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High energy solutions of modified quasilinear fourth-order elliptic equation

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Abstract

This paper focuses on the following modified quasilinear fourth-order elliptic equation:

$$\begin{cases} \Delta^2 u-(a+b\int_{\mathbb{R}^3}|\nabla u|^2\,dx)\Delta u+\lambda V(x)u-\frac{1}{2}\Delta(u^2)u=f(x,u), & \text{in } \mathbb{R}^3,\\ u(x)\in H^2(\mathbb{R}^3), \end{cases}$$

where $\Delta^2=\Delta(\Delta)$ is the biharmonic operator, $a>0$, $b\ge 0$, $\lambda\ge 1$ is a parameter, $V\in C(\mathbb{R}^3,\mathbb{R})$, $f(x,u)\in C(\mathbb{R}^3\times\mathbb{R},\mathbb{R})$. $V(x)$ and $f(x,u)u$ are both allowed to be sign-changing. Under the weaker assumption $\lim_{|t|\to\infty}\frac{\int_0^t f(x,s)\,ds}{|t|^3}=\infty$ uniformly in $x\in\mathbb{R}^3$, a sequence of high energy weak solutions for the above problem are obtained.

MSC: 35J25; 35J20; 35J60; 35J61

Keywords: Super-quadratic; High energy solutions; Sign-changing potential; Fountain theorem

1 Introduction and main results

In this paper, we consider the following elliptic equation:

$$\begin{cases} \Delta^2 u-(a+b\int_{\mathbb{R}^3}|\nabla u|^2\,dx)\Delta u+\lambda V(x)u-\frac{1}{2}\Delta(u^2)u=f(x,u), & \text{in } \mathbb{R}^3,\\ u(x)\in H^2(\mathbb{R}^3), \end{cases} \tag{1.1}$$

where $\Delta^2=\Delta(\Delta)$ is the biharmonic operator, the constants $a>0$, $b\ge 0$, and $\lambda\ge 1$ is a parameter. $V(x):\mathbb{R}^3\to\mathbb{R}$ and $f:\mathbb{R}^3\times\mathbb{R}\to\mathbb{R}$ satisfies the following assumptions:

(V) $V\in C(\mathbb{R}^3,\mathbb{R})$, $\inf_{\mathbb{R}^3}V>-\infty$ and there exists a constant $r>0$ such that

$$\lim_{|y|\to\infty}\operatorname{meas}\{x\in\mathbb{R}^3:|x-y|\le r, V(x)\le M\}=0,\quad \forall M>0;$$

(F_1) $f\in C(\mathbb{R}^3\times\mathbb{R},\mathbb{R})$ and there exists positive constant C_0 and $p>4$ such that

$$|f(x,t)|\le C_0(|t|+|t|^{p-1}),\quad \forall(x,t)\in\mathbb{R}^3\times\mathbb{R}.$$

(F_2) $\lim_{|t|\to\infty}\frac{F(x,t)}{|t|^3}=\infty$ uniformly in $x\in\mathbb{R}^3$, where $F(x,t)=\int_0^t f(x,s)\,ds$.

(F_3) There exists a constant $\alpha \geq 0$ such that

$$f(x,t)t - 4F(x,t) \geq -\alpha t^2, \quad \forall (x,t) \in \mathbb{R}^3 \times \mathbb{R}.$$

(F_4) $f(x,-t) = -f(x,t)$ for all $(x,t) \in \mathbb{R}^3 \times \mathbb{R}$.

The Kirchhoff's model considers the changes in length of the string produced by transverse vibrations. It was pointed out in [1–4] that (1.1) models several physical and biological systems where u describes a process which relies on the mean of itself such as the population density. For more mathematical and physical background on Kirchhoff-type problems, we refer the reader to [1, 5–8] and the references therein. It is well known that fourth-order elliptic equation has been widely studied since Lazer and Mckenna [9] first proposed to study periodic oscillations and traveling waves in a suspension bridge.

In te recent years, many scholars widely studied the Schrödinger equation under variant assumptions on $V(x)$ and $f(x,u)$, such as [3, 4, 10–13]. In [10], Wu considered the following Schrödinger–Kirchhoff-type problem:

$$-\left(a + b\int_{\mathbb{R}^N} |\nabla u|^2\,dx\right)\triangle u + V(x)u = f(x,u), \quad \text{in } \mathbb{R}^N (N \leq 3) \tag{1.2}$$

under these hypotheses:

(V') $V \in C(\mathbb{R}^N, \mathbb{R})$ satisfies $\inf V(x) \geq a_1 > 0$ and for each $M > 0$, $\operatorname{meas}\{x \in \mathbb{R}^N : V(x) \leq M\} < +\infty$, where a_1 is a constant and meas denotes the Lebesgue measure in $\mathbb{R}^N$.

