Liu and Wang *Boundary Value Problems* (2018) 2018:62
https://doi.org/10.1186/s13661-018-0982-2

Boundary Value Problems
a SpringerOpen Journal

RESEARCH

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Some properties of solutions for an isothermal viscous Cahn–Hilliard equation with inertial term

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Abstract

In this paper, we study the global existence and blow-up of solutions for an isothermal viscous Cahn–Hilliard equation with inertial term, which arises in isothermal fast phase separation processes. Based on the Galerkin method and the compactness theorem, we establish the existence of the global generalized solution. Using a lemma on the ordinary differential inequality of second order, we prove the blow-up of the solution for the initial-boundary problem.

MSC: Primary 35L25; secondary 35L35; 35B44

Keywords: Cahn–Hilliard equation with inertial term; Blow-up; Global existence

1 Introduction

In this paper, we are concerned with the following initial-boundary problem:

$$\delta u_{tt} + u_t - k\Delta u_t + \Delta^2 u = \Delta f(u), \quad x \in \Omega, t > 0, \tag{1.1}$$

$$u|_{\partial\Omega} = 0, \qquad \Delta u|_{\partial\Omega} = 0, \quad t \geq 0, \tag{1.2}$$

$$u(x,0) = \varphi(x), \qquad u_t(x,0) = \psi(x), \quad x \in \Omega, \tag{1.3}$$

where $\Omega \subset \mathbb{R}^n$ ($n \leq 3$) is a bounded domain with smooth boundary, $\delta > 0$ is an inertial parameter, $k \geq 0$ is a viscosity coefficient, and $f(s)$ is a given nonlinear function.

Equation (1.1) was proposed in [1] to model rapid spinodal decompositions in a binary alloy. Zheng and Milani [2] proved that the dynamical systems generated by problem (1.1)–(1.3) admit exponential attractors and inertial manifolds. Zheng and Milani [3] show that the dynamical systems admit global attractors and that these global attractors are at least upper-semicontinuous with respect to the vanishing of the perturbation parameter. Gatti et al. [4] considered problem (1.1)–(1.3). Their result is the construction of a robust family of exponential attractors, whose common basins of attraction are the whole phase-space. They [5] also considered the same problem in the three-dimensional setting.

Grasselli et al. [6] studied a differential model describing nonisothermal fast phase separation processes taking place in a three-dimensional bounded domain.

$$\begin{cases} (\vartheta + \chi)_t + \nabla \cdot \mathbf{q} = 0, \\ \sigma \mathbf{q}_t + \mathbf{q} = -\nabla \vartheta, \\ k\chi_{tt} + \chi_t - \Delta(-\Delta \chi + \alpha \chi_t + \phi(\chi) - \vartheta) = 0, \end{cases}$$

where $\sigma \in [0,1]$. This model consists of a viscous Cahn–Hilliard equation characterized by the presence of an inertial term χ_{tt}, χ being the order parameter, which is linearly coupled with an evolution equation for the (relative) temperature ϑ.

The blow-up of solutions for the fourth order equation has been intensively studied. Chen and Lu [7] considered the initial-boundary value problem for the nonlinear wave equation

$$u_{tt} - 2bu_{xxt} + \alpha u_{xxxx} = f(u_x)_x.$$

They obtained the blow-up of the solution and the energy decay of the solutions. Wang [8] studied the equation

$$u_{tt} + \Delta^2 u + \mu u_t + au = |u|^{p-2}u.$$

He gave necessary and sufficient conditions for global existence and finite time blow-up of solutions. Escudero et al. [9] discussed a fourth order parabolic equation involving the Hessian

$$u_t + \Delta^2 u = \det\bigl(D^2 u\bigr) = \lambda f.$$

The authors proved the global existence versus blow-up results. Qu and Zhou [10] studied the following:

$$u_t + D^4 u = |u|^{p-1}u - \fint_\Omega |u|^{p-1}u\,dx.$$

By using the method of potential wells, they obtained a threshold result of global existence and blow-up for the sign-changing weak solutions and the conditions under which the global solutions become extinct in finite time. In this paper, we consider the global existence and blow-up of solutions for problem (1.1)–(1.3). To prove the blow-up of solutions, we establish a new functional and consider the solution of the Bernoulli type equation. Basing on the required estimates and using a lemma on the ordinary differential inequality of second order, we prove the blow-up of the solution for the initial-boundary problem. The main method is nontrivial because of both the nonlinearity of $\Delta f(u)$ and more delicate estimates which are necessary to overcome some delicate technical points.

The plan of this paper is as follows. In Sect. 2, we prove the existence and uniqueness of the global generalized solution for the initial-boundary value problems (1.1)–(1.3) by the Galerkin method. We also give some sufficient conditions of the blow-up of the solutions for the initial-boundary value problems (1.1)–(1.3) in Sect. 3. Finally, in Sect. 4, we discussed the decay rate of energy. For simplicity, we set $\delta = 1$ in this paper.

2 Existence of the global solution

We are going to prove the existence and uniqueness for problems (1.1)–(1.3) by the Galerkin method and the compactness theorem in this section.

Let $y_i(x)$ be the orthonormal basis in $L^2(\Omega)$ composed of the eigenfunctions of the eigenvalue problem

$$\begin{cases} \Delta y + \lambda y = 0, \\ y|_{\partial\Omega} = 0 \end{cases} \tag{2.1}$$

corresponding to eigenvalue λ_i $(i = 1, 2, \ldots)$.

Let

$$u_N(x,t) = \sum_{i=1}^{N} \gamma_{Ni}(t) y_i(x) \tag{2.2}$$

be the Galerkin approximate solution for problem (1.1)–(1.3), where $\gamma_{Ni}(t)$ are the undetermined functions and N is a natural number. Suppose that the initial value functions $\varphi(x)$ may be expressed as

$$\varphi(x) = \sum_{i=1}^{\infty} \mu_i y_i(x), \qquad \psi(x) = \sum_{i=1}^{\infty} \nu_i y_i(x), \tag{2.3}$$

where μ_i and ν_i $(i = 1, 2, \ldots)$ are constants.

