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Existence of solutions for fractional differential equations with infinite point boundary conditions at resonance

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Abstract

In this paper, by using Mawhin's continuation theorem, we establish some sufficient conditions for the existence of at least one solution for a class of fractional infinite point boundary value problem at resonance. Moreover, an example is given to illustrate our results.

MSC: 34A08; 34B15

Keywords: Fractional differential equation; Infinite point boundary value problem; Resonance; Mawhin's continuation theorem

1 Introduction

In the past 20 years, fractional differential equations (FDEs for short) as mathematical models have been successfully used in many fields of science and engineering (see [1–7]), which have attracted considerable attention in studying FDEs. For example, Izhikevich neuron model can be described by two fractional-order differential equations as the form

$$\begin{cases} \tau\, {}^C D^\alpha v(t) = f v^2 + g v + h - u + RI, \\ \tau\, {}^C D^\alpha u(t) = a(bv - u)I, \end{cases}$$

where $\alpha \in (0,1]$, ${}^C D^\alpha$ is the Caputo fractional derivative of order α, $v(t)$ represents the membrane voltage and $u(t)$ express the recovery variable (see [6]). Ates and Zegeling [7] studied the following fractional-order advection–diffusion reaction boundary value problems (BVPs for short):

$$\begin{cases} \varepsilon\, {}^C D^\alpha u + \gamma u' + f(u) = S(x), \quad x \in [0,1], \\ u(0) = u_L, \qquad u(1) = u_R, \end{cases}$$

where $1 < \alpha \le 2$, $0 < \varepsilon \le 1$, $\gamma \in \mathbb{R}$, ${}^C D^\alpha$ is the Caputo fractional derivative of order α. The function $S(x)$ represents a spatially dependent source term.

Recently, fractional differential equations with various kinds of boundary conditions (BCs for short) have been discussed widely and obtained numerous valuable results (see

[8–31]). It is worth mentioning that the discuss of fractional differential equations with infinite point BCs have been attracted many scholars' attention over the past two years (see [18–31]). These work studied on many subjects, such as: existence of solutions, positive solutions, multiple solutions, unique solution. In [18, 19] Zhang and Zhai et al. considered the following fractional differential equation with infinite point BCs:

$$\begin{cases} D_{0+}^{\alpha}u(t) + q(t)f(t,x(t)) = 0, \quad t \in (0,1), \\ u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \\ u^{(i)}(1) = \sum_{j=1}^{\infty} \alpha_j u(\xi_j), \end{cases}$$

respectively, where $\alpha > 2$, $n-1 < \alpha \leq n$, $i \in [1, n-2]$ is a fixed integer, $\alpha_j \geq 0$, $0 < \xi_1 < \xi_2 < \cdots < \xi_{j-1} < \xi_j < \cdots < 1$ $(j = 1, 2 \ldots)$, D_{0+}^{α} is the standard Riemann–Liouville fractional derivative of order α. By employing the fixed-point theorem in cones, Zhang established the existence and multiplicity of positive solutions theorems and Zhai et al. obtained the existence and uniqueness result on positive solutions.

In [20], Guo et al. investigated the following infinite point fractional BVPs:

$$\begin{cases} {}^{C}D_{0+}^{\alpha}u(t) + f(t,u(t),u'(t)) = 0, \quad 0 < t < 1, \\ u(0) = u''(0) = 0, \qquad u'(1) = \sum_{j=1}^{\infty} \eta_j u(\xi_j), \end{cases}$$

where $2 < \alpha \leq 3$, $\eta_j \geq 0$, $0 < \xi_1 < \xi_2 < \cdots < \xi_{j-1} < \xi_j < \cdots < 1$ $(j = 1, 2, \ldots)$, ${}^{C}D_{0+}^{\alpha}$ is the Caputo fractional derivative of order α. The authors obtained the existence of multiple positive solutions by means of Avery–Peterson's fixed-point theorem.

Although many papers dealing with fractional infinite points BVPs, only a few papers consider FDEs with infinite point BCs at resonance (see [21–23]). In [21], Ge et al. discussed the following coupled FDEs with infinitely points BCs at resonance:

$$\begin{cases} D_{0+}^{\alpha}x_1(t) = f_1(t, x_1(t), D_{0+}^{\beta-1}x_2(t)), \\ D_{0+}^{\beta}x_2(t) = f_2(t, x_2(t), D_{0+}^{\alpha-1}x_1(t)), \\ x_1(0) = 0, \quad \lim_{t\to\infty} D_{0+}^{\alpha-1}x_1(t) = \sum_{i=1}^{+\infty} \gamma_i x_1(\eta_i), \\ x_2(0) = 0, \quad \lim_{t\to\infty} D_{0+}^{\beta-1}x_2(t) = \sum_{i=1}^{+\infty} \sigma_i x_2(\xi_i), \end{cases}$$

where $1 < \alpha, \beta \leq 2$, $0 < \eta_1 < \eta_2 < \cdots < \eta_i < \cdots$, $0 < \xi_1 < \xi_2 < \cdots < \xi_i < \cdots$, $\lim_{i\to\infty}\eta_i = \infty$, $\lim_{i\to\infty}\xi_i = \infty$ and $\sum_{i=1}^{+\infty} |\gamma_i|\eta_i^{\alpha} < \infty$, $\sum_{i=1}^{+\infty} |\sigma_i|\xi_i^{\beta} < \infty$, D_{0+}^{α} and D_{0+}^{β} are standard Riemann–Liouville fractional derivative. By using Mawhin's continuous theorem the authors obtained the existence result.

Thus, motivated by the results mentioned, the purpose of this paper is to present the existence of solutions for the following infinite point BVPs by applying Mawhin's continuous theorem:

$$\begin{cases} D_{0+}^{\alpha}x(t) = f(t, x(t), D_{0+}^{\alpha-2}x(t), D_{0+}^{\alpha-1}x(t)), \quad t \in (0,1), \\ x(0) = 0, \quad D_{0+}^{\alpha-1}x(0) = \sum_{i=1}^{+\infty} \alpha_i D_{0+}^{\alpha-1}x(\xi_i), \\ D_{0+}^{\alpha-1}x(1) = \sum_{i=1}^{+\infty} \beta_i D_{0+}^{\alpha-1}x(\gamma_i), \end{cases} \tag{1.1}$$

where $2 < \alpha \le 3$, D_{0+}^{α} is the standard Riemann–Liouville fractional derivative of order α, $f \in [0,1] \times \mathbb{R}^3 \to \mathbb{R}$ is a Carathéodory function, $\xi_i, \gamma_i \in (0,1)$ and $\{\xi_i\}_{i=1}^{+\infty}$, $\{\gamma_i\}_{i=1}^{+\infty}$ are two monotonic sequence with $\lim_{i\to+\infty}\xi_i = a$, $\lim_{i\to+\infty}\gamma_i = b$, $a, b \in (0,1)$, $\alpha_i, \beta_i \in \mathbb{R}$.

