pi}{r_0})^2 T}} \omega_n(r) \right\|^2
$$

$$
= \sum_{n=N+1}^{\infty} \left(\frac{g_n - \int_0^T e^{-(\frac{n\pi}{r_0})^2(T-\tau)} f_n(\tau) \, d\tau}{e^{-(\frac{n\pi}{r_0})^2 T}} \right)^2
$$

$$= \sum_{n=N+1}^{\infty} \left(\left(e^{-(\frac{n\pi}{r_0})^2 T}\right)^{-p} \left(1+n^2\right)^{-\frac{p}{2}} \left(1+n^2\right)^{\frac{p}{2}} \frac{g_n - \int_0^T e^{-(\frac{n\pi}{r_0})^2(T-\tau)} f_n(\tau)\,d\tau}{e^{-(\frac{n\pi}{r_0})^2 T}} \right)^{\frac{2}{p+1}}$$

$$\times \left(g_n - \int_0^T e^{-(\frac{n\pi}{r_0})^2(T-\tau)} f_n(\tau)\,d\tau \right)^{\frac{2p}{p+1}}$$

$$\leq \left(\sum_{n>N}^{\infty} \left(\left(e^{-(\frac{n\pi}{r_0})^2 T}\right)^{\frac{-2p}{p+1}} \left(1+n^2\right)^{-\frac{p}{p+1}} \left(1+n^2\right)^{\frac{p}{p+1}} \right. \right.$$

$$\left. \times \left(\frac{g_n - \int_0^T e^{-(\frac{n\pi}{r_0})^2(T-\tau)} f_n(\tau)\,d\tau}{e^{-(\frac{n\pi}{r_0})^2 T}} \right)^{\frac{2}{p+1}} \right)^{\frac{p+1}{2}} \right)^{\frac{2}{p+1}}$$

$$\times \left(\sum_{n>N}^{\infty} \left(\left(g_n - \int_0^T e^{-(\frac{n\pi}{r_0})^2(T-\tau)} f_n(\tau)\,d\tau \right)^{\frac{2p}{p+1}} \right)^{\frac{p+1}{2p}} \right)^{\frac{2p}{p+1}}$$

$$\leq \max_{n>N} \left(\left(\frac{C_1}{n\pi} \right)^{-p} n^{-p} \right)^{\frac{2}{p+1}} \sum_{n=1}^{\infty} \left(\left(1+n^2\right)^{\frac{p}{2}} \left(\frac{g_n - \int_0^T e^{-(\frac{n\pi}{r_0})^2(T-\tau)} f_n(\tau)\,d\tau}{e^{-(\frac{n\pi}{r_0})^2 T}} \right) \right)^{\frac{2}{p+1}}$$

$$\times \left((I - P_N) \left(g_n - \int_0^T e^{-(\frac{n\pi}{r_0})^2(T-\tau)} f_n(\tau)\,d\tau \right) \right)^{\frac{2p}{p+2}}$$

$$\leq \left(\frac{\pi}{C_1} \right)^{\frac{2p}{p+1}} E^{\frac{2}{p+1}} \left(\left\| (I - P_{N-1}) \left(g^\delta(r) - \int_0^T e^{-(\frac{n\pi}{r_0})^2(T-\tau)} f^\delta(r,\tau)\,d\tau \right) \right\| \right.$$

$$\left. + \left\| (I - P_{N-1}) \left(g^\delta(r) - g(r) \right) - \int_0^T e^{-(\frac{n\pi}{r_0})^2(T-\tau)} \left(f^\delta(r,\tau) - f(r,\tau) \right)\,d\tau \right\| \right)^{\frac{2p}{p+1}}$$

$$\leq \left(\frac{\pi(\tau + \sqrt{2(M+1)})}{C_1} \right)^{\frac{2p}{p+1}} E^{\frac{2}{p+1}} \delta^{\frac{2p}{p+1}}.$$

Thus we obtain

$$\left\| \varphi^N(\cdot) - \varphi(\cdot) \right\| \leq \left(\frac{\pi(\tau + \sqrt{2(M+1)})}{C_1} \right)^{\frac{p}{p+1}} E^{\frac{1}{p+1}} \delta^{\frac{p}{p+1}}. \tag{3.14}$$

