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The existence of nontrivial solution for a class of sublinear biharmonic equations with steep potential well

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Abstract

In this paper, we study the following biharmonic equation:

$$\begin{cases} \Delta^2 u - \Delta u + \lambda V(x)u = \alpha(x)f(u) + \mu K(x)|u|^{q-2}u & \text{in } \mathbb{R}^N, \\ u \in H^2(\mathbb{R}^N), \end{cases}$$

where $\Delta^2 u = \Delta(\Delta u)$, $N > 4$, $\lambda > 0$, $1 < q < 2$ and $\mu \in [0, \mu_0]$. By using Ekeland's variational principle and Gigliardo–Nirenberg's inequality, we prove the existence of nontrivial solution for the above problem.

MSC: 35J50; 35J60

Keywords: Biharmonic equation; Variational method; Steep potential well

1 Introduction

In this paper, we consider the biharmonic equation as follows:

$$\begin{cases} \Delta^2 u - \Delta u + \lambda V(x)u = \alpha(x)f(u) + \mu K(x)|u|^{q-2}u & \text{in } \mathbb{R}^N, \\ u \in H^2(\mathbb{R}^N), \end{cases} \tag{B_μ}$$

where $\Delta^2 u = \Delta(\Delta u)$, $N > 4$, $\lambda > 0$, $1 < q < 2$ and $\mu \in [0, \mu_0]$, $0 < \mu_0 < \infty$. The continuous function f verifies the assumptions:

(f_1) $f(s) = o(|s|)$ as $s \to 0$;

(f_2) $f(s) = o(|s|)$ as $|s| \to \infty$;

(f_3) $F(u_0) > 0$ for some $u_0 > 0$, where $F(u) = \int_0^u f(t)\,dt$.

According to hypotheses (f_1)–(f_3), the number $c_f = \max_{s\neq 0} |\frac{f(s)}{s}| > 0$ is well defined (see [1]). The continuous functions α and K verify the assumptions:

(α_1) $0 < \alpha(x) \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ and $c_f\|\alpha\|_\infty < 1$;

(K_1) $0 < K(x) \in L^{\frac{2}{2-q}}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$.

We require the potential $V : \mathbb{R}^N \to \mathbb{R}$ to satisfy the following assumptions:

(V_1) $V(x)$ is a nonnegative continuous function on $\mathbb{R}^N$, there exists a constant $c_0 > 0$ such that the set $\{V < c_0\} := \{x \in \mathbb{R}^N | V(x) < c_0\}$ has finite positive Lebesgue measure;

(V_2) $\Omega = \operatorname{int}\{x \in \mathbb{R}^N | V(x) = 0\}$ is nonempty and has smooth boundary with $\overline{\Omega} = \{x \in \mathbb{R}^N | V(x) = 0\}$;

(V_3) $|\{V < c_0\}| < (\frac{1-c_f\|\alpha\|_\infty}{S^2 c_f\|\alpha\|_\infty})^{\frac{N}{4}}$, where $|\cdot|$ is the Lebesgue measure, and S is the best constant for the Sobolev embedding $H^2(\mathbb{R}^N) \hookrightarrow L^{2_*}(\mathbb{R}^N)$, $2_* = \frac{2N}{N-4}$;

(V_4) There exists $R_0 > 0$ such that $\inf\{V(x)||x| \geq R_0\} > c_0$.

The biharmonic equations can be used to describe some phenomena appearing in physics and engineering. For example, the problem of nonlinear oscillation in a suspension bridge [2–4] and the problem of the static deflection of an elastic plate in a fluid [5]. In the last decades, the existence and multiplicity of nontrivial solutions for biharmonic equations have begun to receive much attention. Under the hypotheses (V_1) and (V_2), $\lambda V(x)$ is called the steep potential well whose depth is controlled by the parameter λ. So far, the steep potential well has been introduced to the study of many types of nonlinear differential equations such as Kirchhoff type equations [6], Hamiltonian systems [7], Schrödinger–Poisson systems [8], and biharmonic equation [9, 10].

Wang and Zhang [11] studied a class of biharmonic equations without Laplacian as follows:

$$\begin{cases} \Delta^2 u + V_\lambda(x)u = f(u) \quad \text{in } \mathbb{R}^N, \\ u \in H^2(\mathbb{R}^N), \end{cases} \tag{1}$$

where $N \geq 5$, $V_\lambda(x) = 1 + \lambda g(x)$ is a steep potential well. When $f(u)$ is asymptotically linear at infinity on u and $f(u)/|u|$ is nondecreasing, they obtained the existence of nontrivial solution for problem (1) with λ being large enough.

Liu et al. [12] studied the following biharmonic equations:

$$\begin{cases} \Delta^2 u - \Delta u + \lambda V(x)u = f(x,u) \quad \text{in } \mathbb{R}^N, \\ u \in H^2(\mathbb{R}^N), \end{cases} \tag{2}$$

where $N \geq 1$. They obtained the existence and multiplicity of nontrivial solutions for problem (2) when $f(x,u)$ is subcritical and superlinear on u at infinity, and $V(x)$ is steep potential well, and $\lambda > 0$ is large enough. Ye and Tang [13] unified and improved the results in [12], and proved the existence of infinitely many solutions for problem (2) for $\lambda > 0$ large enough. In [12, 13], by using Brezis–Lieb's lemma and $\lambda > 0$ is sufficiently large, the authors showed that any bounded Cerami sequence has a convergent subsequence.

Sun et al. [9] studied a class of biharmonic equations with p-Laplacian as follows:

$$\begin{cases} \Delta^2 u - \beta\Delta_p u + \lambda V(x)u = f(x,u) \quad \text{in } \mathbb{R}^N, \\ u \in H^2(\mathbb{R}^N), \end{cases} \tag{3}$$

where $N \geq 1$, $V(x)$ is a steep potential well. They obtained the existence and multiplicity of nontrivial solution for problem (3) with λ large enough. Specially, they considered the case of $f(x,u) = K(x)|u|^{q-2}u$ $(1 < q < 2)$, nonlinearity term is a sublinear case (sublinear at origin and infinity), they obtained two nontrivial solutions with λ large enough and $\beta < 0$.

It is natural for us to pose a question as follows:

- If $\beta = 1 > 0$ in problem (3), and λ is just larger than a certain constant Λ, but $\lambda \nrightarrow \infty$, then we would much like to know whether equation (B_μ) admits one nontrivial solution.

