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Polynomial energy decay of a wave–Schrödinger transmission system

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Abstract

We study in this paper a wave–Schrödinger transmission system for its stability. By analyzing carefully Green's functions for the infinitesimal generator of the semigroup associated with the system under consideration, we obtain a useful resolvent estimate on this generator which can be applied to derive the decaying property. Our study is inspired by L. Lu & J.-M. Wang [Appl. Math. Lett., 54:7–14, 2016] whose energy decay result is improved upon in our paper. Our method, different from the one used in the previous reference, can be adapted to study stability problems for other 1-D transmission systems.

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Keywords: Wave–Schrödinger transmission system; Polynomial energy decay; Resolvent estimate; Green's functions

1 Introduction

Thanks to its wide applicability, the Schrödinger equation

$$\mathrm{i}\partial_t u + \Delta u + f(\nabla u, u, x, t) = 0, \quad (x,t) \in \mathbb{R}^n \times \mathbb{R},$$

where $\Delta = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}$ is the Laplacian on $\mathbb{R}^n$, has been receiving extensive attention from the mathematical control community; see [2–8] and the references cited therein. Specifically, the systems described by the Schrödinger equation have received extensive studies for their stability in the past three decades. Among the vast references in this direction, Lagnese [9] proved a stability result via "connecting" it to the stability property of the plate equation $\partial_t^2 u + \Delta^2 u + \text{l.o.t} = 0$ (while the study of the stability and stabilization of the plate equation has a relatively long history). Machtyngier and Zuazua [4] studied the boundary and internal stabilization problem via the multiplier method (the main idea has originated from stability studies for wave equations). In [7, 10], some collocated boundary stabilization problems were investigated. Zuazua [2] provided a nice survey on the recent studies on the control properties for the Schrödinger equation.

This paper is devoted to the study of the stabilization of the Schrödinger equation via a damped wave equation through a common end point. More precisely, we are concerned in this paper with the system

$$\begin{cases} \mathrm{i}\partial_t u + \partial_x^2 u = 0 & \text{in } (0,1)\times(0,\infty), \\ \partial_t^2 v - \partial_x^2 v + b\partial_t v = 0 & \text{in } (0,1)\times(0,\infty), \\ u(1,\cdot) = v(1,\cdot) = u(0,\cdot) - k\partial_t v(0,\cdot) & \\ \qquad = \partial_x v(0,\cdot) - \mathrm{i}k\partial_x u(0,\cdot) = 0 & \text{in } (0,\infty), \end{cases} \tag{1.1}$$

where $\mathrm{i} = \sqrt{-1}$ is the imaginary unit, and $k \in \mathbb{R}\setminus\{0\}$ and $b \in (0,\infty)$ are fixed arbitrarily. System (1.1) was recently studied by Lu and Wang [1] with the intension to understand better the transmission of dissipation effect from a damped wave equation to a damping-free Schrödinger equation where the energy can be exchanged by $(1.1)_1$.

The natural phase space for system (1.1) is

$$H = \bigl\{(f,g,h) \in L^2(0,1;\mathbb{C}) \times H^1(0,1;\mathbb{C}) \times L^2(0,1;\mathbb{C}); g(1) = 0\bigr\}. \tag{1.2}$$

Let us define an unbounded linear operator A in H by

$$\begin{cases} \mathcal{D}(A) = \{(f,g,h) \in H^2(0,1;\mathbb{C}^2) \times H^1(0,1;\mathbb{C}); \\ \qquad f(1) = g(1) = h(1) = f(0) - kh(0) = g'(0) - \mathrm{i}kf'(0) = 0\}, \\ A(f,g,h) = (\mathrm{i}f'', h, g'' - bh), \quad \forall (f,g,h) \in \mathcal{D}(A). \end{cases} \tag{1.3}$$

We can prove as in [1] that A is the infinitesimal generator of a strongly continuous semigroup $\{e^{tA}\}$ on H. Therefore, (1.1) admits for every triple $(u^0,u^1,v^0) \in H$ a unique solution $(u,v) \in \mathbb{S}^0$; if further $(u^0,u^1,v^0) \in \mathcal{D}(A)$, then $(u,v) \in \mathbb{S}^1$. Here $\mathbb{S}^0$ and $\mathbb{S}^1$ are defined by

$$\mathbb{S}^0 := \mathcal{C}\bigl([0,\infty);L^2(0,1)\bigr) \times \bigl[\mathcal{C}\bigl([0,\infty);H^1(0,1)\bigr) \cap \mathcal{C}^1\bigl([0,\infty);L^2(0,1)\bigr)\bigr] \quad \text{and} \tag{1.4}$$

$$\begin{aligned} \mathbb{S}^1 := {}& \bigl[\mathcal{C}\bigl([0,\infty);H^2(0,1)\bigr) \cap \mathcal{C}^1\bigl([0,\infty);L^2(0,1)\bigr)\bigr] \\ & \times \bigl[\mathcal{C}\bigl([0,\infty);H^2(0,1)\bigr) \cap \mathcal{C}^1\bigl([0,\infty);H^1(0,1)\bigr)\bigr]. \end{aligned} \tag{1.5}$$

We associate with system (1.1) the following energy functional:

$$E(t) = \frac{1}{2}\int_0^1 \bigl(\bigl|u(x,t)\bigr|^2 + \bigl|\partial_x v(x,t)\bigr|^2 + \bigl|\partial_t v(x,t)\bigr|^2\bigr)\,dx, \quad \forall t \in [0,\infty). \tag{1.6}$$

As indicated before, the study of this paper is directly inspired by [1]. And therefore, it is worth recalling the main results in [1] as follows.

Theorem A (see [1]) *Let A be defined as in* (1.3), *E as in* (1.6), *and H as in* (1.2).

- *A is the infinitesimal generator of a strongly continuous semigroup $\{e^{tA}\}_{t\in[0,\infty)}$ of contractions on H. In particular, we have: For every triple $(u^0,v^0,v^1) \in H$, the boundary value problem* (1.1) *admits a unique solution $(u,v) \in \mathbb{S}^0$ such that $u(\cdot,0) = u^0$, $v(\cdot,0) = v^0$ and $\partial_t v(\cdot,0) = v^1$; if, in addition, $(u^0,v^0,v^1) \in \mathcal{D}(A)$ (see* (1.3)), *then $(u,v) \in \mathbb{S}^1$.*

- *The spectrum* $\sigma(A)$ *of* A *consists merely of eigenvalues of* A, *and is distributed as follows*:

$$\left.\begin{array}{l}\lambda_{1j} = -\frac{b}{2} \pm \frac{\sqrt{b^2-4j^2\pi^2}}{2} + \mathcal{O}(j^{-1}),\\ \lambda_{2j} = -|j-\frac{1}{2}|^2\pi^2\mathrm{i} + \mathcal{O}(j^{-1}), \quad \Re\mathfrak{e}\,\lambda_{2j} < 0\end{array}\right\}, \quad \text{as } j \nearrow \infty. \tag{1.7}$$

- $E(t) \searrow 0$ *as* $t \nearrow \infty$.

Note especially that Lu and Wang [1] proved that $E(t)$ decreases to 0 as $t \to +\infty$. But due to the fact that $\lim_{j\to\infty} \Re\mathfrak{e}\,\lambda_{2j} = 0$, $E(t)$ cannot decay uniformly (see the last section of the paper for a brief proof of this statement). Recently, the non-uniform decay properties have been investigated extensively in the literature for PDEs; see [11, 12]. Our main result gives a more accurate decay rate for the energy $E(t)$.

Theorem 1.1 *Let* E, *defined as in* (1.6), *be the energy associated with system* (1.1). *There exists* $M \in (0,\infty)$ *such that, for every solution* $(u,v) \in \mathbb{S}^1$ *with* $u(\cdot,0) = u^0$, $v(\cdot,0) = v^0$, *and* $\partial_t v(\cdot,0) = v^1$,

$$E(t) \le \frac{M}{t+1}\left(\left\|u^0\right\|^2_{H^2(0,1)} + \left\|v^0\right\|^2_{H^2(0,1)} + \left\|v^1\right\|^2_{H^1(0,1)}\right), \quad \forall t \in [0,\infty). \tag{1.8}$$

By [13, Theorem 2.4], this theorem follows immediately from the following theorem.

Theorem 1.2 *Let* A *be defined by* (1.3). *There exists* $C \in (0,\infty)$ *such that*[a]

$$\left\|R(\mathrm{i}\gamma;A)\right\|_{\mathcal{L}(H)} \le C\langle\lambda\rangle^2, \quad \forall\gamma \in \mathbb{R}. \tag{1.9}$$

Throughout this paper, C is a generic constant which can assume a different value at each occurrence.

The rest of the paper is organized as follows. With the aid of the idea of Green's functions, we provide in Sect. 2 an explicit formulae for the resolvent $R(\mathrm{i}\gamma;A)$. The main results of this paper are proved in Sect. 3. Some concluding remarks are included in Sect. 4.

