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Gradient estimates for the Fisher–KPP equation on Riemannian manifolds

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Abstract

In this paper, we consider positive solutions to the Fisher–KPP equation on complete Riemannian manifolds. We derive the gradient estimate. Using the estimate, we get the classic Harnack inequality which extends the recent result of Cao, Liu, Pendleton, and Ward (Pac. J. Math. 290(2):273–300, 2017).

MSC: 58J35

Keywords: Fisher–KPP equation; Gradient estimate; Harnack inequality

1 Introduction

Let (M,g) be a complete Riemannian manifold. We consider the parabolic equation

$$u_t = \Delta u + cu(1-u) \tag{1.1}$$

on $M \times [0,\infty)$, where c is a positive constant. In the pioneering work of Fisher in 1937 [2], he proposed equation (1.1) to study the propagation of advantageous genes in a population, where $u = u(x,t)$ stands for the population density. In another well-known paper [3], Kolmogorov, Petrovsky, and Piskunov also described the solution to (1.1). Since then, the equation is often referred to as the Fisher–KPP equation and has been widely used in the study of traveling wave solutions and propagation problems (refer to [4–6] and so on).

Recently, Cao et al. [1] derived differential Harnack estimates for positive solutions to (1.1) on Riemannian manifolds with nonnegative Ricci curvature. The idea comes from [7, 8] where a systematic method was developed to find a Harnack inequality for geometric evolution equations. In the complete noncompact case, they obtained the following theorem.

Theorem A (Cao et al.) *Let (M,g) be an n-dimensional complete noncompact Riemannian manifold with nonnegative Ricci curvature, and let $u(x,t): M \times [0,\infty) \to \mathbb{R}$ be a positive solution to* (1.1), *where u is C^2 in x and C^1 in t.*

Let $f = \log u$, then we have

$$\Delta f + \alpha|\nabla f|^2 + \beta e^f + \phi(t) \geq 0 \tag{1.2}$$

for all x and t, provided that

(i) $0 < \alpha < 1$,
(ii) $\beta < \frac{-cn(1+\alpha)}{4\alpha^2-4\alpha+2n} < 0$,
(iii) $\frac{-cn(2+\sqrt{2})}{4(1-\alpha)} < \beta < \frac{-cn(2-\sqrt{2})}{4(1-\alpha)}$,

where

$$\phi(t) = \frac{\mu(\frac{e^{2\mu\omega t}}{\nu-\omega} - \frac{1}{\mu+\omega})}{1 - e^{2\mu\omega t}},$$

with

$$\mu = \beta c\sqrt{\frac{2(1-\alpha)}{c(-cn - 8\beta(1-\alpha))}},$$

$$\nu = \left(\frac{4\beta(1-\alpha)}{n} + c\right) \cdot \sqrt{\frac{2(1-\alpha)}{c(-cn - 8\beta(1-\alpha))}},$$

$$\omega = \sqrt{\frac{2(1-\alpha)}{n}}.$$

Using Theorem A, one can integrate along space-time curves to get a Harnack inequality, but it is different from the classical Li–Yau Harnack [9] in form.

Gradient estimates play an important role in studying elliptic and parabolic operators. The method originated first in [10] and [11], and was further developed by Li and Yau [9], Li [12], Negrin [13], Souplet and Zhang [14], Yang [15], etc. Recent gradient estimates under the geometric flow include [16] and [17]. For more results on the nonlinear PDEs, one may refer to [18, 19].

In this paper, following the line in [12], we prove the following theorems.

Theorem 1.1 *Let M be a complete Riemannian manifold with boundary ∂M (possibly empty). We denote by $B_p(2R)$ the geodesic ball of radius $2R$ around $P \in M$ not intersecting the boundary ∂M. Suppose that the Ricci curvature of M is bounded from below by $-K(2R)$ in $B_p(2R)$, and $K(2R) \geq 0$. Denote $K = K(2R)$. If $u(x,t)$ is a positive smooth solution of* (1.1) *on $M \times [0,\infty)$, then we have*

$$\begin{aligned}
&\frac{|\nabla u|^2}{u^2} + sc(1-u) - s\frac{u_t}{u} \\
&\quad\leq \frac{ns^2}{2(1-\varepsilon)}\frac{1}{t} + \frac{ns^2}{2(1-\varepsilon)(s-1)}K + \frac{sc}{q}\sqrt{\frac{n}{2(1-\varepsilon)}}M_1 \\
&\qquad + \frac{ns^2}{2(1-\varepsilon)R^2}\left(\frac{ns^2}{4(1-\varepsilon)(s-1)}C_1 + (n+1)C_1\right. \\
&\qquad \left. + C_1R(n-1)\sqrt{K} + C_2\right)
\end{aligned} \tag{1.3}$$

on $B_p(R) \times (0,+\infty)$, where C_1, C_2 are positive constants and $0 < \varepsilon < 1$, $s > 1$, $q > 0$ such that $\frac{2(1-\varepsilon)}{n}\frac{s-1}{sq} \geq \frac{1}{\varepsilon} - 1 + \frac{(2s-1)^2}{8}$, $M_1 = \sup_{(x,t)\in B_p(2R)\times[0,\infty)} u(x,t)$. In particular, we can choose $q = \frac{2(1-\varepsilon)(s-1)}{ns[\frac{1}{\varepsilon}-1+\frac{(2s-1)^2}{8}]}$.

