Zhan *Boundary Value Problems* (2018) 2018:30
https://doi.org/10.1186/s13661-018-0953-7

Boundary Value Problems
a SpringerOpen Journal

RESEARCH

Open Access

On the stability of the equation with a partial boundary value condition

Huashui Zhan*

*Correspondence:
huashuizhan@163.com
School of Applied Mathematics, Xiamen University of Technology, Xiamen, P.R. China

Abstract

The degenerate parabolic equation with a convection term is considered. Let Ω be a bounded domain with C^2 smooth boundary and $d(x) = \mathrm{dist}(x, \partial\Omega)$ be the distance function from the boundary. If $\Delta d \leq 0$ when x is near to the boundary, then the stability of the entropy solutions is proved independent of the boundary value conditions. The degeneracy of the convection term on the boundary can take place of the usual boundary value condition.

MSC: 35L65; 35L85; 35R35

Keywords: Degenerate parabolic equation; Entropy solution; Kruzkov's bi-variables method; Stability

1 Introduction

The degenerate parabolic equation

$$\frac{\partial u}{\partial t} = \Delta A(u) + \mathrm{div}\big(b(u,x,t)\big), \quad (x,t) \in \Omega \times (0,T), \tag{1.1}$$

comes from many reaction–diffusion problems [1]. It has been widely researched for a long time, the first well-known paper goes back to the work [2] by Vol'pert and Hudjaev in 1967. Let

$$A(u) = \int_0^u a(s)\,ds, \quad a(s) \geq 0, a(0) = 0. \tag{1.2}$$

Then the degeneracy of $a(s)$ may lead to the equation with the hyperbolic characteristic, and the uniqueness of the usual weak solution is not true. In other words, the usual weak solution (for example, the measured value solution) is so weak that it lacks the regularity to ensure the stability or the uniqueness. The entropy condition is considered in this context. Till now, this has been one of the most well-known conditions in the degenerate parabolic equation theory. In the sense of the entropy solution, there are a lot of important papers devoted to equation (1.1), one can refer to [3–15] and the references therein. Based on the papers, we can conclude that the initial value condition

$$u(x,0) = u_0(x), \quad x \in \Omega, \tag{1.3}$$

is always indispensable, while the usual Dirichlet boundary value condition

$$u(x,t)=0,\quad (x,t)\in\partial\Omega\times(0,T), \tag{1.4}$$

might be overdetermined. Instead of (1.4), a partial boundary value condition

$$u(x,t)=0,\quad (x,t)\in\Sigma_1\times(0,T), \tag{1.5}$$

should be imposed, where Σ_1 is a relative open subset of $\partial\Omega$. If

$$b_i(u,x,t)\equiv b_i(u), \tag{1.6}$$

the explicit formula of Σ_1 was studied in our previous works [16–18].

For small $\eta>0$, let

$$S_\eta(s)=\int_0^s h_\eta(\tau)\,d\tau,\quad h_\eta(s)=\frac{2}{\eta}\left(1-\frac{|s|}{\eta}\right)_+.$$

Then $h_\eta(s)\in C(\mathbb{R})$, and

$$\begin{aligned} &h_\eta(s)\ge 0,\qquad \left|sh_\eta(s)\right|\le 1,\qquad \left|S_\eta(s)\right|\le 1;\\ &\lim_{\eta\to 0}S_\eta(s)=\operatorname{sgn}s,\qquad \lim_{\eta\to 0}sS'_\eta(s)=0. \end{aligned} \tag{1.7}$$

The definition of the entropy solution of equation (1.1) is given as follows.

Definition 1.1 A function u is said to be the entropy solution of equation (1.1) with the initial value condition (1.3) if

1. u satisfies

$$u\in \mathrm{BV}(Q_T)\cap L^\infty(Q_T),\qquad \frac{\partial}{\partial x_i}\int_0^u\sqrt{a(s)}\,ds\in L^2(Q_T).$$

2. For any $\varphi\in C_0^2(Q_T)$, $\varphi\ge 0$, for any $k\in\mathbb{R}$, for any small $\eta>0$, u satisfies

$$\begin{aligned} &\iint_{Q_T}\Bigg[I_\eta(u-k)\varphi_t-\sum_{i=1}^N B_\eta^i(u,x,t,k)\varphi_{x_i}+A_\eta(u,k)\Delta\varphi\\ &\qquad -S'_\eta(u-k)\left|\nabla\int_0^u\sqrt{a(s)}\,ds\right|^2\varphi\Bigg]\,dx\,dt\\ &\qquad -\sum_{i=1}^N\iint_{Q_T}\int_k^u b_{ix_i}S'_\eta(s-k)\,ds\varphi\,dx\,dt\\ &\quad\ge 0. \end{aligned} \tag{1.8}$$

3. Condition (1.3) is true in the sense that

$$\lim_{t\to 0}\int_\Omega\left|u(x,t)-u_0(x)\right|dx=0,\quad \text{a.e. } x\in\Omega. \tag{1.9}$$

Here,

$$B^i_\eta(u,x,t,k) = \int_k^u \frac{\partial b_i(s,x,t)}{\partial s} S_\eta(s-k)\,ds,$$
$$A_\eta(u,k) = \int_k^u a(s)S_\eta(s-k)\,ds,$$

and

$$I_\eta(u-k) = \int_0^{u-k} S_\eta(s)\,ds.$$

We would like to suggest that these kinds of entropy solutions were introduced by the author in [14, 15]. For any small $\eta > 0$, any $k \in \mathbb{R}$, by multiplying with $\varphi S_\eta(u-k)$ in equation (1.1), we can obtain the entropy inequality (1.8).