Substituting the approximate solution $u_N(x,t)$ into Eq. (1.1), multiplying both sides by $y_s(x)$, we obtain

$$\bigl(u_{Ntt} + u_{Nt} - k\Delta u_{Nt} + \Delta^2 u_N, y_s\bigr) = \bigl(\Delta f(u_N), y_s\bigr), \quad s = 1, 2, \ldots, N, \tag{2.4}$$

where $(\cdot,\cdot)$ denotes the inner product of $L^2(\Omega)$.

Substituting the approximate solution $u_N(x,t)$ and the approximations

$$\varphi_N(x) = \sum_{i=1}^{N} \mu_i y_i(x), \qquad \psi_N(x) = \sum_{i=1}^{N} \nu_i y_i(x)$$

of the initial value functions into the initial condition (1.3), we get

$$\gamma_{Ns}(0) = \mu_s, \qquad \dot{\gamma}_{Ns}(0) = \nu_s, \quad s = 1, 2, \ldots, N, \tag{2.5}$$

where $\dot{\gamma}_{Ns}(t) = \frac{d}{dt}\gamma_{Ns}(t)$.

In order to prove the existence of the global generalized solution for problem (1.1)–(1.3), we make a series of estimations for the approximate solution $u_N(x,t)$.

Lemma 2.1 *Suppose that $\varphi \in H^2(\Omega)$ and $\psi \in L^2(\Omega)$ satisfy the boundary condition* (1.2), *$f \in C^1(R)$, $0 \le F(s) = \int_0^s f(\eta)\,d\eta$, and $|f'(s)| \le C_1|s|^2 + C_2$, where $C_1 > 0$, $C_2 > 0$ are constants. Then the following estimate holds*:

$$\bigl\|u_N(\cdot,t)\bigr\|_{H^2}^2 + \bigl\|u_{Nt}(\cdot,t)\bigr\|^2 \le C, \quad t \in [0,T], \tag{2.6}$$

where and in the sequel $C > 0$ is a constant which only depends on T.

Proof Let w_n be the unique solution of the problem

$$\Delta w_n = u_n,$$
$$w_n|_{\partial\Omega} = 0.$$

Substituting the approximate solution $u_N(x,t)$ into Eq. (1.1), multiplying both sides by $2w_{Nt}$, we obtain

$$\left(u_{Ntt} + u_{Nt} - k\Delta u_{Nt} + \Delta^2 u_N, 2w_{Nt}\right) = \left(\Delta f(u_N), 2w_{Nt}\right).$$

Integrating by parts with respect to x on Ω, we have

$$\frac{d}{dt}\left[\left\|\nabla w_{Nt}(\cdot,t)\right\|^2 + \left\|\nabla u_N(\cdot,t)\right\|^2 + 2\int_0^1 F(u_n)\,dx\right] + 2\left\|\nabla w_{Nt}(\cdot,t)\right\|^2 + 2k\left\|u_{Nt}(\cdot,t)\right\|^2 \le 0.$$

Hence, we know

$$\|\nabla w_{Nt}\| \le C, \tag{2.7}$$
$$\|\nabla u_N\| \le C. \tag{2.8}$$

By the Sobolev imbedding theorem, it follows from (2.7) and (2.8) that

$$\|u_N\|_{L^q} \le C, \quad \text{for any } q < \infty \ (n = 2), \tag{2.9}$$
$$\|u_N\|_{L^6} \le C \quad (n = 3). \tag{2.10}$$

Multiplying both sides of (2.4) by $2\gamma_{Nst}(t)$, summing up for $s = 1, 2, \dots, N$, we have

$$\left(u_{Ntt} + u_{Nt} - k\Delta u_{Nt} + \Delta^2 u_N, 2u_{Nt}\right) = \left(\Delta f(u_N), 2u_{Nt}\right).$$

Integrating by parts with respect to x on Ω, we get

$$\frac{d}{dt}\left[\left\|u_{Nt}(\cdot,t)\right\|^2 + \left\|\Delta u_N(\cdot,t)\right\|^2\right] + 2\|u_{Nt}\|^2 + 2k\left\|\nabla u_{Nt}(\cdot,t)\right\|^2 \le \frac{1}{4k}\left\|f'(u_N)\nabla u_N(\cdot,t)\right\|^2 + k\left\|\nabla u_{Nt}(\cdot,t)\right\|^2.$$

By $|f'(s)| \le C_1|u|^2 + C_2$, hence

$$\frac{d}{dt}\left[\left\|u_{Nt}(\cdot,t)\right\|^2 + \left\|\Delta u_N(\cdot,t)\right\|^2\right] + 2\|u_{Nt}\|^2 + k\left\|\nabla u_{Nt}(\cdot,t)\right\|^2 \le C\left\|u_N\right\|_\infty^4 + C. \tag{2.11}$$

On the other hand, by the Gagliardo–Nirenberg inequality, (2.9), and (2.10), we see

$$\|u\|_\infty \le C\|\Delta u\|^a\|u\|_q^{1-a} \le C\|\Delta u\|^a, \quad a = \frac{2}{q+2} \ (n = 2),$$
$$\|u\|_\infty \le C\|\Delta u\|^{1/2}\|u\|_6^{1/2} \le C\|\Delta u\|^{1/2} \quad (n = 3).$$

Therefore, by (2.11),

$$\frac{d}{dt}\left[\left\|u_{Nt}(\cdot,t)\right\|^2+\left\|\Delta u_N(\cdot,t)\right\|^2\right]+2\|u_{Nt}\|^2+k\left\|\nabla u_{Nt}(\cdot,t)\right\|^2 \\ \leq C\|\Delta u_N\|^2+C. \tag{2.12}$$

