Li and Xia *Boundary Value Problems* (2018) 2018:66
https://doi.org/10.1186/s13661-018-0984-0

Boundary Value Problems
a SpringerOpen Journal

RESEARCH Open Access

Existence of multiple solutions for a quasilinear Neumann problem with critical exponent

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Abstract

The main purpose of this paper is to establish the existence and multiplicity of nontrivial solutions for a quasilinear Neumann problem with critical exponent. It is shown, by the methods of the Lions concentration-compactness principle and the mountain pass lemma, that under certain conditions, the existence of nontrivial solutions are obtained.

MSC: 35B33; 35J50; 35J62; 35J92

Keywords: Quasilinear elliptic equation; Nontrivial solution; Neumann boundary condition; Critical Sobolev exponent; Lions concentration-compactness principle

1 Introduction

In this paper, we consider the following quasilinear elliptic problem with critical Sobolev exponent:

$$\begin{cases} -\varepsilon^p \Delta_p u + V(x)|u|^{p-2}u = Q(x)|u|^{p^*-2}u + P(x)|u|^{q-2}u, & x \in \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = 0, & x \in \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset R^N$ is a bounded domain with smooth boundary, $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, $\varepsilon > 0$, $1 < p < N$, $p < q < p^* = \frac{Np}{N-p}$, ν denotes the unit outward normal vector with respect to $\partial\Omega$. The weight functions $V(x)$, $Q(x)$ and $P(x)$ are continuous on Ω. Such problems arise in the theory of quasiregular and quasiconformal mapping or in the study of non-Newtonian fluids. In the latter case, the p is a characteristic of the medium. Media with $p > 2$ are called dilatant fluids and those with $p < 2$ are called pseudoplastics. If $p = 2$, they are Newtonian fluids.

The early study of Laplacian elliptic equation with critical Sobolev exponent was Pohozaev [1], the author established the nonexistence of nontrivial solution to the Dirichlet problems when Ω is a star-shaped domain with respect to the origin. Later, Brézis and Nirenberg [2] showed the existence of positive solutions by introducing the low-order perturbation terms, and Struwe [3] also obtained the global compactness result. Since then, the study of these elliptic problems with critical growth terms have been paid wide attentions in recent years (see [4–7]). Set $p = 2$, $\varepsilon = 1$, $P(x) = 0$, $V(x) = \lambda$, then Problem (1.1) reduces to the following semilinear elliptic problem:

$$\begin{cases} -\Delta u + \lambda u = Q(x)|u|^{2^*-2}u, & x \in \Omega, \\ \frac{\partial u}{\partial \nu} = 0, & x \in \Omega. \end{cases} \tag{1.2}$$

Comte and Knaap [8] proved that there exists a nontrivial solution of problem (1.2) by variational method if $Q(x) = 1$ and $\lambda = -\mu$. Chabrowski and Willem [9] studied this problem with the assumption that the function $Q(x)$ is nonnegative and Hölder continuous, they obtained the existence of least energy solutions by solving minimization problem corresponding to

$$S_\lambda = \inf_{u \in H^1(\Omega), \int_\Omega Q(x)|u|^{2^*} \,dx \neq 0} \frac{\int_\Omega (|\nabla u|^2 + \lambda u^2) \,dx}{(\int_\Omega Q(x)|u|^{2^*} \,dx)^{\frac{2}{2^*}}}.$$

Subsequently, Chabrowski and Girão [10] investigated the existence and nonexistence of least energy solutions when the function $Q(x)$ has some symmetry properties. For more relevant information as regards the corresponding problems, the interested reader may refer to [11–21] and the references therein.

As for quasilinear elliptic problems with critical Sobolev exponent, the existence and multiplicity of solutions have also been studied extensively. Abreu et al. [22] studied the following nonhomogeneous Neumann boundary problems:

$$\begin{cases} -\Delta_p u + \lambda u^{p-1} = u^q, & x \in \Omega, \\ u > 0, & x \in \Omega, \\ |\nabla u|^{p-2} \frac{\partial u}{\partial \nu} = \varphi, & x \in \partial\Omega, \end{cases} \tag{1.3}$$

where $p - 1 < q \leq p^* - 1$, $\varphi \in C^\alpha(\overline{\Omega})$, $0 < \alpha < 1$, $\varphi \not\equiv 0$. They proved that there exists a $\lambda^* > 0$ such that problem (1.3) has at least two positive solutions if $\lambda > \lambda^*$, has at least one positive solution if $\lambda = \lambda^*$ and has no positive solution if $\lambda < \lambda^*$ relying on the lower and upper solutions method and variational approach. Zhao et al. [23] discussed the quasilinear elliptic problem of the form

$$\begin{cases} -\Delta_p u + \lambda(x)|u|^{p-2}u = |u|^{p^*-2}u + |u|^{r-2}u, & x \in \Omega, \\ |\nabla u|^{p-2} \frac{\partial u}{\partial \nu} = \eta|u|^{p-2}u, & x \in \partial\Omega, \end{cases} \tag{1.4}$$

they showed that there exists at least a nontrivial solution when $p < r < p^*$ and there exist infinitely many solutions when $1 < r < p$ by using the Mountain pass theorem and the concentration-compactness principle. Some authors also studied the critical Sobolev exponent for quasilinear equations and the corresponding evolution problems with Neumann boundary conditions, the reader may also refer to [24–37].

Motivated by the results of the above papers, we discuss the existence of nontrivial nonnegative solutions to Problem (1.1) by a variational method. The special features of this problem are the following. Firstly, due to the lack of compactness of the embedding of $W^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$, we cannot use the standard variational argument directly. In order to overcome this difficulty and obtain the existence of solutions, we have to add restrictions on the weight functions $Q(x)$ and $P(x)$ to prove the corresponding functional of Problem (1.1) satisfies $(PS)_c$-condition in a suitable range by the Lions concentration-compactness principle. Secondly, the weight function $V(x)$ may be unbounded near the boundary $\partial\Omega$, which leads to the space $W^{1,p}(\Omega)$ is not suitable for our problem. To solve such problem, we have to introduce a suitable weighted Sobolev space.

For the sake of convenience, we introducing a new parameter $\lambda = \varepsilon^{-p}$, then Problem (1.1) may be rewritten as the following problem:

$$\begin{cases} -\Delta_p u + \lambda V(x)|u|^{p-2}u = \lambda Q(x)|u|^{p^*-2}u + \lambda P(x)|u|^{q-2}u, & x \in \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = 0, & x \in \partial\Omega. \end{cases} \tag{1.5}$$

Throughout this paper, we make some assumptions on the weight functions $Q(x)$, $P(x)$, $V(x)$ as the following:

(A1) $Q(x)$, $P(x)$ are continuous on $\overline{\Omega}$, and $Q(x) > 0$, $P(x) \geq 0$ for $x \in \overline{\Omega}$;

(A2) $V(x)$ is continuous in Ω, and $V(x) \geq 0$, $V(x) \not\equiv 0$ for $x \in \Omega$.

Set $Q_m = \max_{x\in\partial\Omega} Q(x)$, $Q_M = \max_{x\in\overline{\Omega}} Q(x)$, $P_M = \max_{x\in\overline{\Omega}} P(x)$. The main results of this paper are the following.