2 Green's functions and the resolvent $R(\mathrm{i}\gamma;A)$

We would like to calculate in this section the resolvent $R(\mathrm{i}\gamma;A)$ with $\gamma \in \mathbb{R}$ by using the idea of Green's functions. Let $(\phi,\psi,\eta) \in H$. Consider the equation $(\lambda\mathrm{id}_H - A)(f,g,h) = (\phi,\psi,\eta)$ with id_H denoting the identity operator on H, or equivalently, the boundary value problem (BVP)

$$\begin{cases} f'' + \lambda\mathrm{i}f = \mathrm{i}\phi, \qquad h = \lambda g - \psi, \qquad g'' - \lambda(\lambda+b)g = -(\lambda+b)\psi - \eta,\\ f(1) = g(1) = f(0) - kh(0) = g'(0) - \mathrm{i}kf'(0) = 0.\end{cases} \tag{2.1}$$

Denote by F^j and G^j, $j = 1,2,3$, the Green's functions for BVP (2.1). By using the idea of Green's functions, every solution (f,g,h) to BVP (2.1) can be expressed by

$$\left.\begin{aligned} f(x) &= \mathrm{i}\int_0^1 F^1(x,\xi)\phi(\xi)\,d\xi + \int_0^1 F^2(x,\xi)\psi(\xi)\,d\xi \\ &\quad + \int_0^1 F^3(x,\xi)\eta(\xi)\,d\xi, \\ g(x) &= \int_0^1 G^1(x,\xi)\phi(\xi)\,d\xi - (\lambda+b)\int_0^1 G^2(x,\xi)\psi(\xi)\,d\xi \\ &\quad - \int_0^1 G^3(x,\xi)\eta(\xi)\,d\xi, \\ h(x) &= \lambda\int_0^1 G^1(x,\xi)\phi(\xi)\,d\xi - \lambda(\lambda+b)\int_0^1 G^2(x,\xi)\psi(\xi)\,d\xi \\ &\quad - \lambda\int_0^1 G^3(x,\xi)\eta(\xi)\,d\xi - \psi(x) \end{aligned}\right\}, \quad \forall x \in [0,1]. \tag{2.2}$$

The Green's functions for BVP (2.1) should assume the form

$$\begin{cases} F^1(x,\xi) = \sigma_{11}e^{(x-\xi)\sqrt{-\lambda\mathrm{i}}} + \sigma_{12}e^{-(x-\xi)\sqrt{-\lambda\mathrm{i}}} \\ \qquad\qquad + \mathfrak{h}(x-\xi)[\breve{\sigma}_{11}e^{(x-\xi)\sqrt{-\lambda\mathrm{i}}} + \breve{\sigma}_{12}e^{-(x-\xi)\sqrt{-\lambda\mathrm{i}}}], \\ F^2(x,\xi) = \sigma_{21}e^{(x-\xi)\sqrt{-\lambda\mathrm{i}}} + \sigma_{22}e^{-(x-\xi)\sqrt{-\lambda\mathrm{i}}}, \\ F^3(x,\xi) = \sigma_{31}e^{(x-\xi)\sqrt{-\lambda\mathrm{i}}} + \sigma_{32}e^{-(x-\xi)\sqrt{-\lambda\mathrm{i}}}, \\ G^1(x,\xi) = \varsigma_{11}e^{(x-\xi)\sqrt{\lambda(\lambda+b)}} + \varsigma_{12}e^{-(x-\xi)\sqrt{\lambda(\lambda+b)}}, \\ G^2(x,\xi) = \varsigma_{21}e^{(x-\xi)\sqrt{\lambda(\lambda+b)}} + \varsigma_{22}e^{-(x-\xi)\sqrt{\lambda(\lambda+b)}} \\ \qquad\qquad + \mathfrak{h}(x-\xi)[\breve{\varsigma}_{21}e^{(x-\xi)\sqrt{\lambda(\lambda+b)}} + \breve{\varsigma}_{22}e^{-(x-\xi)\sqrt{\lambda(\lambda+b)}}], \\ G^3(x,\xi) = \varsigma_{31}e^{(x-\xi)\sqrt{\lambda(\lambda+b)}} + \varsigma_{32}e^{-(x-\xi)\sqrt{\lambda(\lambda+b)}} \\ \qquad\qquad + \mathfrak{h}(x-\xi)[\breve{\varsigma}_{31}e^{(x-\xi)\sqrt{\lambda(\lambda+b)}} + \breve{\varsigma}_{32}e^{-(x-\xi)\sqrt{\lambda(\lambda+b)}}], \end{cases} \tag{2.3}$$

where $\mathfrak{h}$ is the Heaviside function, namely

$$\mathfrak{h}(\mu) = \begin{cases} 0 & \text{if } \mu \le 0, \\ 1 & \text{if } \mu > 0, \end{cases}$$

and the coefficients σ_{jk}, ς_{jk} ($j = 1,2,3$, $k = 1,2$), $\breve{\sigma}_{11}$, $\breve{\sigma}_{12}$, $\breve{\varsigma}_{jk}$ ($j = 2,3$, $k = 1,2$) are yet to be determined later (see (2.4), (2.5), (2.6), and (2.7)). The Green's functions should also satisfy

$$\left.\begin{aligned} f(1) = 0 \quad &\Longrightarrow \quad F^1(1,\xi) = F^2(1,\xi) = F^3(1,\xi) = 0, \\ g(1) = 0 \quad &\Longrightarrow \quad G^1(1,\xi) = G^2(1,\xi) = G^3(1,\xi) = 0, \\ f(0) = kh(0) \quad &\Longrightarrow \quad \mathrm{i}F^1(0,\xi) - \lambda kG^1(0,\xi) \\ &\qquad = F^2(0,\xi) + k\lambda(\lambda+b)G^2(0,\xi) + k\delta_0(\xi) \\ &\qquad = F^3(0,\xi) + \lambda kG^3(0,\xi) = 0, \\ g'(0) = \mathrm{i}kf'(0) \quad &\Longrightarrow \quad k\partial_x F^1(0,\xi) + \partial_x G^1(0,\xi) \\ &\qquad = k\partial_x F^2(0,\xi) - \mathrm{i}(\lambda+b)\partial_x G^2(0,\xi) \\ &\qquad = k\partial_x F^3(0,\xi) - \mathrm{i}\partial_x G^3(0,\xi) = 0 \end{aligned}\right\}, \quad \forall \xi \in [0,1].$$

This, together with the notion of Green's functions, implies

$$\begin{cases} \breve{\varsigma}_{21} + \breve{\varsigma}_{22} = \breve{\varsigma}_{31} + \breve{\varsigma}_{32} = \breve{\sigma}_{11} + \breve{\sigma}_{12} = 0, \\ \breve{\varsigma}_{21}\sqrt{\lambda(\lambda+b)} - \breve{\varsigma}_{22}\sqrt{\lambda(\lambda+b)} = \breve{\varsigma}_{31}\sqrt{\lambda(\lambda+b)} - \breve{\varsigma}_{32}\sqrt{\lambda(\lambda+b)} \\ \qquad = \breve{\sigma}_{11}\sqrt{-\lambda\mathrm{i}} - \breve{\sigma}_{12}\sqrt{-\lambda\mathrm{i}} = \frac{1}{2}, \end{cases} \tag{2.4}$$

$$\left.\begin{array}{l}
\sigma_{11}e^{(1-\xi)\sqrt{-\lambda\mathrm{i}}}+\sigma_{12}e^{-(1-\xi)\sqrt{-\lambda\mathrm{i}}}=-\breve{\sigma}_{11}e^{(1-\xi)\sqrt{-\lambda\mathrm{i}}}-\breve{\sigma}_{12}e^{-(1-\xi)\sqrt{-\lambda\mathrm{i}}}\\
\qquad\qquad\qquad\qquad\qquad\qquad=-\frac{1}{2\sqrt{-\lambda\mathrm{i}}}\sinh((1-\xi)\sqrt{-\lambda\mathrm{i}}),\\
\varsigma_{11}e^{(1-\xi)\sqrt{\lambda(\lambda+b)}}+\varsigma_{12}e^{-(1-\xi)\sqrt{\lambda(\lambda+b)}}=0,\\
\sigma_{11}e^{-\xi\sqrt{-\lambda\mathrm{i}}}+\sigma_{12}e^{\xi\sqrt{-\lambda\mathrm{i}}}+\varsigma_{11}\mathrm{i}\lambda ke^{-\xi\sqrt{\lambda(\lambda+b)}}+\varsigma_{12}\mathrm{i}\lambda ke^{\xi\sqrt{\lambda(\lambda+b)}}=0,\\
\sigma_{11}k\sqrt{-\lambda\mathrm{i}}e^{-\xi\sqrt{-\lambda\mathrm{i}}}-\sigma_{12}k\sqrt{-\lambda\mathrm{i}}e^{\xi\sqrt{-\lambda\mathrm{i}}}\\
\quad+\varsigma_{11}\sqrt{\lambda(\lambda+b)}e^{-\xi\sqrt{\lambda(\lambda+b)}}-\varsigma_{12}\sqrt{\lambda(\lambda+b)}e^{\xi\sqrt{\lambda(\lambda+b)}}=0
\end{array}\right\},\quad\forall\xi\in[0,1],\tag{2.5}$$

