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Symmetric positive solutions for fourth-order n-dimensional m-Laplace systems

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Abstract

This paper investigates the existence, multiplicity, and nonexistence of symmetric positive solutions for the fourth-order n-dimensional m-Laplace system

$$\begin{cases} \phi_m(\mathbf{x}''(t)))'' = \Psi(t)\mathbf{f}(t,\mathbf{x}(t)), \quad 0<t<1, \\ \mathbf{x}(0)=\mathbf{x}(1)=\int_0^1 \mathbf{g}(s)\mathbf{x}(s)\,ds, \\ \phi_m(\mathbf{x}''(0))=\phi_m(\mathbf{x}''(1))=\int_0^1 \mathbf{h}(s)\phi_m(\mathbf{x}''(s))\,ds. \end{cases}$$

The vector-valued function $\mathbf{x}$ is defined by $\mathbf{x}=[x_1,x_2,\ldots,x_n]^\top$, $\Psi(t)=\operatorname{diag}[\psi_1(t),\ldots,\psi_i(t),\ldots,\psi_n(t)]$, where $\psi_i\in L^p[0,1]$ for some $p\geq 1$. Our methods employ the fixed point theorem in a cone and the inequality technique. Finally, an example illustrates our main results.

Keywords: Symmetric positive solutions; n-dimensional system; m-Laplace operator; Matrix theory; Fixed point technique

1 Introduction

Consider the fourth-order n-dimensional m-Laplace system

$$\bigl(\phi_m\bigl(\mathbf{x}''(t)\bigr)\bigr)'' = \Psi(t)\mathbf{f}\bigl(t,\mathbf{x}(t)\bigr), \quad 0<t<1, \tag{1.1}$$

subject to the following boundary conditions:

$$\begin{cases} \mathbf{x}(0)=\mathbf{x}(1)=\int_0^1 \mathbf{g}(s)\mathbf{x}(s)\,ds, \\ \phi_m(\mathbf{x}''(0))=\phi_m(\mathbf{x}''(1))=\int_0^1 \mathbf{h}(s)\phi_m(\mathbf{x}''(s))\,ds, \end{cases} \tag{1.2}$$

where

$$\mathbf{x}(t)=\bigl(x_1(t),x_2(t),\ldots,x_n(t)\bigr)^T,$$
$$\Psi(t)=\operatorname{diag}\bigl[\psi_1(t),\psi_2(t),\ldots,\psi_n(t)\bigr],$$
$$\mathbf{f}(t,\mathbf{x})=\bigl(f_1(t,\mathbf{x}),\ldots,f_i(t,\mathbf{x}),\ldots,f_n(t,\mathbf{x})\bigr)^T,$$
$$\phi_m\bigl(\mathbf{x}''(t)\bigr)=\bigl(\phi_m\bigl(x_1{}''(t)\bigr),\phi_m\bigl(x_2{}''(t)\bigr),\ldots,\phi_m\bigl(x_n{}''(t)\bigr)\bigr)^T,$$

$$\mathbf{g}(s) = \operatorname{diag}\big[g_1(s), g_2(s), \ldots, g_n(s)\big],$$
$$\mathbf{h}(s) = \operatorname{diag}\big[h_1(s), h_2(s), \ldots, h_n(s)\big].$$

Here, we understand that $f_i(t, \mathbf{x})$ means that $f_i(t, x_1, x_2, \ldots, x_n)$, $i = 1, 2, \ldots, n$.

Therefore, system (1.1) means that

$$\begin{pmatrix} (\phi_m(x_1''(t)))'' \\ (\phi_m(x_2''(t)))'' \\ \vdots \\ (\phi_m(x_n''(t)))'' \end{pmatrix} = \begin{pmatrix} \psi_1(t) & 0 & \cdots & 0 \\ 0 & \psi_2(t) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \psi_n(t) \end{pmatrix} \begin{pmatrix} f_1(t, \mathbf{x}) \\ f_2(t, \mathbf{x}) \\ \vdots \\ f_n(t, \mathbf{x}) \end{pmatrix}. \tag{1.3}$$

Similarly, (1.2) means that

$$\begin{cases} \begin{pmatrix} x_1(0) \\ x_2(0) \\ \vdots \\ x_n(0) \end{pmatrix} = \begin{pmatrix} x_1(1) \\ x_2(1) \\ \vdots \\ x_n(1) \end{pmatrix} = \int_0^1 \begin{pmatrix} g_1(s) & 0 & \cdots & 0 \\ 0 & g_2(s) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & g_n(s) \end{pmatrix} \begin{pmatrix} x_1(s) \\ x_2(s) \\ \vdots \\ x_n(s) \end{pmatrix} ds, \\ \begin{pmatrix} \phi_m(x_1''(0)) \\ \phi_m(x_2''(0)) \\ \vdots \\ \phi_m(x_n''(0)) \end{pmatrix} = \begin{pmatrix} \phi_m(x_1''(1)) \\ \phi_m(x_2''(1)) \\ \vdots \\ \phi_m(x_n''(1)) \end{pmatrix} = \int_0^1 \begin{pmatrix} h_1(s) & 0 & \cdots & 0 \\ 0 & h_2(s) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & h_n(s) \end{pmatrix} \begin{pmatrix} \phi_m(x_1''(s)) \\ \phi_m(x_2''(s)) \\ \vdots \\ \phi_m(x_n''(s)) \end{pmatrix} ds. \end{cases} \tag{1.4}$$

And then it follows respectively from (1.3) and (1.4) that

$$\begin{cases} (\phi_m(x_1''(t)))'' = \psi_1(t) f_1(t, x_1(t), \ldots, x_n(t)), & 0 < t < 1, \\ (\phi_m(x_2''(t)))'' = \psi_2(t) f_2(t, x_1(t), \ldots, x_n(t)), & 0 < t < 1, \\ \vdots \\ (\phi_m(x_n''(t)))'' = \psi_n(t) f_n(t, x_1(t), \ldots, x_n(t)), & 0 < t < 1, \end{cases} \tag{1.5}$$

$$\begin{cases} x_1(0) = x_1(1) = \int_0^1 g_1(s) x_1(s)\,ds, \\ \phi_m(x_1''(0)) = \phi_m(x_1''(1)) = \int_0^1 h_1(s) \phi_m(x_1''(s))\,ds, \\ x_2(0) = x_2(1) = \int_0^1 g_2(s) x_2(s)\,ds, \\ \phi_m(x_2''(0)) = \phi_m(x_2''(1)) = \int_0^1 h_2(s) \phi_m(x_2''(s))\,ds, \\ \vdots \\ x_n(0) = x_n(1) = \int_0^1 g_n(s) x_n(s)\,ds, \\ \phi_m(x_n''(0)) = \phi_m(x_n''(1)) = \int_0^1 h_n(s) \phi_m(x_n''(s))\,ds. \end{cases} \tag{1.6}$$

From above, we know that system (1.1)–(1.2) is equivalent to system (1.3)–(1.4), and system (1.3)–(1.4) is equivalent to system (1.5)–(1.6); thus, system (1.1)–(1.2) is equivalent to (1.5)–(1.6).

A vector-valued function $\mathbf{x}$ is called a solution of (1.1)–(1.2) if $\mathbf{x} \in C^2([0,1], \mathcal{R}^n)$ with $\phi_m(\mathbf{x}'') \in C^2((0,1), \mathcal{R}^n)$, and satisfies (1.1) and (1.2). If, for each $i = 1, 2, \ldots, n$, $x_i(t) \geq 0$ for all $t \in (0,1)$ and there is at least one nontrivial component of $\mathbf{x}$, then we say that $\mathbf{x}(t) = (x_1(t), x_2(t), \ldots, x_n(t))^T$ is positive on J.

For the case of $n = 1$ and $\Psi(t) \equiv 1$ for $t \in J$, system (1.1)–(1.2) reduces to the problem studied by Zhang and Liu in [1]. By using the upper and lower solution method and fixed-point theorems, the authors obtained some sufficient conditions for the existence of positive solutions for the above problem. For the case of $n = 1$, $m = 2$, $g(t) \equiv 0$, $h(t) \equiv 0$ for $t \in J$ and $\Psi \in C[0,1]$ not $\Psi \in L^p[0,1]$, system (1.1)–(1.2) reduces to the problem studied by Graef et al. in [2]. By using Krasnosel'skii's fixed-point theorem, the authors obtained some existence and nonexistence results. For other related results on system (1.1)–(1.2), we refer the reader to the references [3–27]. Moreover, for the latest development direction of the fourth order differential equations, see the references [28–31].