Theorem 1.1 *Suppose that* (A1), (A2) *hold*, $H(0) > 0$ *and* $Q_m = Q(0)$. *If functions* $Q(x)$, $V(x)$ *satisfy*

(A3) $Q_M \leq 2^{\frac{p}{N-p}} Q_m$, *and* $|Q(x) - Q(0)| = o(|x|^\alpha)$ *as* $x \to 0$, *where* $1 < \alpha < \frac{N}{p-1}$;

(A4) $\int_{\Omega\cap B(0,\delta)} V^{r'}\,dx < \infty$, *where* $\frac{1}{r} + \frac{1}{r'} = 1$, $1 < r < \frac{N(p-1)}{Np+2p-N-p^2-1}$, $\delta > 0$.

Then Problem (1.5) *has at least one nontrivial solution for every* $\lambda > 0$ *and* $N \geq 2p$, *where* $H(0)$ *will be later determined*.

Theorem 1.2 *Suppose that* (A1), (A2) *hold. If* $Q_M > 2^{\frac{p}{N-p}} Q_m$ *and functions* $P(x)$, $V(x)$ *satisfy*

(A5) $P(x) \not\equiv 0$ *for* $x \in \Omega$, *and* $V \in L^1(\Omega)$.

Then there exists a $\lambda_* > 0$ *such that Problem* (1.5) *has at least one nontrivial solution for* $0 < \lambda < \lambda_*$.

Theorem 1.3 *Suppose that* (A1), (A2) *hold. If* $Q_M > 2^{\frac{p}{N-p}} Q_m$ *and functions* $P(x)$, $V(x)$ *satisfy*

(A6) $P(x) > 0$ *for* $x \in \overline{\Omega}$;

(A7) *there exist* $x_0 \in \Omega$ *and constant* $\delta > 0$ *such that* $V(x) = 0$ *for* $x \in B(x_0,\delta) \subset \Omega$.

Then there exists a $\lambda^* > 0$ *such that Problem* (1.5) *has at least one nontrivial solution for* $\lambda > \lambda^*$.

Theorem 1.4 *Suppose that* (A1), (A2) *hold. If* $Q_M > 2^{\frac{p}{N-p}} Q_m$ *and functions* $P(x)$, $V(x)$ *satisfy the conditions* (A6) *and* (A7). *Then, for every integer n, there exists a constant* $\Lambda_n > 0$ *such that Problem* (1.5) *has at least n pairs of nontrivial solutions for* $\lambda > \Lambda_n$.

2 Preliminaries

Firstly, we define the weighted Sobolev space

$$W^{1,p}_{\lambda,V}(\Omega) = \left\{u; D_i u \in L^p(\Omega), i = 1,2,\ldots,N, \int_\Omega V(x)|u|^p\,dx < +\infty\right\}$$

with norm $\|u\|_{\lambda,V} = (\int_\Omega (|\nabla u|^p + \lambda V(x)|u|^p)\,dx)^{\frac{1}{p}}$. Obviously, norms $\|u\|_{\lambda,V}$ and $\|u\|_V$ are equivalent, $W^{1,p}_{\lambda,V}(\Omega) \hookrightarrow W^{1,p}(\Omega)$, where $\|u\|_V = (\int_\Omega (|\nabla u|^p + V(x)|u|^p)\,dx)^{\frac{1}{p}}$.

By using the Sobolev embedding theorem, we know that there exists a constant $C_q > 0$ such that

$$|u|_q \le C_q\|u\|_V \le C_q\|u\|_{\lambda V}, \quad \text{for } \lambda \ge 1, u \in W^{1,p}_{\lambda,V}(\Omega), \tag{2.1}$$

and

$$|u|_q \le C_q\|u\|_V \le \lambda^{-\frac{1}{p}} C_q\|u\|_{\lambda V}, \quad \text{for } 0 < \lambda < 1, u \in W^{1,p}_{\lambda,V}(\Omega), \tag{2.2}$$

where $|u|_q = (\int_\Omega |u|^q\,dx)^{\frac{1}{q}}$, $q \in (p, p^*)$.

Next, we give the definition of weak solution to Problem (1.5).

Definition 2.1 A function $u \in W^{1,p}_{\lambda,V}(\Omega)$ is said to be a weak solution of Problem (1.5) if it satisfies

$$\begin{aligned}&\int_\Omega |\nabla u|^{p-2}\nabla u \nabla \psi\,dx + \lambda \int_\Omega V(x)|u|^{p-2}u\psi\,dx \\ &= \lambda \int_\Omega Q(x)|u|^{p^*-2}u\psi\,dx + \lambda \int_\Omega P(x)|u|^{q-2}u\psi\,dx, \quad \forall \psi \in W^{1,p}_{\lambda,V}(\Omega).\end{aligned}$$

Thus, the corresponding energy functional of Problem (1.5) is defined in $W^{1,p}_{\lambda,V}(\Omega)$ by

$$J_\lambda(u) = \frac{1}{p}\int_\Omega \big(|\nabla u|^p + \lambda V(x)|u|^p\big)\,dx - \frac{\lambda}{p^*}\int_\Omega Q(x)|u|^{p^*}\,dx - \frac{\lambda}{q}\int_\Omega P(x)|u|^q\,dx.$$

Let S be the best Sobolev constants, namely

$$S = \inf_{D^{1,p}(R^N)\setminus\{0\}} \frac{\int_\Omega |\nabla u|^p\,dx}{(\int_\Omega |u|^{p^*}\,dx)^{\frac{p}{p^*}}}, \tag{2.3}$$

where $D^{1,p}(R^N) = \{u \in L^{p^*}(R^N) : |\nabla u| \in L^p(R^N)\}$. This constant S is achieved by the functional u_ε given by

$$u_\varepsilon(x) = C_{Np}\varepsilon^{\frac{N-p}{p^2}}\big(\varepsilon + |x|^{\frac{p}{p-1}}\big)^{\frac{p-N}{p}},$$

where the constant C_{Np} is chosen such that $-\Delta_p u_\varepsilon = |u_\varepsilon|^{p^*-1}$ in R^N (see [22] for details).

In order to obtain the existence of solutions to Problem (1.5), we need the following lemma.

Lemma 2.1 *For each* $\lambda > 0$,

(i) *there exist constants* $\beta_\lambda, \rho_\lambda > 0$ *such that* $J_\lambda(u) \ge \beta_\lambda$ *for* $\|u\|_{\lambda V} = \rho_\lambda$;

(ii) *there exists an* $u_0 \in W^{1,p}_{\lambda,V}(\Omega)$ *with* $u_0 \not\equiv 0$ *such that* $J_\lambda(u_0) < 0$ *for* $\|u_0\|_{\lambda V} > \rho_\lambda$.

Proof (i) Firstly, we consider the case $\lambda \geq 1$. let $\|u\|_V^p = \rho^p$, then $\rho_\lambda^p = \|u\|_{\lambda V}^p \leq \lambda\rho^p$. Using (2.1) and (2.3), we have

$$\begin{aligned} J_\lambda(u) &\geq \frac{1}{p}\|u\|_{\lambda V}^p - \frac{\lambda}{p^*}Q_M S^{-\frac{p^*}{p}}\left(\int_\Omega |\nabla u|^p \,\mathrm{d}x\right)^{\frac{p^*}{p}} - \frac{\lambda}{q}P_M C_q^q \|u\|_{\lambda V}^q \\ &\geq \frac{1}{p}\|u\|_{\lambda V}^p - \frac{\lambda}{p^*}Q_M S^{-\frac{p^*}{p}}\|u\|_{\lambda V}^{p^*} - \frac{\lambda}{q}P_M C_q^q \|u\|_{\lambda V}^q \\ &\geq \rho_\lambda^p\left(\frac{1}{p} - \frac{\lambda^{\frac{p^*}{p}}}{p^*}Q_M S^{-\frac{p^*}{p}}\rho^{p^*-p} - \frac{\lambda^{\frac{q}{p}}}{q}P_M C_q^q \rho^{q-p}\right). \end{aligned}$$

Since $p < q < p^*$, taking $\rho > 0$ small enough, there exists a $\beta_\lambda > 0$ such that $J_\lambda(u) \geq \beta_\lambda$ for $\|u\|_{\lambda V} = \rho_\lambda$.

