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Global boundedness of solutions in a reaction–diffusion system of Beddington–DeAngelis-type predator–prey model with nonlinear prey-taxis and random diffusion

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Abstract
In this paper, a reaction–diffusion system of a predator–prey model with Beddington–DeAngelis functional response is considered. This model describes a prey-taxis mechanism that is an immediate movement of the predator u in response to a change of the prey v (which leads to the collection of u). We use some approaches to prove the global existence-boundedness of classical solutions and overcome the substantial difficulty of the existence of a nonlinear prey-taxis term.

MSC: Primary 35A01; secondary 35K57

Keywords: Predator–prey; Reaction–diffusion system; Beddington–DeAngelis-type; Boundedness

1 Introduction

In this paper, we investigate the following reaction–diffusion system of a Beddington–DeAngelis-type predator–prey model with nonlinear prey-taxis and random diffusion:

$$\begin{cases} u_t - d_1\Delta u + \nabla\cdot(u\chi(u)\nabla v) = -au + \frac{\delta uv}{\beta u+v+\alpha}, & x\in\Omega, t\in(0,T),\\ v_t - d_2\Delta v = rv - \frac{r}{K}v^2 - \frac{\gamma uv}{\beta u+v+\alpha}, & x\in\Omega, t\in(0,T),\\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & x\in\partial\Omega, t\in(0,T),\\ (u(x,0),v(x,0)) = (u_0(x),v_0(x)) \geq 0, & x\in\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ ($N = 1,2,3$) with smooth boundary $\partial\Omega$, $0 < T \leq +\infty$, initial condition $u_0(x), v_0(x) \in C^{2+\alpha}(\overline{\Omega})$ compatible on $\partial\Omega$, constants $d_1, d_2, a, K, r, \alpha, \beta, \gamma, \delta > 0$, and ν is the outward directional derivative normal to $\partial\Omega$.

Differential equations are supposed to be sufficient in modeling of the countless processes in all fields of science. Many phenomena in physical sciences, chemistry and biology are naturally described by Partial Differential Equations (PDEs). In a population dynamics system, a lot of mathematical models in partial differential equations to study the relationship between predators and preys have been proposed by mathematicians, biologists and ecologists [1, 2]. In fact, there are many well-known predator–prey models such as ratio-dependent predator–prey models (so-called Michaelis–Menten-type models) [3, 4], Holling-type models [5], Holling-type II models [6], Ivlev-type models [7], Lotka–Volterra-type models [8, 9] and so on. To the best of our knowledge, this is the first study to report that the global boundedness of solutions of the Beddington–DeAngelis-type predator–prey model with nonlinear prey-taxis.

There is a Beddington–DeAngelis-type functional response contained in the model (1.1), where u and v describe the population density of the predator and the prey at time t with diffusion rates d_1 and d_2, respectively. The parameter a denotes the death rate of the predator u. As usual, K is called the carrying capacity of the prey v. The constant r is called the intrinsic growth rate of the prey v. The constants δ, γ are the conversion rate of the predator and the prey, respectively. The term βu measures the mutual interference between predators. The reason for the model is that if one takes the parameter β close to 0 then the model can be regarded as a Holling-type II predator–prey model,

$$\begin{cases} u_t - d_1\Delta u + u + \nabla\cdot(u\chi(u)\nabla v) = -au + \frac{\delta uv}{v+\alpha}, & x\in\Omega, t\in(0,T), \\ v_t - d_2\Delta v = rv - \frac{r}{K}v^2 - \frac{\gamma uv}{v+\alpha}, & x\in\Omega, t\in(0,T), \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & x\in\partial\Omega, t\in(0,T), \\ (u(x,0),v(x,0)) = (u_0(x),v_0(x)) \geq 0, & x\in\Omega, \end{cases} \tag{1.2}$$

and also if one takes the parameter α close to 0 then the model can be thought of as a ratio-dependent predator–prey model,

$$\begin{cases} u_t - d_1\Delta u + u + \nabla\cdot(u\chi(u)\nabla v) = -au + \frac{\delta uv}{\beta u+v}, & x\in\Omega, t\in(0,T), \\ v_t - d_2\Delta v = rv - \frac{r}{K}v^2 - \frac{\gamma uv}{\beta u+v}, & x\in\Omega, t\in(0,T), \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & x\in\partial\Omega, t\in(0,T), \\ (u(x,0),v(x,0)) = (u_0(x),v_0(x)) \geq 0, & x\in\Omega. \end{cases} \tag{1.3}$$