Definition 1.2 Let $u(x,t)$ be the entropy solution of equation (1.1) with the initial value condition (1.3). If, moreover, the partial boundary value condition (1.5) is satisfied in the sense of the trace, then we say that $u(x,t)$ is the entropy solution of the initial-boundary value problem of equation (1.1).

If $b_i(u,x,t) \equiv b_i(u)$ is independent of the variables (x,t), the existence of the entropy solution of initial-boundary value problem was obtained in [18]. By the aid of Fichera–Oleinik theory, we conjectured that the partial boundary value condition (1.5) should be

$$u(x,t) = 0, \qquad (x,t) \in \Sigma_1 \times (0,T), \qquad \Sigma_1 = \bigl\{x \in \partial\Omega : b_i(0)n_i < 0\bigr\}, \tag{1.10}$$

where $\vec{n} = \{n_i\}$ is the inner normal vector of Ω, and proved the following theorem.

Theorem 1.3 . *Suppose that $A(s)$ is C^2 and $b_i(s,x,t) \equiv b_i(s)$ is C^1. Let $u(x,t)$ and $v(x,t)$ be two entropy solutions of equation* (1.1) *with the different initial values $u_0(x)$ and $v_0(x)$, respectively, and with the same partial homogeneous boundary value condition*

$$\gamma u = \gamma v = 0, \quad x \in \Sigma_1. \tag{1.11}$$

If Ω is with the property

$$|\triangle d| \le c, \qquad \frac{1}{\lambda}\int_{\Omega_\lambda} dx\,dt \le c,$$

then

$$\int_\Omega \bigl|u(x,t)-v(x,t)\bigr|\,dx \le \int_\Omega |u_0 - v_0|\,dx + \operatorname{ess}\sup_{(x,t)\in\Sigma_2\times(0,T)} \bigl|u(x,t)-v(x,t)\bigr|, \tag{1.12}$$

where $(x,t) \in \mathbb{R}^{N+1}$, $\Sigma_2 = \partial\Omega \setminus \Sigma_1$, $\operatorname{ess}\sup_{(x,t)\in\Sigma_2\times(0,T)} |u(x,t)-v(x,t)|$ is in the sense of N-dimensional Hausdorff measure.

If $\Omega \subset \mathbb{R}^N$ is a bounded domain, Theorem 1.3 can be found in [16]. If $\Omega = \mathbb{R}^N_+$ is the half space, Theorem 1.3 can be found in [17]. Also, if Ω is some special given domains, Theorem 1.3 was obtained in [19].

However, since there is an unknown term $\operatorname{ess\,sup}_{(x,t)\in\Sigma_2\times(0,T)} |u(x,t) - v(x,t)|$ in (1.12), Theorem 1.3 is far from perfection. The root of the problem lies in that equation (1.1) is with strong nonlinearity, it is almost impossible to verify whether conjecture (1.10) is true or not.

In this paper, we leave conjecture (1.10) out of consideration. We suppose that the convection term satisfies $b_i(\cdot,x,t) = 0$ when $x \in \partial\Omega$. By ingeniously choosing the test function φ in the entropy inequality (1.8), we can deduce an explicit formula of Σ_1 and establish the stability of the entropy solutions based on a partial boundary value condition (1.5). This is the following theorem.

Theorem 1.4 *Let the domain Ω be with a C^2 smooth boundary $\partial\Omega$, $A(s)$ be C^2 and $b_i(s,x,t)$ be C^1. Let $u(x,t)$ and $v(x,t)$ be two solutions of equation* (1.1) *with the different initial values $u_0(x)$ and $v_0(x)$, respectively, with the same partial boundary value condition* (1.5), *and*

$$\Sigma_1 = \{x \in \partial\Omega : \Delta d > 0\}. \tag{1.13}$$

If $u_0(x), v_0(x) \in L^\infty(\Omega)$, and $b_i(s,x,t)$ satisfies

$$\left|b_i(u,x,t) - b_i(v,x,t)\right| \le cd(x), \tag{1.14}$$

then

$$\int_\Omega \left|u(x,t) - v(x,t)\right| dx \le c \int_\Omega |u_0 - v_0|\, dx. \tag{1.15}$$

Theorem 1.4 is proved by Kruzkov's bi-variables method. Comparing (1.15) with (1.12), we find that the unperfect term $\operatorname{ess\,sup}_{(x,t)\in\Sigma_2\times(0,T)} |u(x,t) - v(x,t)|$ in (1.12) has disappeared. Considering different typical techniques used in [14–19], the novelty of this paper lies in that we endow condition (1.14), and by this assumption, we can choose a suitable test function φ in (1.8) to obtain the perfect stability (1.15).

A direct corollary of Theorem 1.4 is the following theorem.