If $0 < \lambda < 1$, let $\|u\|_V^p = \rho^p$, then $\rho > \rho_\lambda > \lambda^{\frac{1}{p}}\rho$. Combining (2.2) with (2.3), we see that

$$\begin{aligned} J_\lambda(u) &\geq \frac{1}{p}\|u\|_{\lambda V}^p - \frac{\lambda}{p^*}Q_M S^{-\frac{p^*}{p}}\|u\|_V^{p^*} - \frac{\lambda}{q}P_M C_q^q \|u\|_V^q \\ &> \lambda\rho^p\left(\frac{1}{p} - \frac{1}{p^*}Q_M S^{-\frac{p^*}{p}}\rho^{p^*-p} - \frac{1}{q}P_M C_q^q \rho^{q-p}\right). \end{aligned}$$

Since $p < q < p^*$, taking $\rho > 0$ small enough, there exists a $\beta_\lambda > 0$ such that $J_\lambda(u) \geq \beta_\lambda$ for $\|u\|_{\lambda V} = \rho_\lambda$.

(ii) For $u \in W_{\lambda,V}^{1,p}(\Omega)$ and $u \not\equiv 0$, we define

$$J_\lambda(tu) = \frac{t^p}{p}\|u\|_{\lambda V}^p - \frac{t^{p^*}}{p^*}\lambda\int_\Omega Q(x)|u|^{p^*}\,\mathrm{d}x - \frac{t^q}{q}\lambda\int_\Omega P(x)|u|^q\,\mathrm{d}x, \quad t > 0,$$

it follows from $\lim_{t\to+\infty} J_\lambda(tu) = -\infty$ that there exists a $t_0 > 0$ such that $\|t_0 u\|_{\lambda V} > \rho_\lambda$ and $J_\lambda(t_0 u) < 0$. Letting $u_0 = t_0 u$, then condition (ii) holds. The proof of Lemma 2.1 is completed. □

Define

$$c = \inf_{h\in\Gamma}\sup_{t\in[0,1]} J_\lambda\big(h(t)\big),$$

where $\Gamma = \{h \in C([0,1], W_{\lambda,V}^{1,p}(\Omega)) \mid h(0) = 0, h(1) = t_0 u = u_0\}$. Using Lemma 2.1, we know that the energy functional $J_\lambda(u)$ satisfies the geometry of the mountain pass lemma, then there exists a $(PS)_c$-sequence $\{u_n\} \subset W_{\lambda,V}^{1,p}(\Omega)$ such that $J_\lambda(u_n) \to c$, $J'_\lambda(u_n) \to 0$ as $n \to \infty$.

Lemma 2.2 *Assume* (A1), (A2) *hold, and* $\{u_n\}$ *be a* $(PS)_c$*-sequence at the level of* c *for* J_λ *with* $c < c^* = \min\{\frac{S^{\frac{N}{p}}}{N\lambda^{\frac{N-p}{p}}Q_M^{\frac{N-p}{p}}}, \frac{S^{\frac{N}{p}}}{2N\lambda^{\frac{N-p}{p}}Q_m^{\frac{N-p}{p}}}\}$, *then* $\{u_n\}$ *is relatively compact in* $W_{\lambda,V}^{1,p}(\Omega)$.

Proof Firstly, we prove that $\{u_n\}$ is bounded. Since $J_\lambda(u_n) \to c$, $J'_\lambda(u_n) \to 0$ as $n \to \infty$, we have

$$\begin{aligned} J_\lambda(u_n) &= \frac{1}{p}\int_\Omega \left(|\nabla u_n|^p + \lambda V(x)|u_n|^p\right)\mathrm{d}x - \frac{\lambda}{p^*}\int_\Omega Q(x)|u_n|^{p^*}\,\mathrm{d}x - \frac{\lambda}{q}\int_\Omega P(x)|u_n|^q\,\mathrm{d}x \\ &= c + o(1)\|u_n\|, \end{aligned}$$

$$\int_\Omega \left(|\nabla u_n|^p + \lambda V(x)|u_n|^p\right) \mathrm{d}x - \lambda \int_\Omega Q(x)|u_n|^{p^*} \mathrm{d}x - \lambda \int_\Omega P(x)|u_n|^q \mathrm{d}x$$
$$= o(1)\|u_n\|.$$

Combining (A1) and (A2), one has

$$\begin{aligned} c + o(1)\|u_n\| &= \left(\frac{1}{p} - \frac{1}{q}\right) \int_\Omega \left(|\nabla u_n|^p + \lambda V(x)|u_n|^p\right) \mathrm{d}x + \lambda \left(\frac{1}{q} - \frac{1}{p^*}\right) \int_\Omega Q(x)|u_n|^{p^*} \mathrm{d}x \\ &\geq \left(\frac{1}{p} - \frac{1}{q}\right) \|u_n\|_{\lambda V}^p. \end{aligned}$$

Thus, we can find that $\{u_n\}$ is bounded in $W_{\lambda,V}^{1,p}(\Omega)$.

Next, we prove that $\{u_n\}$ is relatively compact in $W_{\lambda,V}^{1,p}(\Omega)$. Since $\{u_n\}$ is bounded in $W_{\lambda,V}^{1,p}(\Omega)$, there exists a subsequence, still denoted by $\{u_n\}$ and $u \in W_{\lambda,V}^{1,p}(\Omega)$ such that

$$\begin{aligned} &u_n \rightharpoonup u \quad \text{weakly in } W_{\lambda,V}^{1,p}(\Omega), \\ &u_n \rightharpoonup u \quad \text{weakly in } L^{p^*}(\Omega), \\ &u_n \to u \quad \text{strongly in } L^q(\Omega), p \leq q < p^*, \\ &u_n \to u \quad \text{a.e. in } \Omega. \end{aligned}$$

By the Lions concentration-compactness principle [38], there exists at most set J, a set of different points $\{x_j\}_{j\in J} \subset \overline{\Omega}$, sets of nonnegative real numbers $\{\mu_j\}_{j\in J}$, $\{\nu_j\}_{j\in J}$ such that

$$\begin{aligned} &|\nabla u_n|^p \rightharpoonup \mathrm{d}\mu \geq |\nabla u|^p + \sum_{j\in J} \mu_j \delta_{x_j}, \\ &|u_n|^{p^*} \rightharpoonup \mathrm{d}\nu = |u|^{p^*} + \sum_{j\in J} \nu_j \delta_{x_j}, \end{aligned} \tag{2.4}$$

where δ_x is the Dirac mass at x, and the constants μ_j, ν_j satisfying

$$S\nu_j^{\frac{p}{p^*}} \leq \mu_j, \quad \text{where } x_j \in \Omega, \tag{2.5}$$

$$\frac{S}{2^{\frac{p}{N}}} \nu_j^{\frac{p}{p^*}} \leq \mu_j, \quad \text{where } x_j \in \partial\Omega. \tag{2.6}$$