$$\left.\begin{array}{l}
\sigma_{21}e^{(1-\xi)\sqrt{-\lambda\mathrm{i}}}+\sigma_{22}e^{-(1-\xi)\sqrt{-\lambda\mathrm{i}}}=0,\\
\varsigma_{21}e^{(1-\xi)\sqrt{\lambda(\lambda+b)}}+\varsigma_{22}e^{-(1-\xi)\sqrt{\lambda(\lambda+b)}}=-\breve{\varsigma}_{21}e^{(1-\xi)\sqrt{\lambda(\lambda+b)}}-\breve{\varsigma}_{22}e^{-(1-\xi)\sqrt{\lambda(\lambda+b)}}\\
\qquad\qquad\qquad\qquad\qquad\qquad=-\frac{1}{2\sqrt{\lambda(\lambda+b)}}\sinh((1-\xi)\sqrt{\lambda(\lambda+b)}),\\
\sigma_{21}e^{-\xi\sqrt{-\lambda\mathrm{i}}}+\sigma_{22}e^{\xi\sqrt{-\lambda\mathrm{i}}}+\varsigma_{21}k\lambda(\lambda+b)e^{-\xi\sqrt{\lambda(\lambda+b)}}\\
\quad+\varsigma_{22}k\lambda(\lambda+b)e^{\xi\sqrt{\lambda(\lambda+b)}}=-k\delta_0(\xi),\\
\sigma_{21}\mathrm{i}k\sqrt{-\lambda\mathrm{i}}e^{-\xi\sqrt{-\lambda\mathrm{i}}}-\sigma_{22}\mathrm{i}k\sqrt{-\lambda\mathrm{i}}e^{\xi\sqrt{-\lambda\mathrm{i}}}\\
\quad+\varsigma_{21}(\lambda+b)\sqrt{\lambda(\lambda+b)}e^{-\xi\sqrt{\lambda(\lambda+b)}}-\varsigma_{22}(\lambda+b)\sqrt{\lambda(\lambda+b)}e^{\xi\sqrt{\lambda(\lambda+b)}}=0
\end{array}\right\},\quad\forall\xi\in[0,1],\tag{2.6}$$

and

$$\left.\begin{array}{l}
\sigma_{31}e^{(1-\xi)\sqrt{-\lambda\mathrm{i}}}+\sigma_{32}e^{-(1-\xi)\sqrt{-\lambda\mathrm{i}}}=0,\\
\varsigma_{31}e^{(1-\xi)\sqrt{\lambda(\lambda+b)}}+\varsigma_{32}e^{-(1-\xi)\sqrt{\lambda(\lambda+b)}}=-\breve{\varsigma}_{31}e^{(1-\xi)\sqrt{\lambda(\lambda+b)}}-\breve{\varsigma}_{32}e^{-(1-\xi)\sqrt{\lambda(\lambda+b)}}\\
\qquad\qquad\qquad\qquad\qquad\qquad=-\frac{1}{2\sqrt{\lambda(\lambda+b)}}\sinh((1-\xi)\sqrt{\lambda(\lambda+b)}),\\
\sigma_{31}e^{-\xi\sqrt{-\lambda\mathrm{i}}}+\sigma_{32}e^{\xi\sqrt{-\lambda\mathrm{i}}}+\varsigma_{31}\lambda ke^{-\xi\sqrt{\lambda(\lambda+b)}}+\varsigma_{32}\lambda ke^{\xi\sqrt{\lambda(\lambda+b)}}=0,\\
\sigma_{31}\mathrm{i}k\sqrt{-\lambda\mathrm{i}}e^{-\xi\sqrt{-\lambda\mathrm{i}}}-\sigma_{32}\mathrm{i}k\sqrt{-\lambda\mathrm{i}}e^{\xi\sqrt{-\lambda\mathrm{i}}}\\
\quad+\varsigma_{31}\sqrt{\lambda(\lambda+b)}e^{-\xi\sqrt{\lambda(\lambda+b)}}-\varsigma_{32}\sqrt{\lambda(\lambda+b)}e^{\xi\sqrt{\lambda(\lambda+b)}}=0
\end{array}\right\},\quad\forall\xi\in[0,1].\tag{2.7}$$

By Cramer's rule, we can deduce from (2.4) that

$$\breve{\sigma}_{11}=\frac{\det\begin{pmatrix}0&1\\\frac{1}{2}&-\sqrt{-\lambda\mathrm{i}}\end{pmatrix}}{\det\begin{pmatrix}1&1\\\sqrt{-\lambda\mathrm{i}}&-\sqrt{-\lambda\mathrm{i}}\end{pmatrix}}=\frac{1}{4\sqrt{-\lambda\mathrm{i}}},\tag{2.8}$$

$$\breve{\sigma}_{12}=\frac{\det\begin{pmatrix}1&0\\\sqrt{-\lambda\mathrm{i}}&\frac{1}{2}\end{pmatrix}}{\det\begin{pmatrix}1&1\\\sqrt{-\lambda\mathrm{i}}&-\sqrt{-\lambda\mathrm{i}}\end{pmatrix}}=-\frac{1}{4\sqrt{-\lambda\mathrm{i}}},\tag{2.9}$$

$$\breve{\varsigma}_{21}=\breve{\varsigma}_{31}=\frac{\det\begin{pmatrix}0&1\\\frac{1}{2}&-\sqrt{\lambda(\lambda+b)}\end{pmatrix}}{\det\begin{pmatrix}1&1\\\sqrt{\lambda(\lambda+b)}&-\sqrt{\lambda(\lambda+b)}\end{pmatrix}}=\frac{1}{4\sqrt{\lambda(\lambda+b)}},\tag{2.10}$$

$$\breve{\varsigma}_{22}=\breve{\varsigma}_{32}=\frac{\det\begin{pmatrix}1&0\\\sqrt{\lambda(\lambda+b)}&\frac{1}{2}\end{pmatrix}}{\det\begin{pmatrix}1&1\\\sqrt{\lambda(\lambda+b)}&-\sqrt{\lambda(\lambda+b)}\end{pmatrix}}=-\frac{1}{4\sqrt{\lambda(\lambda+b)}}.\tag{2.11}$$

We deduce σ_{11} from (2.5) by Cramer's rule that

$$\sigma_{11}=\frac{\det\begin{pmatrix}-\frac{1}{2\sqrt{-\lambda\mathrm{i}}}\sinh((1-\xi)\sqrt{-\lambda\mathrm{i}})&e^{-(1-\xi)\sqrt{-\lambda\mathrm{i}}}&0&0\\0&0&e^{(1-\xi)\sqrt{\lambda(\lambda+b)}}&e^{-(1-\xi)\sqrt{\lambda(\lambda+b)}}\\0&e^{\xi\sqrt{-\lambda\mathrm{i}}}&\mathrm{i}\lambda ke^{-\xi\sqrt{\lambda(\lambda+b)}}&\mathrm{i}\lambda ke^{\xi\sqrt{\lambda(\lambda+b)}}\\0&-k\sqrt{-\lambda\mathrm{i}}e^{\xi\sqrt{-\lambda\mathrm{i}}}&\sqrt{\lambda(\lambda+b)}e^{-\xi\sqrt{\lambda(\lambda+b)}}&-\sqrt{\lambda(\lambda+b)}e^{\xi\sqrt{\lambda(\lambda+b)}}\end{pmatrix}}{\det\begin{pmatrix}e^{(1-\xi)\sqrt{-\lambda\mathrm{i}}}&e^{-(1-\xi)\sqrt{-\lambda\mathrm{i}}}&0&0\\0&0&e^{(1-\xi)\sqrt{\lambda(\lambda+b)}}&e^{-(1-\xi)\sqrt{\lambda(\lambda+b)}}\\e^{-\xi\sqrt{-\lambda\mathrm{i}}}&e^{\xi\sqrt{-\lambda\mathrm{i}}}&\mathrm{i}\lambda ke^{-\xi\sqrt{\lambda(\lambda+b)}}&\mathrm{i}\lambda ke^{\xi\sqrt{\lambda(\lambda+b)}}\\k\sqrt{-\lambda\mathrm{i}}e^{-\xi\sqrt{-\lambda\mathrm{i}}}&-k\sqrt{-\lambda\mathrm{i}}e^{\xi\sqrt{-\lambda\mathrm{i}}}&\sqrt{\lambda(\lambda+b)}e^{-\xi\sqrt{\lambda(\lambda+b)}}&-\sqrt{\lambda(\lambda+b)}e^{\xi\sqrt{\lambda(\lambda+b)}}\end{pmatrix}}$$