Then, integrating (2.12) on $[0,t]$ and using the Gronwall inequality, we deduce

$$\left\|u_N(\cdot,t)\right\|_{H^2}^2+\left\|u_{Nt}(\cdot,t)\right\|^2\leq Ce^T\left(\|\varphi\|_{H^2}^2+\|\psi\|^2+1\right),\quad t\in[0,T]. \tag{2.13}$$

Immediately, we get (2.6) from (2.13). The proof is completed. □

Lemma 2.2 *Suppose that the conditions of Lemma* 2.1 *hold. If* $f\in C^3(R)$, $\varphi\in H^4(\Omega)$, $\psi\in H^2(\Omega)$, *and* $|f''(s)|\leq C_3|s|+C_4$, $|f'''(s)|\leq C$, *then the approximate solution for problem* (1.1)–(1.3) *satisfies the following estimate*:

$$\left\|u_N(\cdot,t)\right\|_{H^4}^2+\left\|u_{Nt}(\cdot,t)\right\|_{H^2}^2+\left\|u_{Ntt}(\cdot,t)\right\|^2\leq C(T),\quad 0\leq t\leq T. \tag{2.14}$$

Proof Multiplying both sides of (2.4) by $2\lambda_s^2\gamma_{Nst}(t)$, summing up for $s=1,2,\ldots,N$, we have

$$\left(u_{Ntt}+u_{Nt}-k\Delta u_{Nt}+\Delta^2u_N,2\Delta^2u_{Nt}\right)=\left(\Delta f(u_N),2\Delta^2u_{Nt}\right).$$

Integrating by parts with respect to x, we get

$$\frac{d}{dt}\left(\left\|\Delta u_{Nt}(\cdot,t)\right\|^2+\left\|\Delta^2u_N(\cdot,t)\right\|^2\right)+2\left\|\Delta u_{Nt}(\cdot,t)\right\|^2+2k\left\|\nabla\Delta u_{Nt}(\cdot,t)\right\|^2 \\ =2\int_\Omega\Delta f(u_N)\cdot\Delta^2u_{Nt}\,dx.$$

On the other hand, we know

$$2\int_\Omega\Delta f(u_N)\cdot\Delta^2u_{Nt}\,dx=-2\int_\Omega\nabla\Delta f(u_N)\cdot\nabla\Delta u_{Nt}\,dx$$

and

$$\nabla\Delta f(u_N)=f'''(u_N)|\nabla u_N|^2\nabla u_N+3f''(u_N)\nabla u_N\Delta u_N+f'(u_N)\nabla\Delta u_N.$$

By (2.13), we know that $\|u\|_\infty\leq C$, hence

$$2\int_\Omega\Delta f(u_N)\cdot\Delta^2u_{Nt}\,dx \\ \leq C\left(\left\||\nabla u_N|^3\right\|^2+\|\nabla u_N\Delta u_N\|^2+\|\nabla\Delta u_N\|^2\right)+k\left\|\nabla\Delta u_{Nt}(\cdot,t)\right\|^2.$$

Thus,

$$\frac{d}{dt}\left(\left\|\Delta u_{Nt}(\cdot,t)\right\|^2+\left\|\Delta^2u_N(\cdot,t)\right\|^2\right)+2\left\|\Delta u_{Nt}(\cdot,t)\right\|^2+k\left\|\nabla\Delta u_{Nt}(\cdot,t)\right\|^2 \\ \leq C\left(\left\|\nabla u_N(\cdot,t)\right\|_{L^6}^2+\left\|\nabla u_N(\cdot,t)\right\|_{L^4}^2\left\|\Delta u_N(\cdot,t)\right\|_{L^4}^2+\left\|\nabla\Delta u_N(\cdot,t)\right\|^2\right). \tag{2.15}$$

By (2.13) and the Sobolev imbedding theorem, we see that

$$\|\nabla u_N\|_{L^q} \le C, \quad \text{for any } q < \infty \ (n = 2),$$
$$\|\nabla u_N\|_{L^6} \le C \quad (n = 3).$$

Using the Gagliardo–Nirenberg inequality, we conclude

$$\begin{aligned}
\|\Delta u_N(\cdot,t)\|_{L^4}^2 &\le C\|\Delta^2 u_N(\cdot,t)\|^a \|\nabla u_N(\cdot,t)\|_{L^q}^{1-a} \\
&\le C\|\Delta^2 u_N(\cdot,t)\| + C, \quad \text{where } a = \frac{q+4}{4+4q} \ (n = 2),
\end{aligned}$$
$$\|\Delta u_N(\cdot,t)\|_{L^4}^2 \le C\|\Delta^2 u_N(\cdot,t)\|^{3/8} \|\nabla u_N(\cdot,t)\|_{L^6}^{5/8} \le C\|\Delta^2 u_N(\cdot,t)\| + C \quad (n = 3).$$

On the other hand, by boundary conditions (1.2), we obtain

$$\|\nabla \Delta u_N\|^2 \le C\|\Delta^2 u_N\|^2.$$

Substituting the above inequalities into (2.15), we get

$$\frac{d}{dt}\left(\|\Delta u_{Nt}(\cdot,t)\|^2 + \|\Delta^2 u_N(\cdot,t)\|^2\right) \le C(T) + \|\Delta^2 u_N(\cdot,t)\|^2.$$

Integrating the above inequality, and using the Gronwall inequality, we have

$$\|\Delta u_{Nt}(\cdot,t)\|^2 + \|\Delta^2 u_N(\cdot,t)\|^2 \le C(T)\left(\|\varphi\|_{H^4}^2 + \|\psi\|_{H^2}^2 + 1\right), \quad t \in [0,T]. \tag{2.16}$$