Next, we prove $\mu_j = 0$ and $\nu_j = 0$, where $j \in J$. In fact, choosing $\varepsilon > 0$ sufficiently small such that $B_\varepsilon(x_i) \cap B_\varepsilon(x_j) = \varnothing$ for $i \neq j$, $i,j \in J$. Let $\phi_\varepsilon^j(x)$ be a smooth cut off function centered at x_j such that

$$0 \leq \phi_\varepsilon^j(x) \leq 1 \quad \text{for } |x - x_j| < \varepsilon, \quad \phi_\varepsilon^j(x) = \begin{cases} 1, |x - x_j| \leq \frac{\varepsilon}{2}, \\ 0, |x - x_j| \geq \varepsilon, \end{cases} \quad \text{and} \quad \left|\nabla \phi_\varepsilon^j\right| \leq \frac{4}{\varepsilon}.$$

Noting that

$$\begin{aligned} &\left\langle J_\lambda'(u_n), u_n \phi_\varepsilon^j(x)\right\rangle \\ &\quad = \int_\Omega |\nabla u_n|^p \phi_\varepsilon^j(x) \mathrm{d}x + \int_\Omega |\nabla u_n|^{p-2} \nabla u_n \nabla \phi_\varepsilon^j(x) u_n \mathrm{d}x \end{aligned}$$

$$
\begin{aligned}
&+\lambda \int_{\Omega} V(x)|u_n|^p \phi_\varepsilon^j(x)\,\mathrm{d}x - \lambda \int_{\Omega} Q(x)|u_n|^{p^*} \phi_\varepsilon^j(x)\,\mathrm{d}x \\
&- \lambda \int_{\Omega} P(x)|u_n|^q \phi_\varepsilon^j(x)\,\mathrm{d}x,
\end{aligned}
$$

and by (2.4), we have

$$
\begin{aligned}
&\lim_{\varepsilon\to 0}\lim_{n\to\infty} \int_{\Omega} |\nabla u_n|^p \phi_\varepsilon^j(x)\,\mathrm{d}x \geq \lim_{\varepsilon\to 0}\left[\int_{\Omega} |\nabla u|^p \phi_\varepsilon^j(x)\,\mathrm{d}x + \int_{\Omega} \sum_{j\in J} \mu_j \delta_{x_j} \phi_\varepsilon^j(x)\,\mathrm{d}x\right] \geq \mu_j, \\
&\lim_{\varepsilon\to 0}\lim_{n\to\infty} \int_{\Omega} |\nabla u_n|^{p-2}\nabla u_n \nabla \phi_\varepsilon^j(x) u_n\,\mathrm{d}x = 0, \\
&\lim_{\varepsilon\to 0}\lim_{n\to\infty} \int_{\Omega} V(x)|u_n|^p \phi_\varepsilon^j(x)\,\mathrm{d}x = 0, \\
&\lim_{\varepsilon\to 0}\lim_{n\to\infty} \int_{\Omega} Q(x)|u_n|^{p^*} \phi_\varepsilon^j(x)\,\mathrm{d}x = Q(x_j)\nu_j, \\
&\lim_{\varepsilon\to 0}\lim_{n\to\infty} \int_{\Omega} P(x)|u_n|^q \phi_\varepsilon^j(x)\,\mathrm{d}x = 0.
\end{aligned}
$$

Thus,

$$
0 = \lim_{\varepsilon\to 0}\lim_{n\to\infty} \left\langle J_\lambda'(u_n), u_n\phi_\varepsilon^j(x)\right\rangle \geq \mu_j - \lambda Q(x_j)\nu_j.
$$

If $\nu_j \neq 0$, by (2.5) and (2.6), we find that

$$
\begin{aligned}
&\nu_j \geq \frac{S^{\frac{N}{p}}}{\lambda^{\frac{N}{p}} Q^{\frac{N}{p}}(x_j)}, \quad x_j \in \Omega, \\
&\nu_j \geq \frac{S^{\frac{N}{p}}}{2\lambda^{\frac{N}{p}} Q^{\frac{N}{p}}(x_j)}, \quad x_j \in \partial\Omega.
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
c &= \lim_{n\to\infty}\left(J_\lambda(u_n) - \frac{1}{p}\left\langle J_\lambda'(u_n), u_n\right\rangle\right) \\
&= \left(\frac{1}{p} - \frac{1}{p^*}\right)\lambda \int_{\Omega} Q(x)|u|^{p^*}\,\mathrm{d}x + \left(\frac{1}{p} - \frac{1}{q}\right)\lambda \int_{\Omega} P(x)|u|^q\,\mathrm{d}x + \left(\frac{1}{p} - \frac{1}{p^*}\right)\lambda \sum_{j\in J} Q(x_j)\nu_j \\
&\geq \frac{1}{N}\lambda \sum_{j\in J} Q(x_j)\nu_j,
\end{aligned}
$$

consequently,

$$
\begin{aligned}
&c \geq \frac{1}{N}\lambda Q(x_j)\nu_j \geq \frac{S^{\frac{N}{p}}}{N\lambda^{\frac{N-p}{p}} Q_M^{\frac{N-p}{p}}}, \quad x_j \in \Omega, \\
&c \geq \frac{1}{N}\lambda Q(x_j)\nu_j \geq \frac{S^{\frac{N}{p}}}{2N\lambda^{\frac{N-p}{p}} Q_m^{\frac{N-p}{p}}}, \quad x_j \in \partial\Omega,
\end{aligned}
$$

which is a contradiction. Hence, $\mu_j = 0$, $\nu_j = 0$ and we find that $u_n \to u$ strongly in $L^{p^*}(\Omega)$.

Now, we prove that $u_n \to u$ strongly in $W^{1,p}_{\lambda,V}(\Omega)$. We have

$$\begin{aligned}
&\langle J'_\lambda(u_n) - J'_\lambda(u), u_n - u\rangle \\
&\quad = \|u_n\|^p_{\lambda,V} + \|u\|^p_{\lambda,V} - \int_\Omega \bigl(|\nabla u_n|^{p-2}\nabla u_n \nabla u + \lambda V(x)|u_n|^{p-2}u_n u\bigr)\,\mathrm{d}x \\
&\qquad - \int_\Omega \bigl(|\nabla u|^{p-2}\nabla u \nabla u_n + \lambda V(x)|u|^{p-2}u u_n\bigr)\,\mathrm{d}x - I - II,
\end{aligned}$$

where

$$\begin{aligned}
I &= \lambda \int_\Omega Q(x)\bigl(|u_n|^{p^*-2}u_n - |u|^{p^*-2}u\bigr)(u_n - u)\,\mathrm{d}x, \\
II &= \lambda \int_\Omega P(x)\bigl(|u_n|^{q-2}u_n - |u|^{q-2}u\bigr)(u_n - u)\,\mathrm{d}x.
\end{aligned}$$

By the Hölder inequality and Jensen's inequality

$$(a+b)^\alpha (c+d)^{1-\alpha} \geq a^\alpha c^{1-\alpha} + b^\alpha d^{1-\alpha},$$

where $\alpha \in (0,1)$, $a > 0$, $b > 0$, $c > 0$, $d > 0$, we have

$$\begin{aligned}
&\int_\Omega \bigl(|\nabla u_n|^{p-2}\nabla u_n \nabla u + \lambda V(x)|u_n|^{p-2}u_n u\bigr)\,\mathrm{d}x \\
&\quad \leq \left(\int_\Omega |\nabla u_n|^p\,\mathrm{d}x\right)^{\frac{p-1}{p}} \left(\int_\Omega |\nabla u|^p\,\mathrm{d}x\right)^{\frac{1}{p}} \\
&\qquad + \left(\lambda \int_\Omega V(x)|u_n|^p\,\mathrm{d}x\right)^{\frac{p-1}{p}} \left(\lambda \int_\Omega V(x)|u|^p\,\mathrm{d}x\right)^{\frac{1}{p}} \\
&\quad \leq \left(\int_\Omega |\nabla u_n|^p + \lambda V(x)|u_n|^p\,\mathrm{d}x\right)^{\frac{p-1}{p}} \left(\int_\Omega |\nabla u|^p + \lambda V(x)|u|^p\,\mathrm{d}x\right)^{\frac{1}{p}} = \|u_n\|^{p-1}_{\lambda V}\|u\|_{\lambda V}.
\end{aligned}$$