At the same time, we notice that a class of boundary value problems with integral boundary conditions has attracted many authors (see [20, 32–42]). It is an important and interesting problem, which contains two-point, three-point, and multi-point boundary value problems as special cases; for instance, see [43–58] and the references cited therein.

Here we point out that our problem is new in the sense of fourth-order n-dimensional m-Laplace systems with integral boundary conditions introduced here. To the best of our knowledge, the existence of single or multiple positive solutions for fourth-order n-dimensional m-Laplace systems (1.1)–(1.2) has not yet been studied, especially for the case

$$\Psi(t) = \operatorname{diag}\big[\psi_1(t), \ldots, \psi_i(t), \ldots, \psi_n(t)\big],$$

where $\psi_i \in L^p[0,1]$ for some $p \geq 1$. In consequence, our main results of the present work will be a useful contribution to the existing literature on the topic of fourth-order n-dimensional m-Laplace systems with integral boundary conditions. The existence, multiplicity, and nonexistence of symmetric positive solutions for the given problem are new, though they are proved by applying the well-known method based on the fixed point theorem of cone expansion and compression of norm type.

Throughout this paper, we use $i = 1, 2, \ldots, n$, unless otherwise stated.

Let $J = [0,1]$, $\mathcal{R}_+ = [0, +\infty)$, $\mathcal{R}_+^n = \underbrace{\mathcal{R}_+ \times \mathcal{R}_+ \times \cdots \times \mathcal{R}_+}_{n}$, and

$$\mathbf{x} = [x_1, x_2, \ldots, x_n]^\top \in \mathcal{R}_+^n.$$

In addition, let the components of Ψ, $\mathbf{f}$, $\mathbf{h}$, and $\mathbf{g}$ satisfy the following conditions:

(H_1) $\psi_i(t) \in L^p(J)$ for some $1 \leq p \leq +\infty$, and $\psi_i(t)$ is nonnegative, symmetric on J, and there exists $N > 0$ such that $\psi_i(t) \geq N$ a.e. on J;

(H_2) $f_i : J \times \mathcal{R}_+^n \to \mathcal{R}_+$ is continuous, and for all $\mathbf{x} \in \mathcal{R}_+^n$, $f_i(t, \mathbf{x})$ is symmetric on J;

(H_3) $g_i, h_i \in L^1(J)$ are nonnegative, symmetric on J with

$$\mu_i := \int_0^1 g_i(s)\,ds \in [0,1), \qquad \nu_i := \int_0^1 h_i(s)\,ds \in [0,1). \tag{1.7}$$

The organization of this article is as follows. In Sect. 2, we present some new properties of Green's function associated with system (1.1)–(1.2), and we list some definitions and lemmas that will be used to prove our main results. Section 3 is devoted to prove the existence, multiplicity, and nonexistence of symmetric positive solutions for system (1.1)–(1.2). Finally, in Sect. 4, an example illustrating our main results is also presented.

2 Preliminaries

In this part, we give some properties of Green's function associated with system (1.1)–(1.2), and we present some definitions and lemmas which are needed throughout this paper.

Definition 2.1 (see [59]) Let E be a real Banach space over $\mathcal{R}$. A nonempty closed set $P \subset E$ is said to be a cone provided that
(i) $au + bv \in P$ for all $u, v \in P$ and all $a \geq 0$, $b \geq 0$ and
(ii) $u, -u \in P$ implies $u = 0$.
Every cone $P \subset E$ induces a semi-ordering in E given by $u \leq v$ if and only if $v - u \in P$.

Definition 2.2 If $x(t) = x(1-t)$, $t \in J$, then x is said to be symmetric in J.

In our discussion, $\mathbf{x} = (x_1, x_2, \dots, x_n)^T$ is symmetric on J if and only if x_i is symmetric on J.

Next, we reduce system (1.1)–(1.2) to an integral system. It follows from system (1.5)–(1.6) that system (1.1)–(1.2) can be written as follows:

$$\begin{cases} (\phi_m(x''_i(t)))'' = \psi_i(t) f_i(t, x_1(t), \dots, x_n(t)), \quad 0 < t < 1, \\ x_i(0) = x_i(1) = \int_0^1 g_i(s) x_i(s)\, ds, \\ \phi_m(x_i''(0)) = \phi_m(x_i''(1)) = \int_0^1 h_i(s) \phi_m(x_i''(s))\, ds, \end{cases} \tag{2.1}$$

where $\phi_m(s) = |s|^{m-2}s$, $m > 1$, $\phi_{m^*} = \phi_m^{-1}$, $\frac{1}{m} + \frac{1}{m^*} = 1$.

Firstly, by means of the transformation

$$\phi_m\bigl(x_i''(t)\bigr) = -y_i(t), \tag{2.2}$$

we can convert system (2.1) into

$$\begin{cases} y_i''(t) = -\psi_i(t) f_i(t, \mathbf{x}(t)), \quad 0 < t < 1, \\ y_i(0) = y_i(1) = \int_0^1 h_i(s) y_i(s)\, ds, \end{cases} \tag{2.3}$$

and

$$\begin{cases} x_i''(t) = -\phi_{m^*}(y_i(t)), \quad 0 < t < 1, \\ x_i(0) = x_i(1) = \int_0^1 g_i(s) x_i(s)\, ds. \end{cases} \tag{2.4}$$

Lemma 2.1 *Assume that* (H_3) *holds. Then system* (2.4) *has a unique solution* $x_i(t)$ *and* $x_i(t)$ *can be expressed in the form*

$$x_i(t) = -\int_0^1 H^i(t,s) \phi_{m^*}\bigl(y_i(s)\bigr)\, ds, \tag{2.5}$$

where

$$H^i(t,s) = G(t,s) + \frac{1}{1-\mu_i} \int_0^1 G(\tau, s) g_i(\tau)\, d\tau, \tag{2.6}$$

$$G(t,s) = \begin{cases} t(1-s), & 0 \leq t \leq s \leq 1, \\ s(1-t), & 0 \leq s \leq t \leq 1. \end{cases} \tag{2.7}$$

Proof The proof is similar to that of Lemma 2.1 in [60]. □

By (2.6) and (2.7), we can show that $H^i(t,s)$ and $G(t,s)$ have the following properties.

Proposition 2.1 *Assume that* (H_3) *holds. Then we have*

$$\begin{aligned} &H^i(t,s) > 0, \qquad G(t,s) > 0, \quad \forall t,s \in (0,1); \\ &H^i(t,s) \geq 0, \qquad G(t,s) \geq 0, \quad \forall t,s \in J. \end{aligned} \tag{2.8}$$

Proposition 2.2 *For all* $t,s \in J$, *we have*

$$e(t)e(s) \leq G(t,s) \leq G(t,t) = t(1-t) = e(t) \leq \bar{e} = \max_{t\in J} e(t) = \frac{1}{4}, \tag{2.9}$$

$$G(1-t,1-s) = G(t,s). \tag{2.10}$$

Proposition 2.3 *Assume that* (H_3) *holds. Then, for all* $t,s \in J$, *we have*

$$\rho^i e(s) \leq H^i(t,s) \leq \gamma^i s(1-s) = \gamma^i e(s) \leq \frac{1}{4}\gamma^i, \tag{2.11}$$

where

$$\gamma^i = \frac{1}{1-\mu_i}, \qquad \rho^i = \frac{\int_0^1 e(\tau)g_i(\tau)\,d\tau}{1-\mu_i}. \tag{2.12}$$

Proof By (2.6) and (2.9), we have

$$\begin{aligned} H^i(t,s) &= G(t,s) + \frac{1}{1-\mu_i}\int_0^1 G(s,\tau)g_i(\tau)\,d\tau \\ &\geq \frac{1}{1-\mu_i}\int_0^1 G(s,\tau)g_i(\tau)\,d\tau \\ &\geq \frac{\int_0^1 e(\tau)g_i(\tau)\,d\tau}{1-\mu_i} s(1-s) \\ &= \rho^i e(s), \quad t \in J. \end{aligned} \tag{2.13}$$

