Boundary Value Problems
a SpringerOpen Journal

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Dynamical behavior and bifurcation analysis of a homogeneous reaction–diffusion Atkinson system

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Abstract

In this paper, we are concerned with a homogeneous reaction–diffusion Atkinson oscillator system subject to homogeneous Neumann boundary conditions on a bounded spatial domain. Using the comparison principle and the techniques of invariant rectangle, we prove the existence of the attraction region of the solutions. We thus prove that under certain conditions, the solutions of the PDE system converge to the unique positive equilibrium solutions. We also derive precise conditions such that the system does not have nonconstant positive steady-state solutions. Finally, we use the bifurcation technique to show the existence of Turing patterns. The results provide a clearer understanding of the mechanism of formations of patterns.

MSC: 35J25; 34C23; 92E20

Keywords: Reaction–diffusion Atkinson model; Asymptotic behavior; Nonexistence of positive nonconstant steady states; Steady-state bifurcations

1 Introduction

In 1952, Turing [1] proposed a famous idea of "diffusion-induced instability," which says that the destabilization of otherwise stable constant steady state will lead to the emergence of stable nonuniform spatial structures, which are usually called Turing patterns. Over the years, Turing's idea has attracted the attention of a great number of investigators and was successfully developed on theoretical backgrounds. In 1990, De Kepper et al. [2, 3] discovered the formation of stationary three-dimensional (but almost two-dimensional) structures with characteristic wavelengths of 0.2 mm, which is the first experimental evidence to the Turing patterns nearly 40 years after the publication of [1]. Since then, the research of Turing patterns in chemistry has sprung up (see [4–13] and the references therein).

In this paper, we are concerned with the pattern formations of a kind of diffusive Atkinson oscillator model, which is used to characterize the mechanism of pattern formations

of the oscillators [14]:

$$
\begin{cases}
\frac{du}{dt} = w - u, \\
\frac{dv}{dt} = \beta_4 (s - v), \\
\frac{dw}{dt} = \beta_1 (\lambda_1 (1 + \alpha_1 \frac{v^{n_1}}{1 + v^{n_1}}) - w), \\
\frac{ds}{dt} = \beta_3 (\frac{\lambda_2}{1 + u} (1 + \alpha_2 \frac{(v/a)^{n_2}}{1 + (v/a)^{n_2}}) - s).
\end{cases}
$$

To analyze the generation mechanism of periodic solutions, it is assumed that RNA transcripts are fast process, that is, we assume that $dw/dt = ds/dt = 0$. Then the system can be simplified to the second-order planar system:

$$
\frac{du}{dt} = \lambda_1 \left(1 + \alpha_1 \frac{v^{n_1}}{1 + v^{n_1}} \right) - u, \qquad \frac{dv}{dt} = \frac{\lambda_2}{1 + u} \left(1 + \alpha_2 \frac{(v/a)^{n_2}}{1 + (v/a)^{n_2}} \right) - v. \tag{1.1}
$$

Since the chemical reaction obeys a diffusion process, it is natural to add diffusion to the model (1.1), which leads to the following reaction–diffusion system:

$$
\begin{cases}
\frac{\partial u}{\partial t} - d_1 \Delta u = \lambda_1 (1 + \alpha_1 \frac{v^{n_1}}{1 + v^{n_1}}) - u, & t > 0, x \in \Omega, \\
\frac{\partial v}{\partial t} - d_2 \Delta v = \frac{\lambda_2}{1 + u} (1 + \alpha_2 \frac{(v/a)^{n_2}}{1 + (v/a)^{n_2}}) - v, & t > 0, x \in \Omega, \\
\partial_\nu u = \partial_\nu u = 0, & t \geq 0, x \in \partial\Omega, \\
u(x, 0) = u_0(x) > 0, \qquad v(x, 0) = v_0(x) > 0, & x \in \Omega.
\end{cases} \tag{1.2}
$$

The model has been studied extensively by several authors, but most of the research focuses on the corresponding ODE system (1.1). To the best of knowledge, there are few works in the existing literature studying the dynamical behavior and bifurcations of the corresponding reaction–diffusion equations. In this paper, we attempt to study the Atkinson PDE model. We are concerned with the global existence, boundedness of time-independent solutions, the asymptotical behavior of the constant steady states, and the bifurcations from the positive steady-state solutions. In particular, we derive precise conditions under which system (1.2) does not have Turing patterns.

This paper is organized as follows. In Section 2, we study the boundedness and uniqueness of global-in-time solutions of system (1.2). In particular, we show that there exists an invariant rectangle that attracts all the solutions of system (1.2) regardless of the initial values. Then, we consider the long-time behavior of the solutions of system (1.2) and derive precise conditions under which the solutions converge exponentially to a unique constant steady-state solution. In Section 3, we derive conditions under which system (1.2) has no nonconstant positive steady states, including Turing patterns. In Section 4, we use global bifurcation theory to prove the existence of Turing patterns.

2 Global existence, boundedness, and asymptotical behavior of solutions

In this section, we consider the global existence, boundedness, and asymptotical behavior of the solutions. To begin with, it is clear that system (1.2) has a unique positive steady-state solution (u_*, v_*) of the system

$$
u_* = \lambda_1 \left(1 + \alpha_1 \frac{v_*^{n_1}}{1 + v_*^{n_1}} \right), \qquad \lambda_1 \left(1 + \alpha_1 \frac{v_*^{n_1}}{1 + v_*^{n_1}} \right) = \frac{\lambda_2}{v_*} \left(1 + \alpha_2 \frac{(v_*/a)^{n_2}}{1 + (v_*/a)^{n_2}} \right) - 1. \tag{2.1}
$$

We have the following results on the global existence and boundedness of the time-dependent solutions of system (1.2).