Theorem 1.5 *Let the domain Ω be with a C^2 smooth boundary $\partial\Omega$, $A(s)$ be C^2, and $b_i(s,x,t)$ be C^1. Let $u(x,t)$ and $v(x,t)$ be solutions of equation* (1.1) *with the different initial values $u_0(x)$ and $v_0(x)$, respectively, but without any boundary value condition. Suppose $u_0(x), v_0(x) \in L^\infty(\Omega)$, when x is near to the boundary,*

$$\Delta d \le 0, \tag{1.16}$$

and $b_i(s,x,t)$ satisfies (1.14), *then the stability* (1.15) *is true.*

One can see that, since $u_0(x), v_0(x) \in L^\infty(\Omega)$, $b_i(s,x,t)$ is C^1, then

$$\left|b_i(u,x,t) - b_i(v,x,t)\right| \le c|u - v|$$

is always true. If we assume that

$$|u(x,t)| \le cd(x), \qquad |v(x,t| \le cd(x),$$

which is stronger than the homogeneous boundary value condition (1.4), then condition (1.14) is true naturally. However, the condition has its independent significance. In fact, only if

$$|b_i(s,x,t)| \le cd(x), \tag{1.17}$$

is true, when $|s| \le c$ and x is near to the boundary $\partial\Omega$, then condition (1.14) can be guaranteed. Certainly, if condition (1.14) is not true, how to clarify the partial boundary Σ_1 remains an open problem. If possible, we will present a follow-up work in the future.

Let us give two examples of the domains which satisfy condition (1.16). For example, if $\Omega = D_1 = \{x \in \mathbb{R}^N : x_1^2 + x_2^2 + \cdots + x_N^2 < 1\}$ is the unit disc, then

$$d(x) = 1 - \sqrt{(x_1^2 + x_2^2 + \cdots + x_N^2)}. \tag{1.18}$$

For another example, if $\Omega = \{x \in \mathbb{R}^N : 0 < x_i < 1, i = 1,2,\dots,N\}$ is the N-dimensional unit cube, then

$$d(x) = x_i \quad \text{or} \quad (1 - x_i), \tag{1.19}$$

when $x = (x_1, x_2, \dots, x_N)$ is near to the hyperplane $\{x_i = 0\}$ or $\{x_i = 1\}$ respectively. The distance functions (1.18)–(1.19) all satisfy that

$$\Delta d \le 0.$$

The following example shows what Σ_1 is. Let $\Omega = \{x \in \mathbb{R}^N : r_0^2 < x_1^2 + x_2^2 + \cdots + x_N^2 < R_0^2\}$. When x is near to $\{x \in \mathbb{R}^N : x_1^2 + x_2^2 + \cdots + x_N^2 = R_0^2\}$,

$$\Delta d \le 0.$$

But when x is near to $\{x \in \mathbb{R}^N : x_1^2 + x_2^2 + \cdots + x_N^2 = r_0^2\}$,

$$0 < \Delta d = \frac{N-1}{r},$$

then

$$\Sigma_1 = \{x \in \mathbb{R}^N : x_1^2 + x_2^2 + \cdots + x_N^2 = r_0^2\}.$$

In Sect. 2, we will introduce Kruzkov's bi-variables method. Theorem 1.4 and Theorem 1.5 are proved in Sect. 3.

2 Kruzkov's bi-variables method

The context in this section is just a minor version of Kruzkov's bi-variables method used in our previous works [14–19].

Let Γ_u be the set of all jump points of $u \in \mathrm{BV}(Q_T)$, ν be the normal of Γ_u at $X = (x,t)$, $u^+(X)$ and $u^-(X)$ be the approximate limits of u at $X \in \Gamma_u$ with respect to $(\nu, Y - X) > 0$ and $(\nu, Y - X) < 0$, respectively. For a continuous function $p(u,x,t)$ and $u \in \mathrm{BV}(Q_T)$, define

$$\widehat{p}(u,x,t) = \int_0^1 p\bigl(\tau u^+ + (1-\tau)u^-, x, t\bigr)\,d\tau, \tag{2.1}$$

which is called the composite mean value of p. For given t, we denote Γ_u^t, H^t, $(\nu_1^t, \ldots, \nu_N^t)$ and $u_\pm^t$ as all jump points of $u(\cdot,t)$, Hausdorff measure of Γ_u^t, the unit normal vector of Γ_u^t, and the asymptotic limit of $u(\cdot,t)$, respectively. Moreover, if $f(s) \in C^1(\mathbb{R})$, $u \in \mathrm{BV}(Q_T)$, then $f(u) \in \mathrm{BV}(Q_T)$ and

$$\frac{\partial f(u)}{\partial x_i} = \widehat{f'}(u)\frac{\partial u}{\partial x_i}, \quad i = 1,2,\ldots,N,N+1, \tag{2.2}$$

where $x_{N+1} = t$ as usual.

Lemma 2.1 *Let u be a solution of equation* (1.1). *Then*

$$a(s) = 0, \quad s \in I\bigl(u^+(x,t), u^-(x,t)\bigr) \text{ a.e. on } \Gamma_u, \tag{2.3}$$

where $I(\alpha,\beta)$ denotes the closed interval with endpoints α and β, and (2.3) *is in the sense of Hausdorff measure $H_N(\Gamma_u)$.*

Now, let $u(x,t)$ and $v(x,t)$ be two entropy solutions of equation (1.1) with initial values

$$u(x,0) = u_0(x), \qquad v(x,0) = v_0(x),$$

respectively.