Then we estimate the first term of (3.13). Using (3.1), we obtain

$$\left\| \varphi^{N,\delta}(\cdot) - \varphi^N(\cdot) \right\|^2$$

$$= \left\| \sum_{n=1}^{N} \frac{g_n^\delta - \int_0^T e^{-(\frac{n\pi}{r_0})^2(T-\tau)} f_n^\delta(\tau)\,d\tau}{e^{-(\frac{n\pi}{r_0})^2 T}} \omega_n(r) - \sum_{n=1}^{N} \frac{g_n - \int_0^T e^{-(\frac{n\pi}{r_0})^2(T-\tau)} f_n(\tau)\,d\tau}{e^{-(\frac{n\pi}{r_0})^2 T}} \omega_n(r) \right\|^2$$

$$\leq 2 \sum_{n=1}^{N} \left(\frac{g_n^\delta - g_n}{e^{-(\frac{n\pi}{r_0})^2 T}} \right)^2 + 2 \sum_{n=1}^{N} \frac{\left| \int_0^T e^{-(\frac{n\pi}{r_0})^2(T-\tau)} \left(f_n^\delta(\tau) - f_n(\tau) \right)\,d\tau \right|^2}{\left(e^{-(\frac{n\pi}{r_0})^2 T} \right)^2}$$

$$\leq 2 \max_{n\leq N} \left(\frac{n\pi}{C_1} \right)^2 \sum_{n=1}^{\infty} \left(g_n^\delta - g_n \right)^2 + 2M \max_{n\leq N} \left(\frac{n\pi}{C_1} \right)^2 \sum_{n=1}^{\infty} \left(f_n^\delta(\tau) - f_n(\tau) \right)^2$$

$$\leq 2 \frac{N^2 \pi^2 \delta^2}{C_1^2} + 2M \frac{N^2 \pi^2 \delta^2}{C_1^2}$$

$$= 2(M+1) \frac{N^2 \pi^2 \delta^2}{C_1^2}.$$

Using (3.8), we have

$$\left\| \varphi^{N,\delta}(\cdot) - \varphi^N(\cdot) \right\| \leq \frac{\sqrt{2(M+1)}}{C_1} \left(\frac{C_2}{(\tau - \sqrt{2(M+1)})} \right)^{\frac{1}{p+1}} \pi^{\frac{p}{p+1}} E^{\frac{1}{p+1}} \delta^{\frac{p}{p+1}}. \tag{3.15}$$

Combining (3.14) with (3.15), we obtain

$$\left\| \varphi^{N,\delta}(\cdot) - \varphi(\cdot) \right\| \leq (C_3 + C_4) E^{\frac{1}{p+1}} \delta^{\frac{p}{p+1}}, \tag{3.16}$$

where $C_3 := \left(\frac{\pi(\tau + \sqrt{2(M+1)})}{C_1} \right)^{\frac{p}{p+1}}$, $C_4 := \frac{\sqrt{2(M+1)}}{C_1} \left(\frac{C_2}{(\tau - \sqrt{2(M+1)})} \right)^{\frac{1}{p+1}} \pi^{\frac{p}{p+1}}$.

Theorem 3.2 is proved. □

4 Numerical implementation and numerical example

In this section, we present numerical experiment for above regularization method. The exact solution of problem (3.1) is difficult to obtain. So we use a given $\varphi(r)$ to solve the positive problem. We have

$$\begin{cases} u_t - \frac{2}{r} u_r - u_{rr} = f(r,t), & 0 < t < T, 0 < r < r_0, \\ u(r,0) = \varphi(r), & 0 \leq r \leq r_0, \\ u(r_0,t) = 0, & 0 \leq t \leq T, \\ \lim_{r \to 0} u(r,t) \text{ is bounded}, & 0 < t < T, 0 < r < r_0. \end{cases} \tag{4.1}$$