(f_1) $f \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$ and $|f(x,t)| \leq C(1 + |t|^{p-1})$ for some $2 \leq p < 2^*$, where C is a positive constant;

(f_2) $f(x,t) = o(|t|)$ as $|t| \to 0$;

(f_3) $\frac{F(x,t)}{t^4} \to +\infty$ as $|t| \to +\infty$ uniformly in $\forall x \in \mathbb{R}^N$;

(f_4) $tf(x,t) \geq 4F(x,t)$, $\forall x \in \mathbb{R}^N$, $\forall t \in \mathbb{R}$.

Here (f_3) is essential in these references to overcome the missing of compactness. The author got a nontrivial solution of (1.2). In [8], Zhang and Tang also considered the problem (1.2) under the assumption (V), and they obtained infinitely many high energy solutions of the problem (1.2). In [11], Nie studied the following Schrödinger–Kirchhoff-type equation:

$$\begin{cases} -(a + b\int_{\mathbb{R}^3} |\nabla u|^2\,dx)\triangle u + \lambda V(x)u = f(x,u), & \text{in } \mathbb{R}^3, \\ u(x) \to 0 & \text{as } |x| \to \infty, \end{cases} \tag{1.3}$$

under the assumption (V'). They got a sequence of high energy weak solutions whenever $\lambda > 0$ is sufficiently large. In [14], Xu and Chen also used condition (V') to study the problem (1.3).

More recently, Cheng and Tang [15] studied the following elliptic equation:

$$\begin{cases} \triangle^2 u - \triangle u + V(x)u - \frac{1}{2}\triangle(u^2)u = f(x,u), & \text{in } \mathbb{R}^N \\ u(x) \in H^2(\mathbb{R}^N), \end{cases} \tag{1.4}$$

under the assumption (f_3). Clearly, the problem (1.1) is equivalent to (1.4) whenever $N = 3$, $a = 1$, $b = 0$, $\lambda = 1$, and condition (f_3) is stronger than (F_2).

Motivated by the work we discussed above, we will use weaker conditions (F_2), (F_3) instead of the common assumptions (f_3), (f_4), while $V(x)$ and $f(x,u)u$ are both allowed to be sign-changing. We will further study and establish the existence of infinitely many high energy solutions of (1.1) whenever $\lambda \geq 1$, by using the fountain theorem [16, 17] or its other versions [18, 19]. To the best of our knowledge, there is little work concerning this case up to now.

The following are our main results.

Theorem 1.1 *Assume that* (V) *and* (F_1)–(F_4) *are satisfied, then problem* (1.1) *possesses infinitely many high energy solutions whenever* $\lambda \geq 1$.

Corollary 1.2 *Assume that* (V) *and* (F_1)–(F_4) *are satisfied, then problem* (1.3) *possesses infinitely many high energy nontrivial solutions whenever* $\lambda \geq 1$.

Remark 1.3 Obviously, the condition (V) is weaker than (V'); (F_1) is weaker than (f_1) and (f_2); (F_3) is weaker than (f_7) [14] and (f_4); (F_2) is weaker than (g_2) [15]. Furthermore, we do not require λ large enough, but we only need $\lambda \geq 1$. Therefore, our results extend and improve Theorem 1 [10], Theorem 1.2 [11], Theorem 1.3 [14], Theorem 1.1 [8], Theorem 1.4 [15] and so on.

Remark 1.4 There are many functions satisfying assumptions (F_1)–(F_4) not (f_3). For example

$$f(x,u) = 4u^3 - \frac{2u(1+u^2)\ln(1+u^2) + 2u^3 - 2u^3\ln(1+u^2)}{(1+u^2)^2}$$

for all $(x,u) \in \mathbb{R}^3 \times \mathbb{R}$.

Indeed, $F(x,u) = u^4 - \frac{u^2\ln(1+u^2)}{1+u^2}$, then we can find a positive constant α such that

$$f(x,u)u - 4F(x,u) + \alpha u^2 = \frac{2u^2\ln(1+u^2 - 2u^4 + \alpha u^6 + 2\alpha u^4 + \alpha u^2)}{(1+u^2)^2} \geq 0.$$

2 Preliminary lemmas and proof of our main result

In order to apply the variational method, we first recall some related preliminaries and establish a corresponding variational framework for our problem (1.1); then we give the proof of Theorem 1.1.

For $1 < s < +\infty$, define the Sobolev space

$$W^{m,s}(\mathbb{R}^N) = \{u \in L^s(\mathbb{R}^N) \mid D^\alpha u \in L^s(\mathbb{R}^N), |\alpha| \leq m\}$$

equipped with the norm

$$\|u\|_{W^{m,s}(\mathbb{R}^N)} = \left(\sum_{|\alpha|\leq m}\int_{\mathbb{R}^N}|D^\alpha u|^s\,dx\right)^{\frac{1}{s}},$$

where $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_N)$ with $\alpha_i \in \mathbb{Z}^+$ (the set of all non-negative integers), $i = 1, 2, \dots, N$, $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_N$ and

$$D^\alpha u = \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_N^{\alpha_N}}.$$