$$
\begin{aligned}
&= \frac{-\frac{1}{2\sqrt{-\lambda \mathrm{i}}}\sinh((1-\xi)\sqrt{-\lambda \mathrm{i}})\det\begin{pmatrix} 0 & e^{(1-\xi)\sqrt{\lambda(\lambda+b)}} & e^{-(1-\xi)\sqrt{\lambda(\lambda+b)}} \\ e^{\xi\sqrt{-\lambda \mathrm{i}}} & \mathrm{i}\lambda k e^{-\xi\sqrt{\lambda(\lambda+b)}} & \mathrm{i}\lambda k e^{\xi\sqrt{\lambda(\lambda+b)}} \\ -k\sqrt{-\lambda \mathrm{i}}e^{\xi\sqrt{-\lambda \mathrm{i}}} & \sqrt{\lambda(\lambda+b)}e^{-\xi\sqrt{\lambda(\lambda+b)}} & -\sqrt{\lambda(\lambda+b)}e^{\xi\sqrt{\lambda(\lambda+b)}} \end{pmatrix}}{\varDelta} \\
&= -\frac{e^{\xi\sqrt{-\lambda \mathrm{i}}}\sinh((1-\xi)\sqrt{-\lambda \mathrm{i}})[\sqrt{\lambda(\lambda+b)}\cosh(\sqrt{\lambda(\lambda+b)}) - \mathrm{i}\lambda k^2\sqrt{-\lambda \mathrm{i}}\sinh(\sqrt{\lambda(\lambda+b)})]}{\sqrt{-\lambda \mathrm{i}}\varDelta} \\
&= -\frac{1}{4\sqrt{-\lambda \mathrm{i}}} + \frac{\sqrt{\lambda(\lambda+b)}\cosh(\sqrt{\lambda(\lambda+b)})[e^{(2\xi-1)\sqrt{-\lambda \mathrm{i}}} - e^{-\sqrt{-\lambda \mathrm{i}}}]}{2\sqrt{-\lambda \mathrm{i}}\varDelta} \\
&\quad - \frac{\mathrm{i}\lambda k^2\sqrt{-\lambda \mathrm{i}}\sinh(\sqrt{\lambda(\lambda+b)})[e^{(2\xi-1)\sqrt{-\lambda \mathrm{i}}} + e^{-\sqrt{-\lambda \mathrm{i}}}]}{2\sqrt{-\lambda \mathrm{i}}\varDelta},
\end{aligned}
\tag{2.12}
$$

where $\varDelta$ is given by

$$
\begin{aligned}
\varDelta &= \det\begin{pmatrix} e^{(1-\xi)\sqrt{-\lambda \mathrm{i}}} & e^{-(1-\xi)\sqrt{-\lambda \mathrm{i}}} & 0 & 0 \\ 0 & 0 & e^{(1-\xi)\sqrt{\lambda(\lambda+b)}} & e^{-(1-\xi)\sqrt{\lambda(\lambda+b)}} \\ e^{-\xi\sqrt{-\lambda \mathrm{i}}} & e^{\xi\sqrt{-\lambda \mathrm{i}}} & \mathrm{i}\lambda k e^{-\xi\sqrt{\lambda(\lambda+b)}} & \mathrm{i}\lambda k e^{\xi\sqrt{\lambda(\lambda+b)}} \\ k\sqrt{-\lambda \mathrm{i}}e^{-\xi\sqrt{-\lambda \mathrm{i}}} & -k\sqrt{-\lambda \mathrm{i}}e^{\xi\sqrt{-\lambda \mathrm{i}}} & \sqrt{\lambda(\lambda+b)}e^{-\xi\sqrt{\lambda(\lambda+b)}} & -\sqrt{\lambda(\lambda+b)}e^{\xi\sqrt{\lambda(\lambda+b)}} \end{pmatrix} \\
&= \det\begin{pmatrix} e^{(1-\xi)\sqrt{-\lambda \mathrm{i}}} & e^{-(1-\xi)\sqrt{-\lambda \mathrm{i}}} \\ k\sqrt{-\lambda \mathrm{i}}e^{-\xi\sqrt{-\lambda \mathrm{i}}} & -k\sqrt{-\lambda \mathrm{i}}e^{\xi\sqrt{-\lambda \mathrm{i}}} \end{pmatrix}\det\begin{pmatrix} e^{(1-\xi)\sqrt{\lambda(\lambda+b)}} & e^{-(1-\xi)\sqrt{\lambda(\lambda+b)}} \\ \mathrm{i}\lambda k e^{-\xi\sqrt{\lambda(\lambda+b)}} & \mathrm{i}\lambda k e^{\xi\sqrt{\lambda(\lambda+b)}} \end{pmatrix} \\
&\quad - \det\begin{pmatrix} e^{(1-\xi)\sqrt{-\lambda \mathrm{i}}} & e^{-(1-\xi)\sqrt{-\lambda \mathrm{i}}} \\ e^{-\xi\sqrt{-\lambda \mathrm{i}}} & e^{\xi\sqrt{-\lambda \mathrm{i}}} \end{pmatrix}\det\begin{pmatrix} e^{(1-\xi)\sqrt{\lambda(\lambda+b)}} & e^{-(1-\xi)\sqrt{\lambda(\lambda+b)}} \\ \sqrt{\lambda(\lambda+b)}e^{-\xi\sqrt{\lambda(\lambda+b)}} & -\sqrt{\lambda(\lambda+b)}e^{\xi\sqrt{\lambda(\lambda+b)}} \end{pmatrix} \\
&= \sqrt{\lambda(\lambda+b)}\bigl[e^{\sqrt{\lambda(\lambda+b)}+\sqrt{-\lambda \mathrm{i}}} - e^{\sqrt{\lambda(\lambda+b)}-\sqrt{-\lambda \mathrm{i}}} + e^{-\sqrt{\lambda(\lambda+b)}+\sqrt{-\lambda \mathrm{i}}} - e^{-\sqrt{\lambda(\lambda+b)}-\sqrt{-\lambda \mathrm{i}}}\bigr] \\
&\quad - \mathrm{i}\lambda k^2\sqrt{-\lambda \mathrm{i}}\bigl[e^{\sqrt{\lambda(\lambda+b)}+\sqrt{-\lambda \mathrm{i}}} + e^{\sqrt{\lambda(\lambda+b)}-\sqrt{-\lambda \mathrm{i}}} - e^{-\sqrt{\lambda(\lambda+b)}+\sqrt{-\lambda \mathrm{i}}} - e^{-\sqrt{\lambda(\lambda+b)}-\sqrt{-\lambda \mathrm{i}}}\bigr] \\
&= 4\sqrt{\lambda(\lambda+b)}\cosh\bigl(\sqrt{\lambda(\lambda+b)}\bigr)\sinh(\sqrt{-\lambda \mathrm{i}}) \\
&\quad - 4\mathrm{i}\lambda k^2\sqrt{-\lambda \mathrm{i}}\sinh\bigl(\sqrt{\lambda(\lambda+b)}\bigr)\cosh(\sqrt{-\lambda \mathrm{i}}).
\end{aligned}
\tag{2.13}
$$

Similarly, we can deduce from (2.5) that σ_{12}, ς_{11}, ς_{12} can be expressed as follows:

$$
\begin{aligned}
\sigma_{12} &= \frac{\det\begin{pmatrix} e^{(1-\xi)\sqrt{-\lambda \mathrm{i}}} & -\frac{1}{2\sqrt{-\lambda \mathrm{i}}}\sinh((1-\xi)\sqrt{-\lambda \mathrm{i}}) & 0 & 0 \\ 0 & 0 & e^{(1-\xi)\sqrt{\lambda(\lambda+b)}} & e^{-(1-\xi)\sqrt{\lambda(\lambda+b)}} \\ e^{-\xi\sqrt{-\lambda \mathrm{i}}} & 0 & \mathrm{i}\lambda k e^{-\xi\sqrt{\lambda(\lambda+b)}} & \mathrm{i}\lambda k e^{\xi\sqrt{\lambda(\lambda+b)}} \\ k\sqrt{-\lambda \mathrm{i}}e^{-\xi\sqrt{-\lambda \mathrm{i}}} & 0 & \sqrt{\lambda(\lambda+b)}e^{-\xi\sqrt{\lambda(\lambda+b)}} & -\sqrt{\lambda(\lambda+b)}e^{\xi\sqrt{\lambda(\lambda+b)}} \end{pmatrix}}{\varDelta} \\
&= \frac{e^{-\xi\sqrt{-\lambda \mathrm{i}}}\sinh((1-\xi)\sqrt{-\lambda \mathrm{i}})[\sqrt{\lambda(\lambda+b)}\cosh(\sqrt{\lambda(\lambda+b)}) + \mathrm{i}\lambda k^2\sqrt{-\lambda \mathrm{i}}\sinh(\sqrt{\lambda(\lambda+b)})]}{\sqrt{-\lambda \mathrm{i}}\varDelta},
\end{aligned}
\tag{2.14}
$$