Remark 1 Set $h_\nu(x,u) = \alpha(x)f(u) + \nu K(x)|u|^{q-2}u$. 1. Since $f(\cdot)$ is superlinear at the origin, $\nu K(x)|u|^{q-2}u$ is sublinear at the origin, so $h_\nu(x,\cdot)$ is sublinear at the origin; 2. Since both $f(\cdot)$ and $\nu K(x)|u|^{q-2}u$ are sublinear at infinity, so $h_\nu(x,\cdot)$ is sublinear at infinity. For different kinds of nonlinearity, we refer to for example [14–27] and the references therein.

There is a technical difficulty in applying variational methods directly to equation (B_μ).

Problem 1 Under the assumptions of potential $V(x)$ and $\lambda \nrightarrow \infty$, we could not obtain the compactness result. It is difficult to prove that a Cerami sequence is strongly convergent if we seek solution of equation (B_ν) by min–max methods.

For Problem 1. In order to prove that a Cerami sequence is strongly convergent, we overcome this technical difficulty by Lemma 4 and Lemma 6.

Our main results are as follows.

Theorem 1 *Suppose that a continuous function f satisfies (f_1)–(f_3), then the number $c_f = \max_{s\neq 0} |\frac{f(s)}{s}| > 0$ is well defined.*

Theorem 2 *Suppose that conditions (f_1)–(f_3), (V_1)–(V_3), (α_1) and (K_1) hold, there exists a constant $\Lambda > 0$ for all $\lambda \geq \Lambda$, then equation (B_0) has only the trivial solution.*

Theorem 3 *Suppose that conditions (f_1)–(f_3), (V_1)–(V_4), (α_1) and (K_1) hold, there exists a constant $a_1, \Lambda > 0$ for all $\|K\|_{\frac{q}{2-q}} < a_1$, $\lambda \geq \Lambda$ and $\mu \in (0, \mu_0]$, then equation (B_μ) has a nontrivial solution at negative energy, $u_1 \in E_\lambda$ and*

$$\|u_1\|_\lambda \leq \left(\frac{\mu_0 a_1 (1 + S^2 |\{V < c_0\}|^{\frac{4}{N}})^{\frac{q}{2}}}{1 - c_f \|\alpha\|_\infty (1 + S^2 |\{V < c_0\}|^{\frac{4}{N}})} \right)^{\frac{1}{2-q}}.$$

2 Variational framework

The norm of $L^r(\mathbb{R}^N)$ $(r > 1)$ is given by $\|u\|_r = (\int_{\mathbb{R}^N} |u|^r \,\mathrm{d}x)^{\frac{1}{r}}$. The norm of $H^2(\mathbb{R}^N)$ is

$$\|u\|_{H^2}^2 = \int_{\mathbb{R}^N} \left(|\Delta u|^2 + |\nabla u|^2 + |u|^2 \right) \mathrm{d}x.$$

Let

$$E = \left\{ u \in H^2(\mathbb{R}^N) \,\Big|\, \int_{\mathbb{R}^N} |\Delta u|^2 + V(x) u^2 \,\mathrm{d}x < \infty \right\}.$$

For $\lambda > 0$, the inner product and norm of E_λ are given by

$$\langle u, v \rangle_\lambda = \int_{\mathbb{R}^N} \left(\Delta u \Delta v + \nabla u \nabla v + \lambda V(x) uv \right) \mathrm{d}x, \qquad \|u\|_\lambda = \langle u, u \rangle^{\frac{1}{2}}.$$

Let us define the energy functional as follows:

$$I_\mu(u) = \frac{1}{2} \|u\|_\lambda^2 - \int_{\mathbb{R}^N} \alpha(x) F(u) \,\mathrm{d}x - \mu \int_{\mathbb{R}^N} K(x) |u|^q \,\mathrm{d}x$$

and

$$\langle I_\mu'(u), v\rangle = \int_{\mathbb{R}^N} \big(\Delta u \Delta v + \nabla u \nabla v + \lambda V(x) uv\big)\,\mathrm{d}x - \int_{\mathbb{R}^N} \alpha(x) f(u) v\,\mathrm{d}x \\ - \mu \int_{\mathbb{R}^N} K(x)|u|^{q-2} uv\,\mathrm{d}x. \qquad (4)$$

3 Proof of Theorems 1 and 2

Proof of Theorem 1. By condition (f_1), for $\forall \varepsilon > 0$, there exists $\delta(\varepsilon) > 0$, we have

$$\big|f(u)\big| \le \varepsilon|u|, \quad \text{for all } |u| < \delta(\varepsilon).$$

By condition (f_2), there exists $M > 0$, we get

$$\big|f(u)\big| \le |u|, \quad \text{for all } |u| \ge M.$$

Since f is a continuous function, f achieves its maximum and minimum on $[\delta(\varepsilon), M]$, so there exists a positive number $b(\varepsilon)$, we have that

$$\big|f(u)\big| \le b(\varepsilon) \le b(\varepsilon)\frac{|u|}{\delta(\varepsilon)} = C(\varepsilon)|u|, \quad \text{for all } \delta(\varepsilon) \le |u| \le M.$$

Then we obtain that

$$\big|f(u)\big| \le \big(1 + \varepsilon + C(\varepsilon)\big)|u|, \quad \text{for all } u \in \mathbb{R}.$$

Hence, the number $c_f = \max_{s \ne 0} |\frac{f(s)}{s}| > 0$ is well defined. □

Lemma 4 *Assume that* (V_1) *and* (V_2) *hold, for every* $\lambda \ge \Lambda$, *the embedding* $E_\lambda \hookrightarrow L^r(\mathbb{R}^N)$, $r \in [2, 2_*]$ *is continuous.*

