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Existence of nontrivial solutions for a class of biharmonic equations with singular potential in $\mathbb{R}^N$

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Abstract

In this paper, we study a class of biharmonic equations with a singular potential in $\mathbb{R}^N$. Under appropriate assumptions on the nonlinearity, we establish some existence results via the Morse theory and variational methods. We significantly extend and complement some results from the previous literature.

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1 Introduction and main results

This paper is concerned with the existence and multiplicity of nontrivial solutions for the following biharmonic equations in $\mathbb{R}^N$ ($N \geq 5$):

$$\Delta^2 u - \Delta u + V(x)u = f(x,u) \quad \text{in } \mathbb{R}^N, \tag{1.1}$$

where Δ^2 is the biharmonic operator, $V(x)$ is a singular potential, and $f \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$.

In recent years, many authors have paid attention to studying the existence of nontrivial solutions for biharmonic equations (see, e.g., [1–3] and the references therein) since it was first introduced by Lazer and McKenna [4] to furnish a model for studying the traveling waves in suspension bridges. Let us mention some recent mathematical studies related to biharmonic equations. Applying the mountain pass theorem and employing the Morse theory, the authors in [5, 6] studied the existence of multiple nontrivial solutions for the following biharmonic equations:

$$\begin{cases} \Delta^2 u + c\Delta u = f(u) & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where Ω is a smooth bounded domain of $\mathbb{R}^N$, and $c < \lambda_1$, where λ_1 is the first eigenvalue of $-\Delta$ in $H_0^1(\Omega)$. In addition, the existence of infinitely many sign-changing solutions of problem (1.2) was obtained in [7] via the sign-changing critical point theorem.

On the other hand, there also exist a large number of works on the biharmonic equations in the entire space $\mathbb{R}^N$. For example, for problem (1.1) Ye and Tang [8] considered the existence and multiplicity solutions by using the mountain pass theorem when the potential $V(x)$ is a positive function. Their results unify and sharply improve the results of Liu, Chen, and Wu [9]. For problem (1.1) with a sign-changing potential, infinitely many solutions were obtained in [10] via the symmetric mountain pass theorem. Zhang and Costa [11] investigated the existence of a nontrivial solution by applying the mountain pass theorem. For other interesting results of biharmonic equations, we refer to [12–17] and references therein.

Motivated by the papers mentioned, especially by [6, 11], the aim of this paper is to revisit the existence and multiplicity of nontrivial solutions for problem (1.1) with singular potential $V(x)$ satisfying the following condition:

(V) $V(x)$ is a continuous function and satisfies

$$V(r)+(\overline{\lambda}-\alpha)\frac{1}{r^4}\geq 0,\qquad \lim_{r\to 0}r^4V(r)=\lim_{r\to\infty}r^4V(r)=+\infty,$$

where $r=|x|$, $\overline{\lambda}=\frac{N^2(N-4)^2}{16}$, and $\alpha>0$ is a constant.

Before stating our main results, we present the assumptions on the nonlinearity $f(x,u)=f(|x|,u)$ and its primitive $F(x,u)=\int_0^u f(x,t)\,dt$:

$(f1)$ $f(x,u)\in C(\mathbb{R}^N\times\mathbb{R},\mathbb{R})$, and $\lim_{|u|\to\infty}\frac{|f(x,u)|}{|u|^{p-1}}=0$ uniformly in $x\in\mathbb{R}^N$ for $2<p<2_*$;

$(f2)$ $f(x,u)=o(|u|)$ as $|u|\to 0$ uniformly in $x\in\mathbb{R}^N$;

$(f3)$ $\lim_{|u|\to\infty}\frac{|F(x,u)|}{|u|^2}=\infty$ for a.e. $x\in\mathbb{R}^N$, and there exists $r_0\geq 0$ such that

$$F(x,u)\geq 0,\quad \forall(x,u)\in\mathbb{R}^N\times\mathbb{R},|u|\geq r_0;$$

$(f4)$ $\mathcal{F}(x,u):=\frac{1}{2}uf(x,u)-F(x,u)\geq 0$, and there exist $c_2>0$ and $\kappa>1$ such that

$$\left|F(x,u)\right|^{\kappa}\leq c_2|u|^{2\kappa}\mathcal{F}(x,u),\quad \forall(x,u)\in\mathbb{R}^N\times\mathbb{R},|u|\geq r_0;$$

$(f5)$ There exist $r_1,r_2\geq 0$, $c_3>0$, and $\sigma\in(1,2)$ such that

$$F(x,u)\geq c_3u^{\sigma},\quad \forall(x,u)\in\mathbb{R}^N\times\mathbb{R},r_1\leq|u|\leq r_2;$$

$(f6)$ There exist $\theta_0\in(0,1)$ and $K\geq 1$ such that

$$\frac{1-\theta^2}{2}uf(x,u)\geq F(x,u)-KF(x,\theta u),\quad \forall(x,u)\in\mathbb{R}^N\times\mathbb{R},\theta\in[0,\theta_0].$$

Now we can state our main results.