Using Theorem 1.1, we get the classic Harnack inequality.

Theorem 1.2 *Let M be an n-dimensional complete noncompact Riemannian manifold with Ricci tensor $R_{ij} \geq -kg_{ij}$ ($k \geq 0$). If $u(x,t)$ is a positive solution of* (1.1) *and* $0 < u < 1$, *then*

$$u(x_1,t_1) \leq u(x_2,t_2)\left(\frac{t_2}{t_1}\right)^{\frac{ns}{2(1-\varepsilon)}} \times \exp\left(\frac{sr^2}{4(t_2-t_1)} + (t_2-t_1)\left(\frac{nsk}{2(1-\varepsilon)(s-1)} + \frac{c}{q}\sqrt{\frac{n}{2(1-\varepsilon)}}\right)\right),$$

where $x_1, x_2 \in M$, $0 < t_1 < t_2 < \infty$, and $r(x_1,x_2)$ is the geodesic distance between x_1 and x_2. In particular, taking $s = 3/2$ and $\varepsilon = 1/4$, we get

$$u(x_1,t_1) \leq u(x_2,t_2)\left(\frac{t_2}{t_1}\right)^{n} \times \exp\left(\frac{3r^2}{8(t_2-t_1)} + (t_2-t_1)\left(2nk + \frac{7\sqrt{6}n\sqrt{nc}}{3}\right)\right). \tag{1.4}$$

The rest of the paper is arranged as follows. In Sect. 2, we get a technical lemma which is important to the proof. In Sect. 3, we prove Theorems 1.1 and 1.2.

2 Technical lemma

As in [12], we define

$$W(x,t) = u^{-q},$$

where q is a positive constant to be fixed later. A direct computation shows that

$$\nabla W = -qu^{-q-1}\nabla u,$$

$$|\nabla W|^2 = q^2u^{-2q-2}|\nabla u|^2,$$

$$\frac{|\nabla W|^2}{W^2} = q^2u^{-2}|\nabla u|^2, \tag{2.1}$$

$$W_t = -qu^{-q-1}u_t, \tag{2.2}$$

$$\frac{W_t}{W} = -q\frac{u_t}{u}, \tag{2.3}$$

$$\begin{aligned}\Delta W &= q(q+1)u^{-q-2}|\nabla u|^2 - qu^{-q-1}\Delta u \\ &= \frac{q+1}{q}\frac{|\nabla W|^2}{W} + cqW - cqW^{\frac{q-1}{q}} + W_t.\end{aligned} \tag{2.4}$$

Therefore

$$\left(\Delta - \frac{\partial}{\partial t}\right)W = \frac{q+1}{q}\frac{|\nabla W|^2}{W} + cqW - cqW^{\frac{q-1}{q}}. \tag{2.5}$$

We follow the line in [12]. Define three functions:

$$F_0(x,t) = \frac{|\nabla W|^2}{W^2} + \alpha c\bigl(1 - W^{-1/q}\bigr),$$

$$\begin{aligned}
&F_1 = \frac{W_t}{W}, \\
&F = F_0 + \beta F_1,
\end{aligned} \tag{2.6}$$

where α, β are two positive constants to be fixed later.

Let $e_1, e_2, \dots, e_n$ be a local orthonormal frame field. We adopt the notation that subscripts in i, j, and k, with $1 \leq i, j, k \leq n$, denote covariant differentiations in the e_i, e_j, and e_k directions, respectively.