Proof By using (V_1) and (V_2), we have

$$\int_{\mathbb{R}^N} u^2\,\mathrm{d}x \le \int_{\{V \ge c_0\}} u^2\,\mathrm{d}x + \int_{\{V < c_0\}} u^2\,\mathrm{d}x \\ \le \frac{1}{c_0}\int_{\{V \ge c_0\}} V(x) u^2\,\mathrm{d}x + S^2\big|\{V < c_0\}\big|^{\frac{4}{N}} \int_{\{V < c_0\}} |\Delta u|^2\,\mathrm{d}x,$$

where S is the best constant for the Sobolev embedding $H^2(\mathbb{R}^N) \hookrightarrow L^{2_*}(\mathbb{R}^N)$. Then we obtain

$$\|u\|_{H^2}^2 \le \big(1 + S^2\big|\{V < c_0\}\big|^{\frac{4}{N}}\big) \int_{\mathbb{R}^N} |\Delta u|^2\,\mathrm{d}x \\ + \int_{\mathbb{R}^N} |\nabla u|^2\,\mathrm{d}x + \frac{1}{c_0}\int_{\mathbb{R}^N} V(x) u^2\,\mathrm{d}x \\ \le \big(1 + S^2\big|\{V < c_0\}\big|^{\frac{4}{N}}\big) \int_{\mathbb{R}^N} |\Delta u|^2\,\mathrm{d}x + \int_{\mathbb{R}^N} |\nabla u|^2\,\mathrm{d}x$$

$$
\begin{aligned}
&+ \int_{\mathbb{R}^N} \left(1 + S^2 \left|\{V < c_0\}\right|^{\frac{4}{N}}\right) \lambda V(x) u^2 \,\mathrm{d}x \quad \text{for } \lambda \geq \frac{1}{(1 + S^2|\{a < c_0\}|^{\frac{4}{N}}) c_0} \\
\leq &\left(1 + S^2 \left|\{V < c_0\}\right|^{\frac{4}{N}}\right) \|u\|_\lambda^2.
\end{aligned} \tag{5}
$$

This implies that the embedding $E_\lambda \hookrightarrow H^2(\mathbb{R}^N)$ is continuous. By using Hölder's inequality, we obtain

$$
\begin{aligned}
\int_{\mathbb{R}^N} |u|^r \,\mathrm{d}x &\leq \left(\int_{\mathbb{R}^N} |u|^2 \,\mathrm{d}x\right)^{\frac{2N - r(N-4)}{8}} \left(\int_{\mathbb{R}^N} |u|^{2_*} \,\mathrm{d}x\right)^{\frac{N(r-2)}{4} \frac{(N-4)}{2N}} \\
&\leq \|u\|_2^{\frac{2N - r(N-4)}{4}} \|u\|_{2_*}^{\frac{N(r-2)}{4}} \\
&\leq \|u\|_2^{\frac{2N - r(N-4)}{4}} S^{\frac{N(r-2)}{4}} \|\Delta u\|_2^{\frac{N(r-2)}{4}} \\
&\leq \|u\|_{H^2}^{\frac{2N - r(N-4)}{4}} S^{\frac{N(r-2)}{4}} \|u\|_{H^2}^{\frac{N(r-2)}{4}} \\
&= S^{\frac{N(r-2)}{4}} \|u\|_{H^2}^r \\
&\leq S^{\frac{N(r-2)}{4}} \left[1 + S^2 \left|\{V < c_0\}\right|^{\frac{4}{N}}\right]^{\frac{r}{2}} \|u\|_\lambda^r,
\end{aligned} \tag{6}
$$

where $r \in [2, 2_*]$. We set

$$
\Theta_r = S^{\frac{N(r-2)}{4}} \left[1 + S^2 \left|\{V < c_0\}\right|^{\frac{4}{N}}\right]^{\frac{r}{2}} \quad \text{and} \quad \Lambda = \frac{1}{c_0(1 + S^2|\{V < c_0\}|^{\frac{4}{N}})}.
$$

Thus, for any $r \in [2, 2_*]$ and $\lambda \geq \Lambda$, there holds

$$
\int_{\mathbb{R}^N} |u|^r \,\mathrm{d}x \leq \Theta_r \|u\|_\lambda^r. \tag{7}
$$

This implies that the embedding $E_\lambda \hookrightarrow L^r(\mathbb{R}^N)$, $r \in [2, 2_*]$ is continuous. □

Proof of Theorem 2. Let $\mu = 0$, if we choose $v = u$ in (4), we obtain that

$$
\|u\|_\lambda^2 = \int_{\mathbb{R}^N} \alpha(x) f(u) u \,\mathrm{d}x,
$$

we have

$$
\begin{aligned}
\|u\|_\lambda^2 &\leq \|\alpha\|_\infty \int_{\mathbb{R}^N} \left|\frac{f(u)}{u}\right| u^2 \,\mathrm{d}x \\
&\leq \|\alpha\|_\infty c_f \int_{\mathbb{R}^N} u^2 \,\mathrm{d}x \\
&\leq \|\alpha\|_\infty c_f \left(1 + S^2 \left|\{V < c_0\}\right|^{\frac{4}{N}}\right) \|u\|_\lambda^2 \quad \big(\text{by (7)}\big) \\
&< \|\alpha\|_\infty c_f \left[1 + S^2 \left(\frac{1 - c_f \|\alpha\|_\infty}{S^2 c_f \|\alpha\|_\infty}\right)\right] \|u\|_\lambda^2 \quad \big(\text{by } (V_3)\big) \\
&= \|u\|_\lambda^2.
\end{aligned}
$$

Therefore, the inequality gives $u = 0$. □

4 Proof of Theorem 3

Lemma 5 *Assume that the assumptions of Theorem* 3 *hold, for every* $\lambda \geq \Lambda$, *then any Cerami sequence of* I_μ *is bounded in* E_λ.

Proof Let $\{u_n\} \subset E_\lambda$ be a Cerami sequence of I_μ satisfying

$$I_\mu(u_n) \text{ being bounded}, \quad \left(1 + \|u_n\|_\lambda\right) I'_\mu(u_n) \to 0, \quad \text{as } n \to \infty. \tag{8}$$

Argue by contradiction, let $\|u_n\|_\lambda \to \infty$. Due to (f_2), we have that for every $\varepsilon > 0$, there exists $A > 0$ such that $|F(u)| \leq \frac{\varepsilon}{2\|\alpha\|_\infty \Theta_2}|u|^2$ for every $|u| > A$, we have

$$\begin{aligned} \int_{\mathbb{R}^N} \alpha(x)F(u)\,\mathrm{d}x &= \int_{|u|>A} \alpha(x)F(u)\,\mathrm{d}x + \int_{|u|\leq A} \alpha(x)F(u)\,\mathrm{d}x \\ &\leq \frac{\varepsilon}{2}\|u\|_\lambda^2 + \|\alpha\|_1 \sup_{|u|\leq A}\left|F(u)\right|, \end{aligned} \tag{9}$$

and

$$\begin{aligned} &I_\mu(u_n) \\ &= \frac{1}{2}\|u_n\|_\lambda^2 - \int_{\mathbb{R}^N} \alpha(x)F(u_n)\,\mathrm{d}x - \mu \int_{\mathbb{R}^N} K(x)|u|^q\,\mathrm{d}x \\ &\geq \frac{1}{2}\|u_n\|_\lambda^2 - \int_{\mathbb{R}^N} \alpha(x)F(u_n)\,\mathrm{d}x - \frac{\mu_0}{q}\|K\|_{\frac{2}{2-q}} \Theta_2^{\frac{q}{2}} \|u_n\|_\lambda^q \\ &\geq \left(\frac{1}{2} - \frac{\varepsilon}{2}\right)\|u_n\|_\lambda^2 - \frac{\mu_0}{q}\|K\|_{\frac{2}{2-q}} \Theta_2^{\frac{q}{2}} \|u_n\|_\lambda^q - \|\alpha\|_1 \sup_{|u_n|\leq A}\left|F(u_n)\right|. \end{aligned} \tag{10}$$