Theorem 1.1 *Assume that* (V), $(f1)$, $(f2)$, $(f3)$, $(f4)$, *and* $(f5)$ *are satisfied. Then problem* (1.1) *has a nontrivial solution.*

Theorem 1.2 *Assume that* (V), $(f1)$, $(f2)$, $(f3)$, $(f4)$, *and* $(f6)$ *are satisfied. Then problem* (1.1) *has a ground state solution.*

Theorem 1.3 *Let* $f(x,u)$ *be an odd function with respect to* u. *Assume that* (V), $(f1)$, $(f2)$, $(f3)$, *and* $(f4)$ *are satisfied. Then problem* (1.1) *has infinitely many nontrivial solutions.*

Remark 1.1 The authors in [11] showed that $V(x)$ satisfying condition (V) is a singular potential and the function $V(r) = \frac{|\log r|}{r^4}$ satisfies condition (V). This is quite different from [8, 9] since the potentials in those papers are just positive but not singular.

Remark 1.2 In [11], the existence of a nontrivial solution is obtained under the condition

$$0 < 2F(s) < sf(s) \quad \text{for all } s \neq 0. \tag{1.3}$$

However, condition (1.3) cannot imply condition $(f4)$. For example, let

$$f(x,u) = 3|u|^2 \int_0^u |x|^{1+\sin x} x\,dx + |u|^{4+\sin u} u.$$

Then

$$F(x,u) = |u|^3 \int_0^u |x|^{1+\sin x} x\,dx.$$

It is easy to see that $f(x,u)$ satisfies condition (1.3) but does not satisfy $(f4)$.

Remark 1.3 To the best of our knowledge, condition $(f6)$, which was first given by Tang in [18], is weaker than (1.3). For the details, we refer to [18].

Remark 1.4 Take $p = 5$ and $\kappa = 2$. Then by a simple calculation we can easily check that the function

$$f(x,u) = a|\sin x| \cdot |u|^{p-2},$$

where a is a positive constant, satisfies conditions $(f1)$-$(f6)$.

It should be pointed out that our method is quite different from [11], in which the authors used the mountain pass theorem to verify the existence of nontrivial solutions, whereas we apply the Morse theory combining it with local linking methods. On the other hand, we consider the problem with singular potential on the whole space $\mathbb{R}^N$, whereas the authors in [6] dealt with the problem in a smooth bounded domain $\Omega \subset \mathbb{R}^N$. Fortunately, we show that problem (1.1) has at least one nontrivial solution by the Morse theory. Moreover, the ground state solution and infinitely many nontrivial solutions are obtained by using variational methods.

The remainder of this paper is organized as follows. In Section 2, we present some preliminaries and the variational setting and give the proofs of the main results in Section 3.

2 Preliminaries and variational setting

Throughout this paper, let $C_1, C_2, \ldots$ are positive constants, which may vary from line to line. the strong (weak) convergence is denoted by $\to$ ($\rightharpoonup$). By $B_R(y)$ we denote the ball of radius R and center at y. By X^* we denote the dual space of $E_V := X$, which is the weighted Sobolev space defined as the subspace of the radially symmetric function in the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the inner product and norm

$$\langle u,v\rangle = \int_{\mathbb{R}^N} \bigl[\Delta u \Delta v + \nabla u \nabla v + V(r)uv\bigr]\,dx \quad \text{and} \quad \|u\| = \langle u,u\rangle^{\frac{1}{2}}, \quad u,v \in E_V.$$

Recall that the Hardy-Rellich inequality implies that, for all $u \in C_0^\infty(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N} |\Delta u|^2\,dx \ge \overline{\lambda} \int_{\mathbb{R}^N} \frac{u^2}{r^4}\,dx,$$

where $\overline{\lambda}$ and r are defined in (V). Denote by $L^p(\mathbb{R}^N)$ the space with the norm

$$\|u\|_p = \left(\int_{\mathbb{R}^N} u^p\,dx\right)^{\frac{1}{p}}.$$

Then, proceeding in analogy with [11], we get the following result.

Theorem 2.1 *Under condition (V), E_V is continuously embedded into $L^p(\mathbb{R}^N)$ for $2 \le p \le 2_* = \frac{2N}{N-4}$, that is, there exists $r_p > 0$ such that*

$$\|u\|_p \le r_p \|u\|, \quad u \in E_V. \tag{2.1}$$

Furthermore, the embedding from E_V into $L^p(\mathbb{R}^N)$ is compact for $2 \le p < 2_*$.

Proof The proof is almost the same as that of Theorem 1.1 in [11]. We omit it here. □

It follows from $(f1)$ and $(f2)$ that, for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$\left|f(x,u)\right| \le \varepsilon|u| + C_\varepsilon|u|^{p-1}, \quad \forall (x,u) \in \mathbb{R}^N \times \mathbb{R}. \tag{2.2}$$

Then

$$\left|F(x,u)\right| \le \frac{\varepsilon}{2}|u|^2 + \frac{C_\varepsilon}{p}|u|^p, \quad \forall (x,u) \in \mathbb{R}^N \times \mathbb{R}. \tag{2.3}$$

Thus, a straightforward calculation from (2.1) and (2.3) gives

$$\int_{\mathbb{R}^N} \left|F(x,u)\right|dx \le \frac{\varepsilon r_2^2}{2}\|u\|^2 + \frac{C_\varepsilon r_p^p}{p}\|u\|^p. \tag{2.4}$$

Now we define the functional I on E_V by

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^N} \left[|\Delta u|^2 + |\nabla u|^2 + V(r)u^2\right]dx - \int_{\mathbb{R}^N} F(x,u)\,dx, \quad \forall u \in E_V. \tag{2.5}$$

Following (2.4), it is clear that $I(u) \in C^1(E_V, \mathbb{R})$ and

$$\langle I'(u), v\rangle = \int_{\mathbb{R}^N} \left[\Delta u \Delta v + \nabla u \nabla v + V(r)uv\right]dx - \int_{\mathbb{R}^N} f(x,u)v\,dx, \quad \forall v \in E_V. \tag{2.6}$$

Consequently, the critical points of I are weak solutions of problem (1.1).