$$
\begin{aligned}
\varsigma_{11} &= \frac{\det\begin{pmatrix} e^{(1-\xi)\sqrt{-\lambda \mathrm{i}}} & e^{-(1-\xi)\sqrt{-\lambda \mathrm{i}}} & -\frac{1}{2\sqrt{-\lambda \mathrm{i}}}\sinh((1-\xi)\sqrt{-\lambda \mathrm{i}}) & 0 \\ 0 & 0 & 0 & e^{-(1-\xi)\sqrt{\lambda(\lambda+b)}} \\ e^{-\xi\sqrt{-\lambda \mathrm{i}}} & e^{\xi\sqrt{-\lambda \mathrm{i}}} & 0 & \mathrm{i}\lambda k e^{\xi\sqrt{\lambda(\lambda+b)}} \\ k\sqrt{-\lambda \mathrm{i}}e^{-\xi\sqrt{-\lambda \mathrm{i}}} & -k\sqrt{-\lambda \mathrm{i}}e^{\xi\sqrt{-\lambda \mathrm{i}}} & 0 & -\sqrt{\lambda(\lambda+b)}e^{\xi\sqrt{\lambda(\lambda+b)}} \end{pmatrix}}{\varDelta} \\
&= \frac{k e^{-(1-\xi)\sqrt{\lambda(\lambda+b)}}\sinh((1-\xi)\sqrt{-\lambda \mathrm{i}})}{\varDelta},
\end{aligned}
\tag{2.15}
$$

and

$$\begin{aligned} \varsigma_{12} &= \frac{\det \begin{pmatrix} e^{(1-\xi)\sqrt{-\lambda \mathrm{i}}} & e^{-(1-\xi)\sqrt{-\lambda \mathrm{i}}} & 0 & -\frac{1}{2\sqrt{-\lambda \mathrm{i}}} \sinh((1-\xi)\sqrt{-\lambda \mathrm{i}}) \\ 0 & 0 & e^{(1-\xi)\sqrt{\lambda(\lambda+b)}} & 0 \\ e^{-\xi\sqrt{-\lambda \mathrm{i}}} & e^{\xi\sqrt{-\lambda \mathrm{i}}} & \mathrm{i}\lambda k e^{-\xi\sqrt{\lambda(\lambda+b)}} & 0 \\ k\sqrt{-\lambda \mathrm{i}} e^{-\xi\sqrt{-\lambda \mathrm{i}}} & -k\sqrt{-\lambda \mathrm{i}} e^{\xi\sqrt{-\lambda \mathrm{i}}} & \sqrt{\lambda(\lambda+b)} e^{-\xi\sqrt{\lambda(\lambda+b)}} & 0 \end{pmatrix}}{\Delta} \\ &= -\frac{k e^{(1-\xi)\sqrt{\lambda(\lambda+b)}} \sinh((1-\xi)\sqrt{-\lambda \mathrm{i}})}{\Delta}. \end{aligned} \tag{2.16}$$

We can deduce from (2.6) that σ_{21}, σ_{22}, ς_{21}, ς_{22} can be expressed as follows:

$$\sigma_{21} = \frac{k e^{-(1-\xi)\sqrt{-\lambda \mathrm{i}}} [2\delta_0(\xi)\sqrt{\lambda(\lambda+b)} \cosh(\sqrt{\lambda(\lambda+b)}) - \lambda(\lambda+b) \sinh((1-\xi)\sqrt{\lambda(\lambda+b)})]}{\Delta}, \tag{2.17}$$

$$\sigma_{22} = -\frac{k e^{(1-\xi)\sqrt{-\lambda \mathrm{i}}} [2\delta_0(\xi)\sqrt{\lambda(\lambda+b)} \cosh(\sqrt{\lambda(\lambda+b)}) - \lambda(\lambda+b) \sinh((1-\xi)\sqrt{\lambda(\lambda+b)})]}{\Delta}, \tag{2.18}$$

$$\begin{aligned} \varsigma_{21} = &-\frac{2\mathrm{i}k^2\sqrt{-\lambda \mathrm{i}}\delta_0(\xi) e^{-(1-\xi)\sqrt{\lambda(\lambda+b)}} \cosh(\sqrt{-\lambda \mathrm{i}})}{(\lambda+b)\Delta} \\ &- \frac{\sinh(\sqrt{-\lambda \mathrm{i}}) \sinh((1-\xi)\sqrt{\lambda(\lambda+b)}) e^{\xi\sqrt{\lambda(\lambda+b)}}}{\Delta} \\ &+ \frac{\mathrm{i}\lambda k^2 \sqrt{-\lambda \mathrm{i}} \cosh(\sqrt{-\lambda \mathrm{i}}) \sinh((1-\xi)\sqrt{\lambda(\lambda+b)}) e^{\xi\sqrt{\lambda(\lambda+b)}}}{\Delta\sqrt{\lambda(\lambda+b)}}, \end{aligned} \tag{2.19}$$

$$\begin{aligned} \varsigma_{22} = &\frac{2\mathrm{i}k^2\sqrt{-\lambda \mathrm{i}}\delta_0(\xi) e^{(1-\xi)\sqrt{\lambda(\lambda+b)}} \cosh(\sqrt{-\lambda \mathrm{i}})}{(\lambda+b)\Delta} \\ &- \frac{\sinh(\sqrt{-\lambda \mathrm{i}}) \sinh((1-\xi)\sqrt{\lambda(\lambda+b)}) e^{-\xi\sqrt{\lambda(\lambda+b)}}}{\Delta} \\ &- \frac{\mathrm{i}\lambda k^2 \sqrt{-\lambda \mathrm{i}} \cosh(\sqrt{-\lambda \mathrm{i}}) \sinh((1-\xi)\sqrt{\lambda(\lambda+b)}) e^{-\xi\sqrt{\lambda(\lambda+b)}}}{\Delta\sqrt{\lambda(\lambda+b)}}. \end{aligned} \tag{2.20}$$

We can deduce from (2.7) that σ_{31}, σ_{32}, ς_{31}, ς_{32} can be expressed as follows:

$$\sigma_{31} = -\frac{\lambda k e^{-(1-\xi)\sqrt{-\lambda \mathrm{i}}} \sinh((1-\xi)\sqrt{\lambda(\lambda+b)})}{\Delta}, \tag{2.21}$$

$$\sigma_{32} = \frac{\lambda k e^{(1-\xi)\sqrt{-\lambda \mathrm{i}}} \sinh((1-\xi)\sqrt{\lambda(\lambda+b)})}{\Delta}, \tag{2.22}$$

$$\begin{aligned} \varsigma_{31} = &\frac{\mathrm{i}\lambda k^2 \sqrt{-\lambda \mathrm{i}} \cosh(\sqrt{-\lambda \mathrm{i}}) \sinh((1-\xi)\sqrt{\lambda(\lambda+b)}) e^{\xi\sqrt{\lambda(\lambda+b)}}}{\sqrt{\lambda(\lambda+b)}\Delta} \\ &- \frac{\sinh(\sqrt{-\lambda \mathrm{i}}) \sinh((1-\xi)\sqrt{\lambda(\lambda+b)}) e^{\xi\sqrt{\lambda(\lambda+b)}}}{\Delta}, \end{aligned} \tag{2.23}$$

$$\begin{aligned} \varsigma_{32} = &-\frac{\mathrm{i}\lambda k^2 \sqrt{-\lambda \mathrm{i}} \cosh(\sqrt{-\lambda \mathrm{i}}) \sinh((1-\xi)\sqrt{\lambda(\lambda+b)}) e^{-\xi\sqrt{\lambda(\lambda+b)}}}{\sqrt{\lambda(\lambda+b)}\Delta} \\ &- \frac{\sinh(\sqrt{-\lambda \mathrm{i}}) \sinh((1-\xi)\sqrt{\lambda(\lambda+b)}) e^{-\xi\sqrt{\lambda(\lambda+b)}}}{\Delta}. \end{aligned} \tag{2.24}$$

Let us remind that Δ in the above formulas is given explicitly by (2.13).

3 Proof of the main results

We seek to obtain in this section the lower bound for $|\Delta|$ (see (2.13) for the definition of Δ). As mentioned in Sect. 2, we need merely consider the situation $\lambda \in \mathrm{i}\mathbb{R}$. For the sake of clarity, we distinguish λ into two cases.