Similarly, multiplying both sides of (2.4) by $\gamma_{Nstt}(t)$, summing up for $s = 1, 2, \ldots, N$, we deduce

$$\left(u_{Ntt} + u_{Nt} - k\Delta u_{Nt} + \Delta^2 u_N, u_{Ntt}\right) = \left(\Delta f(u_N), u_{Ntt}\right).$$

Integrating by parts with respect to x and using the Cauchy inequality, we have

$$\begin{aligned}
&\|u_{Ntt}(\cdot,t)\|_{L^2}^2 \\
&\quad = -\int_\Omega u_{Nt}u_{Ntt}\,dx + k\int_\Omega \Delta u_{Nt}u_{Ntt}\,dx - \int_\Omega \Delta^2 u_N u_{Ntt}\,dx + \int_\Omega \Delta f(u_N)u_{Ntt}\,dx \\
&\quad \le 2\|u_{Nt}(\cdot,t)\|^2 + 2k^2\|\Delta u_{Nt}(\cdot,t)\|^2 + 2\|\Delta^2 u_N(\cdot,t)\|^2 \\
&\qquad + 2\|\Delta f(u_N)\|^2 + \frac{1}{2}\|u_{Ntt}(\cdot,t)\|^2.
\end{aligned}$$

Therefore, we conclude

$$\|u_{Ntt}(\cdot,t)\|_{L^2}^2 \le C(T), \quad t \in [0,T]. \tag{2.17}$$

Immediately, we get (2.14) from (2.16) and (2.17). This completes the proof. □

Theorem 2.1 *Suppose that* $\varphi \in H^4(\Omega)$ *and* $\psi \in H^2(\Omega)$ *satisfy the boundary conditions* (1.2), $f \in C^3(R)$, $0 \le F(s) = \int_0^s f(\eta)\,d\eta$, $|f'(s)| \le C_1|s|^2 + C_2$, $|f''(s)| \le C_3|s| + C_4$, *and* $|f'''(s)| \le C$. *Then problem* (1.1)–(1.3) *has a unique global generalized solution*

$$u \in C\bigl([0,T];H^4(\Omega)\bigr) \cap C^1\bigl([0,T];H^2(\Omega)\bigr) \cap C^2\bigl([0,T];L^2(\Omega)\bigr). \tag{2.18}$$

Proof From (2.14) we know that $u_N \in C([0,T];H^4(\Omega))$, $u_{Nt} \in C([0,T];H^2(\Omega))$, $u_{Ntt} \in C([0,T];L^2(\Omega))$. Using the Sobolev imbedding theorem, we have $D^k u_N \in C([0,T] \times \Omega)$, $0 \le k \le 2$. It follows from the above two relations and the Ascoli–Arzelá theorem that there exist a function $u(x,t)$ and a subsequence of $u_N(x,t)$, still denoted by $u_N(x,t)$, such that as $N \to \infty$, $u_N(x,t)$ uniformly converges to $u(x,t)$ in $[0,T] \times \Omega$. The corresponding subsequence of $\Delta u_N(x,t)$ also uniformly converges to $\Delta u(x,t)$ in $[0,T] \times \Omega$. According to the compactness theorem, the subsequence $D^k u_N(x,t)$ $(0 \le k \le 4)$, $D^k u_{Nt}(x,t)$ $(0 \le k \le 2)$, and $u_{Ntt}(x,t)$ weakly converge to $D^k u(x,t)$ $(0 \le k \le 4)$, $D^k u_t(x,t)$ $(0 \le k \le 2)$, and $u_{tt}(x,t)$ in $L^2([0,T] \times \Omega)$, respectively. Hence, we know that $u(x,t)$ satisfies (2.18). Therefore $u(x,t)$ is the generalized solution for problem (1.1)–(1.3). It is easy to prove the uniqueness of the solutions for problem (1.1)–(1.3). This completes the proof of the theorem. □

3 Blow-up of solutions

In the previous sections, we have seen that the solution of problem (1.1)–(1.3) is globally existent, provided that $F(s) \ge 0$. In this section, we will prove the blow-up of the solution for $F(s) < 0$. For this purpose, we need the following lemma.

Lemma 3.1 ([7]) *Assume that* $u' = G(t,u)$, $v' \ge G(t,v)$, $G \in C([0,\infty) \times (-\infty,\infty))$, *and* $u(t_0) = v(t_0)$, $t_0 \ge 0$, *then when* $t \ge t_0$, $v(t) \ge u(t)$, *where* $u' = \frac{d}{dt}u(t)$.

Let w be the unique solution of the problem

$$\Delta w = u,$$

$$w|_{\partial\Omega} = 0.$$

We have the following theorem.

Theorem 3.1 *Suppose that*

(1) $f(s)s \le \gamma F(s)$, $F(s) \le -\alpha|s|^{p+1}$, *where* $F(s) = \int_0^s f(u)\,du$, $\gamma > 2$, $\alpha > 0$, *and* $p > 1$ *are constants.*

(2) $\varphi \in H^4(\Omega)$, $\psi \in H^2(\Omega)$ *and*

$$\begin{aligned} E(0) &= \bigl\|\nabla w_t(0)\bigr\|^2 + \|\nabla\varphi\|^2 + 2\int_\Omega F\bigl(\varphi(x)\bigr)\,dx \\ &\le \frac{-2^{\frac{2p}{p-1}}}{(\frac{\alpha(\gamma-2)}{p+3})^{\frac{2}{p-1}}(1 - e^{-\frac{p-1}{4}})^{\frac{4}{p-1}}} < 0, \end{aligned}$$

then the generalized solution $u(x,\,t)$ *of problem* (1.1)–(1.3) *blows up in finite time, i.e.,*

$$\|\nabla w\|^2 + \int_0^t \bigl\|u(\cdot,\tau)\bigr\|^2\,d\tau + \int_0^t \bigl\|\nabla w(\cdot,\tau)\bigr\|^2\,d\tau + \int_0^t \int_0^\tau \|u\|^2\,ds\,d\tau \to \infty, \quad \text{as } t \to T^*.$$