In what follows, we recall some facts of the Morse theory.

Let X be a real Banach space, $I \in C^1(X,\mathbb{R})$, and $K = \{u \in X : I'(u) = 0\}$. Then the qth critical group of I at an isolated critical point $u \in K$ with $I(u) = c$ is defined by

$$C_q(I,u) := H_q\left(I^c \cap U, I^c \cap U \setminus \{u\}\right), \quad q \in \mathbb{N} := \{0,1,2,\ldots\},$$

where $I^c = \{u \in X : I(u) \leq c\}$, U is any neighborhood of u, and $H_q(\cdot,\cdot)$ denotes a singular relative homology group of pair $(\cdot,\cdot)$ with integer coefficients.

Definition 2.1 The functional I satisfies the Cerami condition at the level $c \in \mathbb{R}$ ($(C)_c$ for short) if any sequence $\{u_n\} \subset X$ satisfying

$$I(u_n) \to c \quad \text{and} \quad \bigl(1 + \|u_n\|\bigr)\bigl\|I'(u_n)\bigr\|_{X^*} \to 0$$

has a convergent subsequence. Thew functional I satisfies (C) condition if I satisfies $(C)_c$ at any $c \in \mathbb{R}$.

If I satisfies (C) condition and the critical values of I are bounded from below by some $a < \inf I(K)$, then the critical groups of I at infinity introduced by Bartsch and Li [19] as

$$C_q(I,\infty) := H_q\bigl(X, I^c\bigr), \quad q \in \mathbb{N},$$

do not depend on the choice of c.

Remark 2.1 Based on the Morse theory [20, 21], we can easily get that if $K = \{0\}$, then $C_q(I,\infty) \cong C_q(I,0)$ for all $q \in \mathbb{N}$. It follows that if $C_q(I,\infty) \ncong C_q(I,0)$ for some $q \in \mathbb{N}$, then I must have a nontrivial critical point.

We say that $u \in K$ is a homological nontrivial critical point of I if at least one of its critical groups is nontrivial.

Proposition 2.1 ([22]) *Assume that I has a critical point $u = 0$ with $I(0) = 0$. Suppose that I has a local linking at 0 with respect to $X = V \oplus W$, $k = \dim V < \infty$, that is, there exists $\rho > 0$ small enough such that*

$$I(u) \leq 0, \quad u \in V, \|u\| \leq \rho \quad \textit{and} \quad I(u) > 0, \quad u \in W, 0 < \|u\| \leq \rho.$$

Then $C_k(I,0) \ncong 0$. Hence, 0 is a homological nontrivial critical point of I.

Here, we present the following symmetric mountain pass theorem, which will be used later.

Proposition 2.2 ([20]) *Let X be an infinite-dimensional Banach space, $X = V \oplus W$, where V is finite dimensional. Suppose that $I \in C^1(X,\mathbb{R})$ satisfies $(C)_c$ condition for all $c > 0$, and*

- (I_1) *$I(0) = 0$, $I(-u) = -I(u)$ for all $u \in X$;*
- (I_2) *there exist constants $\rho, \alpha > 0$ such that $I|_{\partial B_\rho \geq \alpha}$;*
- (I_3) *for any finite-dimensional subspace $\widetilde{X} \subset X$, there is $R = R(\widetilde{X}) > 0$ such that $I(u) \leq 0$ on $\widetilde{X} \backslash B_R$.*

Then I possesses an unbounded sequence of critical values.

3 Proofs of main results

We begin this section by computing the critical groups of the functional I at infinity and zero.

Lemma 3.1 *Assume that conditions* (V), $(f1)$, $(f2)$, $(f3)$, *and* $(f4)$ *hold. Then* $C_q(I,\infty) \cong 0$ *for all* $q \in \mathbb{N}$.

Proof Let $S := \{u \in E_V : \|u\| = 1\}$. Applying the Fatou lemma and $(f3)$ yields that

$$\lim_{t\to\infty} \int_{\mathbb{R}^N} \frac{F(x,tu)}{t^2}\,dx \geq \int_{\mathbb{R}^N} \lim_{t\to\infty} \frac{F(x,tu)}{|tu|^2}|u|^2\,dx = +\infty, \quad \forall u \in S,$$

which, combined with (2.5), implies that

$$\begin{aligned} I(tu) &= \frac{1}{2}t^2 - \int_{\mathbb{R}^N} F(x,tu)\,dx \\ &\leq t^2\left[\frac{1}{2} - \int_{\mathbb{R}^N} \lim_{t\to\infty} \frac{F(x,tu)}{|tu|^2}|u|^2\,dx\right] \to -\infty, \quad \forall u \in S. \end{aligned}$$