For $s = 2$, $H^m(\mathbb{R}^N) = W^{m,2}(\mathbb{R}^N)$ is a Hilbert space equipped with the scalar product

$$\langle u, v\rangle_{H^m} = \sum_{|\alpha| \le m} \int_{\mathbb{R}^N} D^\alpha u D^\alpha v \, dx$$

and the norm

$$\|u\|_{H^m} = \langle u, u\rangle_{H^m}^{\frac{1}{2}} = \left(\sum_{|\alpha| \le m} \int_{\mathbb{R}^N} \left| D^\alpha u \right|^2 dx \right)^{\frac{1}{2}}.$$

Moreover, for $m = 2$ one has

$$\begin{aligned} \langle u, v\rangle_{H^2} &= \int_{\mathbb{R}^N} (\triangle u \triangle v + \nabla u \nabla v + uv) \, dx, \\ \|u\|_{H^2}^2 = \langle u, v\rangle_{H^2} &= \int_{\mathbb{R}^N} \left(|\triangle u|^2 + |\nabla u|^2 + u^2 \right) dx, \end{aligned}$$

whenever $u, v \in H^2(\mathbb{R}^N)$.

Under assumption (V), we can find $V_0 \geq 0$ such that $\widetilde{V}(x) = V(x) + V_0 \geq 1$ for all $x \in \mathbb{R}^3$. Then

$$E_\lambda = \left\{ u \in H^2\left(\mathbb{R}^3\right) : \int_{\mathbb{R}^3} \left(a|\nabla u|^2 + \lambda \widetilde{V}(x) u^2 \right) dx < \infty \right\}$$

is a Hilbert space endowed with the norm

$$\|u\|_\lambda = \left(\int_{\mathbb{R}^3} \left(|\triangle u|^2 + a|\nabla u|^2 + \lambda \widetilde{V}(x) u^2 \right) dx \right)^{\frac{1}{2}}.$$

Let

$$\begin{aligned} \Phi_\lambda(u) = {} & \frac{1}{2} \int_{\mathbb{R}^3} \left(|\triangle u|^2 + a|\nabla u|^2 + \lambda V(x) u^2 \right) dx + \frac{b}{4} \left(\int_{\mathbb{R}^3} |\nabla u|^2 dx \right)^2 \\ & + \frac{1}{2} \int_{\mathbb{R}^3} u^2 |\nabla u|^2 dx - \int_{\mathbb{R}^3} F(x, u) \, dx, \quad \forall u \in E_\lambda. \end{aligned} \tag{2.1}$$

By condition (V), (F_1) and the fact $\int_{\mathbb{R}^3} u^2 |\nabla u|^2 dx < \infty$ (see Lemma 2.2 in [20]), Φ_λ is a well-defined class C^1 functional. For all $u, v \in E_\lambda$

$$\begin{aligned} \left\langle \Phi'_\lambda(u), v \right\rangle = {} & \int_{\mathbb{R}^3} \left(\triangle u \triangle v + a \nabla u \nabla v + \lambda V(x) uv \right) dx + b \int_{\mathbb{R}^3} |\nabla u|^2 dx \int_{\mathbb{R}^3} \nabla u \nabla v \, dx \\ & + \int_{\mathbb{R}^3} \left(uv|\nabla u|^2 + u^2 \nabla u \nabla v \right) dx - \int_{\mathbb{R}^3} f(x, u) v \, dx. \end{aligned} \tag{2.2}$$

Clearly, seeking a weak solution of problem (1.1) is equivalent to finding a critical point of the functional Φ_λ.

Definition 2.1 A sequence $\{u_n\} \subset E_\lambda$ is said to be a $(C)_c$ sequence if

$$\Phi_\lambda(u_n) \to c, \qquad \|\Phi'_\lambda(u_n)\|_\lambda\bigl(1 + \|u_n\|_\lambda\bigr) \to 0.$$

Φ_λ is said to satisfy the $(C)_c$ condition if any $(C)_c$ sequence possesses a convergent subsequence.

Let $E'_\lambda = \{u \in H^2(\mathbb{R}^N) : \int_{\mathbb{R}^N}(a|\nabla u|^2 + \lambda\widetilde{V}(x)u^2)\,dx < \infty\}$.

Lemma 2.2 *Under assumption* (V), *the embedding* $E'_\lambda \hookrightarrow L^s(\mathbb{R}^N)$ *is compact for* $2 \le s < 2_*$, *where* $2_* = \frac{2N}{N-4}$, *if* $N > 4$; $2_* = +\infty$, *if* $N \le 4$.

Proof Define

$$E = \left\{u \in H^1\bigl(\mathbb{R}^N\bigr) : \int_{\mathbb{R}^N}\bigl(a|\nabla u|^2 + \lambda\widetilde{V}(x)u^2\bigr)\,dx < \infty\right\}.$$

By Propositions 3.1 and 3.3 in [13], we know that the embedding $E \hookrightarrow L^s(\mathbb{R}^N)$ is compact for $2 \le s < 2_*$ due to the condition (V), and the embedding $E'_\lambda \hookrightarrow E$ is continuous, therefore, the embedding $E'_\lambda \hookrightarrow L^s(\mathbb{R}^N)$ is compact for $2 \le s < 2_*$. □

Lemma 2.3 *Under assumptions* (V), (F_1), *any bounded* $(C)_c$ *sequence of* Φ_λ *has a strongly convergent subsequence in* E_λ.