Similarly, we get

$$\begin{aligned}
&\int_\Omega \bigl(|\nabla u|^{p-2}\nabla u \nabla u_n + \lambda V(x)|u|^{p-2}u u_n\bigr)\,\mathrm{d}x \leq \|u\|^{p-1}_{\lambda V}\|u_n\|_{\lambda V}, \\
&|I| \leq \lambda Q_M\left[\int_\Omega |u_n|^{p^*-1}|u_n - u|\,\mathrm{d}x + \int_\Omega |u|^{p^*-1}|u_n - u|\,\mathrm{d}x\right] \\
&\quad \leq \lambda Q_M\left(\int_\Omega |u_n|^{p^*}\,\mathrm{d}x\right)^{\frac{p^*-1}{p^*}} \left(\int_\Omega |u_n - u|^{p^*}\,\mathrm{d}x\right)^{\frac{1}{p^*}} \\
&\qquad + \lambda Q_M\left(\int_\Omega |u|^{p^*}\,\mathrm{d}x\right)^{\frac{p^*-1}{p^*}} \left(\int_\Omega |u_n - u|^{p^*}\,\mathrm{d}x\right)^{\frac{1}{p^*}}, \\
&|II| \leq \lambda P_M\left[\int_\Omega |u_n|^{q-1}|u_n - u|\,\mathrm{d}x + \int_\Omega |u|^{q-1}|u_n - u|\,\mathrm{d}x\right] \\
&\quad \leq \lambda P_M\left(\int_\Omega |u_n|^q\,\mathrm{d}x\right)^{\frac{q-1}{q}} \left(\int_\Omega |u_n - u|^q\,\mathrm{d}x\right)^{\frac{1}{q}} \\
&\qquad + \lambda P_M\left(\int_\Omega |u|^q\,\mathrm{d}x\right)^{\frac{q-1}{q}} \left(\int_\Omega |u_n - u|^q\,\mathrm{d}x\right)^{\frac{1}{q}}.
\end{aligned}$$

We have

$$0 = \lim_{n\to\infty}\langle J_\lambda'(u_n) - J_\lambda'(u), u_n - u\rangle \geq \lim_{n\to\infty}\left(\|u_n\|_{\lambda V}^{p-1} - \|u\|_{\lambda V}^{p-1}\right)\left(\|u_n\|_{\lambda V} - \|u\|_{\lambda V}\right) \geq 0.$$

Hence, $u_n \to u$ strongly in $W^{1,p}_{\lambda,V}(\Omega)$. □

Since $0 \in \partial\Omega$ and $\partial\Omega \in C^2$, the boundary $\partial\Omega$ near the origin can be represented $x_N = h(x') = \frac{1}{2}\sum_{i=1}^{N-1}\lambda_i x_i^2 + o(|x'|^2)$, where $x' = (x_1, x_2, \ldots, x_{N-1}) \in D(0,\delta) = B(0,\delta) \cap \{x_N = 0\}$, λ_i $(i = 1, 2, \ldots, N-1)$ are the principal curvatures of $\partial\Omega$ at 0 and the mean curvatures $H(0) = \frac{1}{N-1}\sum_{i=1}^{N-1}\lambda_i > 0$. Then the following lemma holds.

Lemma 2.3 ([22])

(1) *For $N > 2p-1$ and $\varepsilon > 0$ small enough,*

$$\int_\Omega |\nabla u_\varepsilon|^p \,\mathrm{d}x = \int_{R^N_+} |\nabla u_\varepsilon|^p \,\mathrm{d}x - K_1(\varepsilon) + o\left(\varepsilon^{\frac{p-1}{p}}\right),$$
$$\int_\Omega |u_\varepsilon|^{p^*} \,\mathrm{d}x = \int_{R^N_+} |u_\varepsilon|^{p^*} \,\mathrm{d}x - K_2(\varepsilon) + o\left(\varepsilon^{\frac{p-1}{p}}\right),$$

where $K_1(\varepsilon)$, $K_2(\varepsilon)$ satisfy

$$\lim_{\varepsilon\to 0}\varepsilon^{-\frac{p-1}{p}}K_1(\varepsilon) = \frac{1}{2}H(0)C_{Np}^p\left(\frac{N-p}{p-1}\right)^p\int_{R^{N-1}}\left(1 + |x'|^{\frac{p}{p-1}}\right)^{-N}|x'|^{\frac{3p-2}{p-1}}\,\mathrm{d}x' = K_1,$$
$$\lim_{\varepsilon\to 0}\varepsilon^{-\frac{p-1}{p}}K_2(\varepsilon) = \frac{1}{2}H(0)C_{Np}^{p^*}\int_{R^{N-1}}\left(1 + |x'|^{\frac{p}{p-1}}\right)^{-N}|x'|^2\,\mathrm{d}x' = K_2.$$

(2)

$$\int_\Omega |u_\varepsilon|^p \,\mathrm{d}x = \begin{cases} O(\varepsilon^{\frac{N-p}{p}}), & N < p^2, \\ O(\varepsilon^{\frac{N-p}{p}}|\ln\varepsilon|), & N = p^2, \\ O(\varepsilon^{p-1}), & N > p^2. \end{cases}$$

(3)

$$\int_\Omega |u_\varepsilon|^q \,\mathrm{d}x = \begin{cases} O(\varepsilon^{\frac{q(N-p)}{p^2}}), & q < \frac{N(p-1)}{N-p}, \\ O(\varepsilon^{\frac{q(N-p)}{p^2}}|\ln\varepsilon|), & q = \frac{N(p-1)}{N-p}, \\ O(\varepsilon^{\frac{(p-1)(Np-q(N-p))}{p^2}}), & q > \frac{N(p-1)}{N-p}. \end{cases}$$

3 Proof of main results

Let $\varphi(x) \in C_0^\infty(R^N)$ be a smooth cut off function such that

$$0 \leq \varphi(x) \leq 1, \quad \frac{\delta}{2} \leq |x| \leq \delta;$$
$$\varphi(x) = 1, \quad |x| < \frac{\delta}{2};$$
$$\varphi(x) = 0, \quad |x| > \delta.$$

Define $\omega_\varepsilon = \varphi u_\varepsilon$, then we have the following lemma about ω_ε.

Lemma 3.1 *Suppose $N \geq 2p$, $0 \in \partial\Omega$. If the function $V(x)$ satisfies $\int_{\Omega\cap B(0,\delta)} V^{r'}\,\mathrm{d}x < \infty$, then*

$$\int_{\Omega\cap B(0,\delta)} V\omega_\varepsilon^p\,\mathrm{d}x = O\bigl(\varepsilon^{\frac{N-p}{p}+p-N+\frac{N(p-1)}{pr}}\bigr),$$

where $\frac{1}{r}+\frac{1}{r'}=1$, $1<r<\frac{N(p-1)}{Np+2p-N-p^2-1}$.