In the present paper, motivated by [10–13], we will prove the global boundedness of classical solutions to (1.1). In a reaction–diffusion system of predator–prey model, the prey-taxis mechanism means a direct movement of the predator u in response to a variation of the prey v (which results in the aggregation of u), and the involved factor is assumed to satisfy $\chi(u)\in C^1([0,+\infty))$, $\chi(u)\equiv 0$ for $u\geq M$, and $|\chi'(u_1)-\chi'(u_2)|\leq L|u_1-u_2|$ for $u_1,u_2\in[0,+\infty)$, with $L,M>0$. Here the assumption $\chi(u)\equiv 0$ for $u\geq M$ says that there is a threshold value M for the accumulation of u, over which the prey-tactic cross-diffusion $\chi(u)$ vanishes. That is the statement that the author proves.

Theorem 1.1 *Suppose that* $\chi(u)$ *satisfies*:
(i) $\chi(u)\in C^1([0,+\infty))$;
(ii) $\chi(u)\equiv 0$ *for* $u\geq M$, *with* $M>0$;
(iii) $|\chi'(u_1)-\chi'(u_2)|\leq L|u_1-u_2|$ *for* $u_1,u_2\in[0,+\infty)$, *with* $L>0$,
then we see that the solutions to (1.1) *are global and uniformly bounded in time*.

The remainder of this paper is organized as follows. In Sect. 2, we propose some preliminary results, which are essential to the proof of Theorem 1.1. Section 3 illustrates the proof of our main theorem, the global boundedness of solutions, based, mainly, on the use of the standard Moser iterative technique.

2 Preliminaries

In this section we state the following lemmas which are essential in the proofs of Theorem 1.1. The first is the global existence of classical solutions to (1.1).

Lemma 2.1 *Under the assumptions for $\chi(u)$ and initial data in the paper, there exists a unique solution $(u,v) \in (C^{2+\alpha,1+\frac{\alpha}{2}}(\Omega \times (0,T)))^2$ of (1.1) for any given $T > 0$.*

The proof of Lemma 2.1 is similar to the proof of Theorem 3.5 in [14]. Hence, we omit it. The second is on the boundedness of v.

Lemma 2.2 *Let $(u,v) \in (C^{2+\alpha,1+\frac{\alpha}{2}}(\Omega \times (0,T)))^2$ be a solution of (1.1). Then $u \geq 0$ and $0 \leq v \leq K_0 = \max\{\max_{\overline{\Omega}} v_0(x), K\}$.*

The proof of the lemma is based on the comparison principle of Ordinary Differential Equations (ODEs). Refer to the proof of Lemma 3.1 in [14] for the details.

The next lemma is the well-known classical $L^p - L^q$ estimate for the Neumann heat semigroup on bounded domains.

Lemma 2.3 *Suppose $(e^{t\Delta})_{t>0}$ is the Neumann heat semigroup in Ω, and $\lambda_1 > 0$ denotes the first nonzero eigenvalue of $-\Delta$ in Ω under Neumann boundary conditions. Then the following $L^p - L^q$ estimates hold with $C_1, C_2 > 0$ only depending on Ω:*

(i) *if $1 \leq q \leq p \leq +\infty$, then*

$$\left\|\nabla e^{t\Delta} w\right\|_{L^p(\Omega)} \leq C_1\left(1 + t^{-\frac{1}{2}-\frac{n}{2}(\frac{1}{q}-\frac{1}{p})}\right)e^{-\lambda_1 t}\|w\|_{L^p(\Omega)}, \quad t > 0$$

for all $w \in L^q(\Omega)$;

(ii) *if $2 \leq q \leq p < +\infty$, then*

$$\left\|\nabla e^{t\Delta} w\right\|_{L^p(\Omega)} \leq C_2\left(1 + t^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{p})}\right)e^{-\lambda_1 t}\|\nabla w\|_{L^p(\Omega)}, \quad t > 0$$

for all $w \in W^{1,q}(\Omega)$.

Lemma 2.4 *Suppose that $T \in (0,\infty]$ and that $\Omega \subset \mathbb{R}^n$, $n \geq 1$, is a bounded domain, and that D, f and g comply with $D \in C^1(\overline{\Omega} \times [0,T) \times [0,\infty))$ and $D \geq 0$, $f \in C^0((0,T); C^0(\overline{\Omega}) \cap C^1(\Omega))$ and $g \in C^0(\Omega \times (0,T))$ with $f \cdot \nu \leq 0$ on $\partial\Omega \times (0,T)$. Moreover, assume that $D(x,t,s) \geq \delta s^{m-1}$, $f \in L^\infty((0,T); L^{q_1}(\Omega))$ and $g \in L^\infty((0,T); L^{q_2}(\Omega))$ for all $x \in \Omega$, $t \in (0,T)$, $\delta > 0$ and $s \geq s_0$ and for some $\delta > 0$, $m \in \mathbb{R}$ and $s_0 \geq 1$, and some $q_1 > n+2$ and $q_2 > \frac{n+2}{2}$. Then if $u \in C^0(\overline{\Omega} \times [0,T)) \cap C^{2,1}(\overline{\Omega} \times [0,T))$ is a nonnegative function satisfying*