Proof Let $\{u_n\} \subset E_\lambda$ hold with

$$\sup_n \|u_n\|_\lambda < +\infty. \tag{2.3}$$

Then up to a subsequence, there exists a constant $c \in \mathbb{R}$ such that

$$\Phi_\lambda(u_n) \to c, \qquad \Phi'_\lambda(u_n) \to 0. \tag{2.4}$$

According to Lemma 2.2, going if necessary to a subsequence, we can assume that there exists $u \in E_\lambda$ such that

$$\begin{aligned} &u_n \rightharpoonup u \quad \text{in } E_\lambda, \\ &u_n \to u \quad \text{in } L^s\bigl(\mathbb{R}^3\bigr)\ (2 \le s < +\infty), \\ &u_n \to u \quad \text{a.e. in } \mathbb{R}^3. \end{aligned} \tag{2.5}$$

By an elementary computation,

$$\begin{aligned} &\bigl\langle \Phi'_\lambda(u_n) - \Phi'(u), u_n - u\bigr\rangle \\ &\quad \ge \|u_n - u\|_\lambda^2 - \lambda V_0\int_{\mathbb{R}^3}|u_n - u|^2\,dx \end{aligned}$$

$$+ b\left(\int_{\mathbb{R}^3} |\nabla u_n|^2\,dx - \int_{\mathbb{R}^3} |\nabla u|^2\,dx\right)\int_{\mathbb{R}^3} \nabla u_n \nabla(u_n - u)\,dx$$
$$+ \int_{\mathbb{R}^3} \big(u_n|\nabla u_n|^2 - u|\nabla u|^2\big)(u_n - u)\,dx + \int_{\mathbb{R}^3} \big(u_n^2 - u^2\big)\nabla u \nabla(u_n - u)\,dx$$
$$+ \int_{\mathbb{R}^3} \big(f(x,u) - f(x,u_n)\big)(u_n - u)\,dx. \tag{2.6}$$

Clearly, $\lambda V_0 \int_{\mathbb{R}^3} |u_n - u|^2\,dx \to 0$, and $\langle \Phi_\lambda'(u_n) - \Phi'(u), u_n - u\rangle \to 0$. Then, since $\{u_n\} \subset E_\lambda$ is bounded, we have

$$\begin{aligned}
&\left|b\left(\int_{\mathbb{R}^3} |\nabla u_n|^2\,dx - \int_{\mathbb{R}^3} |\nabla u|^2\,dx\right)\int_{\mathbb{R}^3} \nabla u_n \nabla(u_n - u)\,dx\right| \\
&\quad\le \left|b\left(\int_{\mathbb{R}^3} |\nabla u_n|^2\,dx - \int_{\mathbb{R}^3} |\nabla u|^2\,dx\right)\int_{\mathbb{R}^3} \nabla u \nabla(u_n - u)\,dx\right| \\
&\qquad + \left|b\left(\int_{\mathbb{R}^3} |\nabla u_n|^2\,dx - \int_{\mathbb{R}^3} |\nabla u|^2\,dx\right)\int_{\mathbb{R}^3} \big|\nabla(u_n - u)\big|^2\,dx\right| \\
&\quad\to 0.
\end{aligned} \tag{2.7}$$

Note that $E_\lambda \hookrightarrow H^2(\mathbb{R}^3) \hookrightarrow W^{1,s}(\mathbb{R}^3)$ for $2 \le s \le +\infty$,

$$\begin{aligned}
\int_{\mathbb{R}^3} |\nabla u_n|^3\,dx &\le \int_{\mathbb{R}^3} \left(|u_n|^2 + \sum_{i=1}^3 \left|\frac{\partial u_n}{\partial x_i}\right|^2\right)^{\frac{3}{2}} dx \\
&\le \int_{\mathbb{R}^3} \left(|u_n| + \sum_{i=1}^3 \left|\frac{\partial u_n}{\partial x_i}\right|\right)^3 dx \\
&\le \int_{\mathbb{R}^3} \left[4\max\left\{|u_n|, \left|\frac{\partial u_n}{\partial x_1}\right|, \left|\frac{\partial u_n}{\partial x_2}\right|, \left|\frac{\partial u_n}{\partial x_3}\right|\right\}\right]^3 dx \\
&\le 4^3 \int_{\mathbb{R}^3} \left(|u_n|^3 + \sum_{i=1}^3 \left|\frac{\partial u_n}{\partial x_i}\right|^3\right) dx \\
&= 4^3 \|u_n\|^3_{W^{1,3}(\mathbb{R}^3)} \\
&\le 4^3 S_3^3 \|u_n\|^3_\lambda,
\end{aligned} \tag{2.8}$$

where

$$S_s = \sup_{u\in E_\lambda, \|u\|_\lambda = 1} \|u\|_{W^{1,s}}, \quad \forall 2 \le s \le +\infty.$$