Calculate

$$\nabla F_0(x,t) = \frac{2W_iW_{ij}}{W^2} - \frac{2W_i^2W_j}{W^3} + \frac{\alpha c}{q}W^{-(q+1)/q}W_j, \tag{2.7}$$

$$\begin{aligned}
\Delta F_0(x,t) = {}& \frac{2W_{ij}^2}{W^2} + \frac{2W_iW_{ijj}}{W^2} - 8\frac{W_iW_{ij}W_j}{W^3} + 6\frac{W_i^4}{W^4} - 2\frac{W_i^2W_{jj}}{W^3} \\
& - \frac{\alpha c(q+1)}{q^2}\frac{W^{-1/q}}{W^2}W_j^2 + \frac{\alpha c}{q}W^{-(q+1)/q}W_{jj},
\end{aligned} \tag{2.8}$$

$$\frac{\partial F_0(x,t)}{\partial t} = \frac{2W_iW_{it}}{W^2} - \frac{2W_i^2W_t}{W^3} + \frac{\alpha c}{q}W^{-\frac{q+1}{q}}W_t, \tag{2.9}$$

$$\begin{aligned}
\nabla F_1 &= \frac{W_{ti}W - W_tW_i}{W^2}, \\
\Delta F_1 &= \frac{WW_{iit} - 2W_iW_{ti} - W_tW_{ii}}{W^2} + \frac{2W_tW_i^2}{W^3}, \\
\frac{\partial F_1}{\partial t} &= \frac{W_{tt}W - W_t^2}{W^2}, \\
\left(\Delta - \frac{\partial}{\partial t}\right)F_1 &= \frac{W(\Delta W - W_t)_t - W_t(\Delta W - W_t) - 2W_iW_{ti}}{W^2} + \frac{2W_tW_i^2}{W^3} \\
&= \frac{2}{q}\nabla \log W \cdot \nabla F_1 + cW^{-\frac{1}{q}-1}W_t.
\end{aligned} \tag{2.10}$$

We denote the Ricci tensor of M by R_{jj}:

$$\frac{2W_iW_{ijj}}{W^2} = \frac{2W_iW_{jji}}{W^2} + \frac{2R_{ij}W_iW_j}{W^2}.$$

It follows that

$$\begin{aligned}
\frac{2W_iW_{ijj}}{W^2} - \frac{2W_iW_{it}}{W^2} &= \frac{2W_i}{W^2}(\Delta W - W_t)_i + \frac{2R_{ij}W_iW_j}{W^2} \\
&= \frac{4(q+1)}{q}\frac{W_iW_{ij}W_j}{W^3} - \frac{2(q+1)}{q}\frac{W_i^4}{W^4} \\
&\quad + \frac{2cW_i^2}{W^2}\left[q - (q-1)W^{-1/q}\right] + \frac{2R_{ij}W_iW_j}{W^2}.
\end{aligned} \tag{2.11}$$

Equalities (2.2) and (2.4) yield

$$\begin{aligned}
-\frac{2W_i^2W_{jj}}{W^3} + \frac{2W_i^2W_t}{W^3} &= -\frac{2W_i^2}{W^3}(\Delta W - W_t) \\
&= -\frac{2(q+1)}{q}\frac{W_i^4}{W^4} - 2cq\frac{W_i^2}{W^2} + 2cqW^{-\frac{1}{q}}\frac{W_i^2}{W^2}.
\end{aligned} \tag{2.12}$$

By Hölder's inequality, we have

$$\frac{2\varepsilon W_{ij}^2}{W^2} + \frac{2}{\varepsilon} \cdot \frac{W_i^4}{W^4} \geq 4\frac{W_i W_{ij} W_j}{W^3}.$$

So,

$$\frac{2W_{ij}^2}{W^2} - 8\frac{W_i W_{ij} W_j}{W^3} + 6\frac{W_i^4}{W^4} \geq \frac{2(1-\varepsilon)W_{ij}^2}{W^2} - 4\frac{W_i W_{ij} W_j}{W^3} + \left(6 - \frac{2}{\varepsilon}\right)\frac{W_i^4}{W^4},$$

where $0 < \varepsilon < 1$.

Noting the inequality $W_{ij}^2 \geq \frac{1}{n}(W_{ii})^2$, we obtain

$$\begin{aligned} &\frac{2W_{ij}^2}{W^2} - 8\frac{W_i W_{ij} W_j}{W^3} + 6\frac{W_i^4}{W^4} \\ &\quad \geq \frac{2(1-\varepsilon)}{n}\left(\frac{\Delta W}{W}\right)^2 - 4\left(\frac{W_i W_{ij} W_j}{W^3} - \frac{W_i^4}{W^4}\right) - 2\left(\frac{1}{\varepsilon} - 1\right)\frac{W_i^4}{W^4}. \end{aligned} \tag{2.13}$$

By (2.7), we have

$$\nabla F_0 \cdot \nabla \log W = \frac{2W_i W_{ij} W_j}{W^3} - \frac{2W_i^4}{W^4} + \frac{\alpha c}{q} W^{-\frac{1}{q}} \frac{W_j^2}{W^2}. \tag{2.14}$$