Take $a < \min\{\inf_{\|u\|=1} I(u), 0\}$. Then, for $u \in S$, there exists $t_0 > 1$ such that $I(tu) \leq a$ for $t \geq t_0$. Thus,

$$\frac{1}{2}t^2 \leq a + \int_{\mathbb{R}^N} F(x,tu)\,dx. \tag{3.1}$$

From (2.5), (3.1), and $(f4)$ we deduce that

$$\begin{aligned} \frac{d}{dt}I(tu) &= \frac{1}{t}\left[t^2 - \int_{\mathbb{R}^N} f(x,tu)tu\,dx\right] \\ &\leq \frac{1}{t}\left[2\int_{\mathbb{R}^N} F(x,tu)\,dx - \int_{\mathbb{R}^N} f(x,tu)tu\,dx + 2a\right] \\ &= \frac{1}{t}\left[-2\int_{\mathbb{R}^N} \mathcal{F}(x,tu)\,dx + 2a\right] < 0. \end{aligned}$$

Then by the implicit function theorem there exists a unique $T \in C(S,\mathbb{R})$ such that $I(T(u)u) = a$. As in [23], we can use the function T to construct a strong deformation retract from $E_V \setminus \{0\}$ to I^a. Therefore,

$$C_q(I,\infty) = H_q\big(E_V, I^a\big) = H_q\big(E_V, E_V \setminus \{0\}\big) = 0, \quad \forall q \in \mathbb{N}.$$

The proof is completed. □

We now choose an orthogonal basis $\{e_j\}$ of $X := E_V$ and define $X_j := \operatorname{span}\{e_j\}$, $j = 1, 2, \ldots$. Set

$$Y_k := \bigoplus_{j=1}^{k} X_j, \qquad Z_k = \bigoplus_{j=k+1}^{\infty} X_j, \quad \text{and} \quad Z_k = Y_k^{\perp}, \quad k \in \mathbb{Z}.$$

Then $X = Y_k \oplus Z_k$.

Lemma 3.2 *Assume that conditions* (V), $(f1)$, $(f2)$, *and* $(f5)$ *hold. Then there exists* $k \in \mathbb{N}$ *such that* $C_k(I,0) \not\cong 0$.

Proof Let $V = Y_k$ and $W = Z_k$. Then $\dim V < \infty$. It follows from (2.5) and $F(x,u)$ that $I(0) = 0$. On one hand, taking $0 < \varepsilon < r_2^{-2}$, by (2.4) and (2.5) we have

$$I(u) \geq \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^N} \bigl|F(x,u)\bigr|\,dx \geq C_1\|u\|^2 - C_2\|u\|^p, \quad \forall u \in W.$$

Thus, for $p > 2$, we can choose $0 < \|u\| \leq \rho$ sufficiently small such that $I(u) > 0$ for all $u \in W$.

On the other hand, for any $u \in V$ with $|u|$ small, combining (2.5) with $(f5)$ yields

$$I(u) \leq \frac{1}{2}\|u\|^2 - c_3 \int_{\mathbb{R}^N} |u|^\sigma\,dx. \tag{3.2}$$

Note that V is a finite-dimensional subspace and all norms on a finite-dimensional space are equivalent. So, it can be deduced from (3.2) that

$$I(u) \leq \frac{1}{2}\|u\|^2 - C_3\|u\|^\sigma, \tag{3.3}$$

which implies that $I(u) \leq 0$, for all $u \in V$ with $\|u\| \leq \rho$ small since $\sigma \in (1,2)$.

Therefore, it follows from Proposition 2.1 that $C_k(I,0) \ncong 0$, that is, 0 is a homological nontrivial critical point of I. The proof is completed. □

Lemma 3.3 *If conditions* (V), $(f1)$, $(f2)$, $(f3)$, *and* $(f4)$ *are satisfied, then the functional* I *satisfies* (C) *condition on* E_V.

Proof The proof is analogous to that of Lemma 2.4 in [24], but we give it here for completeness. Let $\{u_n\} \subset E_V$ be a (C) sequence. We divide the proof into two steps.

Step 1. We first show that $\{u_n\}$ is bounded. Supposing the contrary, we can assume that

$$\|u_n\| \to \infty, \qquad I(u_n) \to c, \quad \text{and} \quad \bigl\langle I'(u_n), u_n\bigr\rangle \to 0. \tag{3.4}$$

Define $v_n := \frac{u_n}{\|u_n\|}$. For n large enough, we deduce from (3.4) and $(f4)$ that

$$c + 1 \geq I(u_n) - \frac{1}{2}\bigl\langle I'(u_n), u_n\bigr\rangle = \int_{\mathbb{R}^N} \mathcal{F}(x,u_n)\,dx. \tag{3.5}$$

A direct calculation from (2.4) and (3.4) yields

$$\begin{aligned} \frac{1}{2} &= \frac{I(u_n)}{\|u_n\|^2} + \frac{1}{\|u_n\|^2}\int_{\mathbb{R}^N} F(x,u_n)\,dx \\ &\leq \limsup_{n\to\infty}\left[\frac{I(u_n)}{\|u_n\|^2} + \frac{1}{\|u_n\|^2}\int_{\mathbb{R}^N} \bigl|F(x,u_n)\bigr|\,dx\right] \\ &= \limsup_{n\to\infty}\int_{\mathbb{R}^N} \frac{|F(x,u_n)|}{\|u_n\|^2}\,dx. \end{aligned} \tag{3.6}$$

For $0 \leq a < b$, let $\Lambda_n(a,b) := \{x \in \mathbb{R}^N : a \leq |u_n(x)| < b\}$. Going if necessary to a subsequence, we may assume that $v_n \rightharpoonup v$ in E_V. Then by Theorem 2.1 we have $v_n \to v$ in $L^q(\mathbb{R}^N)$ for $q \in [2, 2_*)$ and $v_n \to v$ a.e. on $\mathbb{R}^N$.