Proof According to the Hölder inequality and the definition of ω_ε, we have

$$\begin{aligned}
\int_{\Omega\cap B(0,\delta)} V\omega_\varepsilon^p\,\mathrm{d}x &\leq \biggl(\int_{\Omega\cap B(0,\delta)} V^{r'}\,\mathrm{d}x\biggr)^{\frac{1}{r'}}\biggl(\int_{\Omega\cap B(0,\delta)} \omega_\varepsilon^{pr}\,\mathrm{d}x\biggr)^{\frac{1}{r}} \\
&\leq \varepsilon^{\frac{(N-p)}{p}} C_{Np}^p\biggl(\int_{\Omega\cap B(0,\delta)} V^{r'}\,\mathrm{d}x\biggr)^{\frac{1}{r'}}\biggl(\int_{B(0,\delta)} \bigl(\varepsilon+|x|^{\frac{p}{p-1}}\bigr)^{r(p-N)}\,\mathrm{d}x\biggr)^{\frac{1}{r}} \\
&= \varepsilon^{\frac{N-p}{p}+p-N+\frac{N(p-1)}{pr}} C_{Np}^p\biggl(\int_{\Omega\cap B(0,\delta)} V^{r'}\,\mathrm{d}x\biggr)^{\frac{1}{r'}} \\
&\quad\times\biggl(\int_{B(0,\delta\varepsilon^{-\frac{p-1}{p}})} \bigl(1+|x|^{\frac{p}{p-1}}\bigr)^{r(p-N)}\,\mathrm{d}x\biggr)^{\frac{1}{r}}.
\end{aligned}$$

Noting that $\frac{N(p-1)}{(N-p)p} \leq 1 < r$, a series of computations yield

$$\int_{\Omega\cap B(0,\delta)} V\omega_\varepsilon^p\,\mathrm{d}x = O\bigl(\varepsilon^{\frac{N-p}{p}+p-N+\frac{N(p-1)}{pr}}\bigr).$$

□

Lemma 3.2 *Suppose that* (A1), (A2) *hold and* $0 \in \partial\Omega$, $H(0) > 0$, $Q_m = Q(0)$. *If the functions* $Q(x)$, $V(x)$ *satisfy the conditions* (A3), (A4), *then there exists a nonnegative function* $v \in W_{\lambda,V}^{1,p}(\Omega)$, $v \not\equiv 0$, *such that*

$$\sup_{t\geq 0} J_\lambda(tv) < c^* \tag{3.1}$$

for each $\lambda > 0$, $N \geq 2p$.

Proof We divide the proof into three steps.

(i) We consider the functional

$$\begin{aligned}
g(t) &= J_\lambda(t\omega_\varepsilon) \\
&= \frac{t^p}{p}\int_\Omega \bigl(|\nabla\omega_\varepsilon|^p + \lambda V(x)|\omega_\varepsilon|^p\bigr)\,\mathrm{d}x - \frac{t^{p*}}{p^*}\lambda\int_\Omega Q(x)|\omega_\varepsilon|^{p^*}\,\mathrm{d}x \\
&\quad - \frac{t^q}{q}\lambda\int_\Omega P(x)|\omega_\varepsilon|^q\,\mathrm{d}x, \quad t > 0.
\end{aligned}$$

Noting that $\lim_{t\to\infty} g(t) = -\infty$, $g(0) = 0$, $g(t) > 0$ for $t \to 0^+$, we know that there exists a $t_\varepsilon > 0$ such that $\sup_{t>0} g(t)$ is attained for t_ε and t_ε is uniformly bounded for $\varepsilon > 0$ sufficiently small. Thus,

$$g(t_\varepsilon) = \sup_{t\geq 0} J_\lambda(t\omega_\varepsilon)$$

$$\begin{aligned}
&\leq \sup_{t\geq 0}\biggl[\frac{t^p}{p}\int_\Omega |\nabla\omega_\varepsilon|^p\,\mathrm{d}x - \frac{t^{p*}}{p^*}\lambda\int_\Omega Q(x)|\omega_\varepsilon|^{p^*}\,\mathrm{d}x\biggr] \\
&\quad + \frac{t_\varepsilon^p}{p}\int_\Omega \lambda V(x)|\omega_\varepsilon|^p\,\mathrm{d}x - \frac{t_\varepsilon^q}{q}\lambda\int_\Omega P(x)|\omega_\varepsilon|^q\,\mathrm{d}x \\
&= \frac{1}{N}\biggl[\frac{\int_\Omega |\nabla\omega_\varepsilon|^p\,\mathrm{d}x}{(\lambda\int_\Omega Q(x)|\omega_\varepsilon|^{p^*}\,\mathrm{d}x)^{\frac{N-p}{N}}}\biggr]^{\frac{N}{p}} + \frac{t_\varepsilon^p}{p}\int_\Omega \lambda V(x)|\omega_\varepsilon|^p\,\mathrm{d}x \\
&\quad - \frac{t_\varepsilon^q}{q}\lambda\int_\Omega P(x)|\omega_\varepsilon|^q\,\mathrm{d}x. \qquad (3.2)
\end{aligned}$$

(ii) When $\varepsilon > 0$ is sufficiently small, we have

$$\begin{aligned}
&\int_\Omega Q(x)|\omega_\varepsilon|^{p^*}\,\mathrm{d}x = Q_m\int_\Omega |u_\varepsilon|^{p^*}\,\mathrm{d}x + o\bigl(\varepsilon^{\frac{p-1}{p}}\bigr), \\
&\int_\Omega |\nabla\omega_\varepsilon|^p\,\mathrm{d}x \leq \int_\Omega |\nabla u_\varepsilon|^p\,\mathrm{d}x + o\bigl(\varepsilon^{\frac{p-1}{p}}\bigr), \\
&\int_\Omega |\omega_\varepsilon|^q\,\mathrm{d}x = \int_\Omega |u_\varepsilon|^q\,\mathrm{d}x + o\bigl(\varepsilon^{\frac{p-1}{p}}\bigr), \\
&\int_\Omega |\omega_\varepsilon|^{p^*}\,\mathrm{d}x = \int_\Omega |u_\varepsilon|^{p^*}\,\mathrm{d}x + o\bigl(\varepsilon^{\frac{p-1}{p}}\bigr).
\end{aligned} \qquad (3.3)$$

We firstly prove the first formula. Since $|Q(x) - Q(0)| = o(|x|^\alpha)$ for $x \to 0$, there exists a $0 < \delta_0 \leq \delta$ such that $|Q(x) - Q(0)| \leq C|x|^\alpha$ for $|x| < \delta_0$, where $C > 0$ is constant. Moreover

$$\begin{aligned}
&\int_\Omega \bigl|Q(x) - Q(0)\bigr||\omega_\varepsilon|^{p^*}\,\mathrm{d}x \\
&\quad\leq \int_{\Omega\cap|x|\leq\delta_0} \bigl|Q(x) - Q(0)\bigr||\omega_\varepsilon|^{p^*}\,\mathrm{d}x + \int_{\Omega\cap|x|\geq\delta_0} \bigl|Q(x) - Q(0)\bigr||\omega_\varepsilon|^{p^*}\,\mathrm{d}x \\
&\quad\leq C\int_{|x|\leq\delta_0} |x|^\alpha|\omega_\varepsilon|^{p^*}\,\mathrm{d}x + 2Q_M\int_{\Omega\cap|x|\geq\delta_0} |\omega_\varepsilon|^{p^*}\,\mathrm{d}x \\
&\quad\leq CC_{Np}^{p^*}\varepsilon^{\frac{(p-1)\alpha}{p}}\int_{|x|\leq\frac{\delta_0}{\varepsilon^{\frac{p-1}{p}}}} |x|^\alpha\bigl(1 + |x|^{\frac{p}{p-1}}\bigr)^{-N}\,\mathrm{d}x \\
&\qquad + 2Q_M C_{Np}^{p^*}\varepsilon^{\frac{N}{p}}\int_{\Omega\cap|x|\geq\delta_0} \bigl(\varepsilon + |x|^{\frac{p}{p-1}}\bigr)^{-N}\,\mathrm{d}x \\
&\quad= O\bigl(\varepsilon^{\frac{(p-1)\alpha}{p}}\bigr) + O\bigl(\varepsilon^{\frac{N}{p}}\bigr).
\end{aligned}$$