Proposition 2.1 *Suppose that $a, \alpha_1, \alpha_2, \lambda_1, \lambda_2 > 0$. For any $d_1, d_2 > 0$, the initial boundary value problem* (1.2) *admits a unique solution $(u(x,t), v(x,t))$ for all $x \in \Omega$ and $t > 0$. Moreover, there exist two positive constants M_1 and M_2, depending on $\alpha_1, \alpha_2, \lambda_1, \lambda_2, u_0(x)$, and $v_0(x)$, such that*

$$M_1 < u(x,t), \qquad v(x,t) < M_2, \quad t > 0, x \in \overline{\Omega}. \tag{2.2}$$

Proof The existence and uniqueness of local-in-time solutions to the initial-boundary value problem (1.2) is well known [15].

For the global existence and the boundedness of the solutions, we partially use the techniques of invariant region [10, 11, 16]. A region $\Re := [U_1, U_2] \times [V_1, V_2]$ in the (u, v) phase plane is called a positively invariant region of system (1.2) if the vector field

$$\left(\lambda_1 \left(1 + \alpha_1 \frac{v^{n_1}}{1 + v^{n_1}} \right) - u, \frac{\lambda_2}{1 + u} \left(1 + \alpha_2 \frac{(v/a)^{n_2}}{1 + (v/a)^{n_2}} \right) - v \right) \tag{2.3}$$

points inward on the boundary of $\Re$ for all $t \geq 0$. We construct the invariant rectangle $\Re := [U_1, U_2] \times [V_1, V_2]$ in the following way:

$$U_1 := \min \left\{ \lambda_1, \min_{x \in \overline{\Omega}} u_0(x) \right\}, \qquad U_2 := \max \left\{ \lambda_1(1 + \alpha_1), \max_{x \in \overline{\Omega}} u_0(x) \right\},$$

$$V_1 := \min \left\{ \frac{\lambda_2}{1 + U_2}, \min_{x \in \overline{\Omega}} v_0(x) \right\}, \qquad V_2 := \max \left\{ \frac{\lambda_2(1 + \alpha_2)}{1 + U_1}, \max_{x \in \overline{\Omega}} v_0(x) \right\}. \tag{2.4}$$

Obviously, $u_0(x)$ and $v_0(x)$ are closed by the rectangle $\Re$. Now we prove that the vector field points inward on the boundary of $\Re$.

On $u = U_1, V_1 \leq v \leq V_2$, by the definition of U_1 we have

$$\lambda_1 \left(1 + \alpha_1 \frac{v^{n_1}}{1 + v^{n_1}} \right) - u = \lambda_1 \left(1 + \alpha_1 \frac{v^{n_1}}{1 + v^{n_1}} \right) - U_1 > \lambda_1 - U_1 \geq 0.$$

On $u = U_2, V_1 \leq v \leq V_2$, by the definition of U_2 we have

$$\lambda_1 \left(1 + \alpha_1 \frac{v^{n_1}}{1 + v^{n_1}} \right) - u = \lambda_1 \left(1 + \alpha_1 \frac{v^{n_1}}{1 + v^{n_1}} \right) - U_2 < \lambda_1(1 + \alpha_1) - U_2 \leq 0.$$

On $v = V_1, U_1 \leq u \leq U_2$, by the definition of V_1 we have

$$\frac{\lambda_2}{1 + u} \left(1 + \alpha_2 \frac{(v/a)^{n_2}}{1 + (v/a)^{n_2}} \right) - v = \frac{\lambda_2}{1 + u} \left(1 + \alpha_2 \frac{(V_1/a)^{n_2}}{1 + (V_1/a)^{n_2}} \right) - V_1 > \frac{\lambda_2}{1 + U_2} - V_1 \geq 0.$$

On $v = V_2, U_1 \leq u \leq U_2$, by the definition of V_2 we have

$$\frac{\lambda_2}{1 + u} \left(1 + \alpha_2 \frac{(v/a)^{n_2}}{1 + (v/a)^{n_2}} \right) - v = \frac{\lambda_2}{1 + u} \left(1 + \alpha_2 \frac{(V_2/a)^{n_2}}{1 + (V_2/a)^{n_2}} \right) - V_2 < \frac{\lambda_2(1 + \alpha_2)}{1 + U_1} - V_2 \leq 0.$$

Thus, $\Re := [U_1, U_2] \times [V_1, V_2]$ is an invariant rectangle for the vector field, where $M_1 := \min\{U_1, V_1\}$ and $M_2 := \max\{U_2, V_2\}$. □

We now prove that system (1.2) has an attraction region, which in fact attracts all solutions of the system, regardless of the initial values [17].

Theorem 2.2 *Suppose that $(u(x,t), v(x,t))$ is the unique solution of system* (1.2). *For any* $x \in \overline{\Omega}$, *we have*:

$$
\begin{aligned}
\lambda_1 &< \liminf_{t\to\infty} u \leq \limsup_{t\to\infty} u < \lambda_1(1+\alpha_1); \\
\frac{\lambda_2}{1+\lambda_1(1+\alpha_1)} &< \liminf_{t\to\infty} v \leq \limsup_{t\to\infty} v < \frac{\lambda_2(1+\alpha_2)}{1+\lambda_1}.
\end{aligned}
\tag{2.5}
$$

Proof (a). Let us prove that $\lambda_1 < \liminf_{t\to\infty} u$. By Proposition 2.1 there exists a sufficiently small number $0 < \varepsilon < \lambda_1\alpha_1 v^{n_1}/(1+v^{n_1})$ for all $x \in \overline{\Omega}$ and $t > 0$. Let u_1 be the unique solution of the following ODE:

$$
\frac{du_1(t)}{dt} = \lambda_1 + \varepsilon - u_1(t), \qquad u_1(0) = (1-\varepsilon)\min_{x\in\overline{\Omega}} u_0(x).
\tag{2.6}
$$