Let $T = 1$, $r_0 = \pi$. Using (1.5), we have

$$\begin{aligned} g(r) &= \sum_{n=1}^{\infty} \left(\varphi_n e^{-n^2} + \int_0^1 e^{-n^2(1-\tau)} f_n(\tau) \, d\tau \right) w_n(r) \\ &= \sum_{n=1}^{\infty} \left((\varphi(r), w_n(r)) e^{-n^2} w_n(r) + \int_0^1 e^{-n^2(1-\tau)} (f(r,\tau), w_n(r)) w_n(r) \, d\tau \right) \\ &= \sum_{n=1}^{\infty} \left(e^{-n^2} \int_0^{\pi} s^2 \varphi(s) \sqrt{\frac{2}{\pi}} \frac{\sin(ns)}{s} \, ds \sqrt{\frac{2}{\pi}} \frac{\sin(nr)}{r} \right. \\ &\quad \left. + \int_0^1 e^{-n^2(1-\tau)} \int_0^{\pi} s^2 f(s,\tau) \sqrt{\frac{2}{\pi}} \frac{\sin(ns)}{s} \, ds \, d\tau \sqrt{\frac{2}{\pi}} \frac{\sin(ns)}{s} \right). \end{aligned}$$

Using (3.1), we have

$$\begin{aligned} \varphi^{N,\delta}(r) &= \sum_{n=1}^{N} \frac{1}{e^{-n^2}} \left(g_n^{\delta} - \int_0^1 e^{-n^2(1-\tau)} f_n^{\delta}(\tau) \, d\tau \right) w_n(r) \\ &= \sum_{n=1}^{N} \frac{1}{e^{-n^2}} \left((g^{\delta}(r), w_n(r)) w_n(r) - \int_0^1 e^{-n^2(1-\tau)} (f^{\delta}(r,\tau), w_n(r)) \, d\tau w_n(r) \right) \\ &= \sum_{n=1}^{N} \frac{1}{e^{-n^2}} \left(\int_0^{\pi} s^2 g^{\delta}(s) \sqrt{\frac{2}{\pi}} \frac{\sin(ns)}{s} \, ds \sqrt{\frac{2}{\pi}} \frac{\sin(nr)}{r} \right. \\ &\quad \left. - \int_0^1 e^{-n^2(1-\tau)} \int_0^{\pi} s^2 f^{\delta}(s,\tau) \sqrt{\frac{2}{\pi}} \frac{\sin(ns)}{s} \, ds \, d\tau \sqrt{\frac{2}{\pi}} \frac{\sin(nr)}{r} \right). \end{aligned}$$

Time and space of grid step size are $\Delta t = \frac{1}{P}$ and $\Delta r = \frac{\pi}{M}$. A grid point on the time interval $[0, T]$ is $t_q = \frac{q-1}{P}$ ($q = 1, 2, \ldots, P+1$). $r_i = \frac{i-1}{M}\pi$ ($i = 1, 2, \ldots, M+1$) is a grid point on the space interval. We have

$$
g(r_j) = \sum_{n=1}^{N} 2\left(e^{-n^2} \sum_{i=1}^{M+1} r_i^2 \varphi(r_i) \frac{\sin(nr_i)}{r_i} \frac{\sin(nr_j)}{r_j} \omega_i \right.
$$
$$
\left. + \sum_{q=1}^{P+1} e^{-n^2(1-t_q)} \sum_{i=1}^{M+1} r_i^2 f(r_i, t_q) \frac{\sin(nr_i)}{r_i} \omega_i y_q \frac{\sin(nr_j)}{r_j} \right),
$$
$$
\varphi_N^\delta(r_j) = \sum_{n=1}^{N} \frac{2}{e^{-n^2}} \left(\sum_{i=1}^{M+1} r_i^2 g^\delta(r_i) \frac{\sin(nr_i)}{r_i} \frac{\sin(nr_j)}{r_j} \omega_i \right.
$$
$$
\left. - \sum_{q=1}^{P+1} e^{-n^2(1-t_q)} \sum_{i=1}^{M+1} r_i^2 f^\delta(r_i, t_q) \frac{\sin(nr_i)}{r_i} \omega_i y(q) \frac{\sin(nr_j)}{r_j} \right),
$$

