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Ground state sign-changing solutions for semilinear Dirichlet problems

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Abstract

In the present paper, we consider the existence of ground state sign-changing solutions for the semilinear Dirichlet problem

$$\begin{cases} -\Delta u + \lambda u = f(x,u), & x \in \Omega; \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{0.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with a smooth boundary $\partial\Omega$, $\lambda > -\lambda_1$ is a constant, λ_1 is the first eigenvalue of $(-\triangle, H_0^1(\Omega))$, and $f \in C(\Omega \times \mathbb{R}, \mathbb{R})$. Under some standard growth assumptions on f and a weak version of Nehari type monotonicity condition that the function $t \mapsto f(x,t)/|t|$ is *non-decreasing* on $(-\infty,0) \cup (0,\infty)$ for every $x \in \Omega$, we prove that (0.1) possesses one ground state sign-changing solution, which has precisely two nodal domains. Our results improve and generalize some existing ones.

MSC: 35J20; 35J65

Keywords: Semilinear Dirichlet problem; Ground state sign-changing solutions; Perturbation method

1 Introduction

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with a smooth boundary $\partial\Omega$. In this paper we are concerned with the existence of sign-changing solutions of the semilinear Dirichlet problem

$$\begin{cases} -\triangle u + \lambda u = f(x,u), & x \in \Omega; \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.1}$$

where $\lambda > -\lambda_1$ is a constant, λ_1 is the first eigenvalue of $(-\triangle, H_0^1(\Omega))$, and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies the following assumptions:

(F1) $f \in \mathcal{C}(\Omega \times \mathbb{R}, \mathbb{R})$ and $f(x,t) = o(t)$ as $t \to 0$ uniformly in $x \in \Omega$;

(F2) there exist constants $\mathcal{C}_0 > 0$ and $p \in (2, 2^*)$ such that

$$\bigl|f(x,t)\bigr| \le \mathcal{C}_0\bigl(1 + |t|^{p-1}\bigr), \quad \forall (x,t) \in \Omega \times \mathbb{R},$$

where $2^* = 2N/(N-2)$ if $N \ge 3$, and $2^* = +\infty$ if $N = 1, 2$;

(F3) $\lim_{|t|\to\infty} \frac{F(x,t)}{t^2} = \infty$ uniformly in $x \in \Omega$, where $F(x,t) = \int_0^t f(x,s)\,ds$;

(F4) The function $t \mapsto f(x,t)/|t|$ is non-decreasing on $(-\infty,0) \cup (0,\infty)$ for every $x \in \Omega$.

Let $H_0^1(\Omega)$ be the Sobolev space, and define $\Phi : H_0^1(\Omega) \to \mathbb{R}$ as follows:

$$\Phi(u) = \frac{1}{2}\int_\Omega \left(|\nabla u|^2 + \lambda u^2\right) \mathrm{d}x - \int_\Omega F(x,u)\,\mathrm{d}x. \tag{1.2}$$

It is a well-known consequence of (F1) and (F2) that $\Phi \in \mathcal{C}^1(H_0^1(\Omega), \mathbb{R})$ and the critical points of Φ are weak solutions of (1.1). Furthermore, if $u \in H_0^1(\Omega)$ is a solution of (1.1) and $u^\pm \neq 0$, then u is a sign-changing solution of (1.1), where

$$u^+(x) := \max\{u(x), 0\} \quad \text{and} \quad u^-(x) := \min\{u(x), 0\}.$$

In order to facilitate the narrative, we set

$$\mathcal{M} := \left\{u \in H_0^1(\Omega) : u^\pm \neq 0, \langle \Phi'(u), u^+\rangle = \langle \Phi'(u), u^-\rangle = 0\right\}, \tag{1.3}$$

$$\mathcal{N} := \left\{u \in H_0^1(\Omega) : u \neq 0, \langle \Phi'(u), u\rangle = 0\right\}, \tag{1.4}$$

and put

$$m_0 := \inf_{u\in\mathcal{M}} \Phi(u), \qquad c_0 := \inf_{u\in\mathcal{N}} \Phi(u). \tag{1.5}$$

Problem (1.1) has been studied extensively, and much progress has been made recently concerning the existence of sign-changing solutions, see [1–13]. In particular, Bartsch and Weth [6] proved that (1.1) has a least energy sign-changing solution $\bar{u}$, i.e., $\Phi(\bar{u}) = m_0$, which has precisely two nodal domains under (F1), (F2) and the following assumptions:

(F5) $f \in \mathcal{C}^1(\Omega \times \mathbb{R}, \mathbb{R})$ and $f'(x,t) > f(x,t)/t$ for all $x \in \Omega$ and $t \neq 0$;

(AR) there exists $\mu > 2$ such that $tf(x,t) \geq \mu F(x,t) > 0$ for all $x \in \Omega$ and large $|t|$.

This result improves the work of Castro et al. [9] as well as the one of Bartsch et al. [2] for (1.1) with $f(x,t) = f(t)$. Moreover, further information is gained on sign-changing solutions, in particular on the nodal structure, extremality properties, and the Morse index with respect to Φ.