By Definition 1.1, for $\varphi \in C_0^2(Q_T)$, we have

$$\begin{aligned}
&\iint_{Q_T}\Biggl[I_\eta(u-k)\varphi_t - \sum_{i=1}^N B_\eta^i(u,x,t,k)\varphi_{x_i} + A_\eta(u,k)\Delta\varphi \\
&\quad - S_\eta'(u-k)\left|\nabla\int_0^u \sqrt{a(s)}\,ds\right|^2\varphi\Biggr]\,dx\,dt \\
&\quad - \sum_{i=1}^N \iint_{Q_T}\int_k^u b_{ix_i}(s,x,t)S_\eta'(s-k)\,ds\varphi\,dx\,dt \\
&\geq 0,
\end{aligned} \tag{2.4}$$

$$\begin{aligned}
&\iint_{Q_T}\Biggl[I_\eta(v-l)\varphi_\tau-\sum_{i=1}^{N}B_\eta^i(v,y,\tau,l)\varphi_{y_i}+A_\eta(v,l)\Delta\varphi \\
&\qquad -S_\eta'(v-l)\biggl|\nabla\int_0^v\sqrt{a(s)}\,ds\biggr|^2\varphi\Biggr]\,dy\,d\tau \\
&\qquad -\sum_{i=1}^{N}\iint_{Q_T}\int_l^v b_{iy_i}(s,y,\tau)S_\eta'(s-l)\,ds\varphi\,dx\,dt \\
&\quad\geq 0.
\end{aligned}\tag{2.5}$$

Let $\psi(x,t,y,\tau)=\phi(x,t)j_h(x-y,t-\tau)$. Here, $\phi(x,t)\geq 0$, $\phi(x,t)\in C_0^\infty(Q_T)$, and

$$j_h(x-y,t-\tau)=\omega_h(t-\tau)\prod_{i=1}^{N}\omega_h(x_i-y_i),\tag{2.6}$$

$$\begin{aligned}
&\omega_h(s)=\frac{1}{h}\omega\biggl(\frac{s}{h}\biggr),\qquad \omega(s)\in C_0^\infty(R),\qquad \omega(s)\geq 0,\\
&\omega(s)=0\quad\text{if } |s|>1,\qquad \int_{-\infty}^{\infty}\omega(s)\,ds=1.
\end{aligned}\tag{2.7}$$

We choose $k=v(y,\tau)$, $l=u(x,t)$, $\varphi=\psi(x,t,y,\tau)$ in (2.4), (2.5), integrate over Q_T respectively, add them together. Then

$$\begin{aligned}
&\iint_{Q_T}\iint_{Q_T}\bigl[I_\eta(u-v)(\psi_t+\psi_\tau)+A_\eta(u,v)\Delta_x\psi+A_\eta(v,u)\Delta_y\psi\bigr] \\
&\qquad -\sum_{i=1}^{N}\bigl[B_\eta^i(u,x,t,v)\psi_{x_i}+B_\eta^i(v,y,\tau,u)\psi_{y_i}\bigr] \\
&\qquad -\sum_{i=1}^{N}\biggl[\int_k^u b_{ix_i}(s,x,t)S_\eta'(s-k)\,ds+\int_l^v b_{iy_i}(s,y,\tau)S_\eta'(s-l)\,ds\biggr] \\
&\qquad -S_\eta'(u-v)\biggl(\biggl|\nabla\int_0^u\sqrt{a(s)}\,ds\biggr|^2+\biggl|\nabla\int_0^v\sqrt{a(s)}\,ds\biggr|^2\biggr)\psi\,dx\,dt\,dy\,d\tau \\
&\quad\geq 0.
\end{aligned}\tag{2.8}$$

Clearly,

$$\begin{aligned}
&\frac{\partial j_h}{\partial t}+\frac{\partial j_h}{\partial\tau}=0,\qquad \frac{\partial j_h}{\partial x_i}+\frac{\partial j_h}{\partial y_i}=0,\quad i=1,\ldots,N;\\
&\frac{\partial\psi}{\partial t}+\frac{\partial\psi}{\partial\tau}=\frac{\partial\phi}{\partial t}j_h,\qquad \frac{\partial\psi}{\partial x_i}+\frac{\partial\psi}{\partial y_i}=\frac{\partial\phi}{\partial x_i}j_h.
\end{aligned}$$