Throughout this paper, we assume that the following condition holds:

(H_1) $\sum_{i=1}^{+\infty}\alpha_i = 1$, $\sum_{i=1}^{+\infty}\beta_i = 1$, $\sum_{i=1}^{+\infty}|\alpha_i| < +\infty$, $\sum_{i=1}^{+\infty}|\beta_i| < +\infty$, $\Delta \ne 0$, where

$$\Delta = a_{11}a_{22} - a_{12}a_{21}, \qquad a_{11} = \sum_{i=1}^{+\infty}\alpha_i\xi_i, \qquad a_{12} = \sum_{i=1}^{+\infty}\alpha_i\xi_i^2,$$
$$a_{21} = 1 - \sum_{i=1}^{+\infty}\beta_i\gamma_i, \qquad a_{22} = 1 - \sum_{i=1}^{+\infty}\beta_i\gamma_i^2.$$

Remark 1.1 Here, if we let $\alpha_i, \beta_i \in \mathbb{R}^+$, $a^2 \le \xi_1$, $\gamma_1^2 \le b$ and assume that $\{\xi_i\}_{i=1}^{+\infty}$, $\{\gamma_i\}_{i=1}^{+\infty}$ are monotonically increasing sequence and monotonically decreasing sequence, respectively, then, by (H_1), we have $\Delta > 0$.

In fact, since $\{\xi_i\}_{i=1}^{+\infty}$ monotone increasing and $\{\gamma_i\}_{i=1}^{+\infty}$ monotone decreasing with $\lim_{i\to+\infty}\xi_i = a$, $\lim_{i\to+\infty}\gamma_i = b$, $a, b \in (0,1)$, then by (H_1) we have

$$\begin{aligned}\Delta &= a_{11}a_{22} - a_{12}a_{21}\\ &= \sum_{i=1}^{+\infty}\alpha_i\xi_i\left(1 - \sum_{i=1}^{+\infty}\beta_i\gamma_i^2\right) - \sum_{i=1}^{+\infty}\alpha_i\xi_i^2\left(1 - \sum_{i=1}^{+\infty}\beta_i\gamma_i\right)\\ &> \xi_1\left(1-\gamma_1^2\right) - a^2(1-b) \ge 0.\end{aligned}$$

A boundary value problem is called resonance if the corresponding homogeneous boundary value problem has a nontrivial solution. We point out that if condition (H_1) holds, the BVP (1.1) happens to be at resonance in the sense that the following infinite point boundary value problem:

$$\begin{cases} D_{0+}^{\alpha}x(t) = 0, \quad t \in (0,1),\\ x(0) = 0, \quad D_{0+}^{\alpha-1}x(0) = \sum_{i=1}^{+\infty}\alpha_i D_{0+}^{\alpha-1}u(\xi_i),\\ D_{0+}^{\alpha-1}x(1) = \sum_{i=1}^{+\infty}\beta_i D_{0+}^{\alpha-1}x(\gamma_i),\end{cases}$$

has $x(t) = a_1t^{\alpha-1} + a_2t^{\alpha-2}$, $a_1, a_2 \in \mathbb{R}$ as a nontrivial solution. That means the linear operator $Lx = D_{0+}^{\alpha}x$ is non-invertible.

Even though in the papers [21, 22] and [23] authors have investigated the resonance case with $\dim\operatorname{Ker} L = 2$, all of them considered with the coupled fractional differential equations, in which the linear operator L defined as $L(x,y) = (L_1x, L_2y)$ and $\dim\operatorname{Ker} L_1 = \dim\operatorname{Ker} L_2 = 1$. In our paper, we study the one fractional differential equation resonance case with $\dim\operatorname{Ker} L = 2$, which is obviously different from papers [21, 22] and [23]. Compared with previous work the main difficulties in this paper are as follows. First, resonance BVPs cannot be dealt with fixed-point theorem directly. Second, the theory of Mawhin's continuation theorem is characterized by the higher dimension of kernel space on resonance BVPs, the more difficult to construct the projections P and Q. Third, we give an example to support our main result, and we should point out that to give an example for BVPs (1.1) is very difficult.

The rest of this paper is organized as follows. In Sect. 2, we recall some preliminary definitions and lemmas. In Sect. 3, based on Mawhin's continuation theorem, we establish an existence theorem for problem (1.1). In Sect. 4, we present an example to demonstrate the results. In the last section, brief conclusions are given.

2 Preliminaries

In this section, we recall some definitions and lemmas which are used throughout this paper. First, we present here the results of coincidence degree theory due to Mawhin which can be found in [32, 33].

Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ are two real Banach spaces. Define $L : \operatorname{dom} L \subset X \to Y$ to be a Fredholm operator with index zero, then there exist two continuous projectors $P : X \to X$ and $Q : Y \to Y$ such that

$$\operatorname{Im} P = \operatorname{Ker} L, \qquad \operatorname{Im} L = \operatorname{Ker} Q, \qquad X = \operatorname{Ker} L \oplus \operatorname{Ker} P, \qquad Y = \operatorname{Im} L \oplus \operatorname{Im} Q,$$

and $L|_{\operatorname{dom} L \cap \operatorname{Ker} P} : \operatorname{dom} L \to \operatorname{Im} L$ is invertible. We denote its inverse by K_p. Let Ω be an open bounded subset of X and $\operatorname{dom} L \cap \bar{\Omega} \neq \emptyset$, then the map $N : X \to Y$ is called L-compact on $\bar{\Omega}$, if $QN(\bar{\Omega})$ is bounded and $K_{P,Q}N = K_p(I - Q)N : \bar{\Omega} \to X$ is compact.

Theorem 2.1 *Let* $L : \operatorname{dom} L \subset X \to Y$ *be a Fredholm operator of index zero and* $N : X \to Y$ *be* L*-compact on* $\bar{\Omega}$. *Assume that the following conditions are satisfied*:

(i) $Lu \neq \lambda Nu$ *for any* $u \in (\operatorname{dom} L \setminus \operatorname{Ker} L) \cap \partial\Omega$, $\lambda \in (0, 1)$;

(ii) $Nu \notin \operatorname{Im} L$ *for any* $u \in \operatorname{Ker} L \cap \partial\Omega$;

(iii) $\deg(QN|_{\operatorname{Ker} L}, \Omega \cap \operatorname{Ker} L, 0) \neq 0$.