We point out that (F5) plays a very crucial role in papers [2, 6, 9], it is a stronger version of the following Nehari type monotonicity assumption:

(Ne) The function $t \mapsto f(x,t)/|t|$ is strictly increasing on $(-\infty, 0) \cup (0, \infty)$ for every $x \in \Omega$.

(Ne) seems to be essential in seeking a ground solution of Nehari type for (1.1), for example, see [14–16]. It is also necessary for the existence of a least energy sign-changing solution. In particular, under (F1), (F2), and (Ne), Bartsch and Weth [6] proved that every weak solution $u \in \mathcal{M}$ of (1.1) with $0 < \Phi(u) \leq m_0$ has precisely two nodal domains, while Bartsch et al. [17] showed that every minimizer of Φ on $\mathcal{M}$ is a critical point of Φ, hence a sign-changing solution of (1.1) with precisely two nodal domains. Under some additional conditions on Ω and f, such as (F5) and (AR), the infimum m_0 of Φ can be attained in $\mathcal{M}$, see [2, 6, 9]. However, it is unknown whether assumptions (F1), (F2), and (Ne) guarantee that the infimum m_0 of Φ is attained in $\mathcal{M}$.

It is a well-known consequence of (Ne) that there is unique $t_u > 0$ such that $t_u u \in \mathcal{N}$ for every $u \in H_0^1(\Omega) \setminus \{0\}$, which implies that Φ has one minimizer on $\mathcal{M}$ at most. Moreover, in Bartsch et al. [17], (Ne) plays a very important role in showing that every minimizer of Φ on $\mathcal{M}$ is a critical point. If $t \mapsto f(x,t)/|t|$ is not strictly increasing, then t_u and the minimizer of Φ on $\mathcal{M}$ may not be unique, and their arguments become invalid. This paper intends to address this problem caused by the dropping of this "strictly increasing" condition on f. Motivated by the works [2, 6, 9, 17–26], we will use variational methods to generalize and improve the existence results on sign-changing solutions in reference to the relaxing assumption (Ne). However, our proof relies more on the specific choice of the (P.S.) sequence than on the appropriate minimax principle.

We are now in a position to state the main results of this paper.

Theorem 1.1 *Assume that* $\lambda > -\lambda_1$ *and* (F1)–(F4) *hold. Then Problem* (1.1) *has a sign-changing solution* $u_0 \in \mathcal{M}$ *such that* $\Phi(u_0) = \inf_{\mathcal{M}} \Phi > 0$. *Furthermore, suppose that*

$$\frac{1}{2}tf(x,t) - F(x,t) > 0, \quad \forall x \in \Omega, t \neq 0. \tag{1.6}$$

Then u_0 *has precisely two nodal domains.*

Theorem 1.2 *Assume that* $\lambda > -\lambda_1$ *and* (F1)–(F4) *hold. Then* $m_0 \geq 2c_0$.

Remark 1.3 Tang [27, Theorem 1.2] has proved that if $\lambda > -\lambda_1$ and (F1)–(F4) hold, then Problem (1.1) has a solution $\bar{u} \in \mathcal{N}$ such that $\Phi(\bar{u}) = \inf_{\mathcal{N}} \Phi = c_0 > 0$.

Remark 1.4 In [3], Bartsch et al. obtained the existence of sign-changing solutions of (1.1) under (F1), (F2), (F3), and (AR) by using variational methods and invariant sets of descent flow. However, the sign-changing solutions obtained in [3] are not the ground state ones.

2 Some preliminary lemmas

In this section, we give some preliminary lemmas which are crucial for proving our results. We introduce a new inner product and a norm on $H_0^1(\Omega)$

$$(u,v) = \int_\Omega (\nabla u \cdot \nabla v + \lambda uv)\,\mathrm{d}x, \qquad \|u\| = (u,u)^{1/2}, \quad \forall u,v \in H_0^1(\Omega),$$

where $(\cdot,\cdot)_2$ and $\|\cdot\|_2$ denote the usual L^2-inner product and the norm, respectively. In view of Sobolev embedding theorem, the norm $\|\cdot\|$ is equivalent to the usual norm in $H_0^1(\Omega)$. Furthermore, for any $s \in [2, 2^*]$, there exists a constant $\gamma_s > 0$ such that $\|u\|_s \leq \gamma_s \|u\|$ for all $u \in H_0^1(\Omega)$. Hence, the energy functional Φ can be rewritten as

$$\Phi(u) = \frac{1}{2}\|u\|^2 - \int_\Omega F(x,u)\,\mathrm{d}x, \quad \forall u \in H_0^1(\Omega). \tag{2.1}$$

Moreover, for any $u, \varphi \in H_0^1(\Omega)$, we have

$$\langle \Phi'(u), \varphi \rangle = (u,\varphi) - \int_\Omega f(x,u)\varphi\,\mathrm{d}x. \tag{2.2}$$

Lemma 2.1 *Assume that* (F1)–(F4) *hold. Then*

$$\begin{aligned} &\Phi(u) \geq \Phi(su^+ + tu^-) + \frac{1-s^2}{2}\langle \Phi'(u), u^+ \rangle + \frac{1-t^2}{2}\langle \Phi'(u), u^- \rangle, \\ &\quad \forall u = u^+ + u^- \in H_0^1(\Omega), s,t \geq 0. \end{aligned} \tag{2.3}$$