Noticing that

$$\begin{aligned}
\lim_{\eta\to 0}B_\eta^i(u,x,t,v)&=\lim_{\eta\to 0}B_\eta^i(v,y,\tau,u)\\
&=\operatorname{sgn}(u-v)\bigl(b_i(u,x,t)-b_i(v,y,\tau)\bigr),
\end{aligned}$$

then

$$\lim_{\eta\to 0}\iint_{Q_T}\iint_{Q_T}\left[B_\eta^i(u,x,t,v)\psi_{x_i}+B_\eta^i(v,y,\tau,u)\psi_{y_i}\right]dx\,dt\,dy\,d\tau$$
$$=\iint_{Q_T}\iint_{Q_T}\operatorname{sgn}(u-v)\left[b_i(u,x,t)-b_i(v,y,\tau)\right]\phi_{x_i}j_h\,dx\,dt\,dy\,d\tau$$

and

$$\lim_{h\to 0}\iint_{Q_T}\iint_{Q_T}\operatorname{sgn}(u-v)\left[b_i(u,x,t)-b_i(v,y,\tau)\right]\phi_{x_i}j_h\,dx\,dt\,dy\,d\tau$$
$$=\iint_{Q_T}\operatorname{sgn}(u-v)\left[b_i(u,x,t)-b_i(v,x,t)\right]\phi_{x_i}\,dx\,dt. \tag{2.9}$$

Once more, we have

$$\begin{aligned}
&\iint_{Q_T}\left[A_\eta(u,v)\Delta_x\psi+A_\eta(v,u)\Delta_y\psi\right]dx\,dt\,dy\,d\tau\\
&=\iint_{Q_T}\iint_{Q_T}\left\{A_\eta(u,v)\left(\Delta_x\phi j_h+2\sum_{i=1}^N\phi_{x_i}j_{hx_i}+\phi\Delta j_h\right)+A_\eta(v,u)\phi\Delta_y j_h\right\}dx\,dt\,dy\,d\tau\\
&=\iint_{Q_T}\iint_{Q_T}\left\{A_\eta(u,v)\Delta_x\phi j_h+\sum_{i=1}^N\left[A_\eta(u,v)\phi_{x_i}j_{hx_i}+A_\eta(v,u)\phi_{x_i}j_{hy_i}\right]\right\}dx\,dt\,dy\,d\tau\\
&\quad-\sum_{i=1}^N\iint_{Q_T}\iint_{Q_T}\left\{\left[\widehat{a(u)S_\eta(u-v)}\frac{\partial u}{\partial x_i}\right.\right.\\
&\quad\left.\left.-\int_u^v\widehat{a(s)S_\eta'(s-v)}\,ds\frac{\partial u}{\partial x_i}\right]\phi j_{hx_i}\right\}dx\,dt\,dy\,d\tau,
\end{aligned}\tag{2.10}$$

where

$$\widehat{a(u)S_\eta(u-v)}=\int_0^1 a\left(su^++(1-s)u^-\right)S_\eta\left(su^++(1-s)u^--v\right)ds,$$
$$\int_u^v\widehat{a(s)S_\eta'(s-v)}\,ds=\int_0^1\int_{su^++(1-s)u^-}^v a(\sigma)S_\eta\left(\sigma-su^+-(1-s)u^-\right)d\sigma\,ds.$$

Notice that

$$\begin{aligned}
&\iint_{Q_T}\iint_{Q_T}S_\eta'(u-v)\left(\left|\nabla_x\int_0^u\sqrt{a(s)}\,ds\right|^2+\left|\nabla_y\int_0^v\sqrt{a(s)}\,ds\right|^2\right)\psi\,dx\,dt\,dy\,d\tau\\
&=\iint_{Q_T}\iint_{Q_T}S_\eta'(u-v)\left(\left|\nabla_x\int_0^u\sqrt{a(s)}\,ds\right|-\left|\nabla_y\int_0^v\sqrt{a(s)}\,ds\right|\right)^2\psi\,dx\,dt\,dy\,d\tau\\
&\quad+2\iint_{Q_T}\iint_{Q_T}S_\eta'(u-v)\nabla_x\int_0^u\sqrt{a(s)}\,ds\cdot\nabla_y\int_0^v\sqrt{a(s)}\,ds\psi\,dx\,dt\,dy\,d\tau,
\end{aligned}\tag{2.11}$$

and, by Lemma 2.1, we can show that

$$\begin{aligned}
&\sum_{i=1}^{N}\iint_{Q_T}\iint_{Q_T}\left(a(u)\widehat{S_\eta(u-v)}\frac{\partial u}{\partial x_i}-\widehat{\int_u^v a(s)S'_\eta(s-u)\,ds}\frac{\partial u}{\partial x_i}\right)j_{hx_i}\phi\,dx\,dt\,dy\,d\tau\\
&\quad+2\iint_{Q_T}\iint_{Q_T}S'_\eta(u-v)\nabla_x\int_0^u\sqrt{a(s)}\,ds\cdot\nabla_y\int_0^v\sqrt{a(s)}\,ds\psi\,dx\,dt\,dy\,d\tau\\
&\quad=-\sum_{i=1}^{N}\iint_{Q_T}\iint_{Q_T}\int_0^1\int_{su^++(1-s)u^-}^{v}\left[\sqrt{a(\sigma)}-\sqrt{a\bigl(su^++(1-s)u^-\bigr)}\right]\\
&\quad\times S'_\eta\bigl(\sigma-su^+-(1-s)u^-\bigr)\,d\sigma\,ds\frac{\partial u}{\partial x_i}j_{hx_i}\phi\,dx\,dt\,dy\,d\tau\\
&\quad\to 0
\end{aligned}\tag{2.12}$$

as $\eta\to 0$.