Applying (2.3)–(2.5) and (2.8), there exist constants $C_1 > 0$ such that

$$\begin{aligned}
&\left|\int_{\mathbb{R}^3} \big(u_n|\nabla u_n|^2 - u|\nabla u|^2\big)(u_n - u)\,dx\right| \\
&\quad\le \int_{\mathbb{R}^3} |u_n||\nabla u_n|^2 |u_n - u|\,dx + \int_{\mathbb{R}^3} |u||\nabla u|^2 |u_n - u|\,dx \\
&\quad\le \left(\int_{\mathbb{R}^3} |u_n|^6\,dx\right)^{\frac{1}{6}} \left(\int_{\mathbb{R}^3} |\nabla u_n|^3\,dx\right)^{\frac{2}{3}} \left(\int_{\mathbb{R}^3} |u_n - u|^6\,dx\right)^{\frac{1}{6}}
\end{aligned}$$

$$
\begin{aligned}
&+\left(\int_{\mathbb{R}^3}|u|^6\,dx\right)^{\frac{1}{6}}\left(\int_{\mathbb{R}^3}|\nabla u|^3\,dx\right)^{\frac{2}{3}}\left(\int_{\mathbb{R}^3}|u_n-u|^6\,dx\right)^{\frac{1}{6}}\\
&\le C_1\|u_n-u\|_{L^6}\to 0,\quad \text{as } n\to\infty,
\end{aligned}
\tag{2.9}
$$

and $C_1'>0$ such that

$$
\begin{aligned}
&\left|\int_{\mathbb{R}^3}(u_n^2-u^2)\nabla u\nabla(u_n-u)\,dx\right|\\
&\quad\le\int_{\mathbb{R}^3}|u_n-u||u_n+u||\nabla u|\left|\nabla(u_n-u)\right|dx\\
&\quad\le\left(\int_{\mathbb{R}^3}|u_n-u|^6\right)^{\frac{1}{6}}\left(\int_{\mathbb{R}^3}|u_n+u|^6\right)^{\frac{1}{6}}\left(\int_{\mathbb{R}^3}|\nabla u|^3\right)^{\frac{1}{3}}\left(\int_{\mathbb{R}^3}\left|\nabla(u_n-u)\right|^3\right)^{\frac{1}{3}}\\
&\quad\le C_1'\|u_n-u\|_{L^6}\to 0,\quad \text{as } n\to\infty.
\end{aligned}
\tag{2.10}
$$

By (F_1) and the Hölder inequality,

$$
\begin{aligned}
&\left|\int_{\mathbb{R}^3}\big(f(x,u)-f(x,u_n)\big)(u_n-u)\,dx\right|\\
&\qquad\le C_0\int_{\mathbb{R}^3}\left[|u|+|u|^{p-1}+|u_n|+|u_n|^{p-1}\right]|u_n-u|\,dx\\
&\qquad\le C_0\left[\big(\|u_n\|_{L^2}+\|u\|_{L^2}\big)\|u_n-u\|_{L^2}+\big(\|u_n\|_{L^p}^{p-1}+\|u\|_{L^p}^{p-1}\big)\|u_n-u\|_{L^p}\right].
\end{aligned}
$$

Then, combining the last inequality with (2.5), we get

$$
\int_{\mathbb{R}^3}\big(f(x,u)-f(x,u_n)\big)(u_n-u)\,dx\to 0,\quad \text{as } n\to\infty.
\tag{2.11}
$$

Hence, the combination of (2.7) and (2.9)–(2.11) implies that

$$
u_n\to u\quad \text{in } E_\lambda.
$$

Therefore, the proof is complete. □

Lemma 2.4 *Assume that (V) and (F_1)–(F_3) hold, then Φ_λ satisfies the $(C)_c$ condition.*

Proof Let $\{u_n\}\subset E_\lambda$ be such that

$$
\Phi_\lambda(u_n)\to c,\qquad \left\|\Phi_\lambda'(u_n)\right\|_\lambda\big(1+\|u_n\|_\lambda\big)\to 0.
\tag{2.12}
$$

First, we prove that $\{u_n\}$ is bounded in E_λ. By (F_3), (2.1), (2.2) and (2.12), one has

$$
\begin{aligned}
c+o(1)&=\Phi_\lambda(u_n)-\frac{1}{4}\langle\Phi_\lambda'(u_n),u_n\rangle\\
&=\frac{1}{4}\int_{\mathbb{R}^3}\big(|\triangle u_n|^2+a|\nabla u_n|^2+\lambda\widetilde{V}(x)u_n^2\big)\,dx\\
&\quad+\int_{\mathbb{R}^3}\left[\frac{1}{4}f(x,u_n)u_n-F(x,u_n)-\frac{\lambda}{4}V_0u_n^2\right]dx
\end{aligned}
$$

$$\geq \frac{1}{4}\|u_n\|_\lambda^2 - \frac{\alpha + \lambda V_0}{4} \int_{\mathbb{R}^3} u_n^2 \, dx. \tag{2.13}$$