Proof It follows from (F4) that

$$\begin{aligned}
&\frac{1-t^2}{2}\tau f(x,\tau)+F(x,t\tau)-F(x,\tau)\\
&\quad=\int_t^1\left[\frac{f(x,\tau)}{\tau}-\frac{f(x,s\tau)}{s\tau}\right]s\tau^2\,\mathrm{d}s\geq 0,\quad \forall t\geq 0,\tau\in\mathbb{R}\setminus\{0\}.
\end{aligned}\tag{2.4}$$

Thus, by (2.1), (2.2), and (2.4), one has

$$\begin{aligned}
\Phi(u)-\Phi\bigl(su^++tu^-\bigr)&=\frac{1}{2}\bigl(\|u\|^2-\bigl\|su^++tu^-\bigr\|^2\bigr)+\int_\Omega\bigl[F\bigl(x,su^++tu^-\bigr)-F(x,u)\bigr]\,\mathrm{d}x\\
&=\frac{1-s^2}{2}\bigl\|u^+\bigr\|^2+\frac{1-t^2}{2}\bigl\|u^-\bigr\|^2\\
&\quad+\int_\Omega\bigl[F\bigl(x,su^+\bigr)+F\bigl(x,tu^-\bigr)-F\bigl(x,u^+\bigr)-F\bigl(x,u^-\bigr)\bigr]\,\mathrm{d}x\\
&=\frac{1-s^2}{2}\bigl\langle\Phi'(u),u^+\bigr\rangle+\frac{1-t^2}{2}\bigl\langle\Phi'(u),u^-\bigr\rangle\\
&\quad+\int_\Omega\left[\frac{1-s^2}{2}f\bigl(x,u^+\bigr)u^++F\bigl(x,su^+\bigr)-F\bigl(x,u^+\bigr)\right]\mathrm{d}x\\
&\quad+\int_\Omega\left[\frac{1-t^2}{2}f\bigl(x,u^-\bigr)u^-+F\bigl(x,tu^-\bigr)-F\bigl(x,u^-\bigr)\right]\mathrm{d}x\\
&\geq\frac{1-s^2}{2}\bigl\langle\Phi'(u),u^+\bigr\rangle+\frac{1-t^2}{2}\bigl\langle\Phi'(u),u^-\bigr\rangle,\quad\forall s,t\geq 0.
\end{aligned}$$

This shows that (2.3) holds. □

From Lemma 2.1, we have the following two corollaries immediately.

Corollary 2.2 *Assume that* (F1)–(F4) *hold. If* $u=u^++u^-\in\mathcal{M}$, *then*

$$\Phi\bigl(u^++u^-\bigr)=\max_{s,t\geq 0}\Phi\bigl(su^++tu^-\bigr).\tag{2.5}$$

Corollary 2.3 *Assume that* (F1)–(F4) *hold. If* $u\in\mathcal{N}$, *then*

$$\Phi(u)=\max_{t\geq 0}\Phi(tu).\tag{2.6}$$

By a standard argument, we can prove the following lemma using (Ne), see [28, Lemma 4.1].

Lemma 2.4 *Assume that* (F1)–(F3), (Ne) *hold. If* $u\in H_0^1(\Omega)$ *with* $u^\pm\neq 0$, *then there exists a unique pair* (s_u,t_u) *of positive numbers such that* $s_uu^++t_uu^-\in\mathcal{M}$.

Lemma 2.5 *Assume that* (F1)–(F3), (Ne) *hold. Then*

$$m_0=\inf_{u\in\mathcal{M}}\Phi(u)=\inf_{u\in H_0^1(\Omega),u^\pm\neq 0}\max_{s,t\geq 0}\Phi\bigl(su^++tu^-\bigr).$$

Proof On the one hand, by Corollary 2.2, one has

$$\inf_{u\in H_0^1(\Omega),u^\pm\neq 0}\max_{s,t\geq 0}\Phi\bigl(su^+ + tu^-\bigr)\leq \inf_{u\in\mathcal{M}}\max_{s,t\geq 0}\Phi\bigl(su^+ + tu^-\bigr)=\inf_{u\in\mathcal{M}}\Phi(u)=m_0. \tag{2.7}$$

On the other hand, for any $u\in H_0^1(\Omega)$ with $u^\pm\neq 0$, it follows from Lemma 2.4 that

$$\max_{s,t\geq 0}\Phi\bigl(su^+ + tu^-\bigr)\geq \Phi\bigl(s_u u^+ + t_u u^-\bigr)\geq \inf_{v\in\mathcal{M}}\Phi(v)=m_0,$$

which implies

$$\inf_{u\in H_0^1(\Omega),u^\pm\neq 0}\max_{s,t\geq 0}\Phi\bigl(su^+ + tu^-\bigr)\geq \inf_{u\in\mathcal{M}}\Phi(u)=m_0. \tag{2.8}$$

Hence, the conclusion directly follows from (2.7) and (2.8). □

Lemma 2.6 *Assume that* (F1)–(F3), (Ne) *hold. Then* $m_0>0$ *can be achieved.*

Proof Let $\{u_n\}\subset\mathcal{M}$ be such that $\Phi(u_n)\to m_0$. First, we prove that $\{u_n\}$ is bounded in E. Arguing by contradiction, suppose that $\|u_n\|\to\infty$. Let $v_n=u_n/\|u_n\|$, then $\|v_n\|=1$. By Sobolev embedding theorem, passing to a subsequence, we may assume that $v_n\to v$ in $L^s(\Omega)$, $2\leq s<2^*$, $v_n\to v$ a.e. on Ω.