In addition, noticing that $G(t,s) \leq s(1-s)$, we have

$$\begin{aligned} H^i(t,s) &= G(t,s) + \frac{1}{1-\mu_i}\int_0^1 G(s,\tau)g_i(\tau)\,d\tau \\ &\leq s(1-s) + \frac{1}{1-\mu_i}\int_0^1 s(1-s)g_i(\tau)\,d\tau \\ &\leq s(1-s)\left[1 + \frac{1}{1-\mu_i}\int_0^1 g_i(\tau)\,d\tau\right] \\ &= s(1-s)\frac{1}{1-\mu_i} \\ &= \gamma^i e(s), \quad t \in J. \end{aligned} \tag{2.14}$$

□

Proposition 2.4 *Assume that* (H_3) *holds. Then, for all* $t, s \in J$, *we have*

$$H^i(1-t, 1-s) = H^i(t,s). \tag{2.15}$$

Proof The proof is similar to that of Proposition 2.1 of [60]. □

Lemma 2.2 *Assume that* (H_1)–(H_3) *hold. Then system* (2.3) *has a unique solution*

$$y_i(t) = -\int_0^1 H_1^i(t,s)\psi_i(s)f_i\bigl(s, \mathbf{x}(s)\bigr)\,ds, \tag{2.16}$$

where

$$H_1^i(t,s) = G(t,s) + \frac{1}{1-\nu_i}\int_0^1 G(v,s)h_i(v)dv. \tag{2.17}$$

Proof The proof is similar to Lemma 2.1 of [60]. □

Remark 2.1 Assume that (H_3) holds. Then, for all $t, s \in J$, it follows from (2.17) that

$$H_1^i(t,s) \geq 0, \qquad \rho_1^i e(s) \leq H_1^i(t,s) \leq \gamma_1^i s(1-s) \leq \frac{1}{4}\gamma_1^i, \qquad H_1^i(1-t, 1-s) = H_1^i(t,s),$$

where

$$\rho_1^i = \frac{\int_0^1 e(\tau)h_i(\tau)\,d\tau}{1-\nu_i}, \qquad \gamma_1^i = \frac{1}{1-\nu_i}.$$

Assume that x_i is a solution of system (2.1). Then it follows from Lemma 2.1 that

$$x_i(t) = -\int_0^1 H^i(t,s)\phi_{m^*}\bigl(y_i(s)\bigr)\,ds, \tag{2.18}$$

and then, it follows from Lemma 2.2 that

$$x_i(t) = \int_0^1 H^i(t,s)\phi_{m^*}\left(\int_0^1 H_1^i(s,\tau)\psi_i(\tau)f_i\bigl(\tau, \mathbf{x}(\tau)\bigr)\,d\tau\right)ds. \tag{2.19}$$

Let $E = C[0,1]$, $X = \underbrace{E \times E \times \cdots \times E}_{n}$, and for all $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T \in X$, the norm in X is defined as

$$\|\mathbf{x}\| = \sum_{i=1}^{n} \sup_{t\in J} |x_i|.$$

Then $(X, \|\cdot\|)$ is a real Banach space.

Define a cone K in X by

$$K = \left\{ \mathbf{x} = (x_1, x_2, \ldots, x_n)^T \in X : x_i \geq 0, x_i(t) \text{ is symmetric and concave on } J, \right.$$
$$\left. \min_{t\in J} \sum_{i=1}^{n} x_i(t) \geq \delta\|\mathbf{x}\| \right\}, \tag{2.20}$$

where

$$\delta = \min_{1\leq i\leq n} \delta_i, \qquad \delta_i = \frac{\rho^i(\rho_1^i)^{m^*-1}}{\gamma^i(\gamma_1^i)^{m^*-1}}.$$

We also define two sets K_r, $K_{r,R}$ by

$$K_r = \{x \in K : \|x\| < r\}, \qquad K_{r,R} = \{x \in K : r < \|x\| < R\},$$

where $0 < r < R$.

To make our research significant, let $g_i(t) \not\equiv 0$, $h_i(t) \not\equiv 0$ for any $t \in J$, $i = 1, 2, \dots, n$.

Remark 2.2 By the definition of ρ^i, ρ_1^i, γ^i, γ_1^i, we have $0 < \delta_i < 1$, and then $0 < \delta < 1$.

Let $\mathbf{T} : K \to X$ be a map with components $(T_1, \dots, T_i, \dots, T_n)$. Here, we understand $\mathbf{Tx} = (T_1\mathbf{x}, \dots, T_i\mathbf{x}, \dots, T_n\mathbf{x})^T$, where

$$(T_i\mathbf{x})(t) = \int_0^1 H^i(t,s)\phi_{m^*}\left(\int_0^1 H_1^i(s,\tau)\psi_i(\tau)f_i\bigl(\tau, \mathbf{x}(\tau)\bigr)\,d\tau\right)ds. \tag{2.21}$$

From the proof of Lemma 2.1 and Lemma 2.2, we have the following remark.

Remark 2.3 From (2.21), we know that $\mathbf{x} \in X$ is a solution of system (1.1)–(1.2) if and only if $\mathbf{x}$ is a fixed point of the map $\mathbf{T}$.

Lemma 2.3 *Assume that* (H_1)–(H_3) *hold. Then we have* $\mathbf{T}(K) \subset K$, *and* $\mathbf{T} : K \to K$ *is completely continuous.*

Proof For all $\mathbf{x} \in K$, from (2.21), we know that

$$(T_i\mathbf{x})'' = -\phi_{m^*}\left(\int_0^1 H_1^i(t,s)\psi_i(s)f_i\bigl(s, \mathbf{x}(s)\bigr)\,ds\right) \leq 0, \tag{2.22}$$

which implies that $T_i\mathbf{x}$ is concave on J.

In addition, it follows from (2.21) that

$$(T_i\mathbf{x})(0) = (T_i\mathbf{x})(1) \geq 0.$$

Thus, for all $t \in J$, we have $(T_i\mathbf{x})(t) \geq 0$. Noticing that $\psi_i(t)$ is symmetric on $(0,1)$, $x_i(t)$ is symmetric on J, and $f_i(\cdot, \mathbf{x})$ is symmetric on J, we have

$$\begin{aligned}
(T_i\mathbf{x})(1-t) &= \int_0^1 H^i(1-t,s)\phi_{m^*}\left(\int_0^1 H_1^i(s,\tau)\psi_i(\tau)f_i\bigl(\tau, \mathbf{x}(\tau)\bigr)\,d\tau\right)ds \\
&= \int_1^0 H^i(1-t,1-s)\phi_{m^*}\left(\int_0^1 H_1^i(1-s,\tau)\psi_i(\tau)f_i\bigl(\tau, \mathbf{x}(\tau)\bigr)\,d\tau\right)d(1-s) \\
&= \int_0^1 H^i(t,s)\phi_{m^*}\left(\int_0^1 H_1^i(1-s,\tau)\psi_i(\tau)f_i\bigl(\tau, \mathbf{x}(\tau)\bigr)\,d\tau\right)ds \\
&= \int_0^1 H^i(t,s)\phi_{m^*}
\end{aligned}$$

$$
\begin{aligned}
&\times \left(\int_1^0 H_1^i(1-s,1-\tau)\psi_i(1-\tau) f_i\big(1-\tau,\mathbf{x}(1-\tau)\big)\,d(1-\tau)\right) ds \\
&= \int_0^1 H^i(t,s)\phi_{m^*}\left(\int_0^1 H_1^i(s,\tau)\psi_i(\tau)f_i\big(\tau,\mathbf{x}(\tau)\big)\,d\tau\right) ds \\
&= (T_i\mathbf{x})(t),
\end{aligned}
$$

which shows that $(T_i\mathbf{x})(1-t) = (T_i\mathbf{x})(t)$, $t \in J$. And hence $(T_i\mathbf{x})(t)$ is symmetric on J.