Plugging (2.11), (2.12), (2.13), and (2.14) into (2.8) and (2.9), we have

$$\begin{aligned} \left(\Delta - \frac{\partial}{\partial t}\right)F_0 \geq{}& \frac{2(1-\varepsilon)}{n}\left(\frac{\Delta W}{W}\right)^2 - 2\left(\frac{1}{\varepsilon} - 1\right)\frac{W_i^4}{W^4} \\ &+ \frac{2}{q}\nabla F_0 \cdot \nabla \log W \\ &+ 2c\left(1 - \frac{\alpha}{q^2}\right)\frac{W_i^2}{W^2} W^{-\frac{1}{q}} \\ &+ \frac{2R_{ij} W_i W_j}{W^2} + \alpha c^2 W^{-\frac{1}{q}} - \alpha c^2 W^{-\frac{2}{q}}. \end{aligned} \tag{2.15}$$

Setting $\beta = \alpha/q$ and combining (2.10), we conclude that

$$\begin{aligned} \left(\Delta - \frac{\partial}{\partial t}\right)F \geq{}& \frac{2(1-\varepsilon)}{n}\left(\frac{\Delta W}{W}\right)^2 - 2\left(\frac{1}{\varepsilon} - 1\right)\frac{W_i^4}{W^4} \\ &+ \frac{2}{q}\nabla F \cdot \nabla \log W \\ &+ 2c\left(1 - \frac{\alpha}{q^2}\right)\frac{W_i^2}{W^2} W^{-\frac{1}{q}} \\ &+ \frac{2R_{ij} W_i W_j}{W^2} + \alpha c^2 W^{-\frac{1}{q}} - \alpha c^2 W^{-\frac{2}{q}} \\ &+ \frac{\alpha c}{q} W^{-\frac{1}{q}-1} W_t. \end{aligned} \tag{2.16}$$

By (2.4) and (2.6), we arrive at

$$\frac{\Delta W}{W} = \frac{q}{\alpha}F + \left(\frac{q+1}{q} - \frac{q}{\alpha}\right)\frac{|\nabla W|^2}{W^2}. \tag{2.17}$$

Setting $\alpha = sq^2$ yields

$$\begin{aligned}\frac{\Delta W}{W} &= \frac{1}{sq}F + \left(\frac{q+1}{q} - \frac{1}{sq}\right)\frac{|\nabla W|^2}{W^2}\\ &= \frac{1}{sq}F + \left(\frac{q+1-1/s}{q}\right)\frac{|\nabla W|^2}{W^2}.\end{aligned} \tag{2.18}$$

Substituting (2.18) into (2.16), we obtain

$$\begin{aligned}\left(\Delta - \frac{\partial}{\partial t}\right)F \geq{}& \frac{2(1-\varepsilon)}{n}\frac{1}{s^2q^2}F^2\\ &+ \left[\frac{2(1-\varepsilon)}{n}\frac{(sq+s-1)^2}{s^2q^2} - 2\left(\frac{1}{\varepsilon}-1\right)\right]\frac{|\nabla W|^4}{W^4}\\ &+ \frac{4(1-\varepsilon)}{n}\frac{(sq+s-1)}{s^2q^2}F\frac{|\nabla W|^2}{W^2} + \frac{2}{q}\nabla F\cdot\nabla\log W\\ &+ 2c(1-s)\frac{|\nabla W|^2}{W^2}W^{-\frac{1}{q}} + \frac{2R_{ij}W_iW_j}{W^2} + sq^2c^2W^{-\frac{1}{q}}\\ &- sq^2c^2W^{-\frac{2}{q}} + sqcW^{-\frac{1}{q}-1}W_t.\end{aligned} \tag{2.19}$$

An immediate consequence is the following lemma.

Lemma 2.1 *Let M be an n-dimensional complete Riemannian manifold with Ricci tensor R_{ij}. If F is defined by* (2.6) *where $\beta = \alpha/q$, $\alpha = sq^2$, then we have*

$$\begin{aligned}\left(\Delta - \frac{\partial}{\partial t}\right)F \geq{}& \frac{2(1-\varepsilon)}{n}\frac{1}{s^2q^2}F^2\\ &+ \left[\frac{2(1-\varepsilon)}{n}\frac{(sq+s-1)^2}{s^2q^2} - 2\left(\frac{1}{\varepsilon}-1\right)\right]\frac{|\nabla W|^4}{W^4}\\ &+ \frac{4(1-\varepsilon)}{n}\frac{(sq+s-1)}{s^2q^2}F\frac{|\nabla W|^2}{W^2}\\ &+ \frac{2}{q}\nabla F\cdot\log W + (c-2cs)\frac{|\nabla W|^2}{W^2}W^{-\frac{1}{q}}\\ &+ cW^{-\frac{1}{q}}F + \frac{2R_{ij}W_iW_j}{W^2}.\end{aligned} \tag{2.20}$$