If $v=0$, then $v_n\to 0$ in $L^s(\Omega)$ for $2\leq s<2^*$. Fix $R>[2(1+m_0)]^{1/2}$. By (F1) and (F2), there exists $C_1>0$ such that

$$\limsup_{n\to\infty}\int_\Omega F(x,Rv_n)\,\mathrm{d}x\leq R^2\lim_{n\to\infty}\|v_n\|_2^2+C_1R^p\lim_{n\to\infty}\|v_n\|_p^p=0. \tag{2.9}$$

Let $t_n=R/\|u_n\|$. Hence, by (2.1), (2.9), and Corollary 2.3, one has

$$\begin{aligned}
m_0+o(1)=\Phi(u_n)&\geq\Phi(t_nu_n)\\
&=\frac{t_n^2}{2}\|u_n\|^2-\int_\Omega F(x,t_nu_n)\,\mathrm{d}x\\
&=\frac{R^2}{2}-\int_\Omega F(x,Rv_n)\,\mathrm{d}x\\
&=\frac{R^2}{2}+o(1)>m_0+1+o(1),
\end{aligned}$$

which is a contradiction. Thus $v\neq 0$.

For $x\in\{z\in\mathbb{R}^N: v(z)\neq 0\}$, we have $\lim_{n\to\infty}|u_n(x)|=\infty$. Hence, it follows from (F3), (Ne), and Fatou's lemma that

$$\begin{aligned}
0&=\lim_{n\to\infty}\frac{m_0+o(1)}{\|u_n\|^2}=\lim_{n\to\infty}\frac{\Phi(u_n)}{\|u_n\|^2}\\
&=\lim_{n\to\infty}\left[\frac{1}{2}\|v_n\|^2-\int_\Omega\frac{F(x,u_n)}{u_n^2}v_n^2\,\mathrm{d}x\right]\\
&\leq\frac{1}{2}-\liminf_{n\to\infty}\int_\Omega\frac{F(x,u_n)}{u_n^2}v_n^2\,\mathrm{d}x\leq\frac{1}{2}-\int_\Omega\liminf_{n\to\infty}\frac{F(x,u_n)}{u_n^2}v_n^2\,\mathrm{d}x\\
&=-\infty.
\end{aligned}$$

This contradiction shows that $\{u_n\}$ is bounded in $H_0^1(\Omega)$. Thus there exists $u_0 \in H_0^1(\Omega)$ such that $u_n^{\pm} \rightharpoonup u_0^{\pm}$ in $H_0^1(\Omega)$, which implies that $u_n^{\pm} \to u_0^{\pm}$ in $L^s(\Omega)$ for $s \in [2, 2^*)$ and $u_n^{\pm} \to u_0^{\pm}$ a.e. on Ω.

Next, we prove that $u_0 \in \mathcal{M}$ and $\Phi(u_0) = m_0$. Since $\inf_{\mathcal{N}} \Phi = c_0 > 0$, $u_n \in \mathcal{M}$, and $u_n^{\pm} \in \mathcal{N}$, then it follows from (2.1), (2.2), and the weak semicontinuity of norm that

$$\begin{aligned}
\langle \Phi'(u_0), u_0^{\pm} \rangle &= \|u_0^{\pm}\|^2 - \int_\Omega f(x, u_0^{\pm}) u_0^{\pm}\,\mathrm{d}x \\
&\le \liminf_{n\to\infty} \left[\|u_n^{\pm}\|^2 - \int_\Omega f(x, u_n^{\pm}) u_n^{\pm}\,\mathrm{d}x \right] \\
&= \liminf_{n\to\infty} \langle \Phi'(u_n), u_n^{\pm} \rangle = 0
\end{aligned}$$

and

$$\begin{aligned}
\int_\Omega \left[\frac{1}{2} f(x, u_0^{\pm}) - F(x, u_0^{\pm}) \right] \mathrm{d}x &= \lim_{n\to\infty} \int_\Omega \left[\frac{1}{2} f(x, u_n^{\pm}) - F(x, u_n^{\pm}) \right] \mathrm{d}x \\
&= \lim_{n\to\infty} \left[\Phi(u_n^{\pm}) - \frac{1}{2} \langle \Phi'(u_n^{\pm}), u_n^{\pm} \rangle \right] \\
&= \lim_{n\to\infty} \Phi(u_n^{\pm}) \ge c_0 > 0.
\end{aligned}$$

These, together with (2.4) ($t = 0$), show

$$u_0^{\pm} \ne 0, \qquad \langle \Phi'(u_0), u_0^{\pm} \rangle \le 0. \tag{2.10}$$