In addition, according to (2.14), we know that

$$
\begin{aligned}
(T_i\mathbf{x})(t) &= \int_0^1 H^i(t,s)\phi_{m^*}\left(\int_0^1 H_1^i(s,\tau)\psi_i(\tau)f_i\big(\tau,\mathbf{x}(\tau)\big)\,d\tau\right) ds \\
&\le \gamma^i\big(\gamma_1^i\big)^{m^*-1}\int_0^1 e(s)\phi_{m^*}\left(\int_0^1 e(\tau)\psi_i(\tau)f_i\big(\tau,\mathbf{x}(\tau)\big)\,d\tau\right) ds, \quad \forall t \in J.
\end{aligned}
$$

Then

$$
\|\mathbf{Tx}\| = \sum_{i=1}^n \sup_{t\in J}(T_i\mathbf{x})(t) \le \sum_{i=1}^n \gamma^i\big(\gamma_1^i\big)^{m^*-1}\int_0^1 e(s)\phi_{m^*}\left(\int_0^1 e(\tau)\psi_i(\tau)f_i\big(\tau,\mathbf{x}(\tau)\big)\,d\tau\right) ds.
$$

Similarly, according to (2.13), we know that

$$
\begin{aligned}
\min_{t\in J}\sum_{i=1}^n (T_i\mathbf{x})(t) &= \min_{t\in J}\sum_{i=1}^n \int_0^1 H^i(t,s)\phi_{m^*}\left(\int_0^1 H_1^i(s,\tau)\psi_i(\tau)f_i\big(\tau,\mathbf{x}(\tau)\big)\,d\tau\right) ds \\
&\ge \sum_{i=1}^n \rho^i\big(\rho_1^i\big)^{m^*-1}\int_0^1 e(s)\phi_{m^*}\left(\int_0^1 e(\tau)\psi_i(\tau)f_i\big(\tau,\mathbf{x}(\tau)\big)\,d\tau\right) ds \\
&= \sum_{i=1}^n \delta_i\gamma^i\big(\gamma_1^i\big)^{m^*-1}\int_0^1 e(s)\phi_{m^*}\left(\int_0^1 e(\tau)\psi_i(\tau)f_i\big(\tau,\mathbf{x}(\tau)\big)\,d\tau\right) ds \\
&\ge \delta\|\mathbf{Tx}\|.
\end{aligned}
$$

Thus, we have $T_i\mathbf{x} \in K$, then there is $\mathbf{T}(K) \subset K$.

Next, we show $\mathbf{T}$ is completely continuous, and we need to show T_i is completely continuous.

Let $l > 0$ and define

$$
\widehat{f_l^i} = \sup_{t\in J}\big\{f_i\big(t,\mathbf{x}(t)\big) : \mathbf{x} \in \mathcal{R}_+^n, \|\mathbf{x}\| \le l\big\} > 0.
$$

We show that T_i is compact.

For each $l > 0$, let $B_l = \{\mathbf{x} \in K : \|\mathbf{x}\| \le l\}$. Then B_l is a bounded closed convex set in K. $\forall(\mathbf{x}_m)_{m\in\mathcal{N}} \in K$, it follows from (2.21) that

$$
\begin{aligned}
|T_i\mathbf{x}_m| &= \left|\int_0^1 H^i(t,s)\phi_{m^*}\left(\int_0^1 H_1^i(s,\tau)\psi_i(\tau)f_i\big(\tau,\mathbf{x}_m(\tau)\big)\,d\tau\right) ds\right| \\
&\le \frac{1}{4}\gamma^i\left(\frac{1}{4}\gamma_1^i\right)^{m^*-1}\left|\int_0^1 \phi_{m^*}\left(\int_0^1 \psi_i(\tau)f_i\big(\tau,\mathbf{x}_m(\tau)\big)\,d\tau\right) ds\right|
\end{aligned}
$$

$$
\begin{aligned}
&\le \frac{1}{4}\gamma^i\left(\frac{1}{4}\gamma_1^i\right)^{m^*-1}\left(\widehat{f_l^i}\right)^{m^*-1}\left|\int_0^1 \phi_{m^*}\left(\int_0^1 \psi_i(\tau)\,d\tau\right)ds\right| \\
&\le \frac{1}{4}\gamma^i\left(\frac{1}{4}\gamma_1^i\right)^{m^*-1}\left(\widehat{f_l^i}\right)^{m^*-1}\left|\int_0^1 \phi_{m^*}\left(\|\psi_i\|_1\right)ds\right| \\
&= \frac{1}{4}\gamma^i\left(\frac{1}{4}\gamma_1^i\right)^{m^*-1}\left(\widehat{f_l^i}\right)^{m^*-1}\left(\|\psi_i\|_1\right)^{m^*-1} \\
&= \left(\frac{1}{4}\gamma_1^i\right)^{m^*}\left(\widehat{f_l^i}\right)^{m^*-1}\left(\|\psi_i\|_1\right)^{m^*-1}.
\end{aligned}
$$

Therefore, $(T_i(B_l))$ is uniformly bounded.

Next we show the equicontinuity of $(T_i\mathbf{x}_m)_{m\in\mathcal{N}}$. Due to $H^i(t,s)$ is continuous on $J\times J$, then $H^i(t,s)$ is uniformly continuous. Thus, for any $\varepsilon>0$, there exist $l_1>0$, $t_1,t_2\in J$, if $|t_1-t_2|<l_1$, we have

$$
\begin{aligned}
\left|(T_i\mathbf{x}_m)(t_2)-(T_i\mathbf{x}_m)(t_1)\right| &= \left|\int_0^1 H^i(t_2,s)\phi_{m^*}\left(\int_0^1 H_1^i(s,\tau)\psi_i(\tau)f_i\big(\tau,\mathbf{x}_m(\tau)\big)\,d\tau\right)ds\right. \\
&\quad \left. -\int_0^1 H^i(t_1,s)\phi_{m^*}\left(\int_0^1 H_1^i(s,\tau)\psi_i(\tau)f_i\big(\tau,\mathbf{x}_m(\tau)\big)\,d\tau\right)ds\right| \\
&= \left|\int_0^1 \left[H^i(t_2,s)-H^i(t_1,s)\right]\right. \\
&\quad \left.\times\,\phi_{m^*}\left(\int_0^1 H_1^i(s,\tau)\psi_i(\tau)f_i\big(\tau,\mathbf{x}_m(\tau)\big)\,d\tau\right)ds\right| \\
&\le \left(\frac{1}{4}\gamma_1^i\right)^{m^*-1}\left(\|\psi_i\|_1\right)^{m^*-1}\left(\widehat{f_l^i}\right)^{m^*-1}\int_0^1\left|H^i(t_1,s)-H^i(t_2,s)\right|ds \\
&\le \varepsilon,
\end{aligned}
$$

which shows that $(T_i\mathbf{x}_m)_{m\in\mathcal{N}}$ is equicontinuous on J. Therefore, it follows from the *Arzelà–Ascoli* theorem that there exist a function $T_i^1\in C[0,1]$ and a subsequence of $(T_i\mathbf{x}_m)_{m\in\mathcal{N}}$ converging uniformly to T_i^1 on J.

We prove the continuity of T_i. Let $(\mathbf{x}_m)_{m\in\mathcal{N}}$ be any sequence converging on K to $\mathbf{x}\in K$, and let $L>0$ be such that $\|\mathbf{x}_m\|\le L$ for all $m\in\mathcal{N}$. Note that $f_i(t,\mathbf{x})$ is continuous on $J\times K_L$. It is not difficult to see that the dominated convergence theorem guarantees that

$$
\lim_{m\to\infty}(T_i\mathbf{x}_m)(t)=(T_i\mathbf{x})(t) \tag{2.23}
$$

for each $t\in J$. Moreover, the compactness of T_i implies that $(T_i\mathbf{x}_m)(t)$ converges uniformly to $(T_i\mathbf{x})(t)$ on J. If not, then there exist $\varepsilon_0>0$ and a subsequence $(\mathbf{x}_{m_j})_{j\in\mathcal{N}}$ of $(\mathbf{x}_m)_{m\in\mathcal{N}}$ such that

$$
\sup_{t\in J}\left|(T_i\mathbf{x}_{m_j})(t)-(T_i\mathbf{x})(t)\right|\ge\varepsilon_0,\quad j\in\mathcal{N}. \tag{2.24}
$$

Now, it follows from the compactness of T_i that there exists a subsequence of $(\mathbf{x}_{m_j})_{j\in\mathcal{N}}$ (without loss of generality, assume that the subsequence is $(\mathbf{x}_{m_j})_{j\in\mathcal{N}}$) such that $(T_i\mathbf{x}_{m_j})_{j\in\mathcal{N}}$

converges uniformly to $y_0 \in C[0,1]$. Thus, from (2.24), we easily see that

$$\sup_{t\in J}\bigl|y_0(t)-(T_i\mathbf{x})(t)\bigr| \ge \varepsilon_0, \quad j\in\mathcal{N}. \tag{2.25}$$

On the other hand, from the pointwise convergence (2.23) we obtain

$$y_0(t) = (T_i\mathbf{x})(t), \quad t\in J.$$

This is a contradiction to (2.25). Therefore T_i is continuous.