Proof Let

$$E(t) = \int_\Omega |\nabla w_t|^2\,dx + 2k\int_0^t\int_\Omega \left|u_\tau(x,\tau)\right|^2\,dx\,d\tau + \int_\Omega |\nabla u|^2\,dx + 2\int_\Omega F(u)\,dx + 2\int_0^t\int_\Omega \left|\nabla w_\tau(x,\tau)\right|^2\,dx\,d\tau.$$

A simple calculation shows that

$$\begin{aligned}\frac{dE(t)}{dt} &= 2\int_\Omega \nabla w_t\nabla w_{tt}\,dx + 2k\int_\Omega |u_t|^2\,dx + 2\int_\Omega \nabla u\nabla u_t\,dx \\ &\quad + 2\int_\Omega f(u)u_t\,dx + 2\int_\Omega |\nabla w_t|^2\,dx \\ &= -2\int_\Omega u_t w_{tt}\,dx + 2k\int_\Omega u_t\Delta w_t\,dx - 2\int_\Omega \Delta u u_t\,dx \\ &\quad + 2\int_\Omega f(u)u_t\,dx - 2\int_\Omega w_t u_t\,dx \\ &= -2\int_\Omega \left[w_{tt} - k\Delta w_t + \Delta u - f(u) + w_t\right]u_t\,dx.\end{aligned}$$

Noticing equation (1.1), we know that

$$\frac{dE(t)}{dt} = \int_\Omega \nabla\left[w_{tt} - k\Delta w_t + \Delta u - f(u) + w_t\right]^2\,dx = 0,$$

which implies

$$E(t) = E(0), \quad t > 0. \tag{3.1}$$

Moreover, we easily see

$$\begin{aligned}\gamma\int_\Omega F(u)\,dx &= 2E(0) - 2\|\nabla w_t\|^2 - 4k\int_0^t \|u_\tau\|^2\,d\tau - 2\|\nabla u\|^2 \\ &\quad - 2\int_0^t |\nabla w_t|^2\,d\tau + (\gamma-2)\int_\Omega F(u)\,dx.\end{aligned}\tag{3.2}$$

Now, we define

$$H(t) = \|\nabla w\|^2 + \int_0^t \left\|u(\cdot,\tau)\right\|^2\,d\tau + \int_0^t \left\|\nabla w(\cdot,\tau)\right\|^2\,d\tau + \int_0^t\int_0^\tau \|u\|^2\,ds\,d\tau. \tag{3.3}$$

It is obvious that

$$\begin{aligned}\frac{dH(t)}{dt} &= 2\int_\Omega \nabla w\nabla w_t\,dx + \int_\Omega \left|u(\cdot,t)\right|^2\,dx \\ &\quad + \int_\Omega \left|\nabla w(\cdot,t)\right|^2\,dx + \int_0^t\int_\Omega |u|^2\,dx\,d\tau.\end{aligned}\tag{3.4}$$

Further, we have

$$\begin{aligned}\frac{d^2H(t)}{dt^2} &= 2\int_\Omega\left[|\nabla w_t|^2 - wu_{tt} + u(\cdot,t)u_t + \nabla w\nabla w_t + \frac{1}{2}|u|^2\right]dx\\
&= 2\int_\Omega\left[|\nabla w_t|^2 - wu_{tt} + w(\cdot,t)\Delta u_t + \nabla w\nabla w_t + \frac{1}{2}|u|^2\right]dx\\
&= 2\int_\Omega\left[|\nabla w_t|^2 + w\bigl(u_t + \Delta^2 u - \Delta f(u)\bigr) + \nabla w\nabla w_t + \frac{1}{2}|u|^2\right]dx\\
&= 2\int_\Omega\left[|\nabla w_t|^2 - \nabla w\nabla w_t - |\nabla u|^2 - f(u)u + \nabla w\nabla w_t + \frac{1}{2}|u|^2\right]dx\\
&\geq 2\|\nabla w_t\|^2 - 2\|\nabla u\|^2 - 2\gamma\int_\Omega F(u)\,dx + \|u\|^2\\
&\geq 2\|\nabla w_t\|^2 - 2\|\nabla u\|^2 + \|u\|^2 - 2E(0) + 2\|\nabla u_t\|^2 + 2\int_0^t\|u_t\|^2\,d\tau\\
&\quad + 2\|\nabla u\|^2 + 2\int_0^t|\nabla w_t|^2\,d\tau - 2(\gamma-2)\int_\Omega F(u)\,dx\\
&\geq 2\|\nabla w_t\|^2 + \|u\|^2 - 2E(0) - 2(\gamma-2)\int_\Omega F(u)\,dx + 2\|\nabla u_t\|^2 > 0. \qquad (3.5)\end{aligned}$$

Integrating (3.5), we conclude that

$$H'(t) \geq -2E(0)t - 2(\gamma-2)\int_0^t\int_\Omega F\bigl(u(x,\tau)\bigr)\,dx\,d\tau + H'(0). \qquad (3.6)$$

Integrating (3.6), we deduce

$$H(t) \geq -2E(0)t^2 - 2(\gamma-2)\int_0^t\int_0^\tau\int_\Omega F\bigl(u(x,s)\bigr)\,dx\,ds\,d\tau + H'(0)t + H(0). \qquad (3.7)$$