$$\begin{cases} u_t \leq \nabla \cdot (D(x,t,u)\nabla u) + \nabla \cdot f(x,t) + g(x,t), & x \in \Omega, t \in (0,T), \\ \partial_\nu u(x,t) \leq 0, & x \in \partial\Omega, t \in (0,T), \end{cases}$$

and if $u \in L^\infty((0,T); L^{p_0}(\Omega))$ is valid for some $p_0 \geq 1$ fulfilling

$$p_0 > 1 - m \cdot \frac{(n+1)q_1 - (n+2)}{q_1 - (n+2)}$$

and

$$p_0 > 1 - \frac{m}{1 - \frac{nq_2}{(n+2)(q_2-1)}}$$

as well as

$$p_0 > \frac{n(1-m)}{2},$$

then there exists $C > 0$, *only depending on* m, δ, Ω, $\|f\|_{L^\infty((0,T);L^{q_1}(\Omega))}$, $\|g\|_{L^\infty((0,T);L^{q_2}(\Omega))}$, $\|u\|_{L^\infty((0,T);L^{q_0}(\Omega))}$ *and* $\|u(0)\|_{L^\infty(\Omega)}$, *such that*

$$\|u(t)\|_{L^\infty(\Omega)} \le C$$

for all $t \in (0, T)$.

Refer to the proof of Lemma A.1 in [15] for the details.

3 Proof of main result

In this section, we will deal with the proof of the main result of the paper.

Proof of Theorem 1.1 The proof consists of four parts.

Part 1: *Boundedness of* $\|u\|_{L^1(\Omega)}$.

Integrate the sum of the γ times of the first equation and the δ times of the second equation in (1.1) on Ω by parts,

$$\frac{d}{dt}\int_\Omega \gamma u + \frac{d}{dt}\int_\Omega \delta v = -a\gamma \int_\Omega u + r\delta \int_\Omega v - \frac{r\delta}{K}\int_\Omega v^2. \tag{3.1}$$

Employing Young's inequality, we have

$$2r\delta \int_\Omega v \le \frac{r\delta}{K}\int_\Omega v^2 + Kr\delta|\Omega|.$$

Setting the last inequality into (3.1), we obtain

$$\begin{aligned} \frac{d}{dt}\int_\Omega \gamma u + \frac{d}{dt}\int_\Omega \delta v &= -a\gamma \int_\Omega u + r\delta \int_\Omega v - \frac{r\delta}{K}\int_\Omega v^2 \\ &\le -a\gamma \int_\Omega u - r\delta \int_\Omega v + Kr\delta|\Omega|. \end{aligned} \tag{3.2}$$

Define

$$y_1(t) = \int_\Omega \gamma u + \int_\Omega \delta v, \quad t > 0.$$

Then

$$y_1'(t) + k_1 y_1(t) \le k_2$$

for all $t > 0$ by (3.2) with $k_1 = \min\{a, r\}$ and $k_2 = Kr\delta|\Omega|$. This ensures

$$y_1(t) \le C_1 = \max\left\{y_1(0), \frac{k_2}{k_1}\right\}$$

for all $t > 0$ by the comparison principle of ordinary differential equations.

Part 2: *Boundedness of* $\|u\|_{L^p(\Omega)}$ *with* $p > 2$.

Multiplying the first equation in (1.1) by u^{p-1} and integrate on Ω by parts. Since $v \le K_0$ by Lemma 2.1, we have

$$\int_\Omega u_t \cdot u^{p-1} - \int_\Omega d_1 \Delta u \cdot u^{p-1} = -a\int_\Omega u^p + \delta \int_\Omega \frac{u^p v}{\beta u + v + \alpha}.$$

Now, we need to prove the following inequality:

$$(p-1)\int_\Omega \chi(u)u^{p-1}\nabla u \cdot \nabla v \le \frac{d_1(p-1)}{2}\int_\Omega u^{p-2}|\nabla u|^2 + \frac{p-1}{2d_1}\int_\Omega \chi^2(u)u^p|\nabla v|^2.$$