Thus, it remains to show that $\{u_n\}$ is bounded in $L^2(\mathbb{R}^3)$. Otherwise, suppose that $\|u_n\|_2 \to \infty$ and then $\|u_n\|_\lambda \to \infty$. Let $\omega_n = \frac{u_n}{\|u_n\|_\lambda}$, then $\|\omega_n\|_\lambda = 1$. According to Lemma 2.2, up to a subsequence, for some $\omega \in E_\lambda$, we obtain

$$\begin{aligned} &\omega_n \rightharpoonup \omega \quad \text{in } E_\lambda, \\ &\omega_n \to \omega \quad \text{in } L^2(\mathbb{R}^3), \\ &\omega_n \to \omega \quad \text{a.e. in } \mathbb{R}^3. \end{aligned}$$

Clearly, we deduce that $\omega \neq 0$ from (2.13). Then, for $x \in \{y \in \mathbb{R}^3 : \omega(y) \neq 0\}$, we have $|u_n(x)| \to \infty$ as $n \to \infty$. For any given $u \in H^2(\mathbb{R}^3)\backslash\{0\}$, define

$$\begin{aligned} g(t) &= \|t^{-1}u(tx)\|_{H^2}^2 - 1 \\ &= \frac{1}{t}\int_{\mathbb{R}^3} |\triangle u|^2 \, dx + \frac{1}{t^3}\int_{\mathbb{R}^3} |\nabla u|^2 \, dx + \frac{1}{t^5}\int_{\mathbb{R}^3} u^2 \, dx - 1, \quad \forall t > 0. \end{aligned}$$

By an elementary computation, there exists a unique $T = \widetilde{t}(u) > 0$ such that

$$g(T) = 0, \quad \forall u \in H^2(\mathbb{R}^3)\backslash\{0\}.$$

This implies that $g(t) = 0$ defines a functional $T = \widetilde{t}(u)$ on $H^2(\mathbb{R}^3)\backslash\{0\}$. We define $\widetilde{t}(0) = 0$. It is easy to verify that $T = \widetilde{t}(u)$ is continuous and $\widetilde{t}(u) \to \infty$ as $\|u\|_{H^2} \to \infty$.

Due to the definition of g, for any $u \in H^2(\mathbb{R}^3)\backslash\{0\}$, there exists

$$v(x) = T^{-1}u(Tx) \in H^2(\mathbb{R}^3)$$

such that

$$\|v\|_{H^2} = 1.$$

Note that $u_n \neq 0$ for large $n \in \mathbb{N}$, then there exist

$$v_n(x) = T_n^{-1}u_n(T_n x) \in H^2(\mathbb{R}^3)$$

such that

$$\|v_n\|_{H^2} = 1.$$

That is,

$$u_n(x) = T_n v_n(T_n^{-1}x),$$

with $\|v_n\|_{H^2} = 1$ for large $n \in \mathbb{N}$. Moreover, we have

$$T_n = \widetilde{t}(u_n) \to \infty \quad \text{as } n \to \infty$$

and

$$\{x \in \mathbb{R}^3 : v_n(x) \neq 0\} \neq \emptyset \quad \text{for large } n \in \mathbb{N}.$$

From (F_1)–(F_3), there are $R_0 > 0$ and $C_2 > 0$ such that, for all $x \in \mathbb{R}^3$,

$$f(x,u)u + \alpha u^2 \geq 4F(x,u) \geq 0, \quad \forall |u| \geq R_0, \tag{2.14}$$

and

$$|f(x,u)u| \leq C_2 u^2, \quad \forall |u| \leq R_0. \tag{2.15}$$

Thus, by (F_3), (2.1), (2.2), (2.12)–(2.15) and $\|v_n\|_{H^2} = 1$,

$$\begin{aligned} c + o(1) &= \Phi_\lambda(u_n) - \frac{1}{2}\langle \Phi_\lambda'(u_n), u_n\rangle \\ &\geq -\frac{b}{4}\|\nabla u_n\|_2^4 - \frac{\alpha}{4}\int_{\mathbb{R}^3} u_n^2\,dx - \frac{1}{2}\int_{\mathbb{R}^3} u_n^2|\nabla u_n|^2\,dx + \frac{1}{4}\int_{\mathbb{R}^3} f(x,u_n)u_n\,dx \\ &= -\frac{bT_n^6}{4}\|\nabla v_n\|_2^4 - \frac{\alpha T_n^5}{4}\int_{\mathbb{R}^3} v_n^2\,dx - \frac{T_n^5}{2}\int_{\mathbb{R}^3} v_n^2|\nabla v_n|^2\,dx \\ &\quad + \frac{T_n^3}{4}\int_{|T_nv_n|\leq R_0} f(T_nx, T_nv_n)T_nv_n\,dx + \frac{T_n^6}{4}\int_{|T_nv_n|\geq R_0} \frac{f(T_nx, T_nv_n)T_nv_n}{T_n^3}\,dx \\ &\geq \frac{T_n^6}{4}\biggl\{-b - \frac{\alpha + C_2}{T_n} + \int_{|T_nv_n|\geq R_0} \frac{f(T_nx, T_nv_n)T_nv_n}{T_n^3}\,dx \\ &\quad - \frac{2\int_{\mathbb{R}^3} v_n^2|\nabla v_n|^2\,dx}{T_n}\biggr\}. \end{aligned} \tag{2.16}$$