Therefore $T_i : K \to K$ is completely continuous. This completes the proof of Lemma 2.3. □

In the following lemma, we employ Hölder's inequality to obtain some of the norm inequalities in our main results.

Lemma 2.4 (Hölder) *Let $e\in L^p[a,b]$ with $p>1$, $h\in L^q[a,b]$ with $q>1$, and $\frac{1}{p}+\frac{1}{q}=1$. Then $eh\in L^1[a,b]$, and*

$$\|eh\|_1 \le \|e\|_p\|h\|_q.$$

Let $e\in L^1[a,b]$ and $h\in L^\infty[a,b]$. Then $eh\in L^1[a,b]$, and

$$\|eh\|_1 \le \|e\|_1\|h\|_\infty.$$

Finally, we state the well-known fixed point theorem of cone expansion and compression of norm type.

Lemma 2.5 (see [59]) *Let P be a cone in a real Banach space E. Assume Ω_1, Ω_2 are bounded open sets in E with $0\in\overline{\Omega}_1\subset\Omega_2$, $A : P\cap(\overline{\Omega}_2\setminus\Omega_1)\to P$ is completely continuous such that either*

(i) $\|Ax\|\le\|x\|$, $\forall x\in P\cap\partial\Omega_1$; $\|Ax\|\ge\|x\|$, $\forall x\in P\cap\partial\Omega_2$ *or*

(ii) $\|Ax\|\ge\|x\|$, $\forall x\in P\cap\partial\Omega_1$; $\|Ax\|\le\|x\|$, $\forall x\in P\cap\partial\Omega_2$.

Then A has at least one fixed point in $P\cap(\overline{\Omega}_2\setminus\Omega_1)$.

Remark 2.4 To make it clear for the reader what Ω_1, Ω_2, $\partial\Omega_1$, $\partial\Omega_2$, and $\bar{\Omega}_2\setminus\Omega_1$ mean, we give typical examples of Ω_1 and Ω_2.

$$\Omega_1 = \bigl\{x\in C[a,b] : \|x\| < r\bigr\}, \qquad \Omega_2 = \bigl\{x\in C[a,b] : \|x\| < R\bigr\},$$
$$\overline{\Omega}_2\setminus\Omega_1 = \bigl\{x\in C[a,b] : r\le\|x\|\le R\bigr\},$$

where $0<r<R$, $\|x\| = \max_{t\in[a,b]}|x(t)|$.

3 Main results

In this part, by using Lemmas 2.1–2.5, we show the existence, multiplicity, and nonexistence of symmetric positive solutions for system (1.1)–(1.2) under the following three cases for $\psi_i\in L^p[0,1] : 1<p<\infty$, $p=1$, and $p=\infty$.

For convenience's sake, we introduce the notations:

$$f_i^\beta = \limsup_{\|\mathbf{x}\|\to\beta} \max_{t\in J} \frac{f_i(t,\mathbf{x})}{\phi_m(\|\mathbf{x}\|)}, \qquad f_{i\beta} = \liminf_{\|\mathbf{x}\|\to\beta} \min_{t\in J} \frac{f_i(t,\mathbf{x})}{\phi_m(\|\mathbf{x}\|)},$$

$$D_i = \frac{1}{6} n\gamma^i \bigl(\gamma_1^i \|e\|_q \|\omega_i\|_p\bigr)^{m^*-1}, \qquad D_1^i = \left(\frac{1}{6}\right)^{m^*} n\gamma^i \bigl(\gamma_1^i N\bigr)^{m^*-1},$$

where β is 0 or ∞.

The first existence theorem deals with the case $1 < p < \infty$.

Theorem 3.1 *Assume that* (H_1)–(H_3) *hold. Furthermore, assume that one of the following conditions is satisfied.*

(C_1) *There exist two constants* r, R *with* $0 < r \le \delta R$ *such that* $f_i(t,\mathbf{x}) \le \phi_m(\frac{r}{D_i})$ *for* $t \in J$, $0 \le \|\mathbf{x}\| \le r$, *and* $f_i(t,\mathbf{x}) \ge \phi_m(\frac{R}{\delta_i D_1^i})$ *for* $t \in J$, $\delta R \le \|\mathbf{x}\| \le R$;

(C_2) $f_{i0} > \phi_m(\frac{1}{\delta_i D_1^i})$ *and* $f_i^\infty < \phi_m(\frac{1}{D_i})$ (*particularly,* $f_{i0} = \infty$ *and* $f_i^\infty = 0$).

Then, system (1.1)–(1.2) *has at least one symmetric positive solution.*

Proof Case (1). Considering the condition (C_1), for all $\mathbf{x} \in \partial K_r$, we have $\|\mathbf{x}\| = r$ and $f_i(t,\mathbf{x}(t)) \le \phi_m(\frac{r}{D_i})$, $i = 1,2,\dots,n$. Thus, for all $t \in J$, we have

$$\begin{aligned}
\|\mathbf{Tx}\| &= \sum_{i=1}^n \sup_{t\in J} (T_i\mathbf{x})(t) \\
&= \sum_{i=1}^n \sup_{t\in J} \int_0^1 H^i(t,s)\phi_{m^*}\left(\int_0^1 H_1^i(s,\tau)\psi_i(\tau) f_i\bigl(\tau,\mathbf{x}(\tau)\bigr)\,d\tau\right) ds \\
&\le \sum_{i=1}^n \gamma^i \bigl(\gamma_1^i\bigr)^{m^*-1} \int_0^1 e(s)\phi_{m^*}\left(\int_0^1 e(\tau)\psi_i(\tau)\phi_m\left(\frac{r}{D_i}\right) d\tau\right) ds \\
&\le \sum_{i=1}^n \gamma^i \bigl(\gamma_1^i\bigr)^{m^*-1} \frac{r}{D_i} \int_0^1 e(s)\phi_{m^*}\left(\int_0^1 \|e\|_q \|\psi_i\|_p\,d\tau\right) ds \\
&= \sum_{i=1}^n \frac{1}{6}\gamma^i \bigl(\gamma_1^i \|e\|_q \|\psi_i\|_p\bigr)^{m^*-1} \frac{r}{D_i} \\
&= \sum_{i=1}^n \frac{r}{n} = \|\mathbf{x}\|.
\end{aligned} \tag{3.1}$$

In addition, for all $\mathbf{x} \in \partial K_R$, we have $\|\mathbf{x}\| = R$, and then it follows from (2.20) and (C_1) that

$$\begin{aligned}
\|\mathbf{Tx}\| &= \sum_{i=1}^n \sup_{t\in J} (T_i\mathbf{x})(t) \\
&= \sum_{i=1}^n \sup_{t\in J} \int_0^1 H^i(t,s)\phi_{m^*}\left(\int_0^1 H_1^i(s,\tau)\psi_i(\tau) f_i\bigl(\tau,\mathbf{x}(\tau)\bigr)\,d\tau\right) ds \\
&\ge \sum_{i=1}^n \rho^i \bigl(\rho_1^i N\bigr)^{m^*-1} \int_0^1 e(s)\phi_{m^*}\left(\int_0^1 e(\tau)\phi_m\left(\frac{R}{\delta_i D_1^i}\right) d\tau\right) ds
\end{aligned}$$

$$= \sum_{i=1}^{n} \left(\frac{1}{6}\right)^{m^*} \delta_i \gamma^i \left(\gamma_1^i N\right)^{m^*-1} \frac{R}{\delta_i D_1^i}$$
$$= \sum_{i=1}^{n} \frac{R}{n} = \|\mathbf{x}\|. \tag{3.2}$$