Moreover, since

$$\lim_{\eta\to 0}A_\eta(u,v)=\lim_{\eta\to 0}A_\eta(v,u)=\operatorname{sgn}(u-v)\bigl[A(u)-A(v)\bigr],$$

we have

$$\lim_{\eta\to 0}\bigl[A_\eta(u,v)\phi_{x_i}j_{hx_i}+A_\eta(u,v)\phi_{y_i}j_{hy_i}\bigr]=0.\tag{2.13}$$

Combining (2.8)–(2.13) and letting $\eta\to 0$, $h\to 0$ in (2.8), we get

$$\begin{aligned}
&\iint_{Q_T}\bigl[\bigl|u(x,t)-v(x,t)\bigr|\phi_t+\bigl|A(u)-A(v)\bigr|\Delta\phi\bigr]\,dx\,dt\\
&\quad-\sum_{i=1}^{N}\iint_{Q_T}\operatorname{sgn}(u-v)\bigl[b_i(u,x,t)-b_i(v,x,t)\bigr]\phi_{x_i}\,dx\,dt\\
&\quad-\sum_{i=1}^{N}\iint_{Q_T}\bigl[b_{ix_i}(v,x,t)\operatorname{sgn}(u-v)\phi+b_{ix_i}(u,x,t)\operatorname{sgn}(v-u)\phi\bigr]\,dx\,dt\\
&\quad\ge 0.
\end{aligned}\tag{2.14}$$

By (2.14), by choosing a suitable test function φ, one may obtain the stability of the entropy solution.

3 Proofs of Theorem 1.4 and Theorem 1.5

Proof of Theorem 1.4 For small enough λ, we set

$$\varphi_\lambda(x)=\begin{cases}\sin\frac{d(x)}{\lambda}, & \text{if } 0\le d(x)<\frac{\pi\lambda}{2},\\ 1, & \text{if } d(x)\ge\frac{\pi\lambda}{2}.\end{cases}\tag{3.1}$$

Let $0\le\eta(t)\in C_0^2(t)$ and

$$\phi(x,t)=\eta(t)\varphi_\lambda(x).$$

By (3.1), when $0 \le d(x) < \frac{\pi\lambda}{2}$, we clearly have

$$\partial_{x_i}\phi(x,t) = \eta(t)\partial_{x_i}\varphi_\lambda(x) = \eta(t)\frac{1}{\lambda}\cos\frac{d(x)}{\lambda}d_{x_i}(x) \tag{3.2}$$

and

$$\begin{aligned}\Delta\phi(x,t) &= \frac{1}{\lambda}\eta(t)\left[-\frac{1}{\lambda}\sin\frac{d(x)}{\lambda}\sum_{i=1}^{N}d_{x_i}^2 + \cos\frac{d(x)}{\lambda}\Delta d(x)\right]\\ &= -\frac{1}{\lambda^2}\eta(t)\sin\frac{d(x)}{\lambda}\sum_{i=1}^{N}d_{x_i}^2 + \frac{1}{\lambda}\eta(t)\cos\frac{d(x)}{\lambda}\Delta d(x).\end{aligned} \tag{3.3}$$

In another place, i.e., when $d(x) \ge \frac{\pi\lambda}{2}$,

$$\partial_{x_i}\phi(x,t) = 0 = \Delta\phi(x,t), \quad i = 1,2,\dots,N. \tag{3.4}$$

If we denote $\Omega_{1\lambda} = \{x \in \Omega : d(x) < \frac{\lambda\pi}{2}\}$, by that $|\nabla d| = 1$, according to (3.3)–(3.4), we have

$$\begin{aligned}&\iint_{Q_T}\left|A(u) - A(v)\right|\Delta\phi\,dx\,dt\\ &\quad= -\frac{1}{\lambda^2}\int_0^T\int_{\Omega_{1\lambda}}\left|A(u) - A(v)\right|\eta(t)\sin\frac{d(x)}{\lambda}\,dx\,dt\\ &\qquad+ \frac{1}{\lambda}\int_0^T\int_{\Omega_{1\lambda}}\eta(t)\left|A(u) - A(v)\right|\cos\frac{d(x)}{\lambda}\Delta d(x)\,dx\,dt.\end{aligned} \tag{3.5}$$

Substituting into (2.14), we have

$$\begin{aligned}&\iint_{Q_T}\left|u(x,t) - v(x,t)\right|\eta_t\varphi_\lambda(x)\,dx\,dt\\ &\qquad- \sum_{i=1}^{N}\int_0^T\int_{\Omega_{1\lambda}}\frac{\eta(t)}{\lambda}\operatorname{sgn}(u-v)\big(b_i(u,x,t) - b_i(v,x,t)\big)\cos\frac{d(x)}{\lambda}d_{x_i}(x)\,dx\,dt\\ &\qquad+ \int_0^T\int_{\Omega_{1\lambda}}\frac{\eta(t)}{\lambda}\left|A(u) - A(v)\right|\cos\frac{d(x)}{\lambda}\Delta d(x)\,dx\,dt\\ &\qquad- \sum_{i=1}^{N}\iint_{Q_T}\left[b_{ix_i}(v,x,t)\operatorname{sgn}(u-v) + b_{ix_i}(u,x,t)\operatorname{sgn}(v-u)\right]\eta(t)\varphi_\lambda(x)\,dx\,dt\\ &\quad\ge 0.\end{aligned} \tag{3.6}$$