By the Hölder inequality and the Sobolev embedding inequality, we see that the sequence of integrals $\int_{\mathbb{R}^3} v_n^2|\nabla v_n|^2\,dx < \infty$, since $\|v_n\|_{H^2} = 1$; on the other hand, by (F_2) and (2.14), we have

$$\int_{|T_nv_n|\geq R_0} \frac{f(T_nx, T_nv_n)T_nv_n}{T_n^3}\,dx \to +\infty \quad \text{as } n \to +\infty,$$

which contradicts (2.16). Hence, $\{u_n\}$ is bounded in $L^2(\mathbb{R}^3)$. This shows that $\{u_n\}$ is bounded in E_λ due to (2.13). By Lemma 2.3, $\{u_n\}$ contains a convergent subsequence. □

Next, we define

$$X_j = \mathbb{R}e_j, \qquad Y_k = \bigoplus_{j=1}^{k} X_j, \qquad Z_k = \overline{\bigoplus_{j=k+1}^{\infty} X_j}, \quad k \in \mathbb{Z},$$

where $\{e_j\}$ is an orthonormal basis of E_λ.

Lemma 2.5 *Assume that (V) holds, then, for $2 \leq s < 2_*$,*

$$\beta_k(s) = \sup_{u \in Z_k, \|u\|_\lambda = 1} \|u\|_s \to 0, \quad k \to \infty.$$

Proof By virtue of Lemma 2.2, we can prove the conclusion in a similar way to [16, Lemma 3.8] and [17, Corollary 8.18]. □

Lemma 2.6 *Assume that* (V) *and* (F_1) *hold, then there exist constants* $\rho, \alpha > 0$ *such that* $\Phi|_{\partial B_\rho \cap Z_m} \geq \alpha$.

Proof From (2.1) and (F_1), for all $u \in E_\lambda$ we have

$$\begin{aligned}\Phi_\lambda(u) &= \frac{1}{2}\int_{\mathbb{R}^3}\left(|\triangle u|^2 + a|\nabla u|^2 + \lambda V(x)u^2\right)dx + \frac{b}{4}\left(\int_{\mathbb{R}^3}|\nabla u|^2\,dx\right)^2 \\ &\quad + \frac{1}{2}\int_{\mathbb{R}^3}u^2|\nabla u|^2\,dx - \int_{\mathbb{R}^3}F(x,u)\,dx \\ &\geq \frac{1}{2}\|u\|_\lambda^2 - \left(\frac{\lambda V_0 + C_0}{2}\|u\|_2^2 + \frac{C_0}{p}\|u\|_p^p\right). \end{aligned} \tag{2.17}$$

By virtue of Lemma 2.5, we can choose an integer $m \geq 1$, for all $u \in Z_m$, satisfying

$$\begin{aligned}\|u\|_2^2 &\leq \frac{1}{2(\lambda V_0 + C_0)}\|u\|_\lambda^2, \\ \|u\|_p^p &\leq \frac{p}{4C_0}\|u\|_\lambda^p.\end{aligned}$$

Combining this with (2.17), one has

$$\Phi_\lambda(u) \geq \frac{1}{4}\|u\|_\lambda^2\left(1 - \|u\|_\lambda^{p-2}\right).$$

Note that, if we let $\rho = \|u\|_\lambda > 0$ be sufficiently small, then $\Phi_\lambda(u) \geq \frac{1}{8}\rho^2 > 0$. □

Lemma 2.7 *Assume that* (V), (F_1) *and* (F_2) *hold, then, for any finite dimensional subspace* $E \subset E_\lambda$, *there exists* $R = R(E) > 0$ *such that* $\Phi_\lambda|_{E\setminus B_\rho} < 0$.

Proof According to the proof of Lemma 2.4, we know that, for any $u \in E\setminus\{0\}$, there exists a unique $T = \widetilde{t}(u) > 0$ such that

$$v(x) = T^{-1}u(Tx) \in H^2\left(\mathbb{R}^3\right) \quad \text{and} \quad \|v\|_{H^2} = 1.$$

Hence

$$u(x) = Tv\left(T^{-1}x\right) \quad \text{with } \|v\|_{H^2} = 1 \text{ and } T > 0.$$

By the equivalence of norms in the finite dimensional space E, there exists $C_3 > 0$ such that

$$\min\{a,1\}\|u\|_{H^2}^2 \leq \|u\|_\lambda^2 \leq C_3\|u\|_2^2.$$

Combining this with

$$T = \widetilde{t}(u) \to \infty \quad \text{as } \|u\|_\lambda \to \infty \text{ uniformly in } E,$$

we find that, for any $\delta > 0$, there exists a large $R = R(E, \delta) > 0$ such that

$$T = \widetilde{t}(u) \geq \delta \quad \text{for all } u \in E \text{ with } \|u\|_\lambda \geq R.$$