By Lemma 2.4, there exist $s_0, t_0 > 0$ such that $s_0 u_0^+ + t_0 u_0^- \in \mathcal{M}$. From (2.1), (2.2), (2.10), and Lemma 2.1, we have

$$\begin{aligned}
m_0 &= \lim_{n\to\infty} \left[\Phi(u_n) - \frac{1}{2} \langle \Phi'(u_n), u_n \rangle \right] \\
&= \lim_{n\to\infty} \int_\Omega \left[\frac{1}{2} f(x, u_n) u_n - F(x, u_n) \right] \mathrm{d}x \\
&= \int_\Omega \left[\frac{1}{2} f(x, u_0) u_0 - F(x, u_0) \right] \mathrm{d}x \\
&= \Phi(u_0) - \frac{1}{2} \langle \Phi'(u_0), u_0 \rangle \\
&\ge \Phi(s_0 u_0^+ + t_0 u_0^-) + \frac{1 - s_0^2}{2} \langle \Phi'(u_0), u_0^+ \rangle + \frac{1 - t_0^2}{2} \langle \Phi'(u_0), u_0^- \rangle - \frac{1}{2} \langle \Phi'(u_0), u_0 \rangle \\
&\ge m_0 - \frac{s_0^2}{2} \langle \Phi'(u_0), u_0^+ \rangle - \frac{t_0^2}{2} \langle \Phi'(u_0), u_0^- \rangle,
\end{aligned}$$

which implies

$$\langle \Phi'(u_0), u_0^{\pm} \rangle = 0, \qquad \Phi(u_0) = m_0. \qquad \square$$

Similar to the proof of [17, Proposition 3.1], we can prove the following lemma.

Lemma 2.7 *Assume that* (F1)–(F3), (Ne) *hold. If* $u_0 \in \mathcal{M}$ *and* $\Phi(u_0) = m_0$, *then* u_0 *is a critical point of* Φ.

3 Sign-changing solutions

For any $\epsilon > 0$, let $f_\epsilon(x,t) = f(x,t) + \epsilon p|t|^{p-2}t$ and

$$\Phi_\epsilon(u) = \Phi(u) - \epsilon\|u\|_p^p, \quad \forall u \in H_0^1(\Omega). \tag{3.1}$$

Similarly, we define

$$\mathcal{M}_\epsilon := \{u \in H_0^1(\Omega) : u^\pm \neq 0, \langle \Phi'_\epsilon(u), u^+\rangle = \langle \Phi'_\epsilon(u), u^-\rangle = 0\}, \tag{3.2}$$

$$\mathcal{N}_\epsilon := \{u \in H_0^1(\Omega) : u \neq 0, \langle \Phi'_\epsilon(u), u\rangle = 0\}, \tag{3.3}$$

and

$$m_\epsilon := \inf_{u\in\mathcal{M}_\epsilon} \Phi_\epsilon(u), \qquad c_\epsilon := \inf_{u\in\mathcal{N}_\epsilon} \Phi_\epsilon(u). \tag{3.4}$$

Lemma 3.1 *Assume that* (F1)–(F4) *hold. Then there exists a constant* $\alpha > 0$ *which does not depend on* $\epsilon \in (0,1]$ *such that*

$$\Phi_\epsilon(u) \geq \alpha, \quad \forall u \in \mathcal{N}_\epsilon, \epsilon \in (0,1]. \tag{3.5}$$

Proof By (F1) and (F2), there exists a constant $C_2 > 0$ such that

$$F(x,t) \leq \frac{1}{4\gamma_2^2}t^2 + C_2|t|^p, \quad \forall (x,t) \in \Omega \times \mathbb{R}. \tag{3.6}$$

From (3.1), (3.6), and Corollary 2.3, one has

$$\begin{aligned}
\Phi_\epsilon(u) &= \max_{t\geq 0} \Phi_\epsilon(tu) = \max_{t\geq 0}\left[\frac{t^2}{2}\|u\|^2 - \int_\Omega F(x,tu)\,\mathrm{d}x - \epsilon t^p\|u\|_p^p\right] \\
&\geq \max_{t\geq 0}\left[\frac{t^2}{4}\|u\|^2 - (C_2+1)\gamma_p^p t^p\|u\|^p)\right] \\
&= \frac{p-2}{4p[2(C_2+1)\gamma_p^p p]^{2/(p-2)}} := \alpha > 0, \quad \forall u \in \mathcal{N}_\epsilon, \epsilon \in (0,1].
\end{aligned}$$

□

Proof of Theorem 1.1 Under the conditions of Theorem 1.1, for $\epsilon > 0$, f_ϵ satisfies (F1)–(F3) and (Ne). In view of Lemmas 2.6 and 2.7, there exists $u_\epsilon \in \mathcal{M}_\epsilon$ such that $\Phi_\epsilon(u_\epsilon) = m_\epsilon$ and $\Phi'_\epsilon(u_\epsilon) = 0$.