Let

$$\Omega_+ = \Omega_{1\lambda} \cap \{x \in \Omega : \Delta d > 0\}, \qquad \Omega_- = \Omega_{1\lambda} \cap \{x \in \Omega : \Delta d < 0\}.$$

Then

$$\begin{aligned}
&\iint_{Q_T} \bigl|u(x,t)-v(x,t)\bigr|\eta_t\varphi_\lambda(x)\,dx\,dt \\
&\quad -\sum_{i=1}^{N}\int_0^T\int_{\Omega_{1\lambda}}\frac{\eta(t)}{\lambda}\operatorname{sgn}(u-v)\bigl(b_i(u,x,t)-b_i(v,x,t)\bigr)\cos\frac{d(x)}{\lambda}d_{x_i}(x)\,dx\,dt \\
&\quad +\int_0^T\int_{\Omega_{\lambda+}}\frac{\eta(t)}{\lambda}\bigl|A(u)-A(v)\bigr|\Delta d(x)\,dx\,dt \\
&\quad -\sum_{i=1}^{N}\iint_{Q_T}\bigl[b_{ix_i}(v,x,t)\operatorname{sgn}(u-v)+b_{ix_i}(u,x,t)\operatorname{sgn}(v-u)\bigr]\eta(t)\varphi_\lambda(x)\,dx\,dt \\
&\geq 0. \qquad (3.7)
\end{aligned}$$

In the first place, by that $|d_{x_i}(x)| \leq |\nabla d| = 1$, by condition (1.14),

$$\begin{aligned}
&\left|\int_0^T\int_{\Omega_{1\lambda}}\frac{\eta(t)}{\lambda}\operatorname{sgn}(u-v)\bigl(b_i(u,x,t)-b_i(v,x,t)\bigr)\cos\frac{d(x)}{\lambda}d_{x_i}(x)\right| \\
&\leq c\int_0^T\int_{\Omega_{1\lambda}}\frac{\eta(t)}{\lambda}d(x)\,dx\,dt \\
&\leq c\int_0^T\int_{\Omega_{1\lambda}}\eta(t)\,dx\,dt \qquad (3.8)
\end{aligned}$$

goes to zero when $\lambda \to 0$.

In the second place, by the partial boundary value condition (1.5), we have

$$\begin{aligned}
&\lim_{\lambda\to 0}\frac{1}{\lambda}\int_0^T\int_{\Omega_{\lambda-+}}\frac{\eta(t)}{\lambda}\bigl|A(u)-A(v)\bigr|\cos\frac{d(x)}{\lambda}\Delta d\,dx\,dt \\
&\leq c\lim_{\lambda\to 0}\frac{1}{\lambda}\int_0^T\int_{\Omega_{\lambda_+}}|u-v|\,dx\,dt \\
&= c\int_0^T\int_{\Sigma_1}|u-v|\,d\sigma\,dt = 0. \qquad (3.9)
\end{aligned}$$

Moreover,

$$\begin{aligned}
&\lim_{\lambda\to 0}\left|\iint_{Q_t}\bigl[b_{ix_i}(v,x,t)\operatorname{sgn}(u-v)+b_{ix_i}(u,x,t)\operatorname{sgn}(v-u)\bigr]\eta(t)\varphi_\lambda(x)\,dx\,dt\right| \\
&\leq c\iint_{Q_T}\eta(t)|u-v|\,dx\,dt. \qquad (3.10)
\end{aligned}$$

By (3.6)–(3.10), letting $\lambda \to 0$, we can deduce that

$$\iint_{Q_T}\bigl|u(x,t)-v(x,t)\bigr|\eta_t'\,dx\,dt + c\int_0^T\int_\Omega|u-v|\eta(t)\,dx\,dt. \qquad (3.11)$$

Let $0 < s < \tau < T$, and

$$\eta(t) = \int_{\tau-t}^{s-t}\alpha_\varepsilon(\sigma)\,d\sigma, \quad \varepsilon < \min\{\tau, T-s\}.$$

Here, $\alpha_\varepsilon(t)$ is the kernel of mollifier with $\alpha_\varepsilon(t) = 0$ for $t \notin (-\varepsilon, \varepsilon)$. Then

$$c\int_0^T \eta(t)|u-v|\,dx\,dt + \int_0^T \big[\alpha_\varepsilon(t-s) - \alpha_\varepsilon(t-\tau)\big]|u-v|_{L^1(\Omega)}\,dt \geq 0.$$

Let $\varepsilon \to 0$. Then

$$\big|u(x,\tau) - v(x,\tau)\big|_{L^1(\Omega)} \leq \big|u(x,s) - v(x,s)\big|_{L^1(\Omega)} + c\int_0^t \int_\Omega |u-v|\,dx\,dt.$$

By Gronwall's inequality, we have

$$\int_\Omega \big|u(x,\tau) - v(x,\tau)\big|\,dx \leq c\int_\Omega \big|u(x,s) - v(x,s)\big|\,dx.$$