Case 1 ($\lambda \in \mathbb{C} \setminus \{0\}$ and $\lambda = |\lambda|\mathrm{i}$) In this case, $\sqrt{\lambda(\lambda+b)} = \mathfrak{p}(|\lambda|) + \mathrm{i}\mathfrak{q}(|\lambda|)$, where

$$\begin{aligned} &\mathfrak{p}(\mu) = \frac{b}{\sqrt{2[1+\sqrt{1+(b/\mu)^2}]}} \quad \text{and} \\ &\mathfrak{q}(\mu) = \mu\sqrt{\frac{1+\sqrt{1+(b/\mu)^2}}{2}}, \quad \forall \mu \in (0,\infty). \end{aligned} \tag{3.1}$$

Obviously, we have

$$\begin{aligned} &\mathfrak{p}(\mu) < \frac{b}{2} \quad \text{and} \quad \mathfrak{q}(\mu) > \mu, \quad \forall \mu \in (0,\infty), \quad \text{and} \\ &\mathfrak{p}(\mu) > \frac{b}{4}, \quad \forall \mu \in \left[\frac{b}{\sqrt{48}}, \infty\right). \end{aligned} \tag{3.2}$$

Mainly using the triangle inequality, we can deduce from (2.13) that

$$\begin{aligned} |\Delta| \ge{}& |\lambda|^{\frac{3}{2}} e^{\mathfrak{p}(|\lambda|)+\sqrt{|\lambda|}} \left\{ -\left| \frac{\mathrm{i}\sqrt{\lambda(\lambda+b)}}{\lambda k^2\sqrt{-\lambda \mathrm{i}}} \right| \left[1 + e^{-2\sqrt{|\lambda|}} + e^{-2\mathfrak{p}(|\lambda|)} + e^{-2\mathfrak{p}(|\lambda|)-2\sqrt{|\lambda|}}\right] \right. \\ &\left. + \left|1 - e^{-2\sqrt{\lambda(\lambda+b)}}\right| - e^{-2\sqrt{|\lambda|}} - e^{-2\mathfrak{p}(|\lambda|)-2\sqrt{|\lambda|}} \right\}. \end{aligned} \tag{3.3}$$

But

$$\begin{aligned} \left|1 - e^{-2\sqrt{\lambda(\lambda+b)}}\right| &\ge 1 - e^{-2\mathfrak{p}(|\lambda|)} \\ &\ge 1 - \exp\left(-\frac{b}{2}\right) \\ &> \frac{b}{2+b} \quad \text{whenever } |\lambda| \ge \frac{b}{\sqrt{48}}, \end{aligned} \tag{3.4}$$

where the "$\ge$" in the second line follows if and only if $|\lambda| \ge \frac{b}{\sqrt{48}}$, and $\mathfrak{p}(\cdot)$ is given by (3.1). And similarly, we have

$$\begin{aligned} &\left| \frac{\mathrm{i}\sqrt{\lambda(\lambda+b)}}{\lambda k^2\sqrt{-\lambda \mathrm{i}}} \right| \left[1 + e^{-2\sqrt{|\lambda|}} + e^{-2\mathfrak{p}(|\lambda|)} + e^{-2\mathfrak{p}(|\lambda|)-2\sqrt{|\lambda|}}\right] \\ &\quad \le \frac{4}{k^2}\sqrt{\frac{1}{|\lambda|} + \frac{b}{|\lambda|^2}} \le \frac{b}{3(2+b)} \quad \text{whenever } |\lambda| \ge \max\left(b, \frac{288(b+2)^2}{k^4 b^2}\right), \end{aligned} \tag{3.5}$$

and

$$e^{-2\sqrt{|\lambda|}} + e^{-2\mathfrak{p}(|\lambda|)-2\sqrt{|\lambda|}} \le \frac{b}{3(2+b)} \quad \text{whenever } |\lambda| \ge \frac{6(2+b)}{b}. \tag{3.6}$$

Therefore,

$$\begin{aligned}
&|\Delta| \geq \frac{b}{3(2+b)}|\lambda|^{\frac{3}{2}}e^{\sqrt{|\lambda|}}, \\
&\quad \forall\lambda \in \mathbb{C} \text{ with } \lambda = |\lambda|\mathrm{i} \text{ and with } |\lambda| \geq \max\left(b, \frac{288(b+2)^2}{k^4b^2}, \frac{6(2+b)}{b}\right).
\end{aligned} \tag{3.7}$$

Case 2 ($\lambda \in \mathbb{C} \setminus \{0\}$ and $\lambda = -|\lambda|\mathrm{i}$) In this case, $\sqrt{\lambda(\lambda+b)} = \mathfrak{p}(|\lambda|) - \mathrm{i}\mathfrak{q}(|\lambda|)$, where $\mathfrak{p}$ and $\mathfrak{q}$ are given by (3.1). Substitute this into (2.13) to obtain

$$\begin{aligned}
\Delta &= 4\mathrm{i}\big[\mathfrak{p}(|\lambda|) - \mathrm{i}\mathfrak{q}(|\lambda|)\big]\cosh\big(\mathfrak{p}(|\lambda|) - \mathrm{i}\mathfrak{q}(|\lambda|)\big)\sin\big(\sqrt{|\lambda|}\big) \\
&\quad - 4\mathrm{i}k^2|\lambda|^{\frac{3}{2}}\sinh\big(\mathfrak{p}(|\lambda|) - \mathrm{i}\mathfrak{q}(|\lambda|)\big)\cos\big(\sqrt{|\lambda|}\big) \\
&= 4\mathrm{i}\big[\mathfrak{p}(|\lambda|) - \mathrm{i}\mathfrak{q}(|\lambda|)\big] \\
&\quad \times \big[\cosh\big(\mathfrak{p}(|\lambda|)\big)\cos\big(\mathfrak{q}(|\lambda|)\big) - \mathrm{i}\sinh\big(\mathfrak{p}(|\lambda|)\big)\sin\big(\mathfrak{q}(|\lambda|)\big)\big]\sin\big(\sqrt{|\lambda|}\big) \\
&\quad - 4\mathrm{i}k^2|\lambda|^{\frac{3}{2}}\big[\sinh\big(\mathfrak{p}(|\lambda|)\big)\cos\big(\mathfrak{q}(|\lambda|)\big) - \mathrm{i}\cosh\big(\mathfrak{p}(|\lambda|)\big)\sin\big(\mathfrak{q}(|\lambda|)\big)\big]\cos\big(\sqrt{|\lambda|}\big) \\
&= 4\alpha\big(|\lambda|\big) + 4\beta\big(|\lambda|\big)\mathrm{i}.
\end{aligned} \tag{3.8}$$

Here α and β are given explicitly as

$$\left.\begin{aligned}
\alpha(\mu) &= \sin(\sqrt{\mu})[\mathfrak{p}(\mu)\sinh(\mathfrak{p}(\mu))\sin(\mathfrak{q}(\mu)) + \mathfrak{q}(\mu)\cosh(\mathfrak{p}(\mu))\cos(\mathfrak{q}(\mu))] \\
&\quad - k^2\mu^{\frac{3}{2}}\cos(\sqrt{\mu})\cosh(\mathfrak{p}(\mu))\sin(\mathfrak{q}(\mu)), \\
\beta(\mu) &= \sin(\sqrt{\mu})[\mathfrak{p}(\mu)\cosh(\mathfrak{p}(\mu))\cos(\mathfrak{q}(\mu)) - \mathfrak{q}(\mu)\sinh(\mathfrak{p}(\mu))\sin(\mathfrak{q}(\mu))] \\
&\quad - k^2\mu^{\frac{3}{2}}\cos(\sqrt{\mu})\sinh(\mathfrak{p}(\mu))\cos(\mathfrak{q}(\mu))
\end{aligned}\right\}, \quad \forall\mu \in [0,\infty), \tag{3.9}$$

and satisfy

$$\begin{aligned}
\big|\alpha(\mu)\big|^2 + \big|\beta(\mu)\big|^2 &= \big|\mathfrak{q}(\mu)\sin(\sqrt{\mu})\cos\big(\mathfrak{q}(\mu)\big) - k^2|\lambda|^{\frac{3}{2}}\cos(\sqrt{\mu})\sin\big(\mathfrak{q}(\mu)\big)\big|^2 \\
&\quad - \frac{\mathfrak{p}(\mu)k^2\mu^{\frac{3}{2}}\sin(2\sqrt{\mu})\sinh(2\mathfrak{p}(\mu))}{2} \\
&\quad + \frac{[\cosh(2\mathfrak{p}(\mu)) - 1][|\mathfrak{q}(\mu)|^2\sin^2(\sqrt{\mu}) + k^4\mu^3\cos^2(\sqrt{\mu})]}{2} \\
&\quad + \frac{|\mathfrak{p}(\mu)|^2\sin^2(\sqrt{\mu})[\cosh(2\mathfrak{p}(\mu)) + \cos(2\mathfrak{q}(\mu))]}{2} \\
&\geq \frac{b^2[\mu^2\sin^2(\sqrt{\mu}) + \mu^2\cos^2(\sqrt{\mu})]}{16} - \frac{be^bk^2\mu^{\frac{3}{2}}}{8} \\
&= \frac{\mu^2b^2}{32} + \frac{\mu^{\frac{3}{2}}b^2}{32}\left(\sqrt{\mu} - \frac{4e^bk^2}{b}\right) \\
&\geq \frac{\mu^2b^2}{32} \quad \text{whenever } \mu \geq \max\left(\frac{16e^{2b}k^4}{b^2}, \frac{1}{k^4}, \frac{b}{\sqrt{48}}\right),
\end{aligned} \tag{3.10}$$

where the "=" in the first line follows from a series of elementary calculations and rearrangements, the "≥" in the third line follows from (3.2) and

$$\begin{aligned} &\mu^3 \geq \frac{\mu^2}{k^4} \quad \text{whenever } \mu \geq \frac{1}{k^4}, \quad \text{and} \\ &\bigl|\mathfrak{p}(\mu)k^2\mu^{\frac{3}{2}}\sin(2\sqrt{\mu})\sinh\bigl(2\mathfrak{p}(\mu)\bigr)\bigr| \leq \frac{be^bk^2\mu^{\frac{3}{2}}}{4}, \quad \forall\mu \in (0,\infty). \end{aligned}$$

By (3.10), we deduce from (3.8) that

$$\begin{aligned} |\Delta|^2 &= 16\bigl|\alpha\bigl(|\lambda|\bigr)\bigr|^2 + 16\bigl|\beta\bigl(|\lambda|\bigr)\bigr|^2 \\ &\geq \frac{b^2|\lambda|^2}{2} \quad \text{with } \lambda = -|\lambda|\mathrm{i}, |\lambda| \geq \max\biggl(\frac{16e^{2b}k^4}{b^2}, \frac{1}{k^4}, \frac{b}{\sqrt{48}}\biggr). \end{aligned} \tag{3.11}$$

Having the above analysis results at our disposal, we are now in a position to prove the main results.