By (F1)–(F3), one can easily prove that $\mathcal{M}_0 \neq \emptyset$. Let $u_0 \in \mathcal{M}_0$. Then $\Phi(u_0) := c^* > 0$ and $\langle \Phi'(u_0), u_0^\pm\rangle = 0$. By Lemma 2.4, there exist $s_\epsilon > 0$ and $t_\epsilon > 0$ such that $s_\epsilon u_0^+ + t_\epsilon u_0^- \in \mathcal{M}_\epsilon$. Hence, from Corollary 2.2 and Lemma 3.1, we have

$$\begin{aligned}
c^* &= \Phi(u_0) \\
&\geq \Phi(s_\epsilon u_0^+ + t_\epsilon u_0^-) \geq \Phi_\epsilon(s_\epsilon u_0^+ + t_\epsilon u_0^-) \\
&\geq m_\epsilon \geq \hat{\kappa}, \quad \forall \epsilon \in (0,1).
\end{aligned} \tag{3.7}$$

Hence, we can choose a sequence $\{\epsilon_n\}$ such that $\epsilon_n \searrow 0$ as $n \to \infty$, and

$$u_{\epsilon_n} \in \mathcal{M}_{\epsilon_n}, \qquad \Phi_{\epsilon_n}(u_{\epsilon_n}) = m_{\epsilon_n} \to \bar{m}, \qquad \Phi'_{\epsilon_n}(u_{\epsilon_n}) = 0. \tag{3.8}$$

First, we prove that $\{u_{\epsilon_n}\}$ is bounded in $H_0^1(\Omega)$. Arguing by contradiction, suppose that $\|u_{\epsilon_n}\| \to \infty$. Let $v_n = u_{\epsilon_n}/\|u_{\epsilon_n}\|$, then $\|v_n\| = 1$. By Sobolev embedding theorem, passing to a subsequence, we may assume that $v_n \to v$ in $L^s(\Omega)$, $2 \le s < 2^*$, $v_n \to v$ a.e. on Ω.

If $v = 0$, then $v_n \to 0$ in $L^s(\Omega)$ for $2 \le s < 2^*$. Fix $R > [2(1 + \bar{m})]^{1/2}$. By (F1) and (F2), there exists $C_3 > 0$ such that

$$\limsup_{n\to\infty} \int_\Omega F(x, Rv_n)\,\mathrm{d}x \le R^2 \lim_{n\to\infty} \|v_n\|_2^2 + C_3 R^p \lim_{n\to\infty} \|v_n\|_p^p = 0. \tag{3.9}$$

Let $t_n = R/\|u_{\epsilon_n}\|$. Hence, using (3.1), (3.8), (3.9), and Corollary 2.3, one has

$$\begin{aligned}
m_{\epsilon_n} &= \Phi_{\epsilon_n}(u_{\epsilon_n}) \ge \Phi_{\epsilon_n}(t_n u_{\epsilon_n}) \\
&= \frac{t_n^2}{2}\|u_{\epsilon_n}\|^2 - \int_\Omega \bigl[F(x, t_n u_{\epsilon_n}) + \epsilon_n |t_n u_{\epsilon_n}|^p\bigr]\,\mathrm{d}x \\
&= \frac{R^2}{2} - \int_\Omega \bigl[F(x, Rv_n) + \epsilon_n R^p |v_n|^p\bigr]\,\mathrm{d}x \\
&= \frac{R^2}{2} + o(1) > \bar{m} + 1 + o(1),
\end{aligned}$$

which is a contradiction. Thus $v \neq 0$.

For $x \in \{z \in \mathbb{R}^N : v(z) \neq 0\}$, we have $\lim_{n\to\infty} |u_{\epsilon_n}(x)| = \infty$. Hence, it follows from (F3), (F4), (3.8), and Fatou's lemma that

$$\begin{aligned}
0 &= \lim_{n\to\infty} \frac{m_{\epsilon_n}}{\|u_{\epsilon_n}\|^2} = \lim_{n\to\infty} \frac{\Phi_{\epsilon_n}(u_{\epsilon_n})}{\|u_{\epsilon_n}\|^2} \\
&= \lim_{n\to\infty} \left[\frac{1}{2}\|v_n\|^2 - \int_\Omega \frac{F(x, u_{\epsilon_n}) + \epsilon_n |u_{\epsilon_n}|^p}{u_{\epsilon_n}^2} v_n^2\,\mathrm{d}x\right] \\
&\le \frac{1}{2} - \liminf_{n\to\infty} \int_\Omega \frac{F(x, u_{\epsilon_n})}{u_{\epsilon_n}^2} v_n^2\,\mathrm{d}x \le \frac{1}{2} - \int_\Omega \liminf_{n\to\infty} \frac{F(x, u_{\epsilon_n})}{u_{\epsilon_n}^2} v_n^2\,\mathrm{d}x \\
&= -\infty.
\end{aligned}$$

This contradiction shows that $\{u_{\epsilon_n}\}$ is bounded in $H_0^1(\Omega)$. Hence, there exists a subsequence of $\{\epsilon_n\}$ still denoted by $\{\epsilon_n\}$ and $u_0 \in H_0^1(\Omega)$ such that $u_{\epsilon_n} \rightharpoonup u_0$ in $H_0^1(\Omega)$.