By simplifying the problem, we only need to prove

$$\chi(u)u^{p-1}\nabla u \cdot \nabla v \le \frac{d_1}{2}u^{p-2}|\nabla u|^2 + \frac{1}{2d_1}\chi^2(u)u^p|\nabla v|^2.$$

Applying Young's inequality with ε ($ab \le \frac{\varepsilon}{p}a^p + \frac{\varepsilon^{-\frac{q}{p}}}{q}b^q$) and setting $p = q = 2$, $\varepsilon = d_1$, $a = u^{\frac{p-2}{2}}\nabla u$ and $b = \chi(u)u^{\frac{p}{2}}\nabla v$, we obtain

$$\begin{aligned}\chi(u)u^{p-1}\nabla u \cdot \nabla v &= \chi(u)u^{\frac{p-2}{2}+\frac{p}{2}}\nabla u \cdot \nabla v = \left(u^{\frac{p-2}{2}}\nabla u\right)\cdot\left(\chi(u)u^{\frac{p}{2}}\nabla v\right)\\ &\le \frac{d_1}{2}u^{p-2}|\nabla u|^2 + \frac{1}{2d_1}\chi^2(u)u^p|\nabla v|^2.\end{aligned}$$

Multiplying the inequality by $(p-1)$ and integrating on Ω by parts yield

$$(p-1)\int_\Omega \chi(u)u^{p-1}\nabla u \cdot \nabla v \le \frac{d_1(p-1)}{2}\int_\Omega u^{p-2}|\nabla u|^2 + \frac{p-1}{2d_1}\int_\Omega \chi^2(u)u^p|\nabla v|^2.$$

Multiply the first equation of u in (1.1) by u^{p-1} and integrate on Ω by parts. Since $v \le K_0$ by Lemma 2.1, we have

$$\begin{aligned}&\int_\Omega u_t \cdot u^{p-1} - \int_\Omega d_1\Delta u \cdot u^{p-1} + \int_\Omega \nabla\cdot\left(u\chi(u)\nabla v\right)\cdot u^{p-1}\\ &\quad = -a\int_\Omega u^p + \delta\int_\Omega \frac{u^p v}{\beta u + v + \alpha}.\end{aligned}$$

According to

$$\begin{aligned}&\int_\Omega u_t \cdot u^{p-1} = \frac{1}{p}\int_\Omega pu^{p-1}\cdot u_t = \frac{1}{p}\int_\Omega \frac{d}{dt}u^p = \frac{1}{p}\frac{d}{dt}\int_\Omega u^p,\\ &\int_\Omega d_1\cdot\nabla\cdot\left(\nabla u \cdot u^{p-1}\right) = d_1\int_\Omega \Delta u \cdot u^{p-1} + d_1(p-1)\int_\Omega u^{p-2}|\nabla u|^2 = 0,\end{aligned}$$

and

$$\int_\Omega \nabla \cdot \big(u\chi(u)\nabla v\big) \cdot u^{p-1} + (p-1)\int_\Omega \chi(u)u^{p-1}\nabla u \cdot \nabla v = 0,$$

we have

$$\begin{aligned}
&\frac{1}{p}\frac{d}{dt}\int_\Omega u^p + d_1(p-1)\int_\Omega u^{p-2}|\nabla u|^2 \\
&\quad = -a\int_\Omega u^p + \delta\int_\Omega \frac{u^p v}{\beta u + v + \alpha} + (p-1)\int_\Omega \chi(u)u^{p-1}\nabla u \cdot \nabla v \\
&\quad \le -a\int_\Omega u^p + \delta\int_\Omega \frac{u^p v}{v+\alpha} + \frac{d_1(p-1)}{2}\int_\Omega u^{p-2}|\nabla u|^2 + \frac{p-1}{2d_1}\int_\Omega \chi^2(u)u^p|\nabla v|^2 \\
&\quad \le \left(-a + \frac{\delta K_0}{\alpha + K_0}\right)\int_\Omega u^p + \frac{d_1(p-1)}{2}\int_\Omega u^{p-2}|\nabla u|^2 + \frac{p-1}{2d_1}\int_\Omega \chi^2(u)u^p|\nabla v|^2.
\end{aligned}$$