Setting $\rho_1(x,t) = u(x,t) - u_1(t)$, we have

$$
\begin{aligned}
\frac{\partial\rho_1(x,t)}{\partial t} &- d_1\Delta\rho_1(x,t) + \rho_1(x,t) = \lambda_1\alpha_1\frac{v^{n_1}}{1+v^{n_1}} - \varepsilon > 0, \\
\rho_1(x,0) &= u(x,0) - (1-\varepsilon)\min_{x\in\overline{\Omega}} u_0(x) > 0.
\end{aligned}
\tag{2.7}
$$

By the maximum principle for parabolic equations, we have $\rho_1(x,t) > 0$, which means that $u(x,t) > u_1(t)$ for all $x \in \overline{\Omega}$ and $t > 0$. From (2.6) we have $\lim_{t\to\infty} u_1(t) = \lambda_1 + \varepsilon$.
Thus,

$$
\liminf_{t\to\infty} u \geq \lim_{t\to\infty} u_1(t) > \lambda_1.
\tag{2.8}
$$

(b). Let us now prove that $\limsup_{t\to\infty} u < \lambda_1(1+\alpha_1)$. By Proposition 2.1 there exists a sufficiently small number $0 < \eta < \lambda_1\alpha_1/(1+v^{n_1})$ for all $x \in \overline{\Omega}$ and $t > 0$. Let u_2 be the unique solution of the following ODE:

$$
\frac{du_2(t)}{dt} = \lambda_1(1+\alpha_1) - \eta - u_2(t), \qquad u_2(0) = (1+\eta)\max_{x\in\overline{\Omega}} u_0(x).
\tag{2.9}
$$

Setting $\rho_2(x,t) = u(x,t) - u_2(t)$, we have

$$
\begin{aligned}
\frac{\partial\rho_2(x,t)}{\partial t} &- d_1\Delta\rho_2(x,t) + \rho_2(x,t) = -\frac{\lambda_1\alpha_1}{1+v^{n_1}} + \eta < 0, \\
\rho_2(x,0) &= u(x,0) - (1+\eta)\max_{x\in\overline{\Omega}} u_0(x) < 0.
\end{aligned}
\tag{2.10}
$$

By the maximum principle for parabolic equations we have $\rho_2(x,t) < 0$, which means that $u(x,t) < u_2(t)$ for all $x \in \overline{\Omega}$ and $t > 0$. From (2.9) we have $\lim_{t\to\infty} u_2(t) = \lambda_1(1+\alpha_1) - \eta$.
Thus,

$$
\limsup_{t\to\infty} u < \lambda_1(1+\alpha_1).
\tag{2.11}
$$

(c). Let us prove that $\frac{\lambda_2}{1+\lambda_1(1+\alpha_1)} < \liminf_{t\to\infty} v$. By Proposition 2.1 there exists a sufficiently small number $\varphi > 0$ such that $\varphi < (1 + \lambda_1(1 + \alpha_1))v$ for all $x \in \overline{\Omega}$ and $t > 0$. Let v_1 be the unique solution of the following ODE:

$$\frac{dv_1(t)}{dt} = \lambda_2 + \varphi - \big(1 + \lambda_1(1+\alpha_1)\big)v_1(t), \qquad v_1(0) = (1-\varphi)\min_{x\in\overline{\Omega}} v_0(x). \tag{2.12}$$

Setting $\xi_1(x,t) = v(x,t) - v_1(t)$, we have

$$\begin{aligned}
&\frac{\partial \xi_1(x,t)}{\partial t} - d_2 \Delta \xi_1(x,t) + \big(1 + \lambda_1(1+\alpha_1)\big)\xi_1(x,t) \\
&\quad = \frac{\lambda_2}{1+u} - \varphi + \big(1 + \lambda_1(1+\alpha_1)\big)v + \frac{\lambda_2\alpha_2}{1+u}\cdot\frac{(v/a)^{n_2}}{1+(v/a)^{n_2}} \\
&\quad > -\varphi + \big(1 + \lambda_1(1+\alpha_1)\big)v > 0, \\
&\xi_1(x,0) = v(x,0) - (1-\varphi)\min_{x\in\overline{\Omega}} v_0(x) > 0.
\end{aligned} \tag{2.13}$$

By the maximum principle for parabolic equations we have $\xi_1(x,t) > 0$, which means that $v(x,t) > v_1(t)$ for all $x \in \overline{\Omega}$ and $t > 0$. From (2.12) we have $\lim_{t\to\infty} v_1(t) = \frac{\lambda_2+\varphi}{(1+\lambda_1(1+\alpha_1))}$.
Thus,

$$\lim_{t\to\infty}\inf v > \frac{\lambda_2}{1+\lambda_1(1+\alpha_1)}. \tag{2.14}$$

(d). Let us prove that $\limsup_{t\to\infty} v < \frac{\lambda_2(1+\alpha_2)}{1+\lambda_1}$. Since

$$0 < \lambda_1 < \lim_{t\to\infty}\inf u \quad \text{or, equivalently,} \quad \frac{1}{1+\lambda_1} > \lim_{t\to\infty}\sup \frac{1}{1+u} > 0, \tag{2.15}$$

there exists a finite number t_0, depending on (u_0, v_0), such that, for any $x \in \overline{\Omega}$ and $t \geq t_0$,

$$\frac{1}{1+\lambda_1} > \frac{1}{1+u}. \tag{2.16}$$

By Proposition 2.1 and (2.16) there exists a sufficiently small number $\delta > 0$ such that, for all $x \in \overline{\Omega}$ and $t \geq t_0$, we have

$$\frac{1}{1+\lambda_1} - \frac{\delta}{\lambda_2(1+\lambda_1)(1+\alpha_2)} > \frac{1}{1+u}. \tag{2.17}$$