Second, we prove that $\Phi'(u_0) = 0$ and $\Phi(u_0) = m_0$. By Sobolev embedding theorem, $u_{\epsilon_n} \to u_0$ in $L^s(\Omega)$, $2 \le s < 2^*$, $u_{\epsilon_n} \to u_0$ a.e. on Ω. Then, from (2.2), (3.1), and (3.8), one has

$$\begin{aligned}
\langle \Phi'(u_0), \varphi \rangle &= (u_0, \varphi) - \int_\Omega f(x, u_0)\varphi\,\mathrm{d}x \\
&= \lim_{n\to\infty} \left[(u_{\epsilon_n}, \varphi) - \int_\Omega \bigl[f(x, u_{\epsilon_n}) + \epsilon_n p |u_{\epsilon_n}|^{p-2} u_{\epsilon_n}\bigr]\varphi\,\mathrm{d}x\right] \\
&= \lim_{n\to\infty} \langle \Phi'_{\epsilon_n}(u_{\epsilon_n}), \varphi \rangle = 0, \quad \forall \varphi \in C_0^\infty(\Omega).
\end{aligned}$$

This shows $\Phi'(u_0) = 0$. Since $u_{\epsilon_n} \to u_0$ in $L^s(\Omega)$, $2 \le s < 2^*$, by (3.1) and (3.8), we have

$$\begin{aligned}
&\|u_{\epsilon_n} - u_0\|^2 \\
&\quad = \langle \Phi'_{\epsilon_n}(u_{\epsilon_n}) - \Phi'(u_0), u_{\epsilon_n} - u_0\rangle + \epsilon_n p \int_\Omega \big(|u_{\epsilon_n}|^{p-2}u_{\epsilon_n} - |u_0|^{p-2}u_0\big)(u_{\epsilon_n} - u_0)\,\mathrm{d}x \\
&\qquad + \int_\Omega \big[f(x,u_{\epsilon_n}) - f(x,u_0)\big](u_{\epsilon_n} - u_0)\,\mathrm{d}x \to 0, \quad \text{as } n \to \infty,
\end{aligned} \tag{3.10}$$

which implies $u_{\epsilon_n} \to u_0$ in $H_0^1(\Omega)$, and so $u_{\epsilon_n}^\pm \to u_0^\pm$ in $H_0^1(\Omega)$. Consequently, it follows from (3.1) and (3.8) that $\Phi(u_0) = \bar{m}$. Again from (3.1) and (3.5), one has

$$\begin{aligned}
\int_\Omega \left[\frac{1}{2}f\big(x,u_0^\pm\big) - F\big(x,u_0^\pm\big)\right]\mathrm{d}x &= \lim_{n\to\infty} \int_\Omega \left[\frac{1}{2}f\big(x,u_{\epsilon_n}^\pm\big) - F\big(x,u_{\epsilon_n}^\pm\big) + \frac{(p-2)\epsilon_n}{2}\big|u_{\epsilon_n}^\pm\big|^p\right]\mathrm{d}x \\
&= \lim_{n\to\infty}\left[\Phi_{\epsilon_n}\big(u_{\epsilon_n}^\pm\big) - \frac{1}{2}\big\langle\Phi'_{\epsilon_n}\big(u_{\epsilon_n}^\pm\big), u_{\epsilon_n}^\pm\big\rangle\right] \\
&= \lim_{n\to\infty} \Phi_{\epsilon_n}\big(u_{\epsilon_n}^\pm\big) \ge \alpha > 0.
\end{aligned} \tag{3.11}$$

This, together with (2.4) ($t = 0$), shows $u_0^\pm \neq 0$. Thus $u_0 \in \mathcal{M}$ and $\bar{m} = \Phi(u_0) \ge m_0$. Next, we prove $\Phi(u_0) = m_0$. Let ε be any positive number. Then there exists $v_\varepsilon \in \mathcal{M}$ such that $\Phi(v_\varepsilon) < m_0 + \varepsilon$. Then (F3) implies that there exists $K_\varepsilon > 0$ such that, for $s \ge K_\varepsilon$ or $t \ge K_\varepsilon$,

$$\begin{aligned}
\Phi_{\epsilon_n}\big(sv_\varepsilon^+ + tv_\varepsilon^-\big) &= \frac{s^2}{2}\big\|v_\varepsilon^+\big\|^2 - \int_\Omega F\big(x,sv_\varepsilon^+\big)\,\mathrm{d}x - \epsilon_n s^p\big\|v_\varepsilon^+\big\|_p^p \\
&\quad + \frac{t^2}{2}\big\|v_\varepsilon^-\big\|^2 - \int_\Omega F\big(x,tv_\varepsilon^-\big)\,\mathrm{d}x - \epsilon_n t^p\big\|v_\varepsilon^-\big\|_p^p \\
&\le \frac{s^2}{2}\big\|v_\varepsilon^+\big\|^2 - \int_\Omega F\big(x,sv_\varepsilon^+\big)\,\mathrm{d}x + \frac{t^2}{2}\big\|v_\varepsilon^-\big\|^2 - \int_\Omega F\big(x,tv_\varepsilon^-\big)\,\mathrm{d}x < 0.
\end{aligned} \tag{3.12}$$