3 Main theorems

Theorem 3.1 *Let M be a complete Riemannian manifold with boundary ∂M (possibly empty). We denote by $B_p(2R)$ the geodesic ball of radius $2R$ around $P \in M$ not intersecting the boundary ∂M. Suppose that the Ricci curvature of M is bounded from below by $-K(2R)$ in $B_p(2R)$, and $K(2R) \geq 0$. Denote $K = K(2R)$. If $u(x,t)$ is a positive smooth solution of* (1.1) *on* $M \times [0,\infty)$, *then we have*

$$\begin{aligned}
&\frac{|\nabla u|^2}{u^2} + sc(1-u) - s\frac{u_t}{u} \\
&\quad\le \frac{ns^2}{2(1-\varepsilon)}\frac{1}{t} + \frac{ns^2}{2(1-\varepsilon)(s-1)}K + \frac{sc}{q}\sqrt{\frac{n}{2(1-\varepsilon)}}M_1 \\
&\qquad + \frac{ns^2}{2(1-\varepsilon)R^2}\biggl(\frac{ns^2}{4(1-\varepsilon)(s-1)}C_1 + (n+1)C_1 \\
&\qquad + C_1R(n-1)\sqrt{K} + C_2\biggr)
\end{aligned} \tag{3.1}$$

on $B_p(R) \times (0,+\infty)$, *where* C_1, C_2 *are positive constants*, $0 < \varepsilon < 1$, $s > 1$, $q > 0$ *such that* $\frac{2(1-\varepsilon)}{n}\frac{s-1}{sq} \ge \frac{1}{\varepsilon} - 1 + \frac{(2s-1)^2}{8}$, *and* $M_1 = \sup_{(x,t)\in B_p(2R)\times[0,\infty)} u(x,t)$. *In particular, we can choose* $q = \frac{2(1-\varepsilon)(s-1)}{ns[\frac{1}{\varepsilon}-1+\frac{(2s-1)^2}{8}]}$.

Proof Let $\chi \in C^2[0,+\infty)$ be a cut-off function such that $\chi(r) = 1$ for $r \le 1$, $\chi(r) = 0$ for $r > 2$, and $0 \le \chi(r) \le 1$. We choose χ satisfying $-\sqrt{C_1}\chi^{1/2}(r) \le \chi'(r) \le 0$, $\chi''(r) \ge -C_2$, where C_1, C_2 are positive constants.

Denote by $r(x)$ the geodesic distance between x and some fixed point P. Set

$$\phi(x) = \chi\left(\frac{r(x)}{R}\right).$$

By the conditions on χ and the Laplacian comparison theorem, we get

$$|\nabla\phi|^2 \le \frac{C_1}{R^2}\phi$$

and

$$\Delta\phi \ge -\frac{C_2 + C_1(n-1)}{R^2} - \frac{C_1(n-1)\sqrt{K}}{R}.$$

Define the function $H(x,t) := tF(x,t)$. Using the argument of Calabi [20], we assume that the function $\phi(x)\cdot H(x,t)$ with support in $B_P(2R)$ is smooth. For any fixed $T > 0$, let (x_0,t_0) be the point where $\phi\cdot H$ achieves its maximum in $B_P(2R)\times[0,T]$. Without loss of generality, we assume that $\phi(x_0)\cdot H(x_0,t_0) > 0$. Otherwise, (3.1) is obviously true. By the maximum principle, at (x_0,t_0), we have

$$\nabla(\phi\cdot H) = 0, \tag{3.2}$$

$$\frac{\partial(\phi\cdot H)}{\partial t} \ge 0, \tag{3.3}$$

$$\Delta(\phi\cdot H) \le 0. \tag{3.4}$$

By (3.2), we have

$$\nabla H = -\frac{\nabla\phi}{\phi}H. \tag{3.5}$$

By (3.4), we have

$$\Delta\phi \cdot H + 2\nabla\phi \cdot \nabla H + \phi\Delta H \le 0. \tag{3.6}$$

It follows from (3.3) and (3.6) that

$$\Delta\phi \cdot H + 2\nabla\phi \cdot \nabla H + \phi\left(\Delta - \frac{\partial}{\partial t}\right)H \le 0. \tag{3.7}$$