Let v_2 be the unique solution of the following ODE:

$$\frac{dv_2(t)}{dt} = \frac{\lambda_2(1+\alpha_2) - \delta}{(1+\lambda_1)} - v_2(t), \quad t > t_0, \qquad v_2(t_0) = (1+\delta)\max_{x\in\overline{\Omega}} v_0(x,t_0). \tag{2.18}$$

Setting $\xi_2(x,t) = v(x,t) - v_2(t)$, by (2.17) and (2.18) we have

$$
\begin{aligned}
\frac{\partial \xi_2(x,t)}{\partial t} &- d_2 \Delta \xi_2(x,t) + \xi_2(x,t) \\
&= \frac{\lambda_2}{1+u}\left(1 + \alpha_2 \frac{(v/a)^{n_2}}{1+(v/a)^{n_2}}\right) - \frac{\lambda_2(1+\alpha_2) - \delta}{(1+\lambda_1)} \\
&< \frac{\lambda_2}{1+u}(1+\alpha_2) - \frac{\lambda_2(1+\alpha_2) - \delta}{1+\lambda_1} < 0, \\
\xi_2(x,t_0) &= v(x,t_0) - (1+\delta)\max_{x\in\overline{\Omega}} v_0(x,t_0) < 0, \quad t > t_0,.
\end{aligned}
\tag{2.19}
$$

By the maximum principle for parabolic equations we have $\xi_2(x,t) < 0$, which means that $v(x,t) < v_2(t)$ for all $x \in \overline{\Omega}$ and $t > t_0$. From (2.18) we have $\lim_{t\to\infty} v_2(t) = \frac{\lambda_2(1+\alpha_2)-\delta}{1+\lambda_1}$.

Thus, we have proved that

$$
\limsup_{t\to\infty} v < \frac{\lambda_2(1+\alpha_2)}{1+\lambda_1}.
\tag{2.20}
$$
□

Next, in the particular case, we derive precise conditions under which (u_*, v_*) is a globally asymptotically stable solution of the corresponding ODEs of system (1.2) and that all the solutions of (1.2) tend to (u_*, v_*) [18]:

Theorem 2.3 *Define*

$$
H_1 := \frac{\lambda_2\alpha_2}{a}, \qquad H_2 := \frac{a\lambda_2\alpha_2(1+\lambda_1)}{(1+\alpha_1)[a(1+\lambda_1)+\lambda_2(1+\alpha_2)]^2}.
\tag{2.21}
$$

Suppose that $n_1 = n_2 = 1$ and that either $H_1 < 2$ or $H_2 > 2$. Then, (u_*, v_*) is a globally asymptotically stable solution of the corresponding ODEs of system (1.2). If, additionally, $\max\{1, d\} > M/\lambda_1$, then every solution of system (1.2) converges exponentially to the unique constant equilibrium solution (u_*, v_*), where μ_1 is the smallest positive eigenvalue of $-\Delta$ on Ω subject to homogeneous Neumann boundary conditions, and

$$
M := \max\left\{\lambda_1\alpha_1 + 1, \frac{\lambda_2^2(1+\alpha_2)^2(2+\lambda_1)}{a} + 1\right\}.
\tag{2.22}
$$

Proof Consider the following ODEs in the particular case $n_1 = n_2 = 1$:

$$
\begin{aligned}
\frac{du}{dt} &= \lambda_1\left(1 + \alpha_1\frac{v}{1+v}\right) - u =: f(u,v), \\
\frac{dv}{dt} &= \frac{\lambda_2}{1+u}\left(1 + \alpha_2\frac{v}{a+v}\right) - v =: g(u,v).
\end{aligned}
\tag{2.23}
$$

The Jacobian matrix $J(u,v)$ at (u,v) is given by

$$
J(u,v) = \begin{pmatrix} -1 & \frac{\lambda_1\alpha_1}{(1+v)^2} \\ -\frac{\lambda_2(1+v+\alpha_2 v)}{(a+v)(1+u)^2} & \frac{a\lambda_2\alpha_2}{(1+u)(a+v)^2} - 1 \end{pmatrix}.
\tag{2.24}
$$

Then, we have

$$\frac{\partial f}{\partial u} + \frac{\partial g}{\partial v} = \frac{a\lambda_2\alpha_2}{(1 + u)(a + v)^2} - 2. \tag{2.25}$$

By (2.5) we have

$$\frac{a\lambda_2\alpha_2}{(1 + u)(a + v)^2} < \frac{a\lambda_2\alpha_2}{(1 + \lambda_1)[a + \frac{\lambda_2}{1+\lambda_1+\lambda_1\alpha_1}]^2} < H_1 := \frac{\lambda_2\alpha_2}{a}. \tag{2.26}$$

On the other hand,

$$\frac{a\lambda_2\alpha_2}{(1 + u)(a + v)^2} > \frac{a\lambda_2\alpha_2}{(1 + \lambda_1(1 + \alpha_1))[a + \frac{\lambda_2(1+\alpha_2)}{1+\lambda_1}]^2}$$

$$> H_2 := \frac{a\lambda_2\alpha_2(1 + \lambda_1)}{(1 + \alpha_1)[a(1 + \lambda_1) + \lambda_2(1 + \alpha_2)]^2}. \tag{2.27}$$