Consequently, together with $\chi(u) \le M_1$ due to $\chi(u) \in C^1$ and $\chi \equiv 0$ for $u \ge M$, we have

$$\begin{aligned}
&\frac{1}{p}\frac{d}{dt}\int_\Omega u^p + \frac{d_1(p-1)}{2}\int_\Omega u^{p-2}|\nabla u|^2 \\
&\quad \le \left(-a + \frac{\delta K_0}{\alpha + K_0}\right)\int_\Omega u^p + \frac{p-1}{2d_1}\int_\Omega \chi^2(u)u^p|\nabla v|^2 \\
&\quad \le \left(-a + \frac{\delta K_0}{\alpha + K_0}\right)\int_\Omega u^p + \frac{(p-1)M_1^2M^p}{2d_1}\int_\Omega |\nabla v|^2. \qquad (3.3)
\end{aligned}$$

Multiply the second equation of v in (1.1) by $-\Delta v$, and integrate on Ω by parts to obtain

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega |\nabla v|^2 + 2d_2\int_\Omega |\Delta v|^2 \\
&\quad = 2r\int_\Omega |\nabla v|^2 - \frac{4r}{K}\int_\Omega v|\nabla v|^2 + 2\gamma\int_\Omega \frac{uv}{\beta u + v + \alpha}\Delta v \\
&\quad \le 2r\int_\Omega |\nabla v|^2 + 2\gamma\int_\Omega \frac{uv}{\beta u + v + \alpha}\Delta v \\
&\quad \le 2r\int_\Omega |\nabla v|^2 + 2\gamma\int_\Omega \frac{uv}{\beta u + v + \alpha}|\Delta v| \\
&\quad \le 2r\int_\Omega |\nabla v|^2 + 2\gamma\int_\Omega \frac{uv}{v + \alpha}|\Delta v| \\
&\quad \le 2r\int_\Omega |\nabla v|^2 + \frac{2\gamma K_0}{K_0 + \alpha}\int_\Omega u|\Delta v|.
\end{aligned}$$

Employing Young's inequality, we have

$$\frac{2\gamma K_0}{K_0+\alpha}\int_\Omega u|\Delta v| \le \frac{\varepsilon}{2}\int_\Omega |\Delta v|^2 + \frac{2\gamma^2K_0^2}{\varepsilon(K_0+\alpha)^2}\int_\Omega u^2.$$

Setting $\varepsilon = 2d_2$, we obtain

$$\frac{d}{dt}\int_\Omega |\nabla v|^2 + d_2\int_\Omega |\Delta v|^2 \le 2r\int_\Omega |\nabla v|^2 \frac{\gamma^2K_0^2}{d_2(K_0+\alpha)^2}\int_\Omega u^2. \qquad (3.4)$$

According to

$$d_1(p-1)\int_\Omega u^{p-2}|\nabla u|^2 = d_1(p-1)\int_\Omega u^{\frac{p-2}{2}\cdot 2}|\nabla u|^2 = \frac{4d_1(p-1)}{p^2}\int_\Omega \left(\frac{p}{2}\right)^2 u^{(\frac{p}{2}-1)\cdot 2}|\nabla u|^2 = \frac{4d_1(p-1)}{p^2}\int_\Omega \left|\nabla u^{\frac{p}{2}}\right|^2,$$

for $p > 2$, we know from (3.4) and the last equality by Young's inequality that

$$\begin{aligned}
&\frac{1}{p}\frac{d}{dt}\int_\Omega u^p + \frac{d}{dt}\int_\Omega |\nabla v|^2 + \frac{2d_1(p-1)}{p^2}\int_\Omega \left|\nabla u^{\frac{p}{2}}\right|^2 + d_2\int_\Omega |\Delta v|^2 \\
&\quad\le \left(-a + \frac{\delta K_0}{\alpha + K_0}\right)\int_\Omega u^p + \frac{(p-1)M_1^2 M^p}{2d_1}\int_\Omega |\nabla v|^2 \\
&\qquad + 2r\int_\Omega |\nabla v|^2 + \frac{\gamma^2 K_0^2}{d_2(K_0+\alpha)^2}\int_\Omega u^2 \\
&\quad\le \left(-a + \frac{\delta K_0}{\alpha + K_0} + 1\right)\int_\Omega u^p + \left(2r + \frac{(p-1)M_1^2 M^p}{2d_1}\right)\int_\Omega |\nabla v|^2 + k_3
\end{aligned} \tag{3.5}$$

with $k_3 = \frac{\gamma^2 K_0^2 M^2|\Omega|}{d_2(K_0+\alpha)^2} > 0$.