Proof of Theorem 1.2 It is equivalent to proving

$$\left.\begin{aligned} \|f\|^2_{L^2(0,1)} &\leq C\langle\lambda\rangle^4(\|\phi\|^2_{L^2(0,1)} + \|\psi\|^2_{H^1(0,1)} + \|\eta\|^2_{L^2(0,1)}), \\ \|g\|^2_{H^1(0,1)} &\leq C\langle\lambda\rangle^4(\|\phi\|^2_{L^2(0,1)} + \|\psi\|^2_{H^1(0,1)} + \|\eta\|^2_{L^2(0,1)}), \\ \|h\|^2_{L^2(0,1)} &\leq C\langle\lambda\rangle^4(\|\phi\|^2_{L^2(0,1)} + \|\psi\|^2_{H^1(0,1)} + \|\eta\|^2_{L^2(0,1)}) \end{aligned}\right\}, \quad \forall\lambda \in \mathrm{i}\mathbb{R}, \tag{3.12}$$

where (f,g,h) and (ϕ,ψ,η) are related by (2.2).

Let us consider first the term $\int_0^1 G^2(x,\xi)\psi(\xi)\,d\xi$. Combine (2.2), (2.3), (2.19), and (2.20), to obtain

$$\begin{aligned} \widehat{\psi}(x) &= \int_0^1 G^2(x,\xi)\psi(\xi)\,d\xi \\ &= -\frac{4\mathrm{i}k^2\sqrt{-\lambda\mathrm{i}}\psi(0)\sinh((x-1)\sqrt{\lambda(\lambda+b)})\cosh(\sqrt{-\lambda\mathrm{i}})}{(\lambda+b)\Delta} \\ &\quad - \frac{2\sinh(\sqrt{-\lambda\mathrm{i}})\cosh(x\sqrt{\lambda(\lambda+b)})}{\Delta}\int_0^1 \psi(\xi)\sinh\bigl((1-\xi)\sqrt{\lambda(\lambda+b)}\bigr)\,d\xi \\ &\quad + \frac{2\mathrm{i}\lambda k^2\sqrt{-\lambda\mathrm{i}}\cosh(\sqrt{-\lambda\mathrm{i}})\sinh(x\sqrt{\lambda(\lambda+b)})}{\Delta\sqrt{\lambda(\lambda+b)}} \\ &\quad \times \int_0^1 \psi(\xi)\sinh\bigl((1-\xi)\sqrt{\lambda(\lambda+b)}\bigr)\,d\xi \\ &\quad + \frac{1}{2\sqrt{\lambda(\lambda+b)}}\int_0^x \psi(\xi)\sinh\bigl((x-\xi)\sqrt{\lambda(\lambda+b)}\bigr)\,d\xi. \end{aligned} \tag{3.13}$$

The derivative $\widehat{\psi}'$ of $\widehat{\psi}$ reads

$$\begin{aligned} \widehat{\psi}'(x) &= -\frac{4\mathrm{i}k^2\sqrt{-\lambda\mathrm{i}}\psi(0)\sqrt{\lambda(\lambda+b)}\cosh((x-1)\sqrt{\lambda(\lambda+b)})\cosh(\sqrt{-\lambda\mathrm{i}})}{(\lambda+b)\Delta} \\ &\quad - \frac{2\sinh(\sqrt{-\lambda\mathrm{i}})\sqrt{\lambda(\lambda+b)}\sinh(x\sqrt{\lambda(\lambda+b)})}{\Delta} \end{aligned}$$

$$\begin{aligned}
&\times \int_0^1 \psi(\xi)\sinh\bigl((1-\xi)\sqrt{\lambda(\lambda+b)}\bigr)\,d\xi \\
&+ \frac{2\mathrm{i}\lambda k^2\sqrt{-\lambda\mathrm{i}}\cosh(\sqrt{-\lambda\mathrm{i}})\cosh(x\sqrt{\lambda(\lambda+b)})}{\Delta} \\
&\times \int_0^1 \psi(\xi)\sinh\bigl((1-\xi)\sqrt{\lambda(\lambda+b)}\bigr)\,d\xi \\
&+ \frac{1}{2}\int_0^x \psi(\xi)\cosh\bigl((x-\xi)\sqrt{\lambda(\lambda+b)}\bigr)\,d\xi. \qquad (3.14)
\end{aligned}$$

Since $g \in H^1(0,1)$ satisfies $g(1) = 0$ in the trace sense, it suffices to estimate $\|g'\|_{L^2(0,1)}$ instead of $\|g\|_{H^1(0,1)}$. Therefore, we only need to analyze $\|\widehat{\psi}'\|_{L^2(0,1)}$.

By a density argument, we can prove

$$\bigl|\psi(0)\bigr| \le \|\psi\|_{H^1(0,1)}. \qquad (3.15)$$

By Young's inequality (see [14, Theorem 2.24, p. 33]), we have

$$\begin{aligned}
&\int_0^1 \left|\int_0^x \psi(\xi)\cosh\bigl((x-\xi)\sqrt{\lambda(\lambda+b)}\bigr)\,d\xi\right|^2 dx \\
&\quad\le \|\psi\|_{L^2(0,1)}^2 \left[\int_0^1 \bigl|\cosh\bigl(x\sqrt{\lambda(\lambda+b)}\bigr)\bigr|\,dx\right]^2 \le e^b\|\psi\|_{L^2(0,1)}^2, \qquad (3.16)
\end{aligned}$$

where the "$\le$" in the second line follows from

$$\begin{aligned}
\left[\int_0^1 \bigl|\cosh\bigl(x\sqrt{\lambda(\lambda+b)}\bigr)\bigr|\,dx\right]^2 &\le \max_{x\in[0,1]}\bigl|\cosh\bigl(x\sqrt{\lambda(\lambda+b)}\bigr)\bigr|^2 \\
&\le \frac{\cosh(2\mathfrak{p}(|\lambda|)) + \cos(2\mathfrak{q}(|\lambda|))}{2} < e^b,
\end{aligned}$$

in which we used (3.2) when we establish the last "$<$".

Mainly using Hölder's inequality, we have

$$\begin{cases}
\int_0^1 |\cosh((x-1)\sqrt{\lambda(\lambda+b)})|^2\,dx \\
\quad = \frac{1}{2}\int_0^1 [\cosh(2(x-1)\mathfrak{p}(|\lambda|)) + \cos(2(x-1)\mathfrak{q}(|\lambda|))]\,dx \\
\quad \le \max_{x\in[0,1]} \frac{\cosh(2(x-1)\mathfrak{p}(|\lambda|))+\cos(2(x-1)\mathfrak{q}(|\lambda|))}{2} < e^b, \\
\int_0^1 |\sinh(x\sqrt{\lambda(\lambda+b)})|^2\,dx < e^b \quad \text{and} \quad \int_0^1 |\cosh(x\sqrt{\lambda(\lambda+b)})|^2\,dx < e^b, \\
|\int_0^1 \psi(\xi)\sinh((1-\xi)\sqrt{\lambda(\lambda+b)})\,d\xi|^2 \\
\quad \le \|\psi\|_{L^2(0,1)}^2 \max_{\xi\in[0,1]} |\sinh((1-\xi)\sqrt{\lambda(\lambda+b)})|^2 \le e^b\|\psi\|_{L^2(0,1)}^2.
\end{cases} \qquad (3.17)$$

By some routine calculations, we have

$$\left.\begin{array}{l}
|\frac{\sqrt{-\lambda\mathrm{i}}\sqrt{\lambda(\lambda+b)}\cosh(\sqrt{-\lambda\mathrm{i}})}{(\lambda+b)\Delta}| \le \frac{6(2+b)}{b\sqrt{|\lambda|}}, \\
|\frac{\sinh(\sqrt{-\lambda\mathrm{i}})\sqrt{\lambda(\lambda+b)}}{\Delta}| \le \frac{6(2+b)}{b}, \\
|\frac{\lambda\sqrt{-\lambda\mathrm{i}}\cosh(\sqrt{-\lambda\mathrm{i}})}{\Delta}| \le \frac{3(2+b)\sqrt{|\lambda|}}{b}
\end{array}\right\}$$

$$\text{whenever } |\lambda| \ge \max\left(\frac{16e^{2b}k^4}{b^2}, \frac{1}{k^4}, b, \frac{288(b+2)^2}{k^4b^2}, \frac{6(2+b)}{b}\right). \qquad (3.18)$$

(3.14), together with (3.15), (3.16), (3.17), and (3.18), implies

$$\|(\lambda+b)\widehat{\psi}'\|^2_{L^2(0,1)} \le \frac{9216(1+k^4)(2+b)^2e^{3b}}{b^2}|\lambda|^3\|\psi\|^2_{H^1(0,1)}, \tag{3.19}$$

whenever $\lambda \in \mathrm{i}\mathbb{R}$ satisfies $|\lambda| \ge \max(\frac{16e^{2b}k^4}{b^2}, \frac{1}{k^4}, b, \frac{288(b+2)^2}{k^4b^2}, \frac{6(2+b)}{b})$.