If either $H_1 < 2$ or either $H_2 > 2$, then we have $\frac{\partial f}{\partial u} + \frac{\partial g}{\partial v} < 0$ (or > 0). Hence, the Poincaré–Bendixson theorem implies that (u_*, v_*) is globally asymptotically stable in (2.23). Define

$$Q := \sup_{(u,v)\in R} \|J(u, v)\|. \tag{2.28}$$

By (2.5) and (2.24) we have

$$Q := \max\left\{ \sup_{(u,v)\in R}\left(1 + \left|\frac{\lambda_1\alpha_1}{(1 + v)^2}\right|\right), \sup_{(u,v)\in R}\left(\left|\frac{\lambda_2(1 + v + \alpha_2 v)}{(a + v)(1 + u)^2}\right| + \left|\frac{a\lambda_2\alpha_2}{(1 + u)(a + v)^2}\right| + 1\right)\right\}$$

$$< M := \max\left\{\lambda_1\alpha_1 + 1, \frac{\lambda_2^2(1 + \alpha_2)^2(2 + \lambda_1)}{a} + 1\right\}. \tag{2.29}$$

By [19], if $\max\{1, d\} > M/\mu_1$, where $d = \min\{d_1, d_2\}$, then every solution of system (1.2) converges exponentially to the unique constant equilibrium solution (u_*, v_*). $\qquad\square$

3 Nonexistence of Turing patterns

In this section, for the particular case where both n_1 and n_2 equal 1, we show the nonexistence of nonconstant positive steady-state solutions of the system

$$\begin{cases} -d_1\Delta u = \lambda_1(1 + \alpha_1\frac{v}{1+v}) - u, & x \in \Omega, \\ -d_2\Delta v = \frac{\lambda_2}{1+u}(1 + \alpha_2\frac{v}{a+v}) - v, & x \in \Omega, \\ \partial_\nu u = \partial_\nu u = 0, & x \in \partial\Omega. \end{cases} \tag{3.1}$$

Lemma 3.1 *Suppose that $(u(x), v(x))$ is any given positive steady-state solution of system* (1.2). *For any $x \in \overline{\Omega}$, we have*:

$$\lambda_1 < u(x) < \lambda_1(1 + \alpha_1), \qquad \frac{\lambda_2}{1 + \lambda_1(1 + \alpha_1)} < v(x) < \frac{\lambda_2(1 + \alpha_2)}{1 + \lambda_1}. \tag{3.2}$$

The lemma is the direct consequence of Theorem 2.1 in [17].

For a steady-state solution pair $(u(x), v(x))$ of system (3.1), we define

$$\bar{u} = \frac{1}{|\Omega|} \int_\Omega u(x)\,dx, \qquad \bar{v} = \frac{1}{|\Omega|} \int_\Omega v(x)\,dx, \tag{3.3}$$

$$\phi(x) = u(x) - \bar{u}, \qquad \psi(x) = v(x) - \bar{v}, \tag{3.4}$$

$$\begin{cases} g_1(u,v) := \frac{1}{(1+\bar{v})(1+v)}, \\ g_2(u,v) := \frac{1}{(1+u)(a+v)} - \frac{\bar{v}}{(1+\bar{u})(a+\bar{v})(a+v)} - \frac{1}{\lambda_2\alpha_2}, \\ g_3(u,v) := \frac{1}{(1+\bar{u})(1+u)}\left(1 + \frac{\alpha_2\bar{v}}{a+v}\right). \end{cases} \tag{3.5}$$

Lemma 3.1 shows that any positive solution (u,v) of system (3.1) satisfies $(u,v) \in R$, where

$$R := \left(\lambda_1, \lambda_1(1+\alpha_1)\right) \times \left(\frac{\lambda_2}{1 + \lambda_1(1+\alpha_1)}, \frac{\lambda_2(1+\alpha_2)}{1+\lambda_1}\right). \tag{3.6}$$

Define

$$G_i := \sup_{(u,v)\in R} \left|g_i(u,v)\right|, \quad i = 1,2,3. \tag{3.7}$$

We now state the following theorem regarding the nonexistence of a nonconstant positive solution of system (3.1).

Lemma 3.2 *Let $g_i(u,v)$ and G_i, $i = 1,2,3$, be defined as in (3.5) and (3.7). Suppose that $d_1 > \frac{1}{\mu_1}$ and $d_2 > \frac{\lambda_2\alpha_2 G_2}{\mu_1}$. Then, we have*

$$\left(\frac{d_1\mu_1 - 1}{\lambda_1\alpha_1 G_1}\right)^2 \int_\Omega |\nabla\phi|^2\,dx \le \int_\Omega |\nabla\psi|^2\,dx \le \left(\frac{\lambda_2 G_3}{d_2\mu_1 - \lambda_2\alpha_2 G_2}\right)^2 \int_\Omega |\nabla\phi|^2\,dx. \tag{3.8}$$

Proof Multiplying the first equation of (3.1) by ϕ and integrating over Ω, we have

$$-\int_\Omega d_1\phi\Delta\phi\,dx = \lambda_1 \int_\Omega \phi\,dx + \lambda_1\alpha_1 \int_\Omega \frac{v}{1+v}\phi\,dx - \int_\Omega \phi u\,dx$$

$$= -\lambda_1\alpha_1 \int_\Omega \frac{1}{1+v}\phi\,dx - \int_\Omega \phi^2\,dx. \tag{3.9}$$