Since $N \geq 2p$, $1 < \alpha < \frac{N}{p-1}$, $\int_\Omega |Q(x) - Q(0)||\omega_\varepsilon|^{p^*}\,\mathrm{d}x = o(\varepsilon^{\frac{p-1}{p}})$, which implies

$$\begin{aligned}
&\int_\Omega Q(x)|\omega_\varepsilon|^{p^*}\,\mathrm{d}x \\
&\quad= Q_m\int_\Omega |\omega_\varepsilon|^{p^*}\,\mathrm{d}x + \int_\Omega \bigl(Q(x) - Q(0)\bigr)|\omega_\varepsilon|^{p^*}\,\mathrm{d}x \\
&\quad= Q_m\int_\Omega |\omega_\varepsilon|^{p^*}\,\mathrm{d}x + o\bigl(\varepsilon^{\frac{p-1}{p}}\bigr).
\end{aligned}$$

Similarly, we can evaluate the rest of formulas and omit the details here.

(iii) $\sup_{t\geq 0} J_\lambda(t\omega_\varepsilon) < c^*$.

Combining (3.3) with Lemma 2.3, one has

$$\int_\Omega |\nabla\omega_\varepsilon|^p \, dx \leq M_1\bigl(1 - M_1^{-1}K_1(\varepsilon) + o\bigl(\varepsilon^{\frac{p-1}{p}}\bigr)\bigr),$$
$$\int_\Omega |\omega_\varepsilon|^{p^*} \, dx = M_2\bigl(1 - M_2^{-1}K_2(\varepsilon) + o\bigl(\varepsilon^{\frac{p-1}{p}}\bigr)\bigr),$$

where $M_1 = \frac{1}{2}\int_{R^N} |\nabla u_\varepsilon|^p \, dx$, $M_2 = \frac{1}{2}\int_{R^N} |u_\varepsilon|^{p^*} \, dx$. Then, using (3.2), (3.3), Lemma 2.3 and Lemma 3.1, we see that

$$\sup_{t\geq 0} J_\lambda(t\omega_\varepsilon) \leq \frac{S^{\frac{N}{p}}}{2N(\lambda Q_m)^{\frac{N-p}{p}}}\left[1 + \frac{N-p}{p}M_2^{-1}K_2(\varepsilon) - \frac{N}{p}M_1^{-1}K_1(\varepsilon) + o\bigl(\varepsilon^{\frac{p-1}{p}}\bigr)\right] + O\bigl(\varepsilon^{\frac{N-p}{p}+p-N+\frac{(p-1)N}{pr}}\bigr).$$

Next, we claim that

$$\lim_{\varepsilon\to 0} \varepsilon^{\frac{p-1}{p}}\left[\frac{N-p}{p}M_2^{-1}K_2(\varepsilon) - \frac{N}{p}M_1^{-1}K_1(\varepsilon)\right] < 0 \tag{3.4}$$

for $\varepsilon > 0$ small enough, which implies (3.1) holds. According to $\lim_{\varepsilon\to 0} \varepsilon^{\frac{p-1}{p}} K_1(\varepsilon) = K_1$, $\lim_{\varepsilon\to 0} \varepsilon^{\frac{p-1}{p}} K_2(\varepsilon) = K_2$, we know that (3.4) is equivalent to $\frac{K_1}{K_2} > \frac{N-p}{N}\frac{M_1}{M_2}$.

From the expressions of K_1, K_2, M_1, M_2 and u_ε, a series of computations yield

$$\frac{K_1}{K_2} = \frac{\frac{1}{2}H(0)C_{Np}^{p}(\frac{N-p}{p-1})^p \int_{R^{N-1}}(1 + |x'|^{\frac{p}{p-1}})^{-N}|x'|^{\frac{3p-2}{p-1}} \, dx'}{\frac{1}{2}H(0)C_{Np}^{p^*} \int_{R^{N-1}}(1 + |x'|^{\frac{p}{p-1}})^{-N}|x'|^{2} \, dx'}$$
$$= C_{Np}^{p-p^*}\left(\frac{N-p}{p-1}\right)^p \frac{\int_0^\infty (1+r^2)^{-N} r^{\frac{2Np+3p-2N-2}{p}} \, dr}{\int_0^\infty (1+r^2)^{-N} r^{\frac{2Np+p-2N-2}{p}} \, dr},$$
$$\frac{N-p}{N}\frac{M_1}{M_2} = \frac{N-p}{N}\frac{\int_{R^N} |\nabla u_\varepsilon|^p \, dx}{\int_{R^N} |u_\varepsilon|^{p^*} \, dx}$$
$$= \frac{N-p}{N}C_{Np}^{p-p^*}\left(\frac{N-p}{p-1}\right)^p \frac{\int_0^\infty (1+r^2)^{-N} r^{\frac{2Np+p-2N}{p}} \, dr}{\int_0^\infty (1+r^2)^{-N} r^{\frac{2Np-p-2N}{p}} \, dr}.$$

Integrating by parts, we have

$$\int_0^\infty \frac{r^\beta}{(1+r^2)^n} \, dr = \frac{\beta-1}{2n-\beta-1}\int_0^\infty \frac{r^{\beta-2}}{(1+r^2)^n} \, dr \quad \text{for } 2 \leq \beta < 2n-1.$$

Then

$$\frac{K_1}{K_2} = C_{Np}^{p-p^*}\left(\frac{N-p}{p-1}\right)^p \frac{(p-1)(N+1)}{N-2p+1},$$
$$\frac{N-p}{N}\frac{M_1}{M_2} = C_{Np}^{p-p^*}\left(\frac{N-p}{p-1}\right)^p (p-1).$$

This implies $\frac{K_1}{K_2} > \frac{N-p}{N}\frac{M_1}{M_2}$. Thus

$$\sup_{t\geq 0} J_\lambda(t\omega_\varepsilon) < \frac{S^{\frac{N}{p}}}{2N(\lambda Q_m)^{\frac{N-p}{p}}} = c^*.$$

The proof of Lemma 3.2 is complete. □

Proof of Theorem 1.1 Applying Lemma 2.1 and Lemma 3.2, we obtain

$$c = \inf_{h\in\Gamma}\max_{t\in[0,1]} J_\lambda\bigl(h(t)\bigr) \leq \sup_{t\geq 0} J_\lambda(t\omega_\varepsilon) < c^*.$$

From Lemma 2.2 and the mountain pass theorem, we know that there exists at least one nontrivial solution to Problem (1.5). Since $J_\lambda(u) \geq J_\lambda(|u|)$, Problem (1.5) has at least one nonnegative nontrivial solution. The proof of Theorem 1.1 is complete. □

Proof of Theorem 1.2 Consider the following function:

$$\begin{aligned} h(t) &= J_\lambda(tu) \\ &= \frac{t^p}{p}\int_\Omega \bigl(|\nabla u|^p + \lambda V(x)|u|^p\bigr)\,\mathrm{d}x - \frac{t^{p*}}{p^*}\lambda\int_\Omega Q(x)|u|^{p^*}\,\mathrm{d}x \\ &\quad - \frac{t^q}{q}\lambda\int_\Omega P(x)|u|^q\,\mathrm{d}x, \quad t>0. \end{aligned}$$

Since $V \in L^1(\Omega)$, we find that

$$\begin{aligned} \sup_{t\geq 0} h(t) &= \sup_{t\geq 0}\left[\frac{t^p}{p}\int_\Omega \lambda V(x)|A|^p\,\mathrm{d}x - \frac{t^{p*}}{p^*}\lambda\int_\Omega Q(x)|A|^{p^*}\,\mathrm{d}x - \frac{t^q}{q}\lambda\int_\Omega P(x)|A|^q\,\mathrm{d}x\right] \\ &\leq \frac{\lambda}{N}\left[\frac{\int_\Omega V(x)\,\mathrm{d}x}{(\int_\Omega Q(x)\,\mathrm{d}x)^{\frac{N-p}{N}}}\right]^{\frac{N}{p}} \quad \text{for } u = A. \end{aligned}$$

Then $\sup_{t\geq 0} J_\lambda(tA) < c^*$ for $\lambda < \frac{S(\int_\Omega Q(x)\,\mathrm{d}x)^{\frac{N-p}{N}}}{Q_M^{\frac{N-p}{N}}\int_\Omega V(x)\,\mathrm{d}x}$.