If $v = 0$, then $v_n \to 0$ in $L^q(\mathbb{R}^N)$ for $q \in [2, 2_*)$ and $v_n \to 0$ a.e. on $\mathbb{R}^N$. Hence, it follows from (2.3) and $v_n := \frac{u_n}{\|u_n\|}$ that

$$\begin{aligned}\int_{\Lambda_n(0,r_0)} \frac{|F(x,u_n)|}{\|u_n\|^2}\,dx &= \int_{\Lambda_n(0,r_0)} \frac{|F(x,u_n)|}{|u_n|^2}|v_n|^2\,dx \\ &\le C_4 \int_{\mathbb{R}^N} |v_n|^2\,dx \to 0. \end{aligned} \tag{3.7}$$

Set $\kappa' = \kappa/(\kappa - 1)$. Then $2\kappa' \in (2, 2_*)$. Hence, a direct calculation from (3.5), $(f4)$, and the Hölder inequality shows that

$$\begin{aligned}\int_{\Lambda_n(r_0,\infty)} \frac{|F(x,u_n)|}{\|u_n\|^2}\,dx &= \int_{\Lambda_n(r_0,\infty)} \frac{|F(x,u_n)|}{|u_n|^2}|v_n|^2\,dx \\ &\le \left[\int_{\Lambda_n(r_0,\infty)} \left(\frac{|F(x,u_n)|}{|u_n|^2}\right)^{\kappa} dx\right]^{1/\kappa} \left[\int_{\Lambda_n(r_0,\infty)} |v_n|^{2\kappa'}\,dx\right]^{1/\kappa'} \\ &\le c_2^{\frac{1}{\kappa}} \left[\int_{\mathbb{R}^N} \mathcal{F}(x,u_n)\,dx\right]^{1/\kappa} \left[\int_{\mathbb{R}^N} |v_n|^{2\kappa'}\,dx\right]^{1/\kappa'} \\ &\le c_2^{\frac{1}{\kappa}} [c+1]^{1/\kappa} \left[\int_{\mathbb{R}^N} |v_n|^{2\kappa'}\,dx\right]^{1/\kappa'} \\ &\le C_5 \left[\int_{\mathbb{R}^N} \left(|v_n|^{2\kappa'}\right) dx\right]^{1/\kappa'} \to 0. \end{aligned} \tag{3.8}$$

Hence, combining (3.7) with (3.8) yields that

$$\int_{\mathbb{R}^N} \frac{|F(x,u_n)|}{\|u_n\|^2}\,dx = \int_{\Lambda_n(0,r_0)} \frac{|F(x,u_n)|}{|u_n|^2}|v_n|^2\,dx + \int_{\Lambda_n(r_0,\infty)} \frac{|F(x,u_n)|}{|u_n|^2}|v_n|^2\,dx \to 0,$$

which contradicts (3.6).

If $v \neq 0$ and we set $A := \{x \in \mathbb{R}^N : v(x) \neq 0\}$, then $\operatorname{meas}(A) > 0$. For a.e. $x \in A$, $\lim_{n\to\infty} |u_n(x)| = \infty$. Hence $A \subset \Lambda_n(r_0, \infty)$ for $n \in \mathbb{N}$ large enough. Using (3.4), (3.7), $(f3)$, and the Fatou lemma, it is clear that

$$\begin{aligned} 0 &= \lim_{n\to\infty} \frac{c+o(1)}{\|u_n\|^2} = \lim_{n\to\infty} \frac{I(u_n)}{\|u_n\|^2} = \lim_{n\to\infty} \left[\frac{1}{2} - \int_{\mathbb{R}^N} \frac{F(x,u_n)}{\|u_n\|^2}\,dx\right] \\ &= \lim_{n\to\infty} \left[\frac{1}{2} - \int_{\Lambda_n(0,r_0)} \frac{F(x,u_n)}{u_n^2} v_n^2\,dx - \int_{\Lambda_n(r_0,\infty)} \frac{F(x,u_n)}{u_n^2} v_n^2\,dx\right] \\ &\le \frac{1}{2} + \limsup_{n\to\infty} \int_{\Lambda_n(0,r_0)} \frac{F(x,u_n)}{u_n^2} v_n^2\,dx - \liminf_{n\to\infty} \int_{\Lambda_n(r_0,\infty)} \frac{F(x,u_n)}{u_n^2} v_n^2\,dx \\ &\le C_7 - \liminf_{n\to\infty} \int_{\Lambda_n(r_0,\infty)} \frac{F(x,u_n)}{u_n^2} v_n^2\,dx \\ &\le C_7 - \int_{\mathbb{R}^N} \liminf_{n\to\infty} \frac{F(x,u_n)}{u_n^2} \left[\chi_{\Lambda_n(r_0,\infty)}(x)\right] v_n^2\,dx \to -\infty, \end{aligned} \tag{3.9}$$

which is a contradiction. Thus $\{u_n\}$ is bounded in E_V.