By (F_1), there exists $C_4 > 0$, for all $x \in \mathbb{R}^N$, $|u| \leq R_0$ such that

$$\left|F(x,u)\right| \leq C_4 u^2,$$

where R_0 is given by (2.15). Combining (2.1) with $\|v\|_{H^2} = 1$, it follows that for all $u \in E \backslash \{0\}$

$$\begin{aligned}
\Phi_\lambda(u) &= \frac{1}{2}\|u\|_\lambda^2 + \frac{b}{4}\|\nabla u\|_2^4 + \frac{1}{2}\int_{\mathbb{R}^3} u^2|\nabla u|^2\,dx - \int_{\mathbb{R}^3}\left[\frac{\lambda V_0}{2}u^2 + F(x,u)\right]dx \\
&\leq \frac{C_3}{2}\|u\|_2^2 + \frac{b}{4}\|\nabla u\|_2^4 + \frac{1}{2}\int_{\mathbb{R}^3} u^2|\nabla u|^2\,dx - \int_{\mathbb{R}^3}\left[\frac{\lambda V_0}{2}u^2 + F(x,u)\right]dx \\
&= \frac{C_3 - \lambda V_0}{2}T^5\|v\|_2^2 + \frac{bT^6}{4}\|\nabla v\|_2^4 - T^3\int_{\mathbb{R}^3} F(Tx, Tv)\,dx + \frac{1}{2}\int_{\mathbb{R}^3} u^2|\nabla u|^2\,dx \\
&\leq T^6\left(\frac{b}{4} + \frac{C_3 + \lambda V_0 + 2C_4}{2T} - \int_{|Tv|\geq R_0}\frac{F(Tx, Tv)}{T^3}\,dx\right) + \frac{1}{2}\int_{\mathbb{R}^3} u^2|\nabla u|^2\,dx \\
&= \Psi(T).
\end{aligned} \tag{2.18}$$

Note that $v \neq 0$, then it follows from (F_2) that

$$\frac{F(Tx, Tv)}{|Tv|^3} \to +\infty \quad \text{as } T \to +\infty.$$

Thus

$$\int_{|Tv|\geq R_0}\frac{F(Tx, Tv)}{T^3} \to +\infty \quad \text{as } T \to +\infty.$$

Combining this with (2.18), we obtain

$$\Psi(T) \to -\infty \quad \text{as } T \to +\infty.$$

Thus, there exists a large $T_0 > 0$ such that

$$\Psi(T) \leq -1$$

for all $T \geq T_0$. Taking $\delta = T_0$, then there exists a large $R = R(E) > 0$ such that

$$T = \widetilde{t}(u) \geq T_0$$

for all $u \in E$ with $\|u\|_\lambda \geq R$.

Hence, $\Phi_\lambda(u) < 0$ for all $u \in E$ with $\|u\|_\lambda \geq R$. □

Proof of Theorem 1.1 Let $X = E_\lambda$, $Y = Y_m$ and $Z = \overline{Z_m}$. Clearly, $\Phi(0) = 0$ and $\Phi(u) = \Phi(-u)$ due to (F_4). By virtue of Lemma 2.4, Lemma 2.6, Lemma 2.7 and the fountain theorem (Theorem 3.6 [16]), problem (1.1) possesses infinitely many high energy solutions. □

Proof of Corollary 1.2 Let us consider the Hilbert space

$$H = \left\{ u \in H^1(\mathbb{R}^3) : \int_{\mathbb{R}^3} \left(a|\nabla u|^2 + \lambda \widetilde{V}(x)u^2 \right) dx < \infty \right\}$$

endowed with the norm

$$\|u\| = \left(\int_{\mathbb{R}^3} \left(a|\nabla u|^2 + \lambda \widetilde{V}(x)u^2 \right) dx \right)^{\frac{1}{2}}.$$

Let

$$\Phi(u) = \frac{1}{2} \int_{\mathbb{R}^3} \left(a|\nabla u|^2 + \lambda V(x)u^2 \right) dx + \frac{b}{4} \left(\int_{\mathbb{R}^3} |\nabla u|^2 \, dx \right)^2 - \int_{\mathbb{R}^3} F(x,u) \, dx, \quad \forall u \in H.$$

Obviously, Φ is a well-defined class C^1 functional, and the embedding $H \hookrightarrow L^s$ is compact for $2 \leq s < 6$ (see the proof of Lemma 2.2). By Lemma 2.4, Lemma 2.6, Lemma 2.7 and the fountain theorem (Theorem 3.6 [16]), problem (1.3) possesses infinitely many high energy solutions. □

Remark 2.8 In the next paper, we wish to consider the sign-changing solutions for the biharmonic problem like in [19, 21] and so on.

3 Conclusions

In this paper, we consider a sequence of high energy weak solutions for the modified quasilinear fourth-order elliptic equation (1.1) under rather weak conditions. We first prove that the energy functional satisfies the Cerami condition in the well-defined Hilbert space and then prove that the fountain theorem holds under the given conditions by a new technique. Our results extend and improve some recent results.

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Competing interests
The authors declare that they have no competing interests.

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