Let $s \to 0$. Then

$$\int_\Omega \big|u(x,\tau) - v(x,\tau)\big|\,dx \leq c\int_\Omega \big|u_0(x) - v_0(x)\big|\,dx.$$

The proof is complete. □

Proof of Theorem 1.5 If, when x is near to the boundary, condition (1.16)

$$\Delta d \leq 0$$

is true, then $\Sigma_1 = \emptyset$. By Theorem 1.4, we have the conclusion of Theorem 1.5, i.e., the stability of the entropy solutions is true independent of the boundary value conditions. □

4 Conclusion

The equation considered in this paper comes from many applied fields. It is of a hyperbolic–parabolic mixed type, and only has a discontinuous solution generally. The most characteristic feature of the equation lies in that the usual Dirichlet boundary value condition may be overdetermined. Using Kruzkov's bi-variables method, by choosing a suitable test function, the stability of the entropy solutions is proved based on the partial boundary value condition, provided that the convection term has the degeneracy on the boundary.

Acknowledgements
The author would like to thank SpringerOpen Accounts Team for kindly agreeing to give me a discount of the paper charge if my paper can be accepted.

Funding
The paper is supported by the Natural Science Foundation of Fujian province, supported by the Science Foundation of Xiamen University of Technology (Grant No. XYK201448), China.

Abbreviations
Not applicable

Availability of data and materials
Not applicable.

Ethics approval and consent to participate
Not applicable.

Competing interests
The author declares that he has no competing interests.

Authors' contributions
The author read and approved the final manuscript.

Publisher's Note

Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 11 September 2017 Accepted: 7 March 2018 Published online: 12 March 2018

References

1. Wu, Z., Zhao, J., Yin, J., Li, H.: Nonlinear Diffusion Equations. World Scientific, Singapore (2001)
2. Vol'pert, A.I., Hudjaev, S.I.: On the problem for quasilinear degenerate parabolic equations of second order. Mat. Sb. **3**, 374–396 (1967) (Russian)
3. Brezis, H., Crandall, M.G.: Uniqueness of solutions of the initial value problem for $u_t - \Delta\varphi(u) = 0$. J. Math. Pures Appl. **58**, 153–163 (1979)
4. Kružkov, S.N.: First order quasilinear equations in several independent variables. Math. USSR Sb. **10**, 217–243 (1970)
5. Cockburn, B., Gripenberg, G.: Continuous dependence on the nonlinearities of solutions of degenerate parabolic equations. J. Differ. Equ. **151**, 231–251 (1999)
6. Chen, G.Q., Perthame, B.: Well-posedness for non-isotropic degenerate parabolic-hyperbolic equations. Ann. Inst. Henri Poincaré, Anal. Non Linéaire **20**(4), 645–668 (2003)
7. Bendahamane, M., Karlsen, K.H.: Reharmonized entropy solutions for quasilinear anisotropic degenerate parabolic equations. SIAM J. Math. Anal. **36**(2), 405–422 (2004)
8. Carrillo, J.: Entropy solutions for nonlinear degenerate problems. Arch. Ration. Mech. Anal. **147**, 269–361 (1999)
9. Li, Y., Wang, Q.: Homogeneous Dirichlet problems for quasilinear anisotropic degenerate parabolic-hyperbolic equations. J. Differ. Equ. **252**, 4719–4741 (2012)
10. Lions, P.L., Perthame, B., Tadmor, E.: A kinetic formation of multidimensional conservation laws and related equations. J. Am. Math. Soc. **7**, 169–191 (1994)
11. Kobayasi, K., Ohwa, H.: Uniqueness and existence for anisotropic degenerate parabolic equations with boundary conditions on a bounded rectangle. J. Differ. Equ. **252**, 137–167 (2012)
12. Mascia, C., Porretta, A., Terracina, A.: Qualitative behavior for one-dimensional strongly degenerate parabolic problems. Interfaces Free Bound. **8**(3), 263–280 (2006)
13. Michel, A., Vovelle, J.: Entropy formulation for parabolic degenerate equations with general Dirichlet boundary conditions and application to the convergence of FV methods. SIAM J. Numer. Anal. **41**(6), 2262–2293 (2003)
14. Zhan, H.: The study of the Cauchy problem of a second order quasilinear degenerate parabolic equation and the parallelism of a Riemannian manifold. Doctor Thesis, Xiamen University, Xiamen, China (2004)
15. Zhao, J., Zhan, H.: Uniqueness and stability of solution for Cauchy problem of degenerate quasilinear parabolic equations. Sci. China Ser. A **48**, 583–593 (2005)
16. Zhan, H.: On a hyperbolic-parabolic mixed type equation. Discrete Contin. Dyn. Syst., Ser. S **10**(3), 605–624 (2017)
17. Zhan, H.: The solutions of a hyperbolic-parabolic mixed type equation on half-space domain. J. Differ. Equ. **259**, 1449–1481 (2015)
18. Zhan, H.: Reaction diffusion equations with boundary degeneracy. Electron. J. Differ. Equ. **2016**, 81 (2016)
19. Zhan, H.: The entropy solution of a hyperbolic-parabolic mixed type equation. SpringerPlus **5**(1), 1–13 (2016)