Step 2. We are going to show that $\{u_n\}$ has a convergent subsequence in E_V. Going if necessary to a subsequence, we may assume that $u_n \rightharpoonup u$ in E_V since $\{u_n\}$ is bounded.

Then by Theorem 2.1 we have $v_n \to v$ in $L^q(\mathbb{R}^N)$ for $q \in [2, 2_*)$. By the same arguments as in the proof of Lemma 3.11 in [25] it follows from $(f1)$ and $(f2)$ that

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} \left|f(x,u_n) - f(x,u)\right| |u_n - u| \, dx \to 0 \quad \text{as } n \to \infty. \tag{3.10}$$

Clearly, from (3.4) we have

$$\left\langle I'(u_n) - I'(u), u_n - u \right\rangle \to 0 \quad \text{as } n \to \infty. \tag{3.11}$$

Therefore, applying (3.10) and (3.11), we obtain that

$$\|u_n - u\|^2 = \left\langle I'(u_n) - I'(u), u_n - u \right\rangle + \int_{\mathbb{R}^N} \left[f(x,u_n) - f(x,u)\right](u_n - u) \, dx \to 0$$

as $n \to \infty$. Then clearly $u_n \to u$ in E_V. The proof is completed. □

Lemma 3.4 *Assume that* (V), $(f1)$, $(f2)$, $(f3)$, *and* $(f4)$ *hold. Then for any finite-dimensional subspace* $\widetilde{X} \subset E_V$, *we have*

$$I(u) \to -\infty, \qquad \|u\| \to \infty, \quad u \in \widetilde{X}.$$

Proof Let $V = Y_k := \widetilde{X}$. Arguing indirectly, we may assume that for some sequence $\{u_n\} \subset V$ with $\|u\| \to \infty$, there is $C > 0$ such that $I(u_n) \geq -C$ for all $n \in \mathbb{N}$. Set $v_n := \frac{u_n}{\|u_n\|}$. Then $\|v_n\| = 1$. Passing to a subsequence, we have $v_n \rightharpoonup v$ in V. Since V is finite dimensional, we have $v_n \to v$ in V and $v_n(x) \to v(x)$ a.e. on $\mathbb{R}^N$, and so $\|v\| = 1$. Hence we can get a contradiction by a similar fashion as (3.9). We complete the proof. □

Corollary 3.1 *Assume that* (V), $(f1)$, $(f2)$, $(f3)$, *and* $(f4)$ *hold. Then for any finite-dimensional subspace* $\widetilde{X} \subset E_V$, *there is* $R = R(\widetilde{X}) > 0$ *such that*

$$I(u) \leq 0, \quad u \in \widetilde{X}, \|u\| \geq R.$$

Lemma 3.5 *Assume that* (V), $(f1)$, $(f2)$, *and* $(f3)$ *hold. Then there exist constants* $\rho, \alpha > 0$ *such that* $I|_{\partial B_\rho \cap Z_k} \geq \alpha$.

Proof Taking $0 < \varepsilon < r_2^{-2}$ and using (2.4) and (2.5), we derive that

$$I(u) \geq C_8 \|u\|^2 - C_9 \|u\|^p. \tag{3.12}$$

Set $\rho := (\frac{C_8}{2C_9})^{\frac{1}{p-2}} > 0$ and $\partial B_\rho := \{u \in E_V : \|u\| = \rho\}$. Then, for any $u \in \partial B_\rho \cap Z_k$, we get from (3.12) that

$$I(u) \geq \frac{C_8}{2} \rho^2 := \alpha > 0.$$

Therefore, the proof is completed. □

Lemma 3.6 ([26]) *Let X be a Banach space, let M_0 be a closed subspace of the metric space M, and let $\Gamma_0 \subset C(M_0, X)$. Define*

$$\Gamma = \{\gamma \in C(M, X) : \gamma|_{M_0} \in \Gamma_0\}.$$

If $J \in C^1(X, \mathbb{R})$ satisfies

$$\infty > b := \inf_{\gamma\in\Gamma} \sup_{t\in M} J(\gamma(t)) > a := \sup_{\gamma_0\in\Gamma_0} \sup_{t\in M_0} J(\gamma_0(t)),$$

then there exists a sequence $\{u_n\} \subset X$ satisfying

$$J(u_n) \to b, \qquad \|J'(u_n)\|(1 + \|u_n\|) \to 0.$$

Lemma 3.7 *Assume that (V), $(f1)$, $(f2)$, and $(f6)$ are satisfied. Then*

$$I(u) \ge I(tu) + \frac{1-t^2}{2}\langle I'(u), u\rangle - (K-1)\int_{\mathbb{R}^N} F(x,u)\,dx, \quad \forall u \in E_V, t \in [0, \theta_0]. \tag{3.13}$$