Then the equation $Lx = Nx$ *has at least one solution in* $\operatorname{dom} L \cap \bar{\Omega}$.

Next, we recall some basic knowledge about the fractional calculus. For more details we refer the reader to [2].

Definition 2.1 The Riemann–Liouville fractional integral of order $\alpha > 0$ for a function $x : (0, +\infty) \to \mathbb{R}$ is given by

$$I_{0+}^{\alpha} x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} x(s)\, ds$$

provided that the right-hand side integral is pointwise defined on $(0, +\infty)$.

Definition 2.2 The Riemann–Liouville fractional derivative of order $\alpha > 0$ for a function $x : (0, +\infty) \to \mathbb{R}$ is given by

$$D_{0+}^{\alpha} x(t) = \frac{d^n}{dt^n} I_{0+}^{n-\alpha} x(t) = \frac{1}{\Gamma(n - \alpha)} \frac{d^n}{dt^n} \int_0^t (t - s)^{n-\alpha-1} x(s)\, ds,$$

where $n = [\alpha] + 1$, provided that the right-hand side integral is pointwise defined on $(0, +\infty)$.

Lemma 2.1 *Let $\alpha > 0$. Assume that $x, D_{0+}^{\alpha}x \in L^1(0,1)$, then*

$$I_{0+}^{\alpha}D_{0+}^{\alpha}x(t) = x(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n},$$

where $n = [\alpha] + 1$, $c_i \in \mathbb{R}$ $(i = 1, 2, \ldots, n)$ are arbitrary constants.

Lemma 2.2 *Let $\alpha > \beta > 0$. Assume that $x \in L^1(0,1)$, then*

$$I_{0+}^{\alpha}I_{0+}^{\beta}x(t) = I_{0+}^{\alpha+\beta}x(t), \qquad D_{0+}^{\beta}I_{0+}^{\alpha}x(t) = I_{0+}^{\alpha-\beta}x(t),$$

in particular $D_{0+}^{\alpha}I_{0+}^{\alpha}x(t) = x(t)$.

Lemma 2.3 *Let $\alpha > 0$, $n \in N$ and $D = d/dx$. If the fractional derivatives $(D_{0+}^{\alpha}x)(t)$ and $(D_{0+}^{\alpha+n}x)(t)$ exist, then*

$$\left(D^n D_{0+}^{\alpha}x\right)(t) = \left(D_{0+}^{\alpha+n}x\right)(t).$$

Lemma 2.4 (see [2, 34]) *Assume that $\alpha > 0$, $\lambda > -1$, $t > 0$, then*

$$I_{0+}^{\alpha}t^{\lambda} = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+1+\alpha)}t^{\alpha+\lambda}, \qquad D_{0+}^{\alpha}t^{\lambda} = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+1-\alpha)}t^{\lambda-\alpha},$$

in particular $D_{0+}^{\alpha}t^{\alpha-m} = 0$, $m = 1, 2, \ldots, n$, where $n = [\alpha] + 1$.

3 Main result

Let

$$X = \left\{x : x, D_{0+}^{\alpha-2}x, D_{0+}^{\alpha-1}x \in C[0,1]\right\}, \qquad Y = L^1[0,1].$$

It is easy to check that X is a Banach space with norm

$$\|x\|_X = \max\left\{\|x\|_\infty, \left\|D_{0+}^{\alpha-2}x\right\|_\infty, \left\|D_{0+}^{\alpha-1}x\right\|_\infty\right\},$$

where $\|x\|_\infty = \sup_{t\in[0,1]}|x(t)|$, and Y is a Banach space with norm $\|y\|_Y = \|y\|_1 = \int_0^1 |y(t)|\,dt$.

Define the linear operator $L : \operatorname{dom} L \subset X \to Y$ and the nonlinear operator $N : X \to Y$ as follows:

$$Lx(t) = D_{0+}^{\alpha}x(t), \quad x(t) \in \operatorname{dom} L, \qquad Nx(t) = f\left(t, x(t), D_{0+}^{\alpha-2}x(t), D_{0+}^{\alpha-1}x(t)\right), \quad x(t) \in X,$$

where

$$\operatorname{dom} L = \left\{x \in X : D_{0+}^{\alpha}x(t) \in Y, x \text{ satisfies boundary value conditions of (1.1)}\right\}.$$

Then BVP (1.1) is equivalent to the operator equation $Lx = Nx$, $x \in \operatorname{dom} L$.

Lemma 3.1 *Assume that* (H_1) *holds, then the operator* $L : \operatorname{dom} L \subset X \to Y$ *satisfies*

$$\operatorname{Ker} L = \{x \in \operatorname{dom} L : x(t) = a_1 t^{\alpha-1} + a_2 t^{\alpha-2}, a_1, a_2 \in \mathbb{R}\}, \tag{3.1}$$

$$\operatorname{Im} L = \{y \in Y : T_1 y = T_2 y = 0\}, \tag{3.2}$$

where

$$T_1 y = \sum_{i=1}^{+\infty} \alpha_i \int_0^{\xi_i} y(s)\,ds, \qquad T_2 y = \sum_{i=1}^{+\infty} \beta_i \int_{\gamma_i}^1 y(s)\,ds.$$

Proof If $Lx = D_{0+}^{\alpha} x = 0$, by Lemma 2.1, we have

$$x(t) = a_1 t^{\alpha-1} + a_2 t^{\alpha-2} + a_3 t^{\alpha-3}, \quad a_1, a_2, a_3 \in \mathbb{R}.$$

Considering that boundary condition $x(0) = 0$, one has $a_3 = 0$, then

$$x(t) = a_1 t^{\alpha-1} + a_2 t^{\alpha-2}.$$

So, $\operatorname{Ker} L \subset \{x \in \operatorname{dom} L : x(t) = a_1 t^{\alpha-1} + a_2 t^{\alpha-2}, a_1, a_2 \in \mathbb{R}\}$. Conversely, for any $a_1, a_2 \in \mathbb{R}$, take $x(t) = a_1 t^{\alpha-1} + a_2 t^{\alpha-2}$, it is easy to check that $D_{0+}^{\alpha} x(t) = 0$ and $x(t)$ satisfies boundary value conditions of (1.1). Thus, (3.1) holds. For $y \in \operatorname{Im} L$, there exists $x \in \operatorname{dom} L$ such that $D_{0+}^{\alpha} x(t) = y(t)$. Again by Lemma 2.1 and combining with the boundary condition $x(0) = 0$, one gets