Case (2). Considering condition (C_2), it follows from the definition of f_{i0} and $f_{i0} > \phi_m(\frac{1}{\delta_i D_1^i})$ that there exists $r_1 > 0$ such that

$$f_i(t,\mathbf{x}) \geq \left(f_{i0} - \varepsilon_1^i\right)\phi_m\left(\|\mathbf{x}\|\right), \quad \forall t \in J, 0 \leq \|\mathbf{x}\| \leq r_1,$$

where $\varepsilon_1^i > 0$ satisfies $D_1^i \delta_i (f_{i0} - \varepsilon_1^i)^{m^*-1} \geq 1$, $i = 1,2,\ldots,n$. Then, for all $t \in J$, $\mathbf{x} \in \partial K_{r_1}$, we have

$$\|\mathbf{T}\mathbf{x}\| = \sum_{i=1}^{n} \sup_{t\in J} (T_i\mathbf{x})(t)$$
$$= \sum_{i=1}^{n} \sup_{t\in J} \int_0^1 H^i(t,s)\phi_{m^*}\left(\int_0^1 H_1^i(s,\tau)\psi_i(\tau)f_i\left(\tau,\mathbf{x}(\tau)\right)d\tau\right)ds$$
$$\geq \sum_{i=1}^{n} \rho^i \left(\rho_1^i N\right)^{m^*-1} \int_0^1 e(s)\phi_{m^*}\left(\int_0^1 e(\tau)\left(f_{i0} - \varepsilon_1^i\right)\phi_m\left(\|\mathbf{x}\|\right)d\tau\right)ds$$
$$= \sum_{i=1}^{n} \left(\frac{1}{6}\right)^{m^*} \delta_i \gamma^i \left(\gamma_1^i N\right)^{m^*-1} \left(f_{i0} - \varepsilon_1^i\right)^{m^*-1} \|\mathbf{x}\|$$
$$\geq \sum_{i=1}^{n} \frac{\|\mathbf{x}\|}{n} = \|\mathbf{x}\|. \tag{3.3}$$

Next, turning to $f_i^{\infty} < \phi_m(\frac{1}{D_i})$, $i = 1,2,\ldots,n$, and we know that there exists $\overline{R}_1 > 0$ such that

$$f_i(t,\mathbf{x}) \leq \left(f_i^{\infty} + \varepsilon_2^i\right)\phi_m\left(\|\mathbf{x}\|\right), \quad \forall t \in J, \|\mathbf{x}\| \geq \overline{R}_1,$$

where $\varepsilon_2^i > 0$ satisfies $D_i (f_i^{\infty} + \varepsilon_2^i)^{m^*-1} \leq 1$, $i = 1,2,\ldots,n$.

Let

$$M^i = \max_{0 \leq \|\mathbf{x}\| \leq \overline{R}_1, t\in J} f_i(t,\mathbf{x}), \quad i = 1,2,\ldots,n.$$

Thus, for all $t \in J$, $\mathbf{x} \in K$, we have $f_i(t,\mathbf{x}) \leq M^i + (f_i^{\infty} + \varepsilon_2^i)\phi_m(\|\mathbf{x}\|)$, $i = 1,2,\ldots,n$.

Letting

$$\frac{R_1}{n} > \max\left\{r_1, \overline{R}_1, \left(M^i\right)^{m^*-1} D_i \left(1 - D_i \left(f_i^{\infty} + \varepsilon_2^i\right)^{m^*-1}\right)^{-1}\right\}$$

then, for all $t \in J$, $\mathbf{x} \in \partial K_{R_1}$, we have

$$\begin{aligned}
\|\mathbf{T}\mathbf{x}\| &= \sum_{i=1}^{n} \sup_{t\in J}(T_i\mathbf{x})(t) \\
&= \sum_{i=1}^{n} \sup_{t\in J} \int_0^1 H^i(t,s)\phi_{m^*}\left(\int_0^1 H_1^i(s,\tau)\psi_i(\tau)f_i\big(\tau,\mathbf{x}(\tau)\big)\,d\tau\right)ds \\
&\le \sum_{i=1}^{n} \gamma^i\big(\gamma_1^i\|e\|_q\|\psi_i\|_p\big)^{m^*-1}\int_0^1 e(s)\phi_{m^*}\left(\int_0^1 \big(M^i + \big(f_i^\infty + \varepsilon_2^i\big)\phi_m\big(\|\mathbf{x}\|\big)\big)\,d\tau\right)ds \\
&\le \sum_{i=1}^{n} \big(M^i\big)^{m^*-1}\frac{D_i}{n} + \big(f_i^\infty + \varepsilon_2^i\big)^{m^*-1}\|\mathbf{x}\|\frac{D_i}{n} \\
&< \sum_{i=1}^{n} \frac{R_1}{n} - \frac{D_i}{n}R_1\big(f_i^\infty + \varepsilon_2^i\big)^{m^*-1} + \big(f_i^\infty + \varepsilon_2^i\big)^{m^*-1}\|\mathbf{x}\|\frac{D_i}{n} \\
&= \sum_{i=1}^{n} \frac{R_1}{n} = \|\mathbf{x}\|.
\end{aligned} \tag{3.4}$$

Applying Lemma 2.5 to (3.1) and (3.2), or (3.3) and (3.4) yields that $\mathbf{T}$ has at least one fixed point $\mathbf{x}^* \in \overline{K}_{r,R}$, or $\mathbf{x}^* \in \overline{K}_{r_1,R_1}$. Thus it follows from Remark 2.3 that system (1.1)–(1.2) has at least one symmetric positive solution. This finishes the proof of Theorem 3.1. □

The following theorem deals with the case $p = \infty$.

Theorem 3.2 *Assume that* (H_1)–(H_3), (C_1) *or* (H_1)–(H_3), (C_2) *hold. Then system* (1.1)–(1.2) *has at least one symmetric positive solution.*

Proof Let $\|e\|_1\|\psi_i\|_\infty$ replace $\|e\|_q\|\psi_i\|_p$ and repeat the argument above. □

Finally, we consider the case of $p = 1$.
Let

$$D_i = n\gamma^i\big(\gamma_1^i\big)^{m^*-1}\big(\|\psi_i\|_1\big)^{m^*-1}\left(\frac{1}{4}\right)^{m^*-1}.$$

Theorem 3.3 *Assume that* (H_1)–(H_3), (C_1) *or* (H_1)–(H_3), (C_2) *hold. Then system* (1.1)–(1.2) *has at least one symmetric positive solution.*

Proof Similar to the proof of Theorem 3.1. For all $t \in J$, $\mathbf{x} \in \partial K_r$, we have

$$\begin{aligned}
\|\mathbf{T}\mathbf{x}\| &= \sum_{i=1}^{n} \sup_{t\in J}(T_i\mathbf{x})(t) \\
&= \sum_{i=1}^{n} \sup_{t\in J} \int_0^1 H^i(t,s)\phi_{m^*}\left(\int_0^1 H_1^i(s,\tau)\psi_i(\tau)f_i\big(\tau,\mathbf{x}(\tau)\big)\,d\tau\right)ds
\end{aligned}$$

$$\begin{aligned}
&\leq \sum_{i=1}^{n} \gamma^i (\gamma_1^i)^{m^*-1} \int_0^1 e(s)\phi_{m^*}\left(\int_0^1 e(\tau)\psi_i(\tau)\phi_m\left(\frac{r}{D_i}\right) d\tau\right) ds \\
&= \sum_{i=1}^{n} \gamma^i (\gamma_1^i)^{m^*-1} \frac{r}{D_i} \left(\|\psi_i\|_1\right)^{m^*-1} \left(\frac{1}{4}\right)^{m^*-1} \\
&= \sum_{i=1}^{n} \frac{r}{n} = \|\mathbf{x}\|.
\end{aligned} \tag{3.5}$$

Next, similar to the proof of Theorem 3.1, we can finish the proof. □

In the following theorems we only consider the case of $1 < p < \infty$.

Theorem 3.4 *Assume that* (H_1)–(H_3) *hold. Furthermore, assume that one of the following conditions is satisfied.*

(C_3) *There exists two constants* r, R, *which satisfy* $0 < r \leq \delta R$, *such that* $f_i(t,\mathbf{x}) \geq \phi_m(\frac{r}{\delta_i D_1^i})$ *for* $t \in J$, $0 \leq \|\mathbf{x}\| \leq r$, *and* $f_i(t,\mathbf{x}) \leq \phi_m(\frac{R}{D_i})$ *for* $t \in J$, $\delta R \leq \|\mathbf{x}\| \leq R$;

(C_4) $f_i^0 < \phi_m(\frac{1}{D_i})$ *and* $f_{i\infty} > \phi_m(\frac{1}{\delta_i D_1^i})$ *(particularly,* $f_i^0 = 0$ *and* $f_{i\infty} = \infty$*),*

where $i = 1, 2, \ldots, n$. *Thus, system* (1.1)–(1.2) *has at least one symmetric positive solution* $\mathbf{x}^*$.