Since $1 < q < 2$, if $\varepsilon < 1$, we have $I_\mu(u_n) \to \infty$ as $\|u_n\|_\lambda \to \infty$. There is a contraction with $I_\mu(u_n)$ bounded. The proof is completed. □

Lemma 6 *Assume that the assumptions of Theorem* 3 *hold, for every* $\lambda \geq \Lambda$, *then any Cerami sequence of* I_μ *has a convergent subsequence in* E_λ.

Proof Step 1. Let $\{u_n\}$ be a Cerami sequence of I_μ and $\{u_n\}$ is bounded in E_λ. For any fixed $R > 0$, let $\xi_R \in C^\infty(\mathbb{R}^N, \mathbb{R})$ such that

$$\xi_R(x) = \begin{cases} 0 & \text{for } |x| \leq \frac{R}{2}, \\ 1 & \text{for } |x| > R, \end{cases} \tag{11}$$

and

$$\xi_R(x) \in [0,1], \qquad \left|\nabla \xi_R(x)\right| \leq \frac{C}{R}, \qquad \left|\Delta \xi_R(x)\right| \leq \frac{C}{R^2}. \tag{12}$$

Step 2. First, for all $n \in \mathbb{N}$ and $R > 2R_0$, we have

$$\begin{aligned}
\int_{\mathbb{R}^N} \left|\Delta(u_n\xi_R)\right|^2 \mathrm{d}x &= \int_{\mathbb{R}^N} \big(|\Delta u_n|^2\xi_R^2 + 2u_n\xi_R\Delta u_n\Delta\xi_R + u_n^2|\Delta\xi_R|^2 \\
&\quad + 4(\xi_R\Delta u_n + u_n\Delta\xi_R)\nabla u_n\nabla\xi_R + 4|\nabla u_n|^2|\nabla\xi_R|^2\big)\,\mathrm{d}x \\
&\le \int_{\mathbb{R}^N} \big(|\Delta u_n|^2\xi_R^2 + 2|\xi_R\Delta u_n|\cdot|u_n\Delta\xi_R| + u_n^2|\Delta\xi_R|^2 + 4|\nabla u_n|^2|\nabla\xi_R|^2 \\
&\quad + 4|\xi_R\Delta u_n|\cdot|\nabla u_n\nabla\xi_R| + 4|u_n\Delta\xi_R|\cdot|\nabla u_n\nabla\xi_R|\big)\,\mathrm{d}x \\
&\le \int_{\mathbb{R}^N} \big(4|\Delta u_n|^2\xi_R^2 + 4u_n^2|\Delta\xi_R|^2 + 8|\nabla u_n|^2|\nabla\xi_R|^2\big)\,\mathrm{d}x \\
&\le 4\int_{\mathbb{R}^N} |\Delta u_n|^2\,\mathrm{d}x + \frac{4C^2}{R^4}\int_{\mathbb{R}^N} u_n^2\,\mathrm{d}x + \frac{8C^2}{R^2}\int_{\mathbb{R}^N} |\nabla u_n|^2\,\mathrm{d}x \\
&\le 4\left(1 + \frac{C^2}{R^4} + \frac{2C^2}{R^2}\right)\int_{\mathbb{R}^N} \big(|\Delta u_n|^2 + |\nabla u_n|^2 + u_n^2\big)\,\mathrm{d}x \\
&\le 4\left(1 + \frac{C^2}{R^4} + \frac{2C^2}{R^2}\right)\big(1 + S^2\left|\{V < c_0\}\right|^{\frac{4}{N}}\big)\|u_n\|_\lambda^2,
\end{aligned}$$

and

$$\begin{aligned}
\int_{\mathbb{R}^N} \left|\nabla(u_n\xi_R)\right|^2 \mathrm{d}x &\le 2\int_{\mathbb{R}^N} \big(|\nabla u_n|^2\xi_R^2 + u_n^2|\nabla\xi_R|^2\big)\,\mathrm{d}x \\
&\le 2\int_{\mathbb{R}^N} |\nabla u_n|^2\,\mathrm{d}x + \frac{2C^2}{R^2}\int_{\mathbb{R}^N} u_n^2\,\mathrm{d}x \\
&\le 2\left(1 + \frac{C^2}{R^2}\right)\int_{\mathbb{R}^N} \big(|\Delta u_n|^2 + |\nabla u_n|^2 + u_n^2\big)\,\mathrm{d}x \\
&\le 2\left(1 + \frac{C^2}{R^2}\right)\big(1 + S^2\left|\{V < c_0\}\right|^{\frac{4}{N}}\big)\|u_n\|_\lambda^2,
\end{aligned}$$

then

$$\begin{aligned}
&\int_{\mathbb{R}^N} \big(\left|\Delta(u_n\xi_R)\right|^2 + \left|\nabla(u_n\xi_R)\right|^2 + \lambda V(x)(u_n\xi_R)^2\big)\,\mathrm{d}x \\
&\quad\le \left(6 + \frac{4C^2}{R^4} + \frac{10C^2}{R^2}\right)\big(1 + S^2\left|\{V < c_0\}\right|^{\frac{4}{N}}\big)\|u_n\|_\lambda^2 + \int_{\mathbb{R}^N} \lambda V(x)(u_n\xi_R)^2\,\mathrm{d}x \\
&\quad\le \left(6 + \frac{4C^2}{R^4} + \frac{10C^2}{R^2}\right)\big(1 + S^2\left|\{V < c_0\}\right|^{\frac{4}{N}}\big)\|u_n\|_\lambda^2 + \int_{\mathbb{R}^N} \lambda V(x)u_n^2\,\mathrm{d}x,
\end{aligned} \tag{13}$$

which implies that

$$\|u_n\xi_R\|_\lambda \le \left[1 + \left(6 + \frac{4C^2}{R^4} + \frac{10C^2}{R^2}\right)\big(1 + S^2\left|\{V < c_0\}\right|^{\frac{4}{N}}\big)\right]^{\frac{1}{2}}\|u_n\|_\lambda. \tag{14}$$