$$x(t) = I_{0+}^{\alpha} y(t) + a_1 t^{\alpha-1} + a_2 t^{\alpha-2}.$$

Noting that

$$D_{0+}^{\alpha-1} x(0) = \sum_{i=1}^{+\infty} \alpha_i D_{0+}^{\alpha-1} x(\xi_i), \qquad D_{0+}^{\alpha-1} x(1) = \sum_{i=1}^{+\infty} \beta_i D_{0+}^{\alpha-1} x(\gamma_i),$$

by Lemmas 2.2 and 2.4, we obtain

$$\begin{aligned} D_{0+}^{\alpha-1} x(0) = a_1 \Gamma(\alpha) &= \sum_{i=1}^{+\infty} \alpha_i D_{0+}^{\alpha-1} x(\xi_i) \\ &= \sum_{i=1}^{+\infty} \alpha_i \left[\int_0^{\xi_i} y(s)\,ds + a_1 \Gamma(\alpha) \right] \\ &= \sum_{i=1}^{+\infty} \alpha_i \int_0^{\xi_i} y(s)\,ds + a_1 \Gamma(\alpha) \end{aligned}$$

and

$$\begin{aligned} D_{0+}^{\alpha-1} x(1) &= \int_0^1 y(s)\,ds + a_1 \Gamma(\alpha) = \sum_{i=1}^{+\infty} \beta_i D_{0+}^{\alpha-1} x(\gamma_i) \\ &= \sum_{i=1}^{+\infty} \beta_i \left[\int_0^{\gamma_i} y(s)\,ds + a_1 \Gamma(\alpha) \right] = \sum_{i=1}^{+\infty} \beta_i \int_0^{\gamma_i} y(s)\,ds + a_1 \Gamma(\alpha). \end{aligned}$$

Thus,

$$T_1y = T_2y = 0, \tag{3.3}$$

that is,

$$\operatorname{Im} L \subset \{y \in Y : T_1y = T_2y = 0\}.$$

Conversely, let $y \in Y$ satisfy (3.3) and take $x(t) = I_{0+}^{\alpha} y(t)$. Obviously, we have $x(t) \in \operatorname{dom} L$ and $Lx(t) = y(t)$. Then, $\{y \in Y : T_1y = T_2y = 0\} \subset \operatorname{Im} L$. Therefore, (3.2) holds. □

Let $Q_1, Q_2 : Y \to Y$ be two linear operators defined as follows:

$$Q_1y = \frac{1}{\Delta}(a_{22}T_1y - a_{12}T_2y), \qquad Q_2y = \frac{2}{\Delta}(a_{11}T_2y - a_{21}T_1y).$$

Lemma 3.2 *Assume that* (H$_1$) *holds, then* $L : \operatorname{dom} L \subset X \to Y$ *is a Fredholm operator of index zero. The linear projector operator* $P : X \to X$ *and* $Q : Y \to Y$ *defined as follows*:

$$\begin{aligned}
(Px)(t) &= \frac{1}{\Gamma(\alpha)} D_{0+}^{\alpha-1} x(0) t^{\alpha-1} + \frac{1}{\Gamma(\alpha-1)} D_{0+}^{\alpha-2} x(0) t^{\alpha-2}, \\
Qy(t) &= Q_1y(t) + \big(Q_2y(t)\big)t.
\end{aligned}$$

Proof By the definition of P we can check that P is a continuous linear projector operator and satisfies $\operatorname{Im} P = \operatorname{Ker} L$, $X = \operatorname{Ker} P \oplus \operatorname{Ker} L$. It is clear that Q is a continuous linear operator and $\dim \operatorname{Im} Q = 2$. By the definitions of Q_1, Q_2, we can calculate the following equations hold:

$$\begin{aligned}
&Q_1\big(Q_1y(t)\big) = Q_1y(t), \qquad Q_1\big(\big(Q_2y(t)\big)t\big) = 0, \\
&Q_2\big(Q_1y(t)\big) = 0, \qquad Q_2\big(\big(Q_2y(t)\big)t\big) = Q_2y(t).
\end{aligned}$$

Thus,

$$\begin{aligned}
Q^2y(t) &= Q\big(Qy(t)\big) = Q_1\big(Qy(t)\big) + \big(Q_2\big(Qy(t)\big)\big)t \\
&= Q_1\big[Q_1y(t) + \big(Q_2y(t)\big)t\big] + \big\{Q_2\big[Q_1y(t) + \big(Q_2y(t)\big)t\big]\big\}t \\
&= Q_1y(t) + \big(Q_2y(t)\big)t = Qy(t).
\end{aligned}$$

So, Q is a projector operator. From Lemma 3.1, we have $\operatorname{Im} L \subset \operatorname{Ker} Q$. Now, we show the fact that $\operatorname{Ker} Q \subset \operatorname{Im} L$. In fact, for $y \in \operatorname{Ker} Q$, i.e., $Qy = 0$, then we get a system of linear equations with respect to T_1y, T_2y as follows:

$$\begin{cases}
a_{11}T_2y - a_{21}T_1y = 0, \\
a_{22}T_1y - a_{12}T_2y = 0,
\end{cases} \tag{3.4}$$

Since the determinant of coefficient for (3.4) is $\Delta \neq 0$, we get $T_1y = T_2y = 0$, thus $\operatorname{Ker} Q \subset \operatorname{Im} L$. Therefore, $\operatorname{Ker} Q = \operatorname{Im} L$. For $y \in Y$, set $y = (y - Qy) + Qy$, then $(y - Qy) \in \operatorname{Ker} Q = \operatorname{Im} L$,

$Qy \in \operatorname{Im} Q$. So, $y = \operatorname{Im} L + \operatorname{Im} Q$. Furthermore, for any $y \in \operatorname{Im} L \cap \operatorname{Im} Q$, there exist constants $c_1, c_2 \in \mathbb{R}$ such that $y(t) = c_1 + c_2 t$ and $T_1 y = T_2 y = 0$. Then we also get a system of linear equations with respect to c_1, c_2 as follows:

$$\begin{cases} 2a_{11}c_1 + a_{12}c_2 = 0, \\ 2a_{21}c_1 + a_{22}c_2 = 0. \end{cases} \tag{3.5}$$

Because the determinant of coefficient for (3.5) is $2\Delta \neq 0$. Thus, $c_1 = c_2 = 0$. It means that $\operatorname{Im} Q \cap \operatorname{Im} L = \{0\}$. Therefore, $Y = \operatorname{Im} Q \oplus \operatorname{Im} L$. Furthermore, $\dim \operatorname{Ker} L = \dim \operatorname{Im} Q = \operatorname{co}\dim \operatorname{Im} L = 2$. So, L is a Fredholm operator of index zero. □

Lemma 3.3 *Assume that* (H_1) *holds, define the linear operator* $K_p : \operatorname{Im} L \to \operatorname{dom} L \cap \operatorname{Ker} P$ *by*

$$(K_p y)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s)\,ds, \quad y \in \operatorname{Im} L,$$

then K_p *is the inverse of* $L|_{\operatorname{dom} L \cap \operatorname{Ker} P}$ *and* $\|K_p y\|_X \leq \|y\|_1$, *for all* $y \in \operatorname{Im} L$.