For $\int_\Omega |\nabla v|^2$, applying the Sobolev interpolation inequality,

$$\left\|D^j v\right\|_{p,\Omega} \le \varepsilon \left\|D^k v\right\|_{p,\Omega} + C\|v\|_{p,\Omega},$$

setting $j = 1$, $k = 2$, $p = 2$, and integrating on Ω by parts, it is easy to see that

$$\int_\Omega |\nabla v|^2 \le \varepsilon_1\int_\Omega |\Delta v|^2 + k_4\int_\Omega |v|^2 \le \varepsilon_1\int_\Omega |\Delta v|^2 + k_4 K_0^2|\Omega| = \varepsilon_1\int_\Omega |\Delta v|^2 + k_5 \tag{3.6}$$

depending on ε_1.

For $\int_\Omega u^p$, by the Gagliardo–Nirenberg inequality with $u \ge 0$, we obtain

$$\begin{aligned}
\int_\Omega u^p = \int_\Omega \left|u^{\frac{p}{2}}\right|^2 &\le k_6\left(\left\|\nabla u^{\frac{p}{2}}\right\|_2^{\frac{2Np-2N}{Np-N+2}} \cdot \left\|u^{\frac{p}{2}}\right\|_{\frac{2}{p}}^{\frac{4}{Np-N+2}} + \left\|u^{\frac{p}{2}}\right\|_{\frac{2}{p}}^2\right) \\
&= k_6\left(\left\|\nabla u^{\frac{p}{2}}\right\|_2^{2\theta} \cdot \left\|u^{\frac{p}{2}}\right\|_{\frac{2}{p}}^{2(1-\theta)} + \left\|u^{\frac{p}{2}}\right\|_{\frac{2}{p}}^2\right)
\end{aligned} \tag{3.7}$$

with $k_6 > 0$ and $0 < \theta = \frac{Np-N}{Np-N+2} < 1$. Applying Young's inequality yields

$$\left\|\nabla u^{\frac{p}{2}}\right\|_2^{2\theta} \cdot \left\|u^{\frac{p}{2}}\right\|_{\frac{2}{p}}^{2(1-\theta)} \le \epsilon\theta\left\|\nabla u^{\frac{p}{2}}\right\|_2^2 + \epsilon^{\frac{\theta}{\theta-1}}(1-\theta)\left\|u^{\frac{p}{2}}\right\|_{\frac{2}{p}}^2$$

with $\epsilon > 0$. Setting the last estimate into (3.7), we see that

$$\begin{aligned}
\int_\Omega u^p = \int_\Omega \bigl|u^{\frac{p}{2}}\bigr|^2 &\le k_6\bigl(\bigl\|\nabla u^{\frac{p}{2}}\bigr\|_2^{2\theta} \cdot \bigl\|u^{\frac{p}{2}}\bigr\|_{\frac{2}{p}}^{2(1-\theta)} + \bigl\|u^{\frac{p}{2}}\bigr\|_{\frac{2}{p}}^2\bigr) \\
&\le k_6\bigl(\epsilon\theta\bigl\|\nabla u^{\frac{p}{2}}\bigr\|_2^2 + \epsilon^{\frac{\theta}{\theta-1}}(1-\theta)\bigl\|u^{\frac{p}{2}}\bigr\|_{\frac{2}{p}}^2 + \bigl\|u^{\frac{p}{2}}\bigr\|_{\frac{2}{p}}^2\bigr) \\
&= k_6\epsilon\theta\bigl\|\nabla u^{\frac{p}{2}}\bigr\|_2^2 + k_6\epsilon^{\frac{\theta}{\theta-1}}(1-\theta)\bigl\|u^{\frac{p}{2}}\bigr\|_{\frac{2}{p}}^2 + k_6\bigl\|u^{\frac{p}{2}}\bigr\|_{\frac{2}{p}}^2 \\
&= k_6\epsilon\theta\bigl\|\nabla u^{\frac{p}{2}}\bigr\|_2^2 + k_6\bigl[\epsilon^{\frac{\theta}{\theta-1}}(1-\theta) + 1\bigr]\bigl\|u^{\frac{p}{2}}\bigr\|_{\frac{2}{p}}^2 \\
&= \varepsilon_2\bigl\|\nabla u^{\frac{p}{2}}\bigr\|_2^2 + k_7\bigl\|u^{\frac{p}{2}}\bigr\|_{\frac{2}{p}}^2 \\
&= \varepsilon_2\bigl\|\nabla u^{\frac{p}{2}}\bigr\|_2^2 + k_7\|u\|_1^p
\end{aligned}$$

for any $\varepsilon_2 > 0$, with $k_7 > 0$ depending on ε_2. Because of $\|u\|_1 \le A_1$ by Step 1, we know that

$$\int_\Omega u^p \le \varepsilon_2\bigl\|\nabla u^{\frac{p}{2}}\bigr\|_2^2 + k_7A_1^p = \varepsilon_2\bigl\|\nabla u^{\frac{p}{2}}\bigr\|_2^2 + k_8 \tag{3.8}$$

with $k_8 = k_7A_1^p > 0$.