According to (8), we know that $\|I_\mu'(u_n)\|_{E_\lambda^{-1}}\|u_n\|_\lambda \to 0$ as $n \to \infty$. For any $\varepsilon > 0$, there exists $n(\varepsilon)$ such that

$$\left\|I_\mu'(u_n)\right\|_{E_\lambda^{-1}}\|u_n\|_\lambda \le \frac{\varepsilon}{[1 + (6 + \frac{4C^2}{R^4} + \frac{10C^2}{R^2})(1 + S^2|\{V < c_0\}|^{\frac{4}{N}})]^{\frac{1}{2}}}, \tag{15}$$

for all $n \geq n(\varepsilon)$. Hence, applying (14) and (15), we know

$$\big|\langle I_{\mu}'(u_n), u_n\xi_R\rangle\big| \leq \big\|I_{\mu}'(u_n)\big\|_{E_{\lambda}^{-1}}\|u_n\xi_R\|_{\lambda} \leq \varepsilon, \tag{16}$$

for all $R > 2R_0$ and $n \geq n(\varepsilon)$.

(ii) Choosing $R > 2R_0$, by the result of (i), $\langle I_{\mu}'(u_n), u_n\xi_R\rangle = o(1)$, that is,

$$\begin{aligned} o(1) = {} & \int_{\mathbb{R}^N}\big(|\Delta u_n|^2 + \lambda V(x)u_n^2 + |\nabla u_n|^2\big)\xi_R \,\mathrm{d}x + \int_{\mathbb{R}^N} u_n\nabla u_n\nabla\xi_R \,\mathrm{d}x \\ & + \int_{\mathbb{R}^N} u_n\Delta u_n\Delta\xi_R \,\mathrm{d}x + 2\int_{\mathbb{R}^N}\Delta u_n\nabla u_n\nabla\xi_R \,\mathrm{d}x \\ & - \int_{\mathbb{R}^N}\alpha(x)f(u_n)u_n\xi_R \,\mathrm{d}x - \mu\int_{\mathbb{R}^N}K(x)|u_n|^q\xi_R \,\mathrm{d}x. \end{aligned} \tag{17}$$

Firstly, we estimate the fifth term in (17)

$$\begin{aligned} & \left|\int_{\mathbb{R}^N}\alpha(x)\frac{f(u_n)}{u_n}|u_n|^2\xi_R \,\mathrm{d}x\right| \\ & \quad \leq c_f\|\alpha\|_{\infty}\int_{|x|\geq\frac{R}{2}} u_n^2\xi_R \,\mathrm{d}x \\ & \quad \leq c_f\|\alpha\|_{\infty}\left(\int_{\{V\geq c_0\}\cap\{|x|\geq\frac{R}{2}\}} u_n^2\xi_R \,\mathrm{d}x + \int_{\{V<c_0\}\cap\{|x|\geq\frac{R}{2}\}} u_n^2\xi_R \,\mathrm{d}x\right) \\ & \quad \leq c_f\|\alpha\|_{\infty}\left(\frac{1}{c_0}\int_{\{V\geq c_0\}\cap\{|x|\geq\frac{R}{2}\}} V(x)u_n^2\xi_R \,\mathrm{d}x + \int_{\{V<c_0\}\cap\{|x|\geq\frac{R}{2}\}} u_n^2\xi_R \,\mathrm{d}x\right). \end{aligned} \tag{18}$$

By using (V_1), (V_4), and $R > 2R_0$, we have $\{V < c_0\} \cap \{|x| \geq \frac{R}{2}\} = \emptyset$. Then

$$\begin{aligned} \left|\int_{\mathbb{R}^N}\alpha(x)f(u_n)u_n\xi_R \,\mathrm{d}x\right| & \leq \frac{c_f\|\alpha\|_{\infty}}{c_0}\int_{\{V\geq c_0\}\cap\{|x|\geq\frac{R}{2}\}} V(x)u_n^2\xi_R \,\mathrm{d}x \\ & \leq \frac{c_f\|\alpha\|_{\infty}}{c_0}\int_{|x|\geq\frac{R}{2}} V(x)u_n^2\xi_R \,\mathrm{d}x. \end{aligned} \tag{19}$$

Next, we estimate the others in (17), we obtain

$$\begin{aligned} & \left|\int_{\mathbb{R}^N} u_n\nabla u_n\nabla\xi_R \,\mathrm{d}x\right| \\ & \quad \leq \frac{C}{R}\int_{R\geq|x|\geq\frac{R}{2}} |u_n||\nabla u_n| \,\mathrm{d}x \quad \big(\text{by using (12)}\big) \\ & \quad \leq \frac{C}{2R}\int_{R\geq|x|\geq\frac{R}{2}}\big(|\nabla u_n|^2 + u_n^2\big) \,\mathrm{d}x \\ & \quad \leq \frac{C}{2R}\int_{R\geq|x|\geq\frac{R}{2}}\big(|\Delta u_n|^2 + |\nabla u_n|^2 + u_n^2\big) \,\mathrm{d}x \\ & \quad \leq \frac{C}{2R}\int_{\{V\geq c_0\}\cap\{R\geq|x|\geq\frac{R}{2}\}}\left(|\Delta u_n|^2 + |\nabla u_n|^2 + \frac{\lambda V(x)}{\Lambda c_0}u_n^2\right) \mathrm{d}x \\ & \qquad + \frac{C}{2R}\int_{\{V<c_0\}\cap\{R\geq|x|\geq\frac{R}{2}\}}\big(|\Delta u_n|^2 + |\nabla u_n|^2 + u_n^2\big) \,\mathrm{d}x \end{aligned}$$

$$\begin{aligned}
&\leq \frac{C}{2R}\left(1+S^2\left|\{V<c_0\}\right|^{\frac{4}{N}}\right)\int_{R\geq|x|\geq\frac{R}{2}}\left(|\Delta u_n|^2+|\nabla u_n|^2+\lambda V(x)u_n^2\right)\mathrm{d}x\\
&\leq \frac{C_1}{2R} \qquad (20)
\end{aligned}$$