Proof For any $u \in E_V$ and $t \in [0, \theta_0]$, by (2.5), (2.6), and $(f6)$ we easily get that

$$\begin{aligned} I(u) - I(tu) &= \frac{1-t^2}{2}\|u\|^2 + \int_{\mathbb{R}^N} [F(x,tu) - F(x,u)]\,dx \\ &= \frac{1-t^2}{2}\langle I'(u), u\rangle + \int_{\mathbb{R}^N}\left[\frac{1-t^2}{2} f(x,u)u + F(x,tu) - F(x,u)\right]dx \\ &= \frac{1-t^2}{2}\langle I'(u), u\rangle + \int_{\mathbb{R}^N}\left[\frac{1-t^2}{2} f(x,u)u + KF(x,tu) - F(x,u)\right]dx \\ &\quad -(K-1)\int_{\mathbb{R}^N} F(x,u)\,dx \\ &\ge \frac{1-t^2}{2}\langle I'(u), u\rangle - (K-1)\int_{\mathbb{R}^N} F(x,u)\,dx, \end{aligned}$$

which shows that (3.13) holds. This completes the proof. □

Define

$$\Gamma = \{\gamma \in C([0,1], E_V) : \gamma(0) = 0, I(\gamma(1)) < 0\}$$

and

$$b := \inf_{\gamma\in\Gamma} \sup_{t\in[0,1]} I(\gamma(t)).$$

Lemma 3.8 *Assume that (V), $(f1)$, $(f2)$, and $(f6)$ are satisfied. Then there exists a sequence $\{u_n\} \subset E_V$ satisfying*

$$I(u_n) \to b, \qquad \|I'(u_n)\|(1 + \|u_n\|) \to 0 \quad \text{as } n \to \infty. \tag{3.14}$$

Proof To prove this lemma, we apply Lemma 3.6. Let $M = [0,1]$, $M_0 = \{0,1\}$, and

$$\Gamma_0 = \{\gamma_0 : M_0 \to E_V | \gamma_0(0) = 0, I(\gamma_0(t)) < 0\}.$$

For any $0 < \varepsilon < r_2^{-2}$, a direct calculation from (2.4) and (2.5) gives that

$$\begin{aligned} I(u) &\geq \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^N} \left[\frac{\varepsilon}{2}|u|^2 + \frac{C_\varepsilon}{p}|u|^p\right] dx \\ &\geq \frac{1}{2}\|u\|^2 - \frac{\varepsilon r_2^2}{2}\|u\|^2 + \frac{C_\varepsilon r_p^p}{p}\|u\|^p \\ &= \frac{1}{2}\left(1 - \varepsilon r_2^2\right)\|u\|^2 + \frac{C_\varepsilon r_p^p}{p}\|u\|^p, \end{aligned}$$

which implies that there exists $\tau > 0$ such that

$$\max_{\|u\| \geq \tau} I(u) = 0, \qquad \inf_{\|u\| = \tau} I(u) > 0.$$

Thus, we have

$$b \geq \inf_{\|u\|=\tau} I(u) > 0 = \sup_{\gamma_0 \in \Gamma_0} \sup_{t \in M_0} I(\gamma_0(t)).$$

By Lemma 3.6 there exists a sequence $\{u_n\} \in E_V$ satisfying (3.14). We complete the proof. □

Lemma 3.9 *Assume that* (V), $(f1)$, $(f2)$, $(f3)$, $(f4)$, *and* $(f6)$ *are satisfied. Then any sequence* $\{u_n\} \in E_V$ *satisfying* (3.14) *is bounded.*

Proof To prove the lemma, arguing by contradiction, suppose that $\|u_n\| \to \infty$. Set $w_n := u_n/\|u_n\|$. Then $\|w_n\| = 1$. Up to a subsequence, we can assume that $w_n \rightharpoonup w$ in E_V, $w_n \to w$ in $L^p(\mathbb{R}^N)$, $2 \leq p < 2^*$, and $w_n \to w$ almost everywhere on $\mathbb{R}^N$. Let

$$\delta := \limsup_{n\to\infty} \sup_{y \in \mathbb{R}^N} \int_{B_1(y)} |w_n|^p \, dx. \tag{3.15}$$

If $\delta = 0$, then Lions' concentration compactness principle [25] implies that

$$w_n \to 0 \quad \text{in } L^p(\mathbb{R}^N), 2 < p < 2^*. \tag{3.16}$$

Take $\varepsilon = (4Kr_2^2)^{-1} > 0$ in (2.3) and note that $\|w_n\| = 1$. Then it follows from (2.3) and (3.16) that

$$\begin{aligned} \limsup_{n\to\infty} \int_{\mathbb{R}^N} F(x, Rw_n) \, dx &\leq \limsup_{n\to\infty} \int_{\mathbb{R}^N} \left[\varepsilon |Rw_n|^2 + C_\varepsilon |Rw_n|^p\right] dx \\ &\leq \varepsilon R^2 \|w_n\|_2^2 + C_\varepsilon R^p \lim_{n\to\infty} \|w_n\|_p^p \\ &\leq \varepsilon R^2 r_2^2 + C_\varepsilon R^p \lim_{n\to\infty} \|w_n\|_p^p \\ &= \frac{R^2}{4K}. \end{aligned} \tag{3.17}$$