Combining (3.5, (3.6) with (3.7), we derive

$$\begin{aligned}&H''(t) + H'(t) + H(t)\\
&\quad\geq 2\alpha(\gamma-2)\left[\int_\Omega|u|^{p+1}\,dx + \int_0^t\int_\Omega|u(x,\tau)|^{p+1}\,dx\,d\tau\right.\\
&\qquad\left.+ \int_0^t\int_0^\tau\int_\Omega|u(x,s)|^{p+1}\,dx\,ds\,d\tau\right] + \|u\|^2 - 2E(0)\left(1 + t + \frac{t^2}{2}\right)\\
&\qquad + H'(0)(1+t) + H(0) + 2\|\nabla w_t\|^2 + 2\|\nabla u_t\|^2. \qquad (3.8)\end{aligned}$$

Substituting (3.4) into (3.8), we get

$$\begin{aligned}&H''(t) + 2\int_\Omega\nabla w\nabla w_t\,dx + \int_\Omega\bigl|u(\cdot,t)\bigr|^2\,dx + \int_\Omega\bigl|\nabla w(\cdot,t)\bigr|^2\,dx\\
&\qquad + \int_0^t\int_\Omega|u|^2\,dx\,d\tau + H(t)\\
&\quad\geq 2\alpha(\gamma-2)\left[\int_\Omega|u|^{p+1}\,dx + \int_0^t\int_\Omega\bigl|u(x,\tau)\bigr|^{p+1}\,dx\,d\tau\right.\end{aligned}$$

$$
\begin{aligned}
&+ \int_0^t \int_0^\tau \int_\Omega \left|u(x,s)\right|^{p+1} dx\,ds\,d\tau \Biggr] + \|u\|^2 - 2E(0)\left(1 + t + \frac{t^2}{2}\right) \\
&+ H'(0)(1+t) + H(0) + 2\|\nabla w_t\|^2 + 2\|\nabla u_t\|^2. \qquad (3.9)
\end{aligned}
$$

Recalling $H''(t) > 0$, $H(t) \geq 0$, and

$$
2\int_\Omega \nabla w \nabla w_t\,dx \leq \|\nabla w\|^2 + \|\nabla w_t\|^2,
$$

therefore, from (3.9) we obtain

$$
\begin{aligned}
&H''(t) + H(t) \\
&\quad\geq \alpha(\gamma - 2)\Biggl[\int_\Omega |u|^{p+1}\,dx + \int_0^t \int_\Omega \left|u(x,\tau)\right|^{p+1} dx\,d\tau \\
&\qquad + \int_0^t \int_0^\tau \int_\Omega \left|u(x,s)\right|^{p+1} dx\,ds\,d\tau \Biggr] \\
&\qquad - E(0)\left(1 + t + \frac{t^2}{2}\right) + \frac{1}{2}H'(0)(1+t) + \frac{1}{2}H(0). \qquad (3.10)
\end{aligned}
$$

On the other hand, the Hölder inequality implies that

$$
\begin{aligned}
&|\Omega|^{\frac{p-1}{2}} \int_\Omega |u|^{p+1}\,dx \geq \|u\|^{p+1}, \\
&\int_0^t \int_\Omega |u|^2\,dx\,d\tau \leq t^{\frac{p-1}{p+1}} |\Omega|^{\frac{p-1}{p+1}} \left(\int_0^t \int_\Omega |u|^{p+1}\,dx\,d\tau\right)^{\frac{2}{p+1}}, \\
&\int_0^t \int_0^\tau \int_\Omega |u|^2\,dx\,ds\,d\tau \leq \left(\frac{t^2}{2}\right)^{\frac{p-1}{p+1}} |\Omega|^{\frac{p-1}{p+1}} \left(\int_0^t \int_0^\tau \int_\Omega |u|^{p+1}\,dx\,ds\,d\tau\right)^{\frac{2}{p+1}}.
\end{aligned}
$$

Thus

$$
\begin{aligned}
&\int_0^t \int_\Omega |u|^{p+1}\,dx\,d\tau \geq t^{\frac{1-p}{2}} |\Omega|^{\frac{1-p}{2}} \left(\int_0^t \int_\Omega |u|^2\,dx\,d\tau\right)^{\frac{p+1}{2}}, \\
&\int_0^t \int_0^\tau \int_\Omega |u|^{p+1}\,dx\,ds\,d\tau \geq 2^{\frac{p-1}{2}} t^{1-p} \left(\int_0^t \int_0^\tau \int_\Omega |u|^2\,dx\,ds\,d\tau\right)^{\frac{p+1}{2}}.
\end{aligned}
$$

Substituting the above inequalities into (3.10), and by the fact $(x+y+z)^n \leq 2^{2(n-1)}(x^n + y^n + z^n)$, $x, y, z > 0$, $n > 1$, we know that

$$
\begin{aligned}
&H''(t) + H(t) \\
&\quad\geq \alpha(\gamma - 2)|\Omega|^{\frac{1-p}{2}} \Biggl[\|u\|^{p+1} + t^{\frac{1-p}{2}} \left(\int_0^t \int_\Omega |u|^2\,dx\,d\tau\right)^{\frac{p+1}{2}} \\
&\qquad + 2^{\frac{p-1}{2}} t^{1-p} \left(\int_0^t \int_0^\tau \int_\Omega |u|^2\,dx\,ds\,d\tau\right)^{\frac{p+1}{2}} \Biggr] \\
&\qquad - E(0)\left(1 + t + \frac{t^2}{2}\right) + \frac{1}{2}H'(0)(1+t) + \frac{1}{2}H(0)
\end{aligned}
$$

$$\begin{aligned} &\geq \alpha(\gamma-2)|\Omega|^{\frac{1-p}{2}} 2^{\frac{1}{2}-\frac{3}{2}p} t^{1-p} H^{\frac{p+1}{2}} - F(0)\left(1+t+\frac{t^2}{2}\right) \\ &\quad + \frac{1}{2}H'(0)(1+t) + \frac{1}{2}H(0), \quad t \geq 1. \end{aligned} \tag{3.11}$$