and

$$\begin{aligned}
&\left|\int_{\mathbb{R}^N}\Delta u_n\nabla u_n\nabla\xi_R\,\mathrm{d}x\right|\\
&\leq \frac{C}{R}\int_{R\geq|x|\geq\frac{R}{2}}|\Delta u_n||\nabla u_n|\,\mathrm{d}x\\
&\leq \frac{C}{2R}\int_{R\geq|x|\geq\frac{R}{2}}\left(|\Delta u_n|^2+|\nabla u_n|^2\right)\mathrm{d}x\\
&\leq \frac{C}{2R}\int_{R\geq|x|\geq\frac{R}{2}}\left(|\Delta u_n|^2+|\nabla u_n|^2+u_n^2\right)\mathrm{d}x\\
&\leq \frac{C}{2R}\int_{\{V\geq c_0\}\cap\{R\geq|x|\geq\frac{R}{2}\}}\left(|\Delta u_n|^2+|\nabla u_n|^2+\frac{\lambda V(x)}{\Lambda c_0}u_n^2\right)\mathrm{d}x\\
&\quad+\frac{C}{2R}\int_{\{V<c_0\}\cap\{R\geq|x|\geq\frac{R}{2}\}}\left(|\Delta u_n|^2+|\nabla u_n|^2+u_n^2\right)\mathrm{d}x\\
&\leq \frac{C_1}{2R} \qquad (21)
\end{aligned}$$

and

$$\begin{aligned}
&\left|\int_{\mathbb{R}^N}u_n\Delta u_n\Delta\xi_R\,\mathrm{d}x\right|\\
&\leq \frac{C}{R^2}\int_{R\geq|x|\geq\frac{R}{2}}|u_n||\Delta u_n|\,\mathrm{d}x\leq\frac{C}{2R^2}\int_{R\geq|x|\geq\frac{R}{2}}\left(|\Delta u_n|^2+u_n^2\right)\mathrm{d}x\\
&\leq \frac{C}{2R^2}\int_{\{V\geq c_0\}\cap\{R\geq|x|\geq\frac{R}{2}\}}\left(|\Delta u_n|^2+|\nabla u_n|^2+\frac{\lambda V(x)}{\Lambda c_0}u_n^2\right)\mathrm{d}x\\
&\quad+\frac{C}{2R^2}\int_{\{V<c_0\}\cap\{R\geq|x|\geq\frac{R}{2}\}}\left(|\Delta u_n|^2+|\nabla u_n|^2+u_n^2\right)\mathrm{d}x\\
&\leq \frac{C}{2R^2}\left(1+S^2\left|\{V<c_0\}\right|^{\frac{4}{N}}\right)\int_{R\geq|x|\geq\frac{R}{2}}\left(|\Delta u_n|^2+|\nabla u_n|^2+\lambda V(x)u_n^2\right)\mathrm{d}x\\
&\leq \frac{C_2}{R^2} \qquad (22)
\end{aligned}$$

and

$$\begin{aligned}
&\left|\int_{\mathbb{R}^N}K(x)|u_n|^q\xi_R\,\mathrm{d}x\right|\\
&\leq \|K\|_{L^{\frac{2}{2-q}}(B^c(0,\frac{R}{2}))}\|u\|^q_{L^2(B^c(0,\frac{R}{2}))}\\
&\leq \left(1+S^2\left|\{V<c_0\}\right|^{\frac{4}{N}}\right)^{\frac{q}{2}}\|K\|_{L^{\frac{2}{2-q}}(B^c(0,\frac{R}{2}))}\|u_n\|^q_{E_\lambda(B^c(0,\frac{R}{2}))}\quad\text{(by using (7))}\\
&\leq C_3\|K\|_{L^{\frac{2}{2-q}}(B^c(0,\frac{R}{2}))}. \qquad (23)
\end{aligned}$$

It follows from (17), (19)–(23) that

$$\begin{aligned}
&\int_{|x|\geq R}\left(|\Delta u_n|^2+\left(\Lambda-\frac{c_f\|\alpha\|_\infty}{c_0}\right)V(x)u_n^2+|\nabla u_n|^2\right)\mathrm{d}x \\
&\quad\leq \int_{|x|\geq R}\left(|\Delta u_n|^2+\left(\lambda-\frac{c_f\|\alpha\|_\infty}{c_0}\right)V(x)u_n^2+|\nabla u_n|^2\right)\mathrm{d}x \\
&\quad\leq \int_{|x|\geq \frac{R}{2}}\left(|\Delta u_n|^2+\left(\lambda-\frac{c_f\|\alpha\|_\infty}{c_0}\right)V(x)u_n^2+|\nabla u_n|^2\right)\xi_R\,\mathrm{d}x \\
&\quad\leq \frac{C_1}{R}+\frac{C_2}{R^2}+C_3\|K\|_{L^{\frac{2}{2-q}}(B^c(0,\frac{R}{2}))}.
\end{aligned}\tag{24}$$

Since $\{u_n\}$ is bounded in E_λ, by using conditions (V_1) and (V_4) again, we have $\{V < c_0\} \cap \{|x| \geq R\} = \emptyset$. Hence $V(x) \geq c_0$ in $\{|x| \geq R\}$, we know

$$\int_{|x|\geq R} V(x)u_n^2\,\mathrm{d}x \geq \int_{|x|\geq R} c_0 u_n^2\,\mathrm{d}x.\tag{25}$$

By using (24), for any $\varepsilon > 0$, there exists $R > 0$ such that, for n large enough, we get

$$\begin{aligned}
\varepsilon &\geq \int_{|x|\geq R}\left(|\Delta u_n|^2+|\nabla u_n|^2+\left(\Lambda-\frac{c_f\|\alpha\|_\infty}{c_0}\right)V(x)u_n^2\right)\mathrm{d}x \\
&\geq \int_{|x|\geq R}\left(|\Delta u_n|^2+|\nabla u_n|^2+\left(\Lambda-\frac{c_f\|\alpha\|_\infty}{c_0}\right)c_0u_n^2\right)\mathrm{d}x \quad \bigl(\text{by using (25)}\bigr) \\
&\geq \min\bigl(1, c_0\Lambda - c_f\|\alpha\|_\infty\bigr)\int_{|x|\geq R}\bigl(|\Delta u_n|^2+|\nabla u_n|^2+u_n^2\bigr)\,\mathrm{d}x.
\end{aligned}$$

So, we have

$$\int_{|x|\geq R}\bigl(|\Delta u_n|^2+|\nabla u_n|^2+u_n^2\bigr)\,\mathrm{d}x \leq \varepsilon.\tag{26}$$