Setting $\beta = \alpha/q$, $\alpha = sq^2$, by Lemma 2.1 we have

$$\begin{aligned}\left(\Delta - \frac{\partial}{\partial t}\right)H &= t\left(\Delta - \frac{\partial}{\partial t}\right)F - F \\ &\ge \frac{2(1-\varepsilon)}{n}\frac{1}{s^2q^2}H^2\frac{1}{t} + \left[\frac{2(1-\varepsilon)}{n}\frac{(sq+s-1)^2}{s^2q^2} - 2\left(\frac{1}{\varepsilon} - 1\right)\right]\frac{|\nabla W|^4}{W^4}t \\ &\quad + \frac{4(1-\varepsilon)}{n}\frac{(sq+s-1)}{s^2q^2}\frac{|\nabla W|^2}{W^2}H + \frac{2}{q}\nabla H \cdot \nabla \log W + cW^{-\frac{1}{q}}H \\ &\quad - c(2s-1)\frac{|\nabla W|^2}{W^2}W^{-1/q}t - 2K\frac{|\nabla W|^2}{W^2}t - \frac{H}{t}. \end{aligned} \tag{3.8}$$

By Hölder's inequality, we get

$$2Kt\frac{|\nabla W|^2}{W^2} \le \frac{2(1-\varepsilon)}{n}\frac{(s-1)^2}{s^2q^2}\frac{|\nabla W|^4}{W^4}t + \frac{n}{2(1-\varepsilon)}\frac{s^2q^2}{(s-1)^2}K^2t \tag{3.9}$$

and

$$c(2s-1)\frac{|\nabla W|^2}{W^2}W^{-1/q}t \le \frac{(2s-1)^2}{4}\frac{|\nabla W|^4}{W^4}t + c^2M_1^2t. \tag{3.10}$$

Substituting (3.9) and (3.10) into (3.8), and choosing $s > 1$ and $q > 0$ such that $\frac{2(1-\varepsilon)}{n}\frac{s-1}{sq} \ge \frac{1}{\varepsilon} - 1 + \frac{(2s-1)^2}{8}$, we have

$$\begin{aligned}\left(\Delta - \frac{\partial}{\partial t}\right)H &\ge \frac{2(1-\varepsilon)}{n}\frac{1}{s^2q^2}H^2\frac{1}{t} + \frac{4(1-\varepsilon)}{n}\frac{(sq+s-1)}{s^2q^2}\frac{|\nabla W|^2}{W^2}H \\ &\quad + \frac{2}{q}\nabla H \cdot \nabla \log W - \frac{H}{t} - \frac{n}{2(1-\varepsilon)}\frac{s^2q^2}{(s-1)^2}K^2t - c^2M_1^2t. \end{aligned} \tag{3.11}$$

Substituting (3.11) into (3.7) and using (3.5), we have

$$\begin{aligned}&\frac{2(1-\varepsilon)}{n}\frac{1}{s^2q^2}H^2\frac{1}{t}\phi + \frac{4(1-\varepsilon)}{n}\frac{(sq+s-1)}{s^2q^2}\frac{|\nabla W|^2}{W^2}H\phi - \frac{2}{q}H\nabla\phi \cdot \frac{\nabla W}{W} \\ &\quad - \frac{H}{t}\phi - \frac{n}{2(1-\varepsilon)}\frac{s^2q^2}{(s-1)^2}K^2\phi t - c^2M_1^2\phi t \\ &\quad - \left(\frac{C_2 + C_1(n+1)}{R^2} + \frac{C_1(n-1)\sqrt{K}}{R}\right)H \le 0, \end{aligned} \tag{3.12}$$

where we have used $2\nabla\phi \cdot \nabla H = -2\frac{|\nabla\phi|^2}{\phi}H \ge -\frac{2C_1}{R^2}H$.

Clearly,

$$
\begin{aligned}
&\frac{2}{q}H\nabla\phi\cdot\frac{\nabla W}{W}\\
&\quad\le\frac{4(1-\varepsilon)}{n}\frac{(sq+s-1)}{s^2q^2}\frac{|\nabla W|^2}{W^2}H\phi+\frac{n}{4(1-\varepsilon)}\frac{s^2H}{(sq+s-1)}\frac{|\nabla\phi|^2}{\phi}.
\end{aligned}
\tag{3.13}
$$

Multiplying through by $t\phi$ at (3.12) and using (3.13), we arrive at

$$
\begin{aligned}
&\frac{2(1-\varepsilon)}{n}\frac{1}{s^2q^2}H^2\phi^2-H\phi\\
&\quad-t\left(\frac{ns^2}{4(1-\varepsilon)(sq+s-1)}\frac{C_1}{R^2}+\frac{C_2+(n+1)C_1}{R^2}+\frac{C_1(n-1)\sqrt{K}}{R}\right)H\phi\\
&\quad-t^2\left(\frac{n}{2(1-\varepsilon)}\frac{s^2q^2}{(s-1)^2}K^2+c^2M_1^2\right)\le 0.
\end{aligned}
\tag{3.14}
$$

Equation (3.14) yields

$$
\begin{aligned}
H\phi\le{}&\frac{ns^2q^2}{2(1-\varepsilon)}\left[1+t\left(\frac{ns^2}{4(1-\varepsilon)(sq+s-1)}\frac{C_1}{R^2}+\frac{C_2+(n+1)C_1}{R^2}+\frac{C_1(n-1)\sqrt{K}}{R}\right)\right]\\
&+t\sqrt{\frac{ns^2q^2}{2(1-\varepsilon)}}\sqrt{\frac{n}{2(1-\varepsilon)}\frac{s^2q^2}{(s-1)^2}K^2+c^2M_1^2}
\end{aligned}
$$

at (x_0,t_0).