Proof For $y \in \operatorname{Im} L$, then $T_1 y = T_2 y = 0$, which is combined with the definition of K_p and Lemma 2.2, we can check that $K_p y \in \operatorname{dom} L \cap \operatorname{Ker} P$. So, K_p is well defined on $\operatorname{Im} L$. Obviously, $(LK_p)y(t) = y(t)$, $\forall y \in \operatorname{Im} L$. For $x(t) \in \operatorname{dom} L$, by Lemma 2.1, we have

$$\begin{aligned} (K_p L)x(t) &= I_{0+}^{\alpha} D_{0+}^{\alpha} x(t) \\ &= x(t) + a_1 t^{\alpha-1} + a_2 t^{\alpha-2}, \quad a_1, a_2 \in \mathbb{R}. \end{aligned}$$

It follows from $P[(K_p L)x(t)] = 0$ and $a_1 t^{\alpha-1} + a_2 t^{\alpha-2} \in \operatorname{Ker} L = \operatorname{Im} P$ that $c_1 t^{\alpha-1} + c_2 t^{\alpha-2} = -Px(t)$. That is, $(K_p L)x(t) = x(t) - Px(t)$. Therefore, if $x(t) \in \operatorname{dom} L \cap \operatorname{Ker} P$, we have $(K_p L)x(t) = x(t)$. So, K_p is the inverse of $L|_{\operatorname{dom} L \cap \operatorname{Ker} P}$. By Lemma 2.2, we have the following inequalities:

$$\begin{aligned} |K_p y| &\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |y(s)|\,ds \leq \frac{1}{\Gamma(\alpha)} \int_0^1 |y(s)|\,ds \leq \|y\|_1, \\ |D_{0+}^{\alpha-2} K_p y| &\leq \int_0^t (t-s) |y(s)|\,ds \leq \int_0^1 |y(s)|\,ds = \|y\|_1, \\ |D_{0+}^{\alpha-1} K_p y| &\leq \int_0^t |y(s)|\,ds \leq \int_0^1 |y(s)|\,ds = \|y\|_1. \end{aligned}$$

So, $\|K_p y\|_X \leq \|y\|_1$, for all $y \in \operatorname{Im} L$. □

Lemma 3.4 *Assume that* (H_1) *holds and* $\Omega \subset X$ *is an open bounded subset with* $\operatorname{dom} L \cap \bar{\Omega} \neq \emptyset$, *then* N *is* L*-compact on* $\bar{\Omega}$.

Proof According to $f \in [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ satisfies the Carathéodory conditions, we can get $QN(\bar{\Omega})$ and $(I-Q)N(\bar{\Omega})$ are bounded almost everywhere on $[0,1]$, that is, there exist constants $m, \tilde{m} > 0$ such that $|QNx(t)| \leq \tilde{m}$, $|(I-Q)Nx(t)| \leq m$, $x \in \bar{\Omega}$, a.e. $t \in [0,1]$. Next, we show that $K_p(I-Q)N : \bar{\Omega} \to X$ is compact. In fact, by Lemma 3.3, $K_p(I-Q)N(\bar{\Omega})$ is uniformly bounded. It follows from the Lebesgue dominated convergence theorem that $K_p(I-Q)N:\bar{\Omega}\to X$ is continuous. For $0\le t_1<t_2\le 1$, $x\in\bar{\Omega}$, we have

$$\begin{aligned}
&\left|K_p(I-Q)Nx(t_1)-K_p(I-Q)Nx(t_2)\right| \\
&= \frac{1}{\Gamma(\alpha)}\left|\int_0^{t_1}(t_1-s)^{\alpha-1}(I-Q)Nx(s)\,ds-\int_0^{t_2}(t_2-s)^{\alpha-1}(I-Q)Nx(s)\,ds\right| \\
&\le \frac{1}{\Gamma(\alpha)}\left|\int_0^{t_1}\left[(t_1-s)^{\alpha-1}-(t_2-s)^{\alpha-1}\right](I-Q)Nx(s)\,ds\right| \\
&\quad+\frac{1}{\Gamma(\alpha)}\left|\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}(I-Q)Nx(s)\,ds\right| \\
&\le \frac{m}{\Gamma(\alpha)}\int_0^{t_1}\left[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\right]ds+\frac{m}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}\,ds \\
&= \frac{m}{\Gamma(\alpha+1)}\left(t_2^\alpha-t_1^\alpha\right).
\end{aligned}$$

Since t^α is uniformly continuous on $[0,1]$, $K_p(I-Q)N(\bar{\Omega})$ is equicontinuous. In addition, by Lemma 2.3 we see that the following equation holds:

$$D_{0+}^{\alpha-2}K_p(I-Q)Nx(t_2)-D_{0+}^{\alpha-2}K_p(I-Q)Nx(t_1)=\int_{t_1}^{t_2}D_{0+}^{\alpha-1}K_p(I-Q)Nx(s)\,ds.$$

So, it suffices to show that $D_{0+}^{\alpha-1}K_p(I-Q)N(\bar{\Omega})$ is equicontinuous. In fact,

$$\begin{aligned}
&\left|D_{0+}^{\alpha-1}K_p(I-Q)Nx(t_1)-D_{0+}^{\alpha-1}K_p(I-Q)Nx(t_2)\right| \\
&= \left|\int_{t_1}^{t_2}(I-Q)Nx(s)\,ds\right|\le m(t_2-t_1).
\end{aligned}$$

Since t is uniformly continuous on $[0,1]$, thus $D_{0+}^{\alpha-1}K_p(I-Q)Nx(\Omega)$ is equicontinuous. By the Ascoli–Arzelà theorem, we see that $K_p(I-Q)N:\bar{\Omega}\to X$ is compact. □

In order to obtain our main results, we suppose that the following conditions are satisfied:

(H_2) There exist nonnegative functions $p(t),q(t),r(t),e(t)\in Y$ such that, for all $(u,v,w)\in\mathbb{R}^3$, $t\in(0,1)$,

$$\left|f(t,u,v,w)\right|\le p(t)|u|+q(t)|v|+r(t)|w|+e(t)$$

and

$$\|p\|_1+\|q\|_1+\|r\|_1<\frac{\Gamma(\alpha-1)}{\Gamma(\alpha-1)+2}.$$

(H_3) There exist constants $k_1,k_2>0$ such that, for all $t\in(0,1)$, $x\in\operatorname{dom}L$, if $|D_{0+}^{\alpha-2}x(t)|>k_1$ or $|D_{0+}^{\alpha-1}x(t)|>k_2$, then either $T_1(Nx(t))\neq 0$ or $T_2(Nx(t))\neq 0$.