Since $\{u_n\}$ is bounded in E_λ, we may assume that for some $u \in E_\lambda$, up to a subsequence, $u_n \rightharpoonup u$ in E_λ, by embedding from $H^2(\mathbb{R}^N)$ into $L^r_{\mathrm{loc}}(\mathbb{R}^N)$, $r \in [2, 2_*)$ is compact, and combining with (26), we know

$$u_n \to u \quad \text{in } L^r\bigl(\mathbb{R}^N\bigr), r \in [2, 2_*).\tag{27}$$

(iii) Since $\langle I'_\mu(u_n), u_n\rangle = o(1)$ and $\langle I'_\mu(u_n), u\rangle = o(1)$, we have

$$\begin{aligned}
o(1) = \bigl\langle I'_\mu(u_n), u_n\bigr\rangle &= \int_{\mathbb{R}^N}\bigl(|\Delta u_n|^2+|\nabla u_n|^2+\lambda V(x)u_n^2\bigr)\,\mathrm{d}x \\
&\quad - \int_{\mathbb{R}^N}\alpha(x)f(u_n)u_n\,\mathrm{d}x - \mu\int_{\mathbb{R}^N}K(x)|u_n|^q\,\mathrm{d}x
\end{aligned}\tag{28}$$

and

$$o(1) = \bigl\langle I'_\mu(u_n), u\bigr\rangle = \int_{\mathbb{R}^N}\bigl(\Delta u_n\Delta u+\nabla u_n\nabla u+\lambda V(x)u_n u\bigr)\,\mathrm{d}x$$

$$- \int_{\mathbb{R}^N} \alpha(x) f(u_n) u \,\mathrm{d}x - \mu \int_{\mathbb{R}^N} K(x) |u_n|^{q-2} u_n u \,\mathrm{d}x. \tag{29}$$

Since $u_n \rightharpoonup u$ in E_λ, that is,

$$\int_{\mathbb{R}^N} \big(\Delta u_n \Delta u + \nabla u_n \nabla u + \lambda V(x) u_n u \big) \,\mathrm{d}x = \|u\|_\lambda + o(1). \tag{30}$$

By (27), we obtain

$$\begin{aligned} &\int_{\mathbb{R}^N} \alpha(x) f(u_n) u \,\mathrm{d}x - \int_{\mathbb{R}^N} \alpha(x) f(u_n) u_n \,\mathrm{d}x \\ &\quad \le \int_{\mathbb{R}^N} \alpha(x) \big| f(u_n) \big| |u - u_n| \,\mathrm{d}x \\ &\quad \le c_f \|\alpha\|_\infty \int_{\mathbb{R}^N} |u_n| |u - u_n| \,\mathrm{d}x \\ &\quad \le c_f \|\alpha\|_\infty \|u_n\|_2 \|u - u_n\|_2 \to 0 \end{aligned} \tag{31}$$

and

$$\begin{aligned} &\int_{\mathbb{R}^N} K(x) |u_n|^{q-2} u_n u \,\mathrm{d}x - \int_{\mathbb{R}^N} K(x) |u_n|^q \,\mathrm{d}x \\ &\quad \le \int_{\mathbb{R}^N} K(x) |u_n|^{q-1} |u - u_n| \,\mathrm{d}x \\ &\quad \le \|K\|_{\frac{2}{2-q}} \|u_n\|_2^{q-1} \|u - u_n\|_2 \to 0. \end{aligned} \tag{32}$$

Combining (28)–(32), we get

$$\begin{aligned} o(1) &= \big\langle I'_\mu(u_n), u_n \big\rangle - \big\langle I'_\mu(u_n), u \big\rangle \\ &= \|u_n\|_\lambda^2 - \|u\|_\lambda^2 - \int_{\mathbb{R}^N} \alpha(x) f(u_n) u_n \,\mathrm{d}x + \int_{\mathbb{R}^N} \alpha(x) f(u_n) u \,\mathrm{d}x \\ &\quad - \mu \int_{\mathbb{R}^N} K(x) |u_n|^q \,\mathrm{d}x + \mu \int_{\mathbb{R}^N} K(x) |u_n|^{q-2} u_n u \,\mathrm{d}x \\ &= \|u_n\|_\lambda^2 - \|u\|_\lambda^2 + o(1). \end{aligned}$$

Therefore, $\{u_n\}$ converges strongly in E_λ and the Cerami condition holds for I_μ. The proof is completed. □

Lemma 7 *Assume that the assumptions of Theorem* 2 *hold, for every* $\lambda \ge \Lambda$, *then* I_μ *is bounded from below on* E_λ, *there holds*

$$I_\mu(u) \ge G := -\frac{\Xi(2-q)}{q} \left(\frac{\mu_0 \|K\|_{\frac{2}{2-q}} \Theta_2^{\frac{q}{2}}}{2\Xi} \right)^{\frac{2}{2-q}} - \|\alpha\|_1 \sup_{|u| \le A} \big| F(u) \big|.$$

Proof By (9) and Lemma 5, fixing $\varepsilon < 1$ and $\Xi := (1-\varepsilon)/2$, we get

$$\begin{aligned} I_\mu(u) &\geq \Xi\|u\|_\lambda^2 - \frac{\mu_0}{q}\|K\|_{\frac{2}{2-q}}\Theta_2^{\frac{q}{2}}\|u\|_\lambda^q - \|\alpha\|_1 \sup_{|u|\leq A}\left|F(u)\right| \\ &\geq -\frac{\Xi(2-q)}{q}\left(\frac{\mu_0\|K\|_{\frac{2}{2-q}}\Theta_2^{\frac{q}{2}}}{2\Xi}\right)^{\frac{2}{2-q}} - \|\alpha\|_1 \sup_{|u|\leq A}\left|F(u)\right|. \end{aligned}$$

We have that I_μ is bounded from below on E_λ and $G < 0$. □

Lemma 8 *Under the assumptions of Theorem* 2, *there exist* $a_1 > 0$ *and* $\rho > 0$ *such that, for all K with* $\|K\|_{\frac{2}{2-q}} < a_1$,

$$I_\mu(u) > 0, \quad \textit{for } u \in E_\lambda \textit{ with } \|u\|_\lambda = \rho.$$