It is easy to see that

$$
\begin{aligned}
&\sup_{x\in B_p(R)}H(x,T)\\
&\quad\le H(x_0,t_0)\phi(x_0)\\
&\quad\le\frac{ns^2q^2}{2(1-\varepsilon)}\left[1+T\left(\frac{ns^2}{4(1-\varepsilon)(sq+s-1)}\frac{C_1}{R^2}+\frac{C_2+(n+1)C_1}{R^2}+\frac{C_1(n-1)\sqrt{K}}{R}\right)\right]\\
&\qquad+T\sqrt{\frac{ns^2q^2}{2(1-\varepsilon)}}\sqrt{\frac{n}{2(1-\varepsilon)}\frac{s^2q^2}{(s-1)^2}K^2+c^2M_1^2}.
\end{aligned}
$$

Then we get

$$
\begin{aligned}
&\frac{|\nabla u|^2}{u^2}+sc(1-u)-s\frac{u_t}{u}\\
&\quad\le\frac{ns^2}{2(1-\varepsilon)}\frac{1}{t}+\frac{ns^2}{2(1-\varepsilon)(s-1)}K+\frac{sc}{q}\sqrt{\frac{n}{2(1-\varepsilon)}}M_1\\
&\qquad+\frac{ns^2}{2(1-\varepsilon)R^2}\left(\frac{ns^2}{4(1-\varepsilon)(s-1)}C_1+(n+1)C_1+C_1R(n-1)\sqrt{K}+C_2\right)
\end{aligned}
$$

on $B_p(R)\times(0,+\infty)$ since $T>0$ is arbitrary. □

Using Theorem 3.1 and letting $R\to+\infty$, we can get the following corollary.

Corollary 3.2 *Let M be an n-dimensional complete Riemannian manifold with Ricci tensor $R_{ij} \geq -kg_{ij}$ ($k \geq 0$). If $u(x,t)$ is a positive solution of* (1.1) *and* $0 < u < 1$, *then*

$$\frac{|\nabla u|^2}{u^2} - s\frac{u_t}{u} \leq \frac{ns^2}{2(1-\varepsilon)}\frac{1}{t} + \frac{ns^2}{2(1-\varepsilon)(s-1)}k + \frac{sc}{q}\sqrt{\frac{n}{2(1-\varepsilon)}}. \tag{3.15}$$

Remark 3.3 Let M be an n-dimensional complete Riemannian manifold with nonnegative Ricci curvature. Suppose that $u(x,t)$ is a positive solution of (1.1) and $0 < u < 1$. Let $f = \log u$. Then we have

$$f_t = \Delta f + |\nabla f|^2 + c\bigl(1 - e^f\bigr).$$

It follows from Corollary 3.2 that

$$\Delta f + \frac{s-1}{s}|\nabla f|^2 + \frac{ns}{2(1-\varepsilon)}\frac{1}{t} + \frac{c}{q}\sqrt{\frac{n}{2(1-\varepsilon)}} \geq 0.$$

In particular, taking $s = 3/2$ and $\varepsilon = 1/4$, we get

$$\Delta f + \frac{1}{3}|\nabla f|^2 + \frac{n}{t} + \frac{7\sqrt{6}n\sqrt{n}c}{3} \geq 0.$$

This estimate is simpler than (1.2) in form.

Theorem 3.4 *Let M be an n-dimensional complete Riemannian manifold with Ricci tensor $R_{ij} \geq -kg_{ij}$ ($k \geq 0$). If $u(x,t)$ is a positive solution of* (1.1) *and* $0 < u < 1$, *then*

$$\begin{aligned} u(x_1,t_1) &\leq u(x_2,t_2)\left(\frac{t_2}{t_1}\right)^{\frac{ns}{2(1-\varepsilon)}} \\ &\quad \times \exp\left(\frac{sr^2}{4(t_2-t_1)} + (t_2-t_1)\left(\frac{nsk}{2(1-\varepsilon)(s-1)} + \frac{c}{q}\sqrt{\frac{n}{2(1-\varepsilon)}}\right)\right), \end{aligned}$$

where $x_1, x_2 \in M$, $0 < t_1 < t_2 < \infty$, and $r(x_1,x_2)$ is the geodesic distance between x_1 and x_2.