(H_4) There exists a constant $g>0$ such that, for all $a_1,a_2\in\mathbb{R}$, $x(t)=a_1t^{\alpha-1}+a_2t^{\alpha-2}\in\operatorname{Ker}L$, if $|a_1|>g$ or $|a_2|>g$ then either

$$a_1T_1Nx(t)+a_2T_2Nx(t)>0, \tag{3.6}$$

or

$$a_1 T_1 Nx(t) + a_2 T_2 Nx(t) < 0. \tag{3.7}$$

Lemma 3.5 *Suppose that* (H_1)–(H_3) *hold, set*

$$\Omega_1 = \{x \in \operatorname{dom} L \setminus \operatorname{Ker} L : Lx = \lambda Nx, \lambda \in (0,1)\}.$$

Then Ω_1 *is bounded.*

Proof For $x \in \Omega_1$, we have $Nx \in \operatorname{Im} L = \operatorname{Ker} Q$. Then $T_1(Nx(t)) = T_2(Nx(t)) = 0$. Thus, from (H_3), there exist $t_0, t_1 \in [0,1]$ such that $|D_{0+}^{\alpha-2}x(t_0)| \le k_1$ and $|D_{0+}^{\alpha-1}x(t_1)| \le k_2$. Then, by Lemma 2.3 we have

$$\begin{aligned}
\left|D_{0+}^{\alpha-1}x(t)\right| &= \left|D_{0+}^{\alpha-1}x(t_1) + \int_{t_1}^{t} D_{0+}^{\alpha}x(s)\,ds\right| \\
&\le \left|D_{0+}^{\alpha-1}x(t_1)\right| + \int_{t_1}^{t} \left|D_{0+}^{\alpha}x(s)\right|ds \\
&\le k_2 + \|Nx\|_1
\end{aligned}$$

and

$$\begin{aligned}
\left|D_{0+}^{\alpha-2}x(0)\right| &= \left|D_{0+}^{\alpha-2}x(t_0) - \int_{0}^{t_0} D_{0+}^{\alpha-1}x(s)\,ds\right| \\
&\le \left|D_{0+}^{\alpha-2}x(t_0)\right| + \int_{0}^{t_0} \left|D_{0+}^{\alpha-1}x(s)\right|ds \\
&\le k_1 + \left\|D_{0+}^{\alpha-1}x\right\|_\infty \le k_1 + k_2 + \|Nx\|_1.
\end{aligned}$$

By the definition of P and Lemma 2.4, we get

$$D_{0+}^{\alpha-2}Px(t) = D_{0+}^{\alpha-1}x(0)t + D_{0+}^{\alpha-2}x(0),$$

and

$$D_{0+}^{\alpha-1}Px(t) = D_{0+}^{\alpha-1}x(0).$$

Thus,

$$\begin{aligned}
&\left|D_{0+}^{\alpha-2}Px\right| \le \left|D_{0+}^{\alpha-1}x(0)\right| + \left|D_{0+}^{\alpha-2}x(0)\right| \le k_1 + 2k_2 + 2\|Nx\|_1, \\
&\left|D_{0+}^{\alpha-1}Px\right| = \left|D_{0+}^{\alpha-1}x(0)\right| \le k_2 + \|Nx\|_1.
\end{aligned}$$

We have

$$\begin{aligned}
|Px| &\le \frac{1}{\Gamma(\alpha)}\left|D_{0+}^{\alpha-1}x(0)\right| + \frac{1}{\Gamma(\alpha-1)}\left|D_{0+}^{\alpha-2}x(0)\right| \\
&\le \frac{1}{\Gamma(\alpha-1)}\left(k_1 + 2k_2 + 2\|Nx\|_1\right).
\end{aligned}$$

Therefore,

$$\begin{aligned}\|Px\|_X &= \max\bigl\{\|Px\|_\infty, \bigl\|D_{0+}^{\alpha-2}Px\bigr\|_\infty, \bigl\|D_{0+}^{\alpha-1}Px\bigr\|_\infty\bigr\} \\ &\leq \frac{1}{\Gamma(\alpha-1)}\bigl(k_1+2k_2+2\|Nx\|_1\bigr).\end{aligned} \tag{3.8}$$

Also, for $x \in \Omega_1$, then $(I-P)x \in \operatorname{dom} L \cap \operatorname{Ker} P$, $LPx = 0$, from Lemma 3.3, we have

$$\bigl\|(I-P)x\bigr\|_X = \bigl\|K_pL(I-P)x\bigr\|_X = \|K_pLx\|_X \leq \|Lx\|_1 \leq \|Nx\|_1. \tag{3.9}$$

Then, from (3.8) and (3.9), we see that

$$\begin{aligned}\|x\|_X &= \bigl\|Px+(I-P)x\bigr\|_X \leq \|Px\|_X + \bigl\|(I-P)x\bigr\|_X \\ &\leq \frac{1}{\Gamma(\alpha-1)}\bigl(k_1+2k_2+2\|Nx\|_1\bigr) + \|Nx\|_1.\end{aligned} \tag{3.10}$$

By (H_2), we have

$$\begin{aligned}\|Nx\|_1 &\leq \|p\|_1\|x\|_\infty + \|q\|_1\bigl\|D_{0+}^{\alpha-2}x\bigr\|_\infty + \|r\|_1\bigl\|D_{0+}^{\alpha-1}x\bigr\|_\infty + \|e\|_1 \\ &\leq \bigl(\|p\|_1+\|q\|_1+\|r\|_1\bigr)\|x\|_X + \|e\|_1.\end{aligned} \tag{3.11}$$

Substituting (3.11) into (3.10), one gets

$$\|x\|_X \leq \frac{k_1+2k_2+(\Gamma(\alpha-1)+2)\|e\|_1}{\Gamma(\alpha-1)-[\Gamma(\alpha-1)+2](\|p\|_1+\|q\|_1+\|r\|_1)}.$$

So, Ω_1 is bounded. □

Lemma 3.6 *Suppose that* (H_3) *holds, set*

$$\Omega_2 = \{x \in \operatorname{Ker} L : Nx \in \operatorname{Im} L\}.$$

Then Ω_2 is bounded.