Proof The proof is similar to that of Theorem 3.1, so we omit it here. □

Next, we discuss the multiplicity of system (1.1)–(1.2).

Theorem 3.5 *Assume that* (H_1)–(H_3) *and the following conditions hold.*

(C_5) $f_{i0} > \phi_m(\frac{1}{\delta_i D_1^i})$ *and* $f_{i\infty} > \phi_m(\frac{1}{\delta_i D_1^i})$ *(particularly,* $f_{i0} = f_{i\infty} = \infty$*);*

(C_6) *there exists a constant* $b > 0$ *such that* $\max_{t\in J,\|\mathbf{x}\|=b} f_i(t,\mathbf{x}) < \phi_m(\frac{b}{D_i})$.

Then system (1.1)–(1.2) *has at least two symmetric positive solutions* $\mathbf{x}^*$, $\mathbf{x}^{**}$ *with*

$$0 < \left\|\mathbf{x}^{**}\right\| < b < \left\|\mathbf{x}^*\right\|. \tag{3.6}$$

Proof Choose two constants r, R with $0 < r < b < R$. It follows from (C_5) that if $f_{i0} > \phi_m(\frac{1}{\delta_i D_1^i})$, then by means of the proof of (3.3), we obtain that

$$\|\mathbf{T}\mathbf{x}\| > \|\mathbf{x}\|, \quad \forall \mathbf{x} \in \partial K_r; \tag{3.7}$$

if $f_{i\infty} > \phi_m(\frac{1}{\delta_i D_1^i})$, then by means of the proof of (3.3), we obtain that

$$\|\mathbf{T}\mathbf{x}\| > \|\mathbf{x}\|, \quad \forall \mathbf{x} \in \partial K_R. \tag{3.8}$$

On the other hand, it follows from (C_6) that

$$\begin{aligned}
\|\mathbf{T}\mathbf{x}\| &= \sum_{i=1}^{n} \sup_{t\in J} \int_0^1 H^i(t,s)\phi_{m^*}\left(\int_0^1 H_1^i(s,\tau)\psi_i(\tau) f_i\big(\tau,\mathbf{x}(\tau)\big)\, d\tau\right) ds \\
&\leq \sum_{i=1}^{n} \gamma^i (\gamma_1^i)^{m^*-1} \int_0^1 e(s)\phi_{m^*}\left(\int_0^1 \|e\|_q \|\psi_i\|_p f_i\big(\tau,\mathbf{x}(\tau)\big)\, d\tau\right) ds
\end{aligned}$$

$$
\begin{aligned}
&< \sum_{i=1}^{n} \gamma^i \bigl(\gamma_1^i\bigr)^{m^*-1} \int_0^1 e(s)\phi_{m^*}\left(\int_0^1 \|e\|_q \|\psi_i\|_p \phi_m\left(\frac{b}{D_i}\right) d\tau\right) ds \\
&= \sum_{i=1}^{n} \frac{1}{6} \gamma^i \bigl(\gamma_1^i \|e\|_q \|\psi_i\|_p\bigr)^{m^*-1} \frac{b}{D_i} \\
&= b = \|\mathbf{x}\|, \quad \forall \mathbf{x} \in \partial K_b. \qquad (3.9)
\end{aligned}
$$

Applying Lemma 2.5 to (3.7), (3.8), and (3.9) yields that $\mathbf{T}$ has a fixed point $\mathbf{x}^{**} \in \overline{K}_{r,b}$ and a fixed point $\mathbf{x}^* \in \overline{K}_{b,R}$. Then it follows from Remark 2.3 that system (1.1)–(1.2) has at least two symmetric positive solutions $\mathbf{x}^*$ and $\mathbf{x}^{**}$. Noticing (3.9), we obtain $\|\mathbf{x}^*\| \neq b$ and $\|\mathbf{x}^{**}\| \neq b$. Therefore (3.6) holds, and the proof of Theorem 3.5 is complete. □

Similarly, the following Theorem 3.6 can be obtained.

Theorem 3.6 *Assume that* (H_1)–(H_3) *and the following conditions hold.*

(C_7) $f_i^0 < \phi_m(\frac{1}{D_i})$ *and* $f_i^\infty < \phi_m(\frac{1}{D_i})$;

(C_8) *There exists a constant* $B > 0$ *such that* $\min_{t\in J,\|\mathbf{x}\|=B} f_i(t,\mathbf{x}) > \phi_m(\frac{B}{\delta_i D_1^i})$.

Then system (1.1)–(1.2) *has at least two symmetric positive solutions* $\mathbf{x}^*$ *and* $\mathbf{x}^{**}$ *with*

$$0 < \|\mathbf{x}^{**}\| < B < \|\mathbf{x}^*\|.$$

Theorem 3.7 *Assume that* (H_1)–(H_3) *hold. If there exist* $2n$ *positive numbers* b_k, d_k, $k = 1,2,\ldots,n$, *with* $b_1 < \delta d_1 < d_1 < b_2 < \delta d_2 < d_2 < \cdots < b_n < \delta d_n < d_n$, *such that*

(C_9) $f_i(t,\mathbf{x}) \le \phi_m(\frac{b_k}{D_i})$ *for* $t \in J$, $\delta b_k \le \|\mathbf{x}\| \le b_k$ *and* $f_i(t,\mathbf{x}) \ge \phi_m(\frac{d_k}{\delta_i D_1^i})$ *for* $t \in J$, $\delta d_k \le \|\mathbf{x}\| \le d_k$, $k = 1,2,\ldots,n$; *or*

(C_{10}) $f_i(t,\mathbf{x}) \ge \phi_m(\frac{b_k}{\delta_i D_1^i})$ *for* $t \in J$, $\delta b_k \le \|\mathbf{x}\| \le b_k$ *and* $f_i(t,\mathbf{x}) \le \phi_m(\frac{d_k}{D_i})$ *for* $t \in J$, $\delta d_k \le \|\mathbf{x}\| \le d_k$, $k = 1,2,\ldots,n$.

Then system (1.1)–(1.2) *has at least n symmetric positive solutions* $\mathbf{x}_k$, *and* $\mathbf{x}_k$ *satisfy*

$$b_k \le \|\mathbf{x}_k\| \le d_k, \quad k = 1,2,\ldots,n.$$

Finally, we discuss the existence result corresponding to the case when system (1.1)–(1.2) has no symmetric positive solutions.

Theorem 3.8 *Assume that conditions* (H_1)–(H_3) *hold and* $f_i(t,\mathbf{x}) < \phi_m(\frac{\|\mathbf{x}\|}{D_i})$, $\forall t \in J$, $\|\mathbf{x}\| > 0$. *Then system* (1.1)–(1.2) *has no positive solution.*

Proof Assume to the contrary that $\mathbf{x}$ is a positive solution of system (1.1)–(1.2), then for any $0 < t < 1$, we have $\mathbf{x} \in K$, $x_i(t) > 0$, and

$$
\begin{aligned}
\|\mathbf{x}\| &= \sum_{i=1}^{n} \sup_{t\in J} x_i(t) \\
&= \sum_{i=1}^{n} \sup_{t\in J} \int_0^1 H^i(t,s)\phi_{m^*}\left(\int_0^1 H_1^i(s,\tau)\psi_i(\tau) f_i\bigl(\tau,\mathbf{x}(\tau)\bigr)\, d\tau\right) ds \\
&\le \sum_{i=1}^{n} \gamma^i \bigl(\gamma_1^i\bigr)^{m^*-1} \int_0^1 e(s)\phi_{m^*}\left(\int_0^1 e(\tau)\psi_i(\tau) f_i\bigl(\tau,\mathbf{x}(\tau)\bigr)\, d\tau\right) ds
\end{aligned}
$$

$$
\begin{aligned}
&< \sum_{i=1}^{n} \gamma^i \big(\gamma_1^i\big)^{m^*-1} \int_0^1 e(s)\phi_{m^*}\left(\int_0^1 e(\tau)\psi_i(\tau)\phi_m\left(\frac{\|\mathbf{x}\|}{D_i}\right) d\tau\right) ds \\
&\leq \sum_{i=1}^{n} \gamma^i \big(\gamma_1^i\big)^{m^*-1} \frac{\|\mathbf{x}\|}{D_i} \int_0^1 e(s)\phi_{m^*}\left(\int_0^1 \|e\|_q \|\psi_i\|_p \, d\tau\right) ds \\
&= \sum_{i=1}^{n} \frac{1}{6}\gamma^i \big(\gamma_1^i \|e\|_q \|\psi_i\|_p\big)^{m^*-1} \frac{\|\mathbf{x}\|}{D_i} \\
&= \|\mathbf{x}\|.
\end{aligned}
$$

This is a contradiction, and this completes the proof. □

Similarly, we have the following results.