In addition, note from (3.6) and (3.7) that $H'(t) \to +\infty$ and $H(t) \to +\infty$ as $t \to \infty$. Therefore, we see that there is $t_0 \geq 1$ such that when $t \geq t_0$, $H'(t) > 0$ and $H(t) > 0$. Multiplying (3.11) by $2H'(t)$, we get

$$\frac{d}{dt}\left[H'^2 + H^2\right] \geq Mt^{1-p}\frac{d}{dt}H^{\frac{p+3}{2}} + I(t), \quad t \geq t_0, \tag{3.12}$$

where

$$\begin{aligned} M &= \frac{\alpha(\gamma-2)|\Omega|^{\frac{1-p}{2}} 2^{\frac{1}{2}-\frac{3}{2}p}}{p+3}, \\ I(t) &= \left[-4F(0)t + 2H'(0)\right]\left[-E(0)\left(1+t+\frac{t^2}{2}\right) + \frac{1}{2}H'(0)(1+t) + \frac{1}{2}H(0)\right]. \end{aligned}$$

It follows from (3.12) that

$$\frac{d}{dt}\left[t^{p-1}\left(H'^2 + H^2\right) - MH^{\frac{p+3}{2}}\right] \geq t^{p-1}I(t), \quad t \geq t_0.$$

Integrating the above inequality over (t_0, t), we easily see

$$\begin{aligned} &t^{p-1}\left(H'^2 + H^2\right) - MH^{\frac{n+3}{2}} \\ &\quad \geq \int_{t_0}^{t} \tau^{p-1} I(\tau)\,d\tau + t_0^{p-1}\left(H'^2(t_0) + H^2(t_0)\right) - MH^{\frac{n+3}{2}}(t_0), \quad t \geq t_0. \end{aligned} \tag{3.13}$$

Note that when $t \to \infty$, the right-hand side of (3.13) approaches positive infinity, hence, there is $t_1 > t_0$ such that when $t \geq t_1$, the right-hand side of (3.13) is larger than or equal to zero. We thus have

$$t^{p-1}\left(H' + H\right)^2 \geq t^{p-1}\left(H'^2 + H^2\right) \geq MH^{\frac{p+3}{2}}(t), \quad t \geq t_0,$$

that is,

$$H' + H \geq t^{\frac{1-p}{2}} M_1 H^{\frac{p+3}{4}}(t), \quad t \geq t_0,$$

where $M_1 = M^{1/2}$.

Now, we consider the initial value problem of the ordinary differential equation

$$\begin{aligned} &S'(t) + S(t) = M_1 t^{\frac{1-p}{2}} \left(S(t)\right)^{\frac{p+3}{4}}(t), \\ &S(t_1) = H(t_1). \end{aligned}$$

Therefore, we conclude that

$$S(t) = e^{-(t-t_1)}\left[H^{\frac{1-p}{4}}(t_1) - \frac{M_1(p-1)}{4}\int_{t_1}^{t} \tau^{\frac{1-p}{2}} e^{-\frac{p-1}{4}(\tau-t_1)}\,d\tau\right]$$
$$\equiv e^{-(t-t_1)}H(t_1)J^{\frac{4}{1-p}}(t),$$

where

$$J(t) = 1 - \frac{M_1(p-1)}{4}H^{\frac{p-1}{4}}(t_1)\int_{t_1}^{t} \tau^{\frac{1-p}{2}} e^{-\frac{p-1}{4}(\tau-t_1)}\,d\tau.$$

It is obvious that $J(t_1) = 1 > 0$, and

$$J(t) \le 1 - \frac{M_1(p-1)}{4}H^{\frac{p-1}{4}}(t_1)(t_1+1)^{\frac{1-p}{2}}\int_{t_1}^{t_1+1} e^{-\frac{p-1}{4}(\tau-t_1)}\,d\tau$$
$$= 1 - M_1 H^{\frac{p-1}{4}}(t_1)(t_1+1)^{\frac{1-p}{2}}\bigl(1-e^{-\frac{p-1}{4}}\bigr).$$

By (3.7), we can take t_1 sufficiently large such that

$$H^{\frac{p-1}{4}}(t_1)(t_1+1)^{\frac{1-p}{2}} \ge \frac{1}{2}\bigl(-E(0)\bigr)^{\frac{p-1}{4}}.$$

Condition (2) of Theorem 3.1 implies

$$J(t) \le 0, \quad t \ge t_1 + 1.$$

Noticing the continuity of $J(t)$, we know that there is a constant T^* ($t_1 < T^* < t_1 + 1$) such that $J(T^*) = 0$. Hence $S(t) \to \infty$, as $t \to T^*$. It follows from Lemma 3.1 that when $t \ge t_1$, $H(t) \ge S(t)$. Thus, $H(t) \to \infty$ as $t \to T^*$. Theorem 3.1 is proved. □

4 Decay rate of energy

In this section, we are going to discuss the decay rate of energy for problem (1.1)–(1.3). We need the following lemma.

Lemma 4.1 ([11]) *Suppose that* $J : [0,\infty) \to [0,\infty)$ *is a non-increasing function and assume that there is a constant* $L > 0$ *such that*

$$\int_t^{\infty} J(s)\,ds \le LJ(t), \quad \forall t \ge 0.$$

Then

$$J(t) \le J(0)e^{1-\frac{t}{L}}, \quad \forall t \ge 0.$$

Theorem 4.1 *Suppose that the assumptions of Theorem* 2.1 *hold and* $2F(s) \le f(s)s$. *Let* $u(x,t)$ *be a global generalized solution for problem* (1.1)–(1.3). *Then we have*

$$G(t) \le G(0)e^{1-Mt},$$

where $G(t) = \int_\Omega |\nabla w_t|^2\,dx + \int_\Omega |\nabla u|^2\,dx + 2\int_\Omega F(u)\,dx$.