Now, we need to consider the value of ε_1 and ε_2. Fix them with

$$\biggl(2r + \frac{(p-1)M_1^2M^p}{2d_1}\biggr)\varepsilon_1 = \frac{d_2}{2}$$

and

$$\biggl(\frac{\delta K_0}{\alpha + K_0} + 1\biggr)\varepsilon_2 = \frac{2d_1(p-1)}{p^2}.$$

We have from (3.5), (3.6) and (3.8) that

$$\begin{aligned}
&\frac{1}{p}\frac{d}{dt}\int_\Omega u^p + \frac{d}{dt}\int_\Omega |\nabla v|^2 + \biggl(\frac{\delta K_0}{\alpha + K_0} + 1\biggr)\varepsilon_2\int_\Omega \bigl|\nabla u^{\frac{p}{2}}\bigr|^2 \\
&\quad + 2\biggl(2r + \frac{(p-1)M_1^2M^p}{2d_1}\biggr)\varepsilon_1\int_\Omega |\Delta v|^2 \\
&\quad\le \biggl(-a + \frac{\delta K_0}{\alpha + K_0} + 1\biggr)\int_\Omega u^p + \biggl(2r + \frac{(p-1)M_1^2M^p}{2d_1}\biggr)\int_\Omega |\nabla v|^2 + k_3 \\
&\quad\le -a\int_\Omega u^p + \biggl(\frac{\delta K_0}{\alpha + K_0} + 1\biggr)\varepsilon_2\bigl\|\nabla u^{\frac{p}{2}}\bigr\|_2^2 + \biggl(\frac{\delta K_0}{\alpha + K_0} + 1\biggr)k_8 \\
&\qquad + \biggl(2r + \frac{(p-1)M_1^2M^p}{2d_1}\biggr)\varepsilon_1\int_\Omega |\Delta v|^2 + \biggl(2r + \frac{(p-1)M_1^2M^p}{2d_1}\biggr)k_5 + k_3.
\end{aligned}$$

Obviously,

$$\begin{aligned}
&\frac{1}{p}\frac{d}{dt}\int_\Omega u^p + \frac{d}{dt}\int_\Omega |\nabla v|^2 \\
&\quad\le -a\int_\Omega u^p - \biggl(2r + \frac{(p-1)M_1^2M^p}{2d_1}\biggr)\varepsilon_1\int_\Omega |\Delta v|^2
\end{aligned}$$

$$
\begin{aligned}
&+ \left[\left(\frac{\delta K_0}{\alpha + K_0} + 1 \right) k_8 + \left(2r + \frac{(p-1)M_1^2 M^p}{2d_1} \right) k_5 + k_3 \right] \\
&= -a \int_\Omega u^p - \left(2r + \frac{(p-1)M_1^2 M^p}{2d_1} \right) \left(\varepsilon_1 \int_\Omega |\Delta v|^2 + k_5 \right) \\
&\quad + \left[\left(\frac{\delta K_0}{\alpha + K_0} + 1 \right) k_8 + 2\left(2r + \frac{(p-1)M_1^2 M^p}{2d_1} \right) k_5 + k_3 \right] \\
&\leq -a \int_\Omega u^p - \left(2r + \frac{(p-1)M_1^2 M^p}{2d_1} \right) \int_\Omega |\nabla v|^2 + k_9
\end{aligned}
$$

with $k_9 = (\frac{\delta K_0}{\alpha+K_0} + 1)k_8 + 2(2r + \frac{(p-1)M_1^2 M^p}{2d_1})k_5 + k_3 > 0$. Therefore, we define the function

$$
y_2(t) = \frac{1}{p} \int_\Omega u^p + \int_\Omega |\nabla v|^2, \quad t > 0,
$$

satisfies

$$
y_2'(t) + k_{10} y_2(t) \leq k_9
$$

for all $t > 0$ with $k_{10} = \min\{2r + \frac{(p-1)M_1^2 M^p}{2d_1}, pa\}$. This also ensures

$$
y_2(t) \leq C_2 = \max\left\{ y_2(0), \frac{k_9}{k_{10}} \right\}
$$

for all $t > 0$ by the comparison principle of ordinary differential equations.

Part 3: *Boundedness of* $\|\nabla v\|_{L^p(\Omega)}$ *with* $p > 2$.