Proof For $x \in \Omega_2$, we have $x(t) = a_1t^{\alpha-1} + a_2t^{\alpha-2}$, $a_1, a_2 \in \mathbb{R}$ and $T_1Nx(t) = T_2Nx(t) = 0$. From (H_3), there exist $t_2, t_3 \in [0,1]$ such that $|D_{0+}^{\alpha-2}x(t_2)| \leq k_1$ and $|D_{0+}^{\alpha-1}x(t_3)| \leq k_2$, that is,

$$\begin{aligned}&\bigl|D_{0+}^{\alpha-1}x(t_3)\bigr| = \bigl|a_1\Gamma(\alpha)\bigr| \leq k_2, \\ &\bigl|D_{0+}^{\alpha-2}x(t_2)\bigr| = \bigl|a_1\Gamma(\alpha)t_2 + a_2\Gamma(\alpha-1)\bigr| \leq k_1.\end{aligned}$$

Thus,

$$|a_1| \leq k_2/\Gamma(\alpha), \qquad |a_2| \leq (k_1+k_2)/\Gamma(\alpha-1).$$

Therefore,

$$\|x\|_\infty \le |a_1| + |a_2| \le \frac{k_2}{\Gamma(\alpha)} + \frac{k_1 + k_2}{\Gamma(\alpha-1)} = \frac{1}{\Gamma(\alpha)}\bigl[k_2 + (\alpha-1)(k_1+k_2)\bigr],$$
$$\bigl\|D_{0+}^{\alpha-2}x\bigr\|_\infty \le \Gamma(\alpha)|a_1| + \Gamma(\alpha-1)|a_2| \le 2k_2 + k_1,$$
$$\bigl\|D_{0+}^{\alpha-1}x\bigr\|_\infty \le \Gamma(\alpha)|a| \le k_2.$$

So, Ω_2 is bounded. □

Lemma 3.7 *Suppose that* (H_4) *holds, we set*

$$\Omega_3 = \bigl\{x \in \operatorname{Ker} L : \vartheta\lambda Jx + (1-\lambda)QNx = 0, \lambda \in [0,1]\bigr\}.$$

Then Ω_3 *is bounded, where* $\vartheta = 1$, *if* (3.6) *holds and* $\vartheta = -1$, *if* (3.7) *holds*, $J : \operatorname{Ker} L \to \operatorname{Im} Q$ *is the linear isomorphism defined by*

$$J\bigl(a_1 t^{\alpha-1} + a_2 t^{\alpha-2}\bigr) = \frac{1}{\Delta}\bigl[(a_{22}a_1 - a_{12}a_2) + 2(a_{11}a_2 - a_{21}a_1)t\bigr], \quad \forall a_1, a_2 \in \mathbb{R}.$$

Proof Without loss of generality, we suppose that (3.7) holds, then, for any $x \in \Omega_3$, there exist constants $a_1, a_2 \in \mathbb{R}$, $\lambda \in [0,1]$ such that $x(t) = a_1 t^{\alpha-1} + a_2 t^{\alpha-2}$ and $-\lambda Jx + (1-\lambda)QNx = 0$. By Lemma 3.6, in order to prove Lemma 3.7, it suffices to show that $|a_1| \le g$, $|a_2| \le g$. In fact, if $\lambda = 0$ then $QNx = 0$, which means $T_1Nx = T_2Nx = 0$. From (H_4), we get $|a_1| \le g$, $|a_2| \le g$. If $\lambda = 1$ then $Jx = 0$, that is, $a_1 = a_2 = 0$. Obviously, $|a_1| \le g$, $|a_2| \le g$. For $\lambda \in (0,1)$, by $\lambda Jx = (1-\lambda)QNx$ one has

$$\begin{cases} \lambda(a_{22}a_1 - a_{12}a_2) = (1-\lambda)(a_{22}T_1Nx - a_{12}T_2Nx), \\ \lambda(a_{11}a_2 - a_{21}a_1) = (1-\lambda)(a_{11}T_2Nx - a_{21}T_1Nx). \end{cases}$$

Because $\Delta \ne 0$, we have

$$\begin{cases} \lambda a_1 = (1-\lambda)T_1Nx, \\ \lambda a_2 = (1-\lambda)T_2Nx. \end{cases}$$

Then, if $|a_1| > g$ or $|a_2| > g$, by (3.7), we get a contradiction,

$$0 < \lambda\bigl(a_1^2 + a_2^2\bigr) = (1-\lambda)(a_1T_1Nx + a_2T_2Nx) < 0.$$

Thus, $|a_1| \le g$, $|a_2| \le g$. So, Ω_3 is bounded. If (3.6) holds, by a similar method, we can see that Ω_3 is bounded. □

Theorem 3.1 *Suppose that* (H_1)–(H_4) *hold. Then problem* (1.1) *has at least on solution in* X.

Proof Let Ω be a bounded open set of X such that $\bigcup_{i=1}^{3} \bar{\Omega}_i \subset \Omega$. By Lemma 3.4, N is L-compact on $\bar{\Omega}$. From Lemmas 3.5 and 3.6, we get

(i) $Lx \neq \lambda Nx$ for any $(x,\lambda) \in [(\operatorname{dom} L \setminus \operatorname{Ker} L) \cap \partial\Omega] \times (0,1)$,
(ii) $Nx \in \operatorname{Im} L$ for any $x \in \operatorname{Ker} L \cap \partial\Omega$.
Thus, we only need to show that (iii) of Theorem 2.1 is satisfied. Take

$$H(x,\lambda) = \vartheta\lambda Jx + (1-\lambda)QNx,$$

where ϑ is defined as before. According to Lemma 3.7, we derive $H(x,\lambda) \neq 0$ for all $x \in \operatorname{Ker} L \cap \partial\Omega$. Thus, it follows from the homotopy of degree that

$$\begin{aligned} \deg\{QN|_{\operatorname{Ker} L}, \Omega \cap \operatorname{Ker} L, 0\} &= \deg\bigl\{H(\cdot,0), \Omega \cap \operatorname{Ker} L, 0\bigr\} \\ &= \deg\bigl\{H(\cdot,1), \Omega \cap \operatorname{Ker} L, 0\bigr\} \\ &= \deg(\vartheta J, \Omega \cap \operatorname{Ker} L, 0) \neq 0. \end{aligned}$$

Then, by Theorem 2.1, we can see that the operator function $Lx = Nx$ has at least one solution in $\operatorname{dom} L \cap \bar{\Omega}$, which, equivalently to problem (1.1), has at least one solution in X. □