By Green's formula we have

$$d_1 \int_\Omega |\nabla\phi|^2\,dx$$

$$= -\lambda_1\alpha_1 \int_\Omega \frac{1}{1+v}\phi\,dx - \int_\Omega \phi^2\,dx = -\lambda_1\alpha_1 \int_\Omega \left(\frac{1}{1+v} - \frac{1}{1+\bar{v}}\right)\phi\,dx - \int_\Omega \phi^2\,dx$$

$$= \lambda_1\alpha_1 \int_\Omega \frac{\phi\psi}{(1+\bar{v})(1+v)}\,dx - \int_\Omega \phi^2\,dx = \lambda_1\alpha_1 \int_\Omega g_1(v)\phi\psi\,dx - \int_\Omega \phi^2\,dx$$

$$\le \lambda_1\alpha_1 G_1 \int_\Omega |\phi\psi|\,dx + \int_\Omega \phi^2\,dx \le \lambda_1\alpha_1 G_1 \int_\Omega |\phi\psi|\,dx + \frac{1}{\mu_1} \int_\Omega |\nabla\phi|^2\,dx. \tag{3.10}$$

On the other hand, we have

$$\int_\Omega |\phi\psi|\, dx \le \left(\int_\Omega |\phi|^2\, dx\right)^{\frac{1}{2}} \left(\int_\Omega |\psi|^2\, dx\right)^{\frac{1}{2}}$$

$$\le \left(\frac{1}{\mu_1}\int_\Omega |\nabla\phi|^2\, dx\right)^{\frac{1}{2}} \left(\frac{1}{\mu_1}\int_\Omega |\nabla\psi|^2\, dx\right)^{\frac{1}{2}}$$

$$= \frac{1}{\mu_1}\left(\int_\Omega |\nabla\phi|^2\, dx\right)^{\frac{1}{2}} \left(\int_\Omega |\nabla\psi|^2\, dx\right)^{\frac{1}{2}}. \tag{3.11}$$

Substituting (3.11) into (3.10), we have

$$d_1 \int_\Omega |\nabla\phi|^2\, dx \le \frac{\lambda_1\alpha_1 G_1}{\mu_1}\left(\int_\Omega |\nabla\phi|^2\, dx\right)^{\frac{1}{2}} \left(\int_\Omega |\nabla\psi|^2\, dx\right)^{\frac{1}{2}} + \frac{1}{\mu_1}\int_\Omega |\nabla\phi|^2\, dx. \tag{3.12}$$

Thus,

$$\left(d_1 - \frac{1}{\mu_1}\right)\int_\Omega |\nabla\phi|^2\, dx \le \frac{\lambda_1\alpha_1 G_1}{\mu_1}\left(\int_\Omega |\nabla\phi|^2\, dx\right)^{\frac{1}{2}} \left(\int_\Omega |\nabla\psi|^2\, dx\right)^{\frac{1}{2}}. \tag{3.13}$$

Since $d_1 > \frac{1}{\mu_1}$, we have

$$\left(d_1 - \frac{1}{\mu_1}\right)^2 \int_\Omega |\nabla\phi|^2\, dx \le \left(\frac{\lambda_1\alpha_1 G_1}{\mu_1}\right)^2 \int_\Omega |\nabla\psi|^2\, dx. \tag{3.14}$$

Thus,

$$\left(\frac{d_1\mu_1 - 1}{\lambda_1\alpha_1 G_1}\right)^2 \int_\Omega |\nabla\phi|^2\, dx \le \int_\Omega |\nabla\psi|^2\, dx. \tag{3.15}$$

Multiplying the second equation of (3.1) by ψ and integrating over Ω, we have

$$d_2 \int_\Omega |\nabla\psi|^2\, dx = \lambda_2 \int_\Omega \frac{\psi}{1+u}\, dx + \lambda_2\alpha_2 \int_\Omega \frac{v\psi}{(1+u)(a+v)}\, dx - \int_\Omega v\psi\, dx$$

$$= -\frac{\lambda_2}{1+\overline{u}}\int_\Omega \frac{\phi\psi}{1+u}\, dx + \lambda_2\alpha_2 \int_\Omega \frac{\psi^2}{(1+u)(a+v)}\, dx$$

$$+ \lambda_2\alpha_2\overline{v} \int_\Omega \frac{\psi}{(1+u)(a+v)}\, dx - \int_\Omega \psi^2\, dx. \tag{3.16}$$

Furthermore,

$$\int_\Omega \frac{\psi}{(1+u)(a+v)}\, dx = \int_\Omega \frac{\psi}{(1+u)(a+v)}\, dx - \int_\Omega \frac{\psi}{(1+\overline{u})(a+\overline{v})}\, dx$$

$$= -\int_\Omega \frac{(a+v)\phi + (1+u)\psi - \phi\psi}{(1+u)(a+v)(1+\overline{u})(a+\overline{v})}\psi\, dx$$

$$= -\frac{1}{(1+\overline{u})(a+\overline{v})}\int_\Omega \frac{(a+\overline{v})\phi\psi + (1+u)\psi^2}{(1+u)(a+v)}\, dx. \tag{3.17}$$

By (3.16) and (3.17) we have

$$
d_2 \int_\Omega |\nabla \psi|^2 \, dx
$$

$$
= \lambda_2 \alpha_2 \int_\Omega g_2(u,v)\psi^2 \, dx - \lambda_2 \int_\Omega g_3(u,v)\phi\psi \, dx
$$

$$
\leq \lambda_2 \alpha_2 G_2 \int_\Omega \psi^2 \, dx + \lambda_2 G_3 \int_\Omega |\phi\psi| \, dx
$$

$$
\leq \frac{\lambda_2 \alpha_2 G_2}{\mu_1} \int_\Omega |\nabla \psi|^2 \, dx + \frac{\lambda_2 G_3}{\mu_1} \left(\int_\Omega |\nabla \phi|^2 \, dx \right)^{\frac{1}{2}} \left(\int_\Omega |\nabla \psi|^2 \, dx \right)^{\frac{1}{2}}. \tag{3.18}
$$