Applying the approach used in deducing (3.19) from (3.14) via the "steps" (3.15), (3.16), (3.17), and (3.18), we can prove

$$\|g\|^2_{H^1(0,1)} \le C\langle\lambda\rangle^3\big(\|\phi\|^2_{L^2(0,1)} + \|\psi\|^2_{H^1(0,1)} + \|\eta\|^2_{L^2(0,1)}\big), \tag{3.20}$$

which, together with $(2.2)_3$, implies

$$\|h\|^2_{L^2(0,1)} \le C\langle\lambda\rangle^4\big(\|\phi\|^2_{L^2(0,1)} + \|\psi\|^2_{H^1(0,1)} + \|\eta\|^2_{L^2(0,1)}\big). \tag{3.21}$$

We can also prove

$$\begin{aligned}&\int_0^1\left|\int_0^1 F^2(x,\xi)\psi(\xi)\,d\xi + \int_0^1 F^3(x,\xi)\eta(\xi)\,d\xi\right|^2 dx\\ &\quad\le C\langle\lambda\rangle^2\big(\|\psi\|^2_{H^1(0,1)} + \|\eta\|^2_{L^2(0,1)}\big),\end{aligned} \tag{3.22}$$

where the constant $C > 0$ is independent of (ϕ,ψ,η) and λ.

Now it remains to analyze the term $\int_0^1 F^1(x,\xi)\phi(\xi)\,d\xi$. But

$$\begin{aligned}\int_0^1 F^1(x,\xi)\phi(\xi)\,d\xi = {}& -\frac{\sqrt{\lambda(\lambda+b)}\cosh(\sqrt{\lambda(\lambda+b)})}{\sqrt{-\lambda\mathrm{i}}}\\ &\times\int_x^1\frac{\phi(\xi)[\cosh((1+x-\xi)\sqrt{-\lambda\mathrm{i}}) - \cosh((1-x-\xi)\sqrt{-\lambda\mathrm{i}})]}{\Delta}\,d\xi\\ &+\frac{\mathrm{i}\lambda k^2\sqrt{-\lambda\mathrm{i}}\sinh(\sqrt{\lambda(\lambda+b)})}{\sqrt{-\lambda\mathrm{i}}}\\ &\times\int_x^1\frac{\phi(\xi)[\sinh((1+x-\xi)\sqrt{-\lambda\mathrm{i}}) - \sinh((1-x-\xi)\sqrt{-\lambda\mathrm{i}})]}{\Delta}\,d\xi\\ &+\frac{e^{-x\sqrt{-\lambda\mathrm{i}}}[\sqrt{\lambda(\lambda+b)}\cosh(\sqrt{\lambda(\lambda+b)}) + \mathrm{i}\lambda k^2\sqrt{-\lambda\mathrm{i}}\sinh(\sqrt{\lambda(\lambda+b)})]}{\sqrt{-\lambda\mathrm{i}}}\\ &\times\int_0^x\frac{\phi(\xi)\sinh((1-\xi)\sqrt{-\lambda\mathrm{i}})}{\Delta}\,d\xi\\ &-\frac{1}{4\sqrt{-\lambda\mathrm{i}}}\int_0^x\phi(\xi)e^{-(x-\xi)\sqrt{-\lambda\mathrm{i}}}\,d\xi\\ &+\frac{\sqrt{\lambda(\lambda+b)}\cosh(\sqrt{\lambda(\lambda+b)})e^{(x-1)\sqrt{-\lambda\mathrm{i}}}}{\sqrt{-\lambda\mathrm{i}}}\\ &\times\int_0^x\frac{\phi(\xi)\sinh(\xi\sqrt{-\lambda\mathrm{i}})}{\Delta}\,d\xi.\end{aligned} \tag{3.23}$$

To provide in detail a way to analyze $\int_0^1 F^1(x,\xi)\phi(\xi)\,d\xi$, we continue as follows:

$$\begin{aligned}
&\left|\int_x^1 \frac{\phi(\xi)[\cosh((1+x-\xi)\sqrt{-\lambda \mathrm{i}}) - \cosh((1-x-\xi)\sqrt{-\lambda \mathrm{i}})]}{\Delta}\,d\xi\right|^2 \\
&\quad\le 2\int_x^1 \left|\phi(\xi)\right|^2 d\xi \int_x^1 \left[\left|\frac{\cosh((1+x-\xi)\sqrt{-\lambda \mathrm{i}})}{\Delta}\right|^2 + \left|\frac{\cosh((1-x-\xi)\sqrt{-\lambda \mathrm{i}})}{\Delta}\right|^2\right] d\xi \\
&\quad\le \frac{C}{\langle\lambda\rangle^2}\int_x^1 \left|\phi(\xi)\right|^2 d\xi.
\end{aligned}$$

Employing the same idea, we analyze the rest of (3.23) term-by-term, and then collect all the information together to obtain

$$\int_0^1 \left|\int_0^1 F^1(x,\xi)\phi(\xi)\,d\xi\right|^2 dx \le C\|\phi\|^2_{L^2(0,1)}.$$

This, together with (3.22), implies

$$\|f\|^2_{L^2(0,1)} \le C\langle\lambda\rangle^2\bigl(\|\phi\|^2_{L^2(0,1)} + \|\psi\|^2_{H^1(0,1)} + \|\eta\|^2_{L^2(0,1)}\bigr), \tag{3.24}$$

where the constant $C > 0$ is independent of (ϕ,ψ,η) and λ.

Combining (3.20), (3.21), and (3.24), we know that (3.12) is proved, so is Theorem 1.2. □

4 Concluding comments and an open question

By analyzing carefully Green's functions for boundary value problems associated with ordinary differential equations (i.e., (2.1)), we prove that the infinitesimal generator of the semigroup associated with system (1.1) satisfies the resolvent estimate (1.9), thereby proving that the energy of system (1.1) decays polynomially.

Having a very simple underlying idea, our method is based on Green's functions and relies on heavy calculations. Our method can be modified to treat other transmission systems of 1-D partial differential equations where one of the equations is damped in the whole interval. However, according to the deductions based on our idea, it seems very hard to find the optimal decay rate of the energy of system (1.1). Therefore, one of our next concerns is to understand better the following question.

Open question Could the decay rate $(t+1)^{-1}$ given in estimate (1.8) be improved?

As indicated before, the above question seems difficult to solve with merely the method used in this paper. To close this section, we prove by a contradiction argument that the energy E (defined in (1.6)) can NOT decay exponentially. Assume to the contrary that $E(t)$ decays exponentially, or equivalently, there exists a pair $(M_0,\gamma_0) \in (0,\infty)^2$ such that, for every $w \in H$,

$$\left\|e^{tA}w\right\|_H \le M_0 e^{-\gamma_0 t}\|w\|_H, \quad \forall t \in [0,\infty), \tag{4.1}$$

where H is given by (1.2), and A by (1.3).

Write, for every $\lambda_0 \in \mathbb{C}$ with $\gamma < \Re\mathfrak{e}\,\lambda < 0$,

$$R_\lambda w = \int_0^\infty e^{-\lambda t}e^{tA}w\,dt, \quad \forall w \in H. \tag{4.2}$$

By (4.1), R_λ is well defined and belongs to $\mathcal{L}(H)$. Moreover, by (4.2), R_λ satisfies

$$(\lambda \mathrm{id}_H - A)R_\lambda w = w, \quad \forall w \in H, \quad \text{and} \quad R_\lambda(\lambda \mathrm{id}_H - A)w = w, \quad \forall w \in \mathcal{D}(A).$$

Therefore, λ belongs to $\rho(A)$, the resolvent set of A, and moreover, $R_\lambda = R(\lambda; A)$, the resolvent of A.

Thus, we proved just now that λ belongs to $\rho(A)$ whenever $\lambda \in \mathbb{C}$ satisfies $\gamma_0 < \Re\mathfrak{e}\,\lambda < 0$. This contradicts $(1.7)_2$. The proof is complete.

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Endnote
[a] Throughout this paper, $\langle\cdot\rangle = \sqrt{1 + |\cdot|^2}$.

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