Proof By conditions (f_1)–(f_3) and $\alpha(x) \in L^1(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N)$, for any $\tilde{\varepsilon} > 0$, there exists $C_{\tilde{\varepsilon}} > 0$, for every $u \in \mathbb{R}$, we have

$$\left|f(u)\right| \leq \frac{\tilde{\varepsilon}}{\|\alpha\|_\infty\Theta_2}|u| + C_{\tilde{\varepsilon}}|u|^{s-1} \tag{33}$$

and

$$\left|F(u)\right| \leq \frac{\tilde{\varepsilon}}{2\|\alpha\|_\infty\Theta_2}|u|^2 + \frac{C_{\tilde{\varepsilon}}}{s}|u|^s \tag{34}$$

for $s \in (2, 2_*)$. Fixing $\tilde{\varepsilon} < 1$ and $\tilde{\Xi} := (1-\tilde{\varepsilon})/2$, we have the following inequality:

$$\begin{aligned} I_\mu(u) &\geq \frac{1}{2}\|u\|_\lambda^2 - \frac{\mu_0}{q}\|K\|_{\frac{2}{2-q}}\Theta_2^{\frac{q}{2}}\|u\|_\lambda^q - \int_{\mathbb{R}^3}\alpha(x)\left(\frac{\tilde{\varepsilon}}{2\|\alpha\|_\infty\Theta_2}|u|^2 + \frac{C_{\tilde{\varepsilon}}}{s}|u|^s\right)\mathrm{d}x \\ &\geq \tilde{\Xi}\|u\|_\lambda^2 - \frac{\mu_0}{q}\|K\|_{\frac{2}{2-q}}\Theta_2^{\frac{q}{2}}\|u\|_\lambda^q - \frac{C_\varepsilon\Theta_s^s}{s}\|\alpha\|_\infty\|u\|_\lambda^s \\ &\geq \left(\tilde{\Xi} - \frac{\mu_0}{q}\|K\|_{\frac{2}{2-q}}\Theta_2^{\frac{q}{2}}\|u\|_\lambda^{q-2} - \frac{C_\varepsilon\Theta_s^s}{s}\|\alpha\|_\infty\|u\|_\lambda^{s-2}\right)\|u\|_\lambda^2. \end{aligned}$$

Let

$$g(t) = \tilde{\Xi} - \frac{\mu_0}{q}\|K\|_{\frac{2}{2-q}}\Theta_2^{\frac{q}{2}}t^{q-2} - \frac{C_\varepsilon\Theta_s^s}{s}\|\alpha\|_\infty t^{s-2} \quad \text{for } t > 0.$$

Since $1 < q < 2 < s < 2_*$, it is easy to see that the function $g(t)$ achieves its maximum on $(0, +\infty)$ at some $t_0 > 0$. Moreover, there exists $a_1 > 0$ such that, for $\|K\|_{\frac{2}{2-q}} < a_1$, the maximum

$$g(t_0) = \max_{t\in(0,\infty)} g(t) > 0.$$

Take $\rho = t_0$ such that the conclusion holds. □

Proof of Theorem 3. For $\rho > 0$ given by Lemma 8, we define

$$\overline{B}(0,\rho) := \left\{u \in E_\lambda \mid \|u\|_\lambda \leq \rho\right\}, \qquad \partial B(0,\rho) := \left\{u \in E_\lambda \mid \|u\|_\lambda = \rho\right\} \tag{35}$$

and

$$I_\mu|_{\partial B(0,\rho)} > 0. \tag{36}$$

By Lemma 7, I_μ is bounded from below on $\overline{B}(0,\rho)$. Let $c_1 := \inf\{I_\mu | u \in \overline{B}(0,\rho)\} > -\infty$. By using condition (K_1) and $1 < q < 2$, it is easy to check that

$$I_\mu(tu) < 0 \quad \text{for } t \text{ small}. \tag{37}$$

Thus $c_1 < 0$. By (36), Lemma 5, Lemma 6, and Ekeland's variational principle, c_1 can be achieved at some inner point $u_1 \in \overline{B}(0,\rho)$ and u_1 is a critical point of I_μ at negative energy.

The norm estimate of u_1. By (4), we know

$$\begin{aligned} 0 &= \|u_1\|_\lambda^2 - \int_{\mathbb{R}^N} \alpha(x) f(u_1) u_1 \,\mathrm{d}x - \mu \int_{\mathbb{R}^N} K(x)|u_1|^q \,\mathrm{d}x \\ &\geq \|u_1\|_\lambda^2 - c_f\|\alpha\|_\infty \|u_1\|_2^2 - \mu_0 a_1 \|u_1\|_2^q \\ &\geq \left(\left(1 - c_f\|\alpha\|_\infty \Theta_2\right)\|u_1\|_\lambda^{2-q} - \mu_0 a_1 \Theta_2^{\frac{q}{2}}\right)\|u_1\|_\lambda^q. \end{aligned} \tag{38}$$

By using condition (V_3), we get

$$\begin{aligned} 1 &= c_f\|\alpha\|_\infty \left(1 + \frac{1 - c_f\|\alpha\|_\infty}{c_f\|\alpha\|_\infty}\right) \\ &> c_f\|\alpha\|_\infty \left(1 + S^2\left|\{V < c_0\}\right|^{\frac{4}{N}}\right) = c_f\|\alpha\|_\infty \Theta_2. \end{aligned} \tag{39}$$

Hence, we have

$$\|u_1\|_\lambda^{2-q} \leq \frac{\mu_0 a_1 \Theta_2^{\frac{q}{2}}}{1 - c_f\|\alpha\|_\infty \Theta_2} = \frac{\mu_0 a_1 (1 + S^2|\{V < c_0\}|^{\frac{4}{N}})^{\frac{q}{2}}}{1 - c_f\|\alpha\|_\infty (1 + S^2|\{V < c_0\}|^{\frac{4}{N}})}. \tag{40}$$

□

5 Conclusion

Biharmonic equations with steep potential well have attracted much attention in recent years. This paper considers a class of sublinear biharmonic equations with steep potential well.

Equation (B_μ) has different solutions when $\mu \in [0,\mu_0]$ takes different values, where $0 < \mu_0 < \infty$. When $\mu = 0$, Equation (B_μ) has only the trivial solution. When $\mu \in (0,\mu_0]$, it has a nontrivial solution u_1 at negative energy, and

$$\|u_1\|_\lambda \leq \left(\frac{\mu_0 a_1 (1 + S^2|\{V < c_0\}|^{\frac{4}{N}})^{\frac{q}{2}}}{1 - c_f\|\alpha\|_\infty (1 + S^2|\{V < c_0\}|^{\frac{4}{N}})}\right)^{\frac{1}{2-q}}.$$

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