Since $d_2 > \frac{\lambda_2 \alpha_2 G_2}{\mu_1}$, we have

$$
\left(d_2 - \frac{\lambda_2 \alpha_2 G_2}{\mu_1} \right)^2 \int_\Omega |\nabla \psi|^2 \, dx \leq \left(\frac{\lambda_2 G_3}{\mu_1} \right)^2 \int_\Omega |\nabla \phi|^2 \, dx. \tag{3.19}
$$

Thus,

$$
\int_\Omega |\nabla \psi|^2 \, dx \leq \left(\frac{\lambda_2 G_3}{d_2 \mu_1 - \lambda_2 \alpha_2 G_2} \right)^2 \int_\Omega |\nabla \phi|^2 \, dx. \tag{3.20}
$$
□

Theorem 3.3 *Let $g_i(u,v)$, G_i, $i = 1,2,3$, and $\varphi(x)$ be defined as in (3.5), (3.7), and (3.22). Then, for any $(d_1, d_2) \in \Re$, system (3.1) has no nonconstant positive solutions, where*

$$
\Re := \left\{ (d_1, d_2) \in \mathbf{R}^2 : d_1 > \frac{1}{\mu_1}, d_2 > \frac{\lambda_2 \alpha_2 G_2}{\mu_1} \right\} \cap \left\{ (d_1, d_2) \in \mathbf{R}^2 : d_2 > \varphi(d_1) \right\} \tag{3.21}
$$

with

$$
\varphi(x) := \frac{\lambda_1 \lambda_2 \alpha_1 G_1 G_3}{\mu_1 x - 1} + \lambda_2 \alpha_2 G_2. \tag{3.22}
$$

Proof By Lemma 3.2 we have

$$
\int_\Omega |\nabla \psi|^2 \, dx \leq \left(\frac{\lambda_2 G_3}{d_2 \mu_1 - \lambda_2 \alpha_2 G_2} \right)^2 \int_\Omega |\nabla \phi|^2 \, dx
$$

$$
\leq \left(\frac{\lambda_1 \lambda_2 \alpha_1 G_1 G_3}{(d_2 \mu_1 - \lambda_2 \alpha_2 G_2)(d_1 \mu_1 - 1)} \right)^2 \int_\Omega |\nabla \psi|^2 \, dx. \tag{3.23}
$$

For any $(d_1, d_2) \in \Re$, we have $\frac{\lambda_1 \lambda_2 \alpha_1 G_1 G_3}{(d_2 \mu_1 - \lambda_2 \alpha_2 G_2)(d_1 \mu_1 - 1)} < 1$. Thus $\nabla \psi \equiv 0$. Similarly, we have $\nabla \phi \equiv 0$. Hence, system (3.1) has no nonconstant positive solutions. □

4 Existence of bifurcating positive nonconstant steady-state solutions

In this section, for the particular case where both n_1 and n_2 equal 1, we use the global bifurcation theory to prove the existence of positive nonconstant steady state of system (3.1). In particular, we are concerned with the existence of Turing patterns.

The linearized operator of system (2.23) evaluated at equilibrium (u_*, v_*) is given by

$$J(u_*, v_*) = \begin{pmatrix} -1 & \lambda_1 \alpha_1 \chi_1 \\ -\lambda_2 \chi_2 & \lambda_2 \alpha_2 \chi_3 - 1 \end{pmatrix}, \tag{4.1}$$

where

$$\chi_1 := (1 + v_*)^2, \qquad \chi_2 := \frac{1 + v_* + \alpha_2 v_*}{(a + v_*)(1 + u_*)^2}, \qquad \chi_3 := \frac{a}{(1 + u_*)(a + v_*)^2}. \tag{4.2}$$

Then the characteristic equation of (4.1) is given by

$$\mu^2 - (\lambda_2 \alpha_2 \chi_3 - 2)\mu + 1 + \lambda_2(\lambda_1 \alpha_1 \chi_1 \chi_2 - \alpha_2 \chi_3) = 0. \tag{4.3}$$

By (2.1) and (4.3) if follows that if

$$\alpha_2 > 1, \qquad \chi_3 < \min\left\{ \frac{2}{\lambda_2 \alpha_2}, \frac{\lambda_1 \alpha_1 \chi_1 \chi_2}{\alpha_2} \right\}, \tag{4.4}$$

then (u_*, v_*) is positive and stable in the ODEs (2.23); Moreover, if $\chi_3 > \frac{1}{\lambda_2 \alpha_2}$, then system (1.2) is a substrate-inhibition system.

The linearized operator of system (3.1) evaluated at (u_*, v_*) is given by (choosing d_1 as the bifurcation parameter)

$$L(d_1) := \begin{pmatrix} d_1 \Delta - 1 & \lambda_1 \alpha_1 \chi_1 \\ -\lambda_2 \chi_2 & d_2 \Delta + \lambda_2 \alpha_2 \chi_3 - 1 \end{pmatrix}. \tag{4.5}$$

Let μ_i and $\xi_i(x), i \in \mathcal{N}_0$, be the eigenvalues and the corresponding eigenfunctions of $-\Delta$ in Ω subject to Neumann boundary conditions. Then the eigenvalues of $L(d_1)$ are given by those of the following operator $L_i(d_1)$:

$$L_i(d_1) = \begin{pmatrix} -d_1 \mu_i - 1 & \lambda_1 \alpha_1 \chi_1 \\ -\lambda_2 \chi_2 & -d_2 \mu_i + \lambda_2 \alpha_2 \chi_3 - 1 \end{pmatrix}, \tag{4.6}$$

the characteristic equation of which is

$$\mu^2 - H_i(d_1)\mu + D_i(d_1) = 0, \quad i \in \mathcal{N}_0, \tag{4.7}$$

where

$$\begin{aligned} H_i(d_1) &= -(d_1 + d_2)\mu_i + \lambda_2 \alpha_2 \chi_3 - 2, \\ D_i(d_1) &= d_1 d_2 \mu_i^2 + (d_1 + d_2 - d_1 \lambda_2 \alpha_2 \chi_3)\mu_i + \lambda_1 \lambda_2 \alpha_1 \chi_1 \chi_2 - \lambda_2 \alpha_2 \chi_3 + 1. \end{aligned} \tag{4.8}$$