where

$$
\varphi(r_i) = \left(\varphi(r_1), \varphi(r_2), \ldots, \varphi(r_{M+1}) \right),
$$
$$
\varphi^\delta(r_i) = \left(\varphi^\delta(r_1), \varphi^\delta(r_2), \ldots, \varphi^\delta(r_{M+1}) \right),
$$
$$
g(r_j) = \left(g(r_1), g(r_2), \ldots, g(r_{M+1}) \right),
$$
$$
g^\delta(r_j) = \left(g^\delta(r_1), g^\delta(r_2), \ldots, g^\delta(r_{M+1}) \right),
$$
$$
f(r_i, t_q) = \left(f(r_i, t_1), f(r_i, t_2), \ldots, f(r_i, t_{P+1}) \right),
$$
$$
f^\delta(r_i, t_q) = \left(f^\delta(r_i, t_1), f^\delta(r_i, t_2), \ldots, f^\delta(r_i, t_{P+1}) \right),
$$
$$
w_i = \begin{cases} \frac{1}{3M}, & i = 1, M+1, \\ \frac{4}{3M}, & i = 2, 4, \ldots, M, \\ \frac{2}{3M}, & i = 3, 5, \ldots, M-1; \end{cases} \tag{4.2}
$$
$$
y_q = \begin{cases} \frac{1}{3y}, & i = 1, y+1, \\ \frac{4}{3y}, & i = 2, 4, \ldots, \\ \frac{2}{3y}, & i = 3, 5, \ldots, y-1. \end{cases} \tag{4.3}
$$

Noise data is generated by adding a random perturbation, that is,

$$
g^\delta(\cdot) = g(\cdot) + \varepsilon \cdot g(\cdot) \cdot \left(2\,\text{rand}(\cdot) - 1 \right),
$$
$$
f^\delta(\cdot, \cdot) = f(\cdot, \cdot) + \varepsilon \cdot f(\cdot, \cdot) \cdot \left(2\,\text{rand}(\cdot, \cdot) - 1 \right),
$$

where ε is relative error level. In the computational procedure, we take the source function $f(r, t) = t\sin(r)$.

Example 1 Take initial function $\varphi(r) = \sin(r)$.

Figure 1 shows the comparisons of the numerical effects between the exact solution and its regularization solution for the prior and posterior regularization parameter choice rules. We can find that the smaller ε, the better the computed approximation is. Moreover,

Figure 1 The comparison of numerical effects between the exact solution and its regularized solution for Example 1.
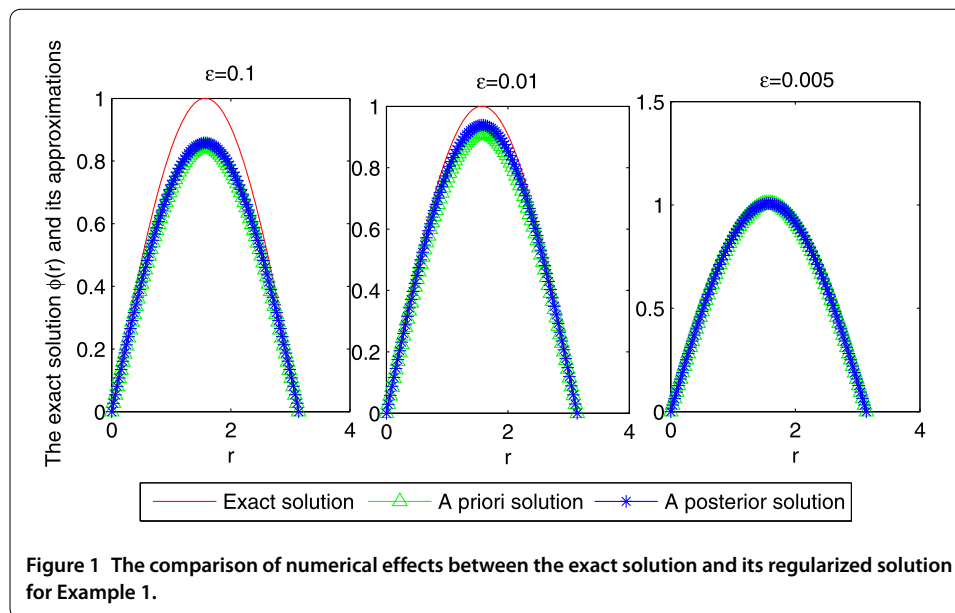
we can also easily find that the posterior parameter choice rule works better than the prior parameter choice rule. This is consistent with our theoretical analysis.

5 Conclusion

We consider an inverse problem to determine an initial date for heat equation with inhomogeneous source on a columnar symmetric domain. Using the truncation method, we construct the regularization solution. Moreover, we obtain the Hölder type error estimate under prior and posterior parameter choice rules. Finally, an example is given to show the effectiveness of the truncation method.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The main idea of this paper was proposed by FY and YRS prepared the manuscript initially and performed all the steps of the proofs in this research. All authors read and approved the final manuscript.

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