In view of Lemma 2.4, there exists a pair (s_n, t_n) of positive numbers such that $s_n v_\varepsilon^+ + t_n v_\varepsilon^- \in \mathcal{M}_{\epsilon_n}$, which, together with (3.12) and $c_{\epsilon_n} > 0$, implies $0 < s_n, t_n < K_\varepsilon$. Hence, from (2.3), (3.1), and $\langle\Phi'(v_\varepsilon), v_\varepsilon^\pm\rangle = 0$, we have

$$\begin{aligned}
m_0 + \varepsilon > \Phi(v_\varepsilon) &= \Phi_{\epsilon_n}(v_\varepsilon) + \epsilon_n\|v_\varepsilon\|_p^p \\
&\ge \Phi_{\epsilon_n}\big(s_n v_\varepsilon^+ + t_n v_\varepsilon^-\big) + \frac{1-s_n^2}{2}\big\langle\Phi'_{\epsilon_n}(v_\varepsilon), v_\varepsilon^+\big\rangle + \frac{1-t_n^2}{2}\big\langle\Phi'_{\epsilon_n}(v_\varepsilon), v_\varepsilon^-\big\rangle \\
&\ge m_{\epsilon_n} - \frac{1+K_\varepsilon^2}{2}\big|\big\langle\Phi'_{\epsilon_n}(v_\varepsilon), v_\varepsilon^+\big\rangle\big| - \frac{1+K_\varepsilon^2}{2}\big|\big\langle\Phi'_{\epsilon_n}(v_\varepsilon), v_\varepsilon^-\big\rangle\big| \\
&= m_{\epsilon_n} - \frac{(1+K_\varepsilon^2)p\epsilon_n}{2}\big\|v_\varepsilon^+\big\|_p^p - \frac{(1+K_\varepsilon^2)p\epsilon_n}{2}\big\|v_\varepsilon^-\big\|_p^p,
\end{aligned}$$

which yields

$$\bar{m} = \lim_{n\to\infty} m_{\epsilon_n} \le m_0 + \varepsilon. \tag{3.13}$$

Since $\varepsilon > 0$ is arbitrary, one has $\bar{m} \le m_0$. Thus, $\bar{m} = m_0$, i.e., $\Phi(u_0) = m_0$.

Finally, we show that u_0 has exactly two nodal domains. Let $u_0 = u_1 + u_2 + u_3$, where

$$\begin{aligned} &u_1 \geq 0, \qquad u_2 \leq 0, \qquad \Omega_1 \cap \Omega_2 = \emptyset, \\ &u_1|_{\Omega\setminus(\Omega_1\cup\Omega_2)} = u_2|_{\Omega\setminus(\Omega_1\cup\Omega_2)} = u_3|_{\Omega_1\cup\Omega_2} = 0, \end{aligned} \tag{3.14}$$

$$\Omega_1 := \{x \in \Omega : u_1(x) > 0\}, \qquad \Omega_2 := \{x \in \Omega : u_2(x) < 0\}, \tag{3.15}$$

and Ω_1, Ω_2 are connected open subsets of Ω.

Setting $v = u_1 + u_2$, we see that $v^+ = u_1$ and $v^- = u_2$, i.e., $v^\pm \neq 0$. Note that $\Phi'(u_0) = 0$, by a simple computation, one has

$$\langle \Phi'(v), v^+ \rangle = \langle \Phi'(v), v^- \rangle = 0. \tag{3.16}$$

From (2.1), (2.2), (2.3), (3.14), and (3.16), we have

$$\begin{aligned} m_0 &= \Phi(u_0) - \frac{1}{2}\langle \Phi'(u_0), u_0 \rangle \\ &= \Phi(v) + \Phi(u_3) - \frac{1}{2}\big[\langle \Phi'(v), v\rangle + \langle \Phi'(u_3), u_3\rangle\big] \\ &\geq \sup_{s,t\geq 0} \Phi(sv^+ + tv^-) + \Phi(u_3) - \frac{1}{2}\langle \Phi'(u_3), u_3\rangle \\ &\geq m_0 + \int_\Omega \left[\frac{1}{2} f(x,u_3)u_3 - F(x,u_3)\right] \mathrm{d}x, \end{aligned}$$

which, together with (1.6), shows $u_3 = 0$. Therefore, u_0 has exactly two nodal domains. □

Proof of Theorem 1.2 In view of Theorem 1.1, there exists $u_0 \in \mathcal{M}$ such that $m_0 = \Phi(u_0)$. Since $u^\pm \in \mathcal{N}$, then one has

$$m_0 = \Phi(u_0) = \Phi(u_0^+) + \Phi(u_0^-) \geq 2c_0.$$ □

4 Conclusion

In this paper, by using the variational methods and a suitable approximating method, we prove that Problem (1.1) has a sign-changing solution $u_0 \in \mathcal{M}$ such that $\Phi(u_0) = \inf_{\mathcal{M}} \Phi > 0$ if $\lambda > -\lambda_1$ and f satisfies (F1)–(F4). Furthermore, if $\frac{1}{2}tf(x,t) - F(x,t) > 0$ for all $x \in \mathbb{R}^N$ and $t \neq 0$, we also prove that u_0 has precisely two nodal domains. Our results improve and generalize some existing ones in the literature.

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Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors jointly worked on the results, and they read and approved the final manuscript.

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