According to [13], if there exist $i \in \mathcal{N}_0$ and $d_1* > 0$ such that

$$D_i(d_1*) = 0, \qquad H_i(d_1*) \neq 0, \qquad H_j(d_1*) \neq 0, \qquad D_j(d_1*) \neq 0 \quad \text{for all } j \neq i, \tag{4.9}$$

and $\frac{d}{dd_1} D_i(d_1*) \neq 0$, then a global steady-state bifurcation occurs at the critical point d_1*.

By (4.4) we have $H_0(d_1) < 0$. Thus, for all $i \in \mathcal{N}_0$, we have $H_i(d_1) < 0$. Solving $D_i(d_1) = 0$, we get the set of critical values of (d_1, d_2) given by the hyperbolic curves C_i with $i \in \mathcal{N} := \mathcal{N}_0 \setminus \{0\}$:

$$(C_i) : d_2^i = \frac{d_1(\lambda_2 \alpha_2 \chi_3 - 1)\mu_i + \lambda_2 \alpha_2 \chi_3 - \lambda_1 \lambda_2 \alpha_1 \chi_1 \chi_2 - 1}{d_1 \mu_i^2 + \mu_i}. \tag{4.10}$$

Suppose that μ_i, $i \in \mathcal{N}$, are the simple eigenvalues of $-\triangle$. Following [20], we call $B := \bigcup_{i=1}^{\infty} C_i$ the bifurcation set with respect to (u_*, v_*) and denote B_0 by the countable set of intersection points of two curves of $\{C_i\}_{i=1}^{\infty}$; also denote $\widehat{B} = B \setminus B_0$.

Clearly, for any fixed $d_2 > 0$, there exists a unique d_1^i such that $(d_1^i, d_2) \in \widehat{B} \cap C_i$, and at $d = d_1^i$, both (4.9) and $\frac{d}{dd_1} D_i(d_1*) \neq 0$ are satisfied.

Then, from [13] we have the following results regarding the existence of Turing patterns.

Theorem 4.1 *Suppose that* (4.4) *holds and that* C_i *is defined in* (4.10), *where* μ_i, $i \in \mathcal{N}$, *is the simple eigenvalue of* $-\triangle$. *Then, for any* $(d_1^i, d_2) \in \widehat{B} \cap C_i$ *with* d_2 *fixed, there is a smooth curve* Γ_i *of positive solutions of* (3.1) *bifurcating from* $(d_1, u, v) = (d_1^i, u_*, v_*)$ *with* Γ_i *contained in a global branch* C_i *of the positive solutions of* (3.1). *Moreover*:

1. *Near* $(d_1, u, v) = (d_1^i, u_*, v_*)$, $\Gamma_i = \{(d_1(s), u(s), v(s) : s \in (-\varepsilon, \varepsilon))\}$, *where*
   $u(s) = u_* + s\mathbf{a}_i \xi_i(x) + so_1(s)$, $v(s) = v_* + s\mathbf{b}_i \xi_i(x) + so_2(s)$ *for* $s \in (-\varepsilon, \varepsilon)$ *for some* C^{∞}
   smooth functions $d_1(s), o_1(s), o_2(s)$ *such that* $d_1(0) = d_1^i$ *and* $o_1(0) = o_2(0) = 0$. *Here* $\mathbf{a}_i$
   and $\mathbf{b}_i$ *satisfy* $L_i(d_1)(\mathbf{a}_i, \mathbf{b}_i)^T = (0, 0)^T$, *and* $\xi_i(\cdot)$ *is the corresponding eigenfunction of the eigenvalue* μ_i *of -$\triangle$.*
2. *The projection of* C_i *onto* d_1^i-*axis contains the interval* $(0, d_1^i)$.

Proof By (4.10) it follows that (4.9) holds. Thus, the local steady-state bifurcation occurs near the positive steady-state solution. Then, by Theorem 3.1 in [13] we can conclude that the local steady-state bifurcation branches are actually global. Furthermore, since our model is in the framework of [20], we can induce from Theorem 2.3 of [20] that the bifurcation branch cannot contain another bifurcation point (d_1^j, u_*, v_*), with $i \neq j$. This completes the proof of the theorem. □

5 Conclusion

In this paper, we considered a class of homogeneous reaction–diffusion Atkinson oscillator system under Neumann boundary conditions. First, we proved the global existence and boundedness of the time-dependent solutions, which converge to the unique positive equilibrium solution under certain conditions. Then we show the nonexistence of non-constant positive steady-state solutions of the system. Finally, we proved the existence of Turing patterns by using the global steady-state bifurcation theory. To get better results, we have only taken into account the particular case $n_1 = n_2 = 1$. Considering the more general situation is the next step of our research.

Abbreviations
LacI,Lipoprotein-associated coagulation inhibito; NRI, nitrogen regulator I; NRI-P, NRI-phosphate.

Availability of data and materials
Not applicable.

Ethics approval and consent to participate
Not applicable.

Competing interests
The authors declare that they have no competing interests.

Consent for publication
Not applicable.

Authors' contributions
All authors read and approved the final manuscript.

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