In view of $\|u_n\| \to \infty$, we can see that $\frac{R}{\|u_n\|} \in [0, \theta_0]$ for large $n \in \mathbb{N}$. Fix $R = [4(b+1)]^{\frac{1}{2}}$. Using (3.14), (3.17), and Lemma 3.7 gives

$$\begin{aligned} b + o(1) &= I(w_n) \geq I(Rw_n) + \left(\frac{1}{2} - \frac{R^2}{2\|w_n\|^2}\right)\langle I'(w_n), w_n\rangle - (K-1)\int_{\mathbb{R}^N} F(x, Rw_n)\,dx \\ &= \frac{R^2}{2} - K\int_{\mathbb{R}^N} F(x, Rw_n)\,dx + \left(\frac{1}{2} - \frac{R^2}{2\|w_n\|^2}\right)\langle I'(w_n), w_n\rangle \\ &\geq \frac{R^2}{2} - \frac{R^2}{4} + o(1) \\ &\geq b + 1 + o(1), \end{aligned}$$

which is a contradiction. So, $\delta > 0$. This means that $w \neq 0$. Then by a similar fashion as (3.9) we get a contradiction. Hence, $\{u_n\}$ is bounded in E_V. □

Lemma 3.10 *Assume that* (V), $(f1)$, $(f2)$, $(f3)$, $(f4)$, *and* $(f6)$ *are satisfied. Then problem* (1.1) *has a nontrivial solution, that is,* $\mathcal{M} \neq \emptyset$, *where* $\mathcal{M} := \{u \in E_V : I'(u) = 0, u \neq 0\}$.

Proof Lemma 3.8 and Lemma 3.9 show that there exists a bounded sequence $\{u_n\} \in E_V$ satisfying (3.14). So, we can assume that $u_n \rightharpoonup u$ in E_V. Then by Theorem 2.1 we have $v_n \to v$ in $L^q(\mathbb{R}^N)$ for $q \in [2, 2_*)$. Set

$$\bar{\delta} := \limsup_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_1(y)} |u_n|^p\,dx = 0.$$

Then we can easily get that $\bar{\delta} > 0$ by the similar argument as proving that $\delta > 0$, where δ is defined in (3.15). This means that $u \neq 0$. As in Step 2 in Lemma 3.3, for every $w \in C_0^\infty(\mathbb{R}^N)$, we have

$$\langle I'(u), w\rangle = \lim_{n\to\infty}\langle I'(u_n), w\rangle = 0. \tag{3.18}$$

Therefore, $I'(u) = 0$, which shows that $u \in \mathcal{M}$ is a nontrivial solution of problem (1.1). This ends the proof. □

Now, we give the proofs of Theorems 1.1, 1.2, and 1.3.

Proof of Theorem 1.1 From Lemma 3.1 and Lemma 3.2, for some $k \in \mathbb{N}$, we have $C_k(I, \infty) \ncong C_k(I, 0)$. Then Theorem 1.1 follows from Lemma 3.3 and Remark 2.1. □

Proof of Theorem 1.2 Lemma 3.10 shows that $\mathcal{M} \neq \emptyset$. So, we can let

$$c_0 = \inf_{\mathcal{M}} I_\lambda(u).$$

By Lemma 3.7 we have $I(u) \geq I(0) = 0$ for all $u \in \mathcal{M}$. Hence, $c_0 \geq 0$. On the other hand, Lemma 3.8 and Lemma 3.9 show that there exists a bounded sequence $\{u_n\} \in E_V$ satisfying (3.14). So we can assume that $u_n \rightharpoonup u$ in E_V. Then by Theorem 2.1 we have $v_n \to v$ in

$L^q(\mathbb{R}^N)$ for $q \in [2, 2_*)$. A direct calculation from (2.5), (2.6), $(f4)$, and Fatou's lemma yields that

$$\begin{aligned} c_0 &= \lim_{n\to\infty}\left[I(u_n) - \frac{1}{2}\langle I'(u_n), u_n\rangle\right] \\ &= \lim_{n\to\infty}\int_{\mathbb{R}^N}\left[\frac{1}{2}f(x,u_n)u_n - F(x,u_n)\right]dx \\ &\geq \int_{\mathbb{R}^N}\lim_{n\to\infty}\left[\frac{1}{2}f(x,u_n)u_n - F(x,u_n)\right]dx \\ &= \int_{\mathbb{R}^N}\left[\frac{1}{2}f(x,u)u - F(x,u)\right]dx \\ &= I(u) - \frac{1}{2}\langle I'(u), u\rangle \\ &= I(u), \end{aligned}$$

which shows that $I(u) \leq c_0$.

Therefore, $I(u) = c_0 = \inf_{\mathcal{M}} I_\lambda(u)$, which means that u is the ground state solution of problem (1.1). The proof is completed. □

Proof of Theorem 1.3 Let $V := Y_k$ and $W := Z_k$. By (2.5), the definition of $F(x,u)$, and the fact that $f(x,u)$ is an odd function with respect to u we have that (I_1) of Proposition 2.2 holds. By Lemma 3.3 the functional I satisfies $(C)_c$ condition. Corollary 3.1 and Lemma 3.5 show that conditions (I_2) and (I_3) of Proposition 2.2 are satisfied. Hence, we complete the proof of Theorem 1.2. □

4 Conclusions

In this paper, we established some existence results for a class of biharmonic equations with singular potentials in whole space by employing the Morse theory and variational methods. We significantly extended and complemented some results from the previous literature.

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