Proof Recalling (3.1), we derive

$$\begin{aligned} G(t) &= \int_\Omega |\nabla w_t|^2\,dx + \int_\Omega |\nabla u|^2\,dx + 2\int_\Omega F(u)\,dx \\ &= \big\|\nabla w_t(0)\big\|^2 + \|\nabla\varphi\|^2 + 2\int_\Omega F\big(\varphi(x)\big)\,dx - 2k\int_0^t\int_\Omega \big|u_\tau(x,\tau)\big|^2\,dx\,d\tau \\ &\quad - 2\int_0^t\int_\Omega \big|\nabla w_\tau(x,\tau)\big|^2\,dx\,d\tau. \end{aligned}$$

A simple computation gives, for any $0 \le t_1 \le t_2 < \infty$,

$$\begin{aligned} G(t_1) - G(t_2) &= 2k\int_0^{t_2}\int_\Omega \big|u_t(x,t)\big|^2\,dx\,dt + 2\int_0^{t_2}\int_\Omega \big|\nabla w_t(x,t)\big|^2\,dx\,dt \\ &\quad - \left(2k\int_0^{t_1}\int_\Omega \big|u_t(x,t)\big|^2\,dx\,dt + 2\int_0^{t_1}\int_\Omega \big|\nabla w_t(x,t)\big|^2\,dx\,dt\right) \\ &= 2k\int_{t_1}^{t_2}\int_\Omega \big|u_t(x,t)\big|^2\,dx\,dt + 2\int_{t_1}^{t_2}\int_\Omega \big|\nabla w_t(x,t)\big|^2\,dx\,dt, \end{aligned} \tag{4.1}$$

which shows that $G(t)$ is non-increasing.

Multiplying (1.1) by $w(x,t)$, integrating over $(t_1,t_2)\times\Omega$, and integrating by parts, we have

$$\begin{aligned} &-\int_{t_1}^{t_2}\int_\Omega \big(|\nabla w_t|^2 + |\nabla u|^2 + 2F(u)\big)\,dx\,dt \\ &\quad = -\left(\int_\Omega \nabla w\nabla w_t\,dx\right)\bigg|_{t_1}^{t_2} - \left(\int_\Omega u^2\,dx\right)\bigg|_{t_1}^{t_2} - \left(\int_\Omega |\nabla w|^2\,dx\right)\bigg|_{t_1}^{t_2} \\ &\qquad + \int_{t_1}^{t_2}\int_\Omega \big(2F(u) - f(u)u\big)\,dx\,dt, \end{aligned}$$

which implies

$$\begin{aligned} \int_{t_1}^{t_2} G(t)\,dt &= 2\int_{t_1}^{t_2}\int_\Omega |\nabla w_t|^2\,dx\,dt \\ &\quad - \left(\int_\Omega \nabla w\nabla w_t\,dx\right)\bigg|_{t_1}^{t_2} - \left(\int_\Omega u^2\,dx\right)\bigg|_{t_1}^{t_2} - \left(\int_\Omega |\nabla w|^2\,dx\right)\bigg|_{t_1}^{t_2} \\ &\quad + \int_{t_1}^{t_2}\int_\Omega \big(2F(u) - f(u)u\big)\,dx. \end{aligned} \tag{4.2}$$

Recalling the assumption $f(s)s \le 2F(s)$, we know

$$\begin{aligned} \int_{t_1}^{t_2} G(t)\,dt &\le 2\int_{t_1}^{t_2}\int_\Omega |\nabla w_t|^2\,dx\,dt \\ &\quad - \left(\int_\Omega \nabla w\nabla w_t\,dx\right)\bigg|_{t_1}^{t_2} - \left(\int_\Omega u^2\,dx\right)\bigg|_{t_1}^{t_2} - \left(\int_\Omega |\nabla w|^2\,dx\right)\bigg|_{t_1}^{t_2}. \end{aligned} \tag{4.3}$$

Using the Poincaré inequality, we obtain

$$\int_\Omega u^2 \le C_* \int_\Omega |\nabla u|^2\,dx \le C_* G(t), \tag{4.4}$$

$$\int_\Omega |\nabla w|^2 \le C_* \int_\Omega |u|^2\,dx \le C_*^2 \int_\Omega |\nabla u|^2\,dx \le C_*^2 G(t). \tag{4.5}$$

The Cauchy inequality yields

$$\left|\int_\Omega \nabla w \nabla w_t\,dx\right| \le \frac{1}{2}\int_\Omega |\nabla w|^2\,dx + \frac{1}{2}\int_\Omega |\nabla w_t|\,dx \le \frac{1}{2}\bigl(C_*^2+1\bigr)G(t). \tag{4.6}$$

On the other hand, by the non-increasing property of $G(t)$, we get

$$2\int_{t_1}^{t_2}\int_\Omega |\nabla w_t|^2\,dx\,dt \le G(t_1) - G(t_2) \le G(t_1). \tag{4.7}$$

Using (4.4)–(4.7), we deduce

$$\int_{t_1}^{t_2} G(t)\,dt \le 2\bigl(C_* + 3C_*^2 + 2\bigr)G(t_1) \equiv \frac{1}{M}G(t_1). \tag{4.8}$$

By Lemma 4.1, we conclude that

$$G(t) \le G(0)e^{1-Mt}, \quad \forall t \ge 0.$$

This completes the proof. □

Acknowledgements
The authors would like to express their deep thanks to the referee for valuable suggestions for the revision and improvement of the manuscript.

Funding
This work is supported by the Jilin Scientific and Technological Development Program [number 20170101143JC].

Abbreviations
Not applicable.

Availability of data and materials
Not applicable.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the manuscript and read and approved the final manuscript.

Publisher's Note

Received: 11 March 2018 Accepted: 11 April 2018 Published online: 25 April 2018

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