Proof If we set $f = \log u$, then

$$|\nabla f|^2 - sf_t \leq \frac{ns^2}{2(1-\varepsilon)}\frac{1}{t} + \frac{ns^2}{2(1-\varepsilon)(s-1)}k + \frac{sc}{q}\sqrt{\frac{n}{2(1-\varepsilon)}} \tag{3.16}$$

for all $(x,t) \in M \times (0,+\infty)$.

Fix points (x_1,t_1) and (x_2,t_2) in $M \times (0,+\infty)$ with $t_1 < t_2$, and let $r : [0,1] \to M$ be the shortest geodesic joining x_1 and x_2 with $r(0) = x_2$ and $r(1) = x_1$.

Define the curve $\eta : [0,1] \to M \times (0,+\infty)$ by $\eta(y) = (r(y), (1-y)t_2 + yt_1)$. It is clear that $\eta(0) = (x_2,t_2)$, $\eta(1) = (x_1,t_1)$ and

$$f(x_1,t_1) - f(x_2,t_2) = \int_0^1 \frac{df(\eta(y))}{dy}\,dy \leq \int_0^1 \bigl(\rho|\nabla f| - (t_2-t_1)f_t\bigr)\,dy, \tag{3.17}$$

where $\rho = r(x_1,x_2)$.

By inequality (3.16), we get

$$-f_t \le -\frac{|\nabla f|^2}{s} + \frac{ns}{2(1-\varepsilon)t} + \frac{nsk}{2(1-\varepsilon)(s-1)} + \frac{c}{q}\sqrt{\frac{n}{2(1-\varepsilon)}}.$$

Thus (3.17) becomes

$$\begin{aligned} & f(x_1,t_1) - f(x_2,t_2) \\ & \quad \le \int_0^1 \biggl(\rho|\nabla f| - (t_2-t_1)\frac{|\nabla f|^2}{s} \\ & \qquad + (t_2-t_1)\frac{ns}{2(1-\varepsilon)t} + (t_2-t_1)\biggl(\frac{nsk}{2(1-\varepsilon)(s-1)} + \frac{c}{q}\sqrt{\frac{n}{2(1-\varepsilon)}}\biggr)\biggr)\,dy, \end{aligned}$$

where $t = (1-y)t_2 + yt_1$.

We can see that as a function of $|\nabla f|$, the quadratic

$$\begin{aligned} & \rho|\nabla f| - (t_2-t_1)\frac{|\nabla f|^2}{s} + (t_2-t_1)\frac{ns}{2(1-\varepsilon)t} \\ & \qquad + (t_2-t_1)\biggl(\frac{nsk}{2(1-\varepsilon)(s-1)} + \frac{c}{q}\sqrt{\frac{n}{2(1-\varepsilon)}}\biggr) \\ & \quad \le \frac{s\rho^2}{4(t_2-t_1)} + (t_2-t_1)\frac{ns}{2(1-\varepsilon)t} + (t_2-t_1)\biggl(\frac{nsk}{2(1-\varepsilon)(s-1)} + \frac{c}{q}\sqrt{\frac{n}{2(1-\varepsilon)}}\biggr). \end{aligned}$$

So,

$$\begin{aligned} f(x_1,t_1) - f(x_2,t_2) \le {}& \frac{s\rho^2}{4(t_2-t_1)} + (t_2-t_1)\biggl(\frac{nsk}{2(1-\varepsilon)(s-1)} + \frac{c}{q}\sqrt{\frac{n}{2(1-\varepsilon)}}\biggr) \\ & + \frac{ns}{2(1-\varepsilon)}\log\biggl(\frac{t_2}{t_1}\biggr), \end{aligned}$$

i.e.,

$$\begin{aligned} u(x_1,t_1) \le {}& u(x_2,t_2)\biggl(\frac{t_2}{t_1}\biggr)^{\frac{ns}{2(1-\varepsilon)}} \\ & \times \exp\biggl(\frac{sr^2}{4(t_2-t_1)} + (t_2-t_1)\biggl(\frac{nsk}{2(1-\varepsilon)(s-1)} + \frac{c}{q}\sqrt{\frac{n}{2(1-\varepsilon)}}\biggr)\biggr). \end{aligned}$$

□

4 Conclusions

In this paper, we use the method of gradient estimates to study the Fisher–KPP equation. We get the local gradient estimate (Theorem 1.1). Since the solution u of (1.1) often describes the density, it is natural to study solutions of which $0 < u < 1$. We get the Harnack estimate if $0 < u < 1$ (Theorem 1.2). Our results can be used to study the solution of (1.1) further. The similar method can be also applied to the following equation:

$$u_t = \Delta u + au^p + bu^q,$$

where a, b, p, q are constants.

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Abbreviations
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Not applicable.

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