4 Example

Example 4.1 Consider the following fractional boundary value problem:

$$\begin{cases} D_{0+}^{2.5}x(t) = f(t, x(t), D_{0+}^{0.5}x(t), D_{0+}^{1.5}x(t)), \quad t \in (0,1), \\ x(0) = 0, \quad D_{0+}^{1.5}x(0) = \sum_{i=1}^{+\infty} \alpha_i D_{0+}^{1.5}x(\xi_i), \\ D_{0+}^{1.5}x(1) = \sum_{i=1}^{+\infty} \beta_i D_{0+}^{1.5}x(\gamma_i), \end{cases} \tag{4.1}$$

where we take

$$\begin{aligned} &\alpha_i = \frac{1}{2^i}, \qquad \beta_i = \frac{1}{2^i}, \qquad \xi_i = \frac{i}{2i+1}, \qquad \gamma_i = \frac{i+1}{2i+1}, \\ &f\bigl(t, x(t), D_{0+}^{0.5}x(t), D_{0+}^{1.5}x(t)\bigr) \\ &\quad = m(t)\left[t^2 + 1 + \frac{1}{8}\cos^2 x(t) + \frac{1}{24}D_{0+}^{0.5}x(t)\right] - \frac{1}{12}\bigl(1 - m(t)\bigr)D_{0+}^{1.5}x(t), \\ &m(t) = \begin{cases} 1, & [0,1/2), \\ 0, & [1/2,1]. \end{cases} \end{aligned}$$

Obviously, we have

$$\alpha_i, \beta_i \in \mathbb{R}^+, \quad a^2 = \frac{1}{4} \leq \frac{1}{3} = \xi_1, \qquad \gamma_1^2 = \frac{4}{9} \leq \frac{1}{2} = b,$$

and $\{\xi_i\}_{i=1}^{+\infty}$ is a monotone increasing sequence and $\{\gamma_i\}_{i=1}^{+\infty}$ is a monotone decreasing sequence. Applying Remark 1.1 we conclude that $\Delta > 0$. Let

$$p(t) = 1/8, \qquad q(t) = 1/24, \qquad r(t) = 1/12, \qquad e(t) = 2.$$

Then

$$\bigl|f(t,u,v,w)\bigr| \leq p(t)|u| + q(t)|v| + r(t)|w| + e(t)$$

and

$$\|p\|_1 + \|q\|_1 + \|r\|_1 = \frac{1}{4} < \frac{\Gamma(\alpha-1)}{\Gamma(\alpha-1)+2} = \frac{\sqrt{\pi}}{\sqrt{\pi}+4}.$$

Take $k_1 = 48$, $k_2 = 24$. Then, if $|D_{0+}^{0.5}x(t)| > k_1$, one has

$$T_1\bigl(Nx(t)\bigr) = \sum_{i=1}^{+\infty}\alpha_i \int_0^{\xi_i} f\bigl(s, x(s), D_{0+}^{0.5}x(s), D_{0+}^{1.5}x(s)\bigr)\,ds \neq 0,$$

and if $|D_{0+}^{1.5}x(t)| > k_2$, one gets

$$T_2\bigl(Nx(t)\bigr) = \sum_{i=1}^{+\infty}\beta_i \int_{\gamma_i}^{1} f\bigl(s, x(s), D_{0+}^{0.5}x(s), D_{0+}^{1.5}x(s)\bigr)\,ds \neq 0.$$

Let $g = 261$. Then if $|a_1| > g$, we have

$$\begin{aligned}
&a_1T_1Nx(t) + a_2T_2Nx(t)\\
&\quad= a_1\sum_{i=1}^{+\infty}\alpha_i \int_0^{\xi_i} f\bigl(s, x(s), D_{0+}^{0.5}x(s), D_{0+}^{1.5}x(s)\bigr)\,ds\\
&\qquad+ a_2\sum_{i=1}^{+\infty}\beta_i \int_{\gamma_i}^{1} f\bigl(s, x(s), D_{0+}^{0.5}x(s), D_{0+}^{1.5}x(s)\bigr)\,ds\\
&\quad= a_1\sum_{i=1}^{+\infty}\alpha_i \int_0^{\xi_i} \biggl[s^2 + 1 + \frac{1}{8}\cos^2 x(s) + \frac{1}{24}\bigl(a_1\Gamma(1.5)s + a_2\Gamma(0.5)\bigr)\biggr]\,ds\\
&\qquad- \frac{a_2}{12}\sum_{i=1}^{+\infty}\beta_i \int_{\gamma_i}^{1} a_1\Gamma(1.5)\,ds\\
&\quad= a_1\sum_{i=1}^{+\infty}\alpha_i \int_0^{\xi_i} \biggl(s^2 + 1 + \frac{1}{8}\cos^2 x(s)\biggr)\,ds + \frac{1}{48}\Gamma(1.5)a_1^2\sum_{i=1}^{+\infty}\alpha_i\xi_i^2\\
&\qquad+ \frac{1}{24}a_1a_2\Gamma(0.5)\sum_{i=1}^{+\infty}\alpha_i\xi_i - \frac{1}{12}a_1a_2\Gamma(1.5)\sum_{i=1}^{+\infty}\beta_i(1-\gamma_i)\\
&\quad= a_1\sum_{i=1}^{+\infty}\alpha_i \int_0^{\xi_i} \biggl(s^2 + 1 + \frac{1}{8}\cos^2 x(s)\biggr)\,ds + \frac{1}{48}\Gamma(1.5)a_1^2\sum_{i=1}^{+\infty}\alpha_i\xi_i^2\\
&\quad\geq \frac{1}{864}a_1^2 - \frac{29}{48}a_1 > 0.
\end{aligned}$$

In view of Theorem 3.1, boundary value problem (4.1) has at least one solution.

Remark 4.1 By the example, we find that it is more difficult to give an infinite series $\sum_{i=1}^{+\infty}\alpha_i$ ($\sum_{i=1}^{+\infty}\alpha_i = 1$, $\sum_{i=1}^{+\infty}|\alpha_i| < +\infty$) than to give a function $g(t) \in L^1[0,1]$ ($\int_0^1 g(t)\,dt = 1$). For example, for $\int_0^1 g(t) = 1$, we can take $g(t) \equiv 1$.

5 Conclusion

In this paper, we are focused on investigating the existence of solutions for a class of FDEs with infinite point BCs. By using Mawhin's continuation theorem, an existence theorem is established and we also give an example to illustrate the application of the theorem. To the best of our knowledge, the issue on the existence of solutions for infinite point BVPs was first studied by Ma (see [35]) and discussed widely in recent years. It is an interesting question and there is some work to be done in the future, such as: discussing the existence of solutions for p-Laplacian FDEs with infinite point BCs at resonance in the case $\dim\operatorname{Ker} L = 2$ for one fractional differential equation.

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