We can define $f(u,v) = rv - \frac{r}{K}v^2 - \frac{\gamma uv}{\beta u + v + \alpha}$. It follows from Part 2 and Lemma 2.3 that there exists $C_3 > 0$ such that

$$
\sup_{t>0} \big\| f(u,v) \big\|_{L^p(\Omega)} \leq C_3 < +\infty.
$$

By the variation-of-constants formula for v, we have

$$
v(\cdot,t) = e^{d_2 t\Delta} v_0 + \int_0^t e^{d_2(t-s)\Delta} f\big(u(s), v(s)\big)\, ds, \quad t > 0.
$$

Because of Lemma 2.3, we can draw the conclusion that

$$
\begin{aligned}
&\|\nabla v\|_{L^p(\Omega)} \\
&\quad = \left\| \nabla e^{d_2 t\Delta} v_0 + \int_0^t \nabla e^{d_2(t-s)\Delta} f\big(u(s), v(s)\big)\, ds \right\|_{L^p(\Omega)} \\
&\quad \leq \big\| \nabla e^{d_2 t\Delta} v_0 \big\|_{L^p(\Omega)} + \left\| \int_0^t \nabla e^{d_2(t-s)\Delta} f\big(u(s), v(s)\big)\, ds \right\|_{L^p(\Omega)} \\
&\quad \leq C_2 \big(1 + d_2 t^{-\frac{n}{2}(\frac{1}{q} - \frac{1}{p})}\big) e^{-\lambda_1 d_2 t} \|\nabla v_0\|_{L^p(\Omega)} + \int_0^t \big\| \nabla e^{d_2(t-s)\Delta} f\big(u(s), v(s)\big) \big\|_{L^p(\Omega)}\, ds \\
&\quad \leq 2C_2 e^{-\lambda_1' t} \|\nabla v_0\|_{L^p(\Omega)} + C_1 \int_0^t \big(1 + d_2^{-\frac{1}{2}} (t-s)^{-\frac{1}{2}}\big) e^{-\lambda_1'(t-s)} \big\| f\big(u(s), v(s)\big) \big\|_{L^p(\Omega)}\, ds
\end{aligned}
$$

$$\leq 2C_2 e^{-\lambda_1' t}\|\nabla v_0\|_{L^p(\Omega)} + C_1 C_3 \int_0^t \bigl(1 + d_2^{-\frac{1}{2}} s^{-\frac{1}{2}}\bigr) e^{-\lambda_1' s}\, ds$$
$$\leq 2C_2 \|\nabla v_0\|_{L^p(\Omega)} + C_1 C_3 \left(\frac{1}{\lambda_1'} + d_2^{-\frac{1}{2}}\left(2 + \frac{1}{\lambda_1'}\right)\right)$$

for all $t > 0$. Therefore, $\|\nabla v\|_{L^p(\Omega)}$ is global bounded.

Part 4: *Global boundedness.*

Based on Part 2, Part 3 and Lemma 2.4, the global boundedness of solutions can be proved by using the standard Moser iterative technique. The proof is complete. □

If we take the parameter β close to 0 then the model can be regarded as a Holling-type II predator–prey model (1.2). Therefore, we can obtain the following corollary.

Corollary 3.1 *Under the assumptions for* $\chi(u)$ *and initial data described above*, *the unique nonnegative classical solution of* (1.2) *is globally bounded*.

On the other hand, if we take the parameter α close to 0 then the model can be though of a ratio-dependent predator–prey model (1.3). That is the statement of the corollary we are trying to illustrate.

Corollary 3.2 *Under the assumptions for* $\chi(u)$ *and initial data described above*, *the unique nonnegative classical solution of* (1.3) *is globally bounded*.

Remark 3.3 It is well known that the global boundedness of solutions in a reaction–diffusion system is a comparatively effortless outcome to the corresponding predator–prey model. The existence of a prey-taxis term in (1.1) makes it massively hard to obtain the global boundedness, and even the global existence of solutions. In addition, the prey-taxis term $\nabla \cdot (u\chi(u)\nabla v)$ contained in the corresponding predator–prey model means that $\chi(u) \equiv 0$ whenever $u \geq M$, where the maximal density M serves as a switch to repulsion at high densities of the predator population, extremely similar to the prevention of overcrowding for chemotaxis or volume-filling effect. Therefore, the global boundedness of solutions in a reaction–diffusion system of Beddington–DeAngelis-type predator–prey model with prey-taxis established by way of Theorem 1.1 should be reasonable and natural.

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Competing interests
The author declares to have no competing interests.

Author's contributions
DML participated in the design of the study, conceived of the study, and drafted the manuscript. The author read and approved the final manuscript.

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