Theorem 3.9 *Assume that* (H_1)–(H_3) *hold and* $f_i(t,\mathbf{x}) > \phi_m(\frac{\|\mathbf{x}\|}{\delta_i D_1^i})$, $\forall t \in J$, $\|\mathbf{x}\| > 0$, $i = 1, 2, \ldots, n$. *Then system* (1.1)–(1.2) *has no positive solution.*

Proof Assume that $\mathbf{x}$ is a positive solution of system (1.1)–(1.2). Then, for any $0 < t < 1$, we have $\mathbf{x} \in K$, $x_i(t) > 0$, and

$$
\begin{aligned}
\|\mathbf{x}\| &= \sum_{i=1}^{n} \sup_{t\in J} x_i(t) \\
&= \sum_{i=1}^{n} \sup_{t\in J} \int_0^1 H^i(t,s)\phi_{m^*}\left(\int_0^1 H_1^i(s,\tau)\psi_i(\tau) f_i\big(\tau, \mathbf{x}(\tau)\big)\, d\tau\right) ds \\
&\geq \sum_{i=1}^{n} \rho^i \big(\rho_1^i N\big)^{m^*-1} \int_0^1 e(s)\phi_{m^*}\left(\int_0^1 e(\tau) f_i\big(\tau, \mathbf{x}(\tau)\big)\, d\tau\right) ds \\
&= \sum_{i=1}^{n} \delta_i \gamma^i \big(\gamma_1^i N\big)^{m^*-1} \int_0^1 e(s)\phi_{m^*}\left(\int_0^1 e(\tau) f_i\big(\tau, \mathbf{x}(\tau)\big)\, d\tau\right) ds \\
&> \sum_{i=1}^{n} \delta_i \gamma^i \big(\gamma_1^i N\big)^{m^*-1} \int_0^1 e(s)\phi_{m^*}\left(\int_0^1 e(\tau)\phi_m\left(\frac{\|\mathbf{x}\|}{\delta_i D_1^i}\right) d\tau\right) ds \\
&\geq \sum_{i=1}^{n} \left(\frac{1}{6}\right)^{m^*} \delta_i \gamma^i \big(\gamma_1^i N\big)^{m^*-1} \frac{\|\mathbf{x}\|}{\delta_i D_1^i} \\
&= \|\mathbf{x}\|.
\end{aligned}
$$

This leads to a contradiction, and this finishes the proof. □

4 An example

In the following example, we select $n = 2$, $m = 2$, $p = 2$, and $N = 1$.

Example 4.1 Consider the following system:

$$
\begin{cases}
(\phi_2(\mathbf{x}''(t)))'' = \Psi(t)\mathbf{f}(t,\mathbf{x}(t)), \quad 0 < t < 1, \\
\mathbf{x}(0) = \mathbf{x}(1) = \int_0^1 \mathbf{g}(s)\mathbf{x}(s)\, ds, \\
\phi_2(\mathbf{x}''(0)) = \phi_2(\mathbf{x}''(1)) = \int_0^1 \mathbf{h}(s)\phi_2(\mathbf{x}''(s))\, ds,
\end{cases}
\tag{4.1}
$$

where

$$\mathbf{g}(t) = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}, \qquad \mathbf{h}(t) = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{3} \end{pmatrix},$$
$$\Psi(t) = \begin{pmatrix} 6 & 0 \\ 0 & 3 \end{pmatrix}, \qquad \mathbf{f}(t,\mathbf{x}) = \begin{pmatrix} (1+\sin\pi t)x_1^6 \\ x_2^4 \end{pmatrix}.$$

Then, by calculations, we obtain that $m^* = 2$, $\mu_1 = \mu_2 = \nu_1 = \frac{1}{2}$, $\nu_2 = \frac{1}{3}$, $\gamma^1 = \gamma^2 = \gamma_1^1 = 2$, $\gamma_1^2 = \frac{3}{2}$, $\rho^1 = \rho^2 = \rho_1^1 = \frac{1}{6}$, $\rho_1^2 = \frac{1}{12}$, $\delta_1 = \frac{1}{144}$, $\delta_2 = \frac{1}{216}$, $\delta = \frac{1}{216}$, $D_1 = \frac{8}{\sqrt{30}}$, $D_2 = \frac{3}{\sqrt{30}}$, $D_1^1 = \frac{2}{9}$, $D_1^2 = \frac{1}{6}$, and

$$H^1(t,s) = H_1^1(t,s) = H^2(t,s) = G(t,s) + \int_0^1 G(s,\tau)\,d\tau,$$
$$H_1^2(t,s) = G(t,s) + \frac{1}{2}\int_0^1 G(s,\tau)\,d\tau,$$

where

$$G(t,s) = \begin{cases} t(1-s), & 0 \le t \le s \le 1, \\ s(1-t), & 0 \le s \le t \le 1. \end{cases}$$

Clearly, conditions (H_1)–(H_3) hold. Next, we show that the condition (C_1) of Theorem 3.1 holds. Choosing $r = \frac{1}{2}$, $R = 6^5 \times \sqrt[6]{3}$, we obtain that

$$\phi_2\left(\frac{r}{D_1}\right) = \frac{1}{2} \times \frac{\sqrt{30}}{8} = \frac{\sqrt{30}}{16}, \qquad \phi_2\left(\frac{R}{\delta_1 D_1^1}\right) = 144 \times \frac{9}{2} \times 6^5 \times \sqrt[6]{3} = 3 \times 6^8 \times \sqrt[6]{3},$$
$$\phi_2\left(\frac{r}{D_2}\right) = \frac{1}{2} \times \frac{\sqrt{30}}{3} = \frac{\sqrt{30}}{6}, \qquad \phi_2\left(\frac{R}{\delta_2 D_1^2}\right) = 216 \times 6 \times 6^5 \times \sqrt[6]{3} = \times 6^9 \times \sqrt[6]{3},$$
$$f_1(t,\mathbf{x}) = (1+\sin\pi t)x_1^6 \le 2 \times r^6 = 2 \times \left(\frac{1}{2}\right)^6 = \frac{1}{32}, \quad \text{for } t \in J, 0 \le \|\mathbf{x}\| \le r,$$
$$f_1(t,\mathbf{x}) = (1+\sin\pi t)x_1^6 \ge (\delta_1 R)^6 = \left(\frac{1}{144} \times 6^5 \times \sqrt[6]{3}\right)^6 = 3 \times (54)^6,$$
$$\text{for } t \in J, \delta R \le \|\mathbf{x}\| \le R,$$
$$f_2(t,\mathbf{x}) = x_2^4 \le r^4 = \left(\frac{1}{2}\right)^4 = \frac{1}{16}, \quad \text{for } t \in J, 0 \le \|\mathbf{x}\| \le r,$$
$$f_1(t,\mathbf{x}) = x_2^4 \ge (\delta_2 R)^4 = \left(\frac{1}{216} \times 6^5 \times \sqrt[6]{3}\right)^4 = 3^{\frac{2}{3}} \times 6^8, \quad \text{for } t \in J, \delta R \le \|\mathbf{x}\| \le R.$$

Therefore, $f_i(t,\mathbf{x}) \le \phi_2(\frac{r}{D_i})$, for all $t \in J$, $0 \le \|\mathbf{x}\| \le r$, and $f_i(t,\mathbf{x}) \ge \phi_2(\frac{R}{\delta_i D_1^i})$, for all $t \in J$, $\delta R \le \|\mathbf{x}\| \le R$, $i = 1, 2$.

Therefore, it follows from Theorem 3.1 that system (4.1) has at least one symmetric positive solution.

5 Conclusion

In this paper, we obtained several sufficient conditions for the existence, multiplicity, and nonexistence of symmetric positive solutions for the fourth-order n-dimensional m-

Laplace system with integral boundary conditions. Our results will be a useful contribution to the existing literature on the topic of fourth-order n-dimensional m-Laplace systems.

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