Similarly,

$$\begin{aligned} J_\lambda(tA) &= \sup_{t\geq 0} h(t) \\ &= \sup_{t\geq 0}\left[\frac{t^p}{p}\int_\Omega \lambda V(x)|A|^p\,\mathrm{d}x - \frac{t^{p*}}{p^*}\lambda\int_\Omega Q(x)|A|^{p^*}\,\mathrm{d}x - \frac{t^q}{q}\lambda\int_\Omega P(x)|A|^q\,\mathrm{d}x\right] \\ &\leq \lambda\left(\frac{q-p}{pq}\right)\frac{(\int_\Omega V(x)\,\mathrm{d}x)^{\frac{q}{q-p}}}{(\int_\Omega P(x)\,\mathrm{d}x)^{\frac{p}{q-p}}} < c^* \end{aligned}$$

for $\lambda < (\frac{pq}{q-p})^{\frac{p}{N}}\frac{S}{N^{\frac{p}{N}}Q_M^{\frac{N-p}{N}}}\frac{(\int_\Omega P(x)\,\mathrm{d}x)^{\frac{p^2}{N(q-p)}}}{(\int_\Omega V(x)\,\mathrm{d}x)^{\frac{pq}{N(q-p)}}}$.

Set

$$\lambda_* = \max\left\{\frac{S(\int_\Omega Q(x)\,dx)^{\frac{N-p}{N}}}{Q_M^{\frac{N-p}{N}}\int_\Omega V(x)\,dx}, \left(\frac{pq}{q-p}\right)^{\frac{p}{N}}\frac{S}{N^{\frac{p}{N}}Q_M^{\frac{N-p}{N}}}\frac{(\int_\Omega P(x)\,dx)^{\frac{p^2}{N(q-p)}}}{(\int_\Omega V(x)\,dx)^{\frac{pq}{N(q-p)}}}\right\},$$

then we have $\sup_{t\geq 0} J_\lambda(tA) < c^*$ for $0 < \lambda < \lambda_*$. Similar to the proof of Theorem 1.1, Problem (1.5) has at least one nonnegative nontrivial solution. The proof of Theorem 1.2 is complete. □

Proof of Theorem 1.3 Define

$$K = \inf_{u\in W_0^{1,p}(B(x_0,\delta))\setminus\{0\}} \frac{\int_{B(x_0,\delta)} |\nabla u|^p\,dx}{(\int_{B(x_0,\delta)} |u|^q\,dx)^{\frac{p}{q}}}.$$

Since $p < q < p^*$, as is well known, there exists a function $w \in W_0^{1,p}(B(x_0,\delta))$ such that

$$K = \frac{\int_{B(x_0,\delta)} |\nabla w|^p\,dx}{(\int_{B(x_0,\delta)} |w|^q\,dx)^{\frac{p}{q}}}.$$

Thus,

$$\begin{aligned}\sup_{t\geq 0} J_\lambda(tw) &\leq \sup_{t\geq 0}\left[\frac{t^p}{p}\int_{B(x_0,\delta)}\left(|\nabla w|^p + \lambda V(x)|w|^p\right)dx - \frac{t^q}{q}\lambda\int_{B(x_0,\delta)} P(x)|w|^q\,dx\right] \\ &\leq \frac{q-p}{pq}\frac{(\int_{B(x_0,\delta)} |\nabla w|^p\,dx)^{\frac{q}{q-p}}}{P_m^{\frac{p}{q-p}}(\int_{B(x_0,\delta)} \lambda|w|^q\,dx)^{\frac{p}{q-p}}} \\ &= \frac{q-p}{pq}\frac{K^{\frac{q}{q-p}}}{\lambda^{\frac{p}{q-p}}P_m^{\frac{p}{q-p}}}.\end{aligned}$$

Let $\lambda^* = (\frac{N(q-p)K^{\frac{q}{q-p}}Q_M^{\frac{N-p}{p}}}{pqS^{\frac{N}{p}}P_m^{\frac{p}{q-p}}})^{\frac{p(q-p)}{Np+pq-Nq}}$, where $P_m = \min_{x\in B(x_0,\delta)} P(x)$, then $\sup_{t\geq 0} J_\lambda(tw) < c^*$ for $\lambda > \lambda^*$. Similar to the proof of Theorem 1.1, Problem (1.5) has at least one nonnegative nontrivial solution for $\lambda > \lambda^*$. The proof of Theorem 1.3 is complete. □

Proof of Theorem 1.4 Fix $n \in N$, let $\varphi_1, \varphi_2, \dots, \varphi_n \in C_0^\infty(R^N)$ be smooth functions such that $\operatorname{supp}\varphi_j \subset B(x_0,\delta)$, $j = 1, 2, \dots, n$, $\operatorname{supp}\varphi_i \cap \operatorname{supp}\varphi_j = \varnothing$, $i \neq j$.

We define $E_n = \operatorname{Span}\{\varphi_1, \varphi_2, \dots, \varphi_n\}$, Σ is the set of all symmetric and closed subsets of $W_V^{1,p}(\Omega)$, $\gamma(A)$ is the Krasnoselski genus,

$$i(A) = \min_{h\in\Gamma} \gamma\left(h(A)\right) \cap \partial B_{\beta_\lambda}), \quad A \in \Sigma,$$

where Γ is the set of all odd homomorphisms $C^1(W_V^{1,p}(\Omega), W_V^{1,p}(\Omega))$.

Set

$$c_j = \inf_{i(A)\geq j}\sup_{u\in A} J_\lambda(u), \quad j = 1, 2, \dots, n.$$

Since $i(E_n) = \dim E_n = n$ and $J_\lambda(u) \geq \beta_\lambda$ for $\|u\|_{\lambda V} = \rho_\lambda$ in Lemma 2.1, we find that

$$\beta_\lambda \leq c_1 \leq c_2 \leq \cdots \leq c_n \leq \sup_{u\in E_n} J_\lambda(u).$$

We now estimate $\sup_{u\in E_n} J_\lambda(u)$. If $u \in E_n$, one has $u = \sum_{j=1}^n \tau_j\varphi_j$ for $\tau_j \in R$. From the properties of φ_j, we obtain

$$\begin{aligned}\sup_{u\in E_n} J_\lambda(u) &= \sup_{u\in E_n}\left(\sum_{j=1}^n J_\lambda(\tau_j\varphi_j)\right)\\ &\leq \sup_{u\in E_n}\sum_{j=1}^n\left[\frac{\tau_j^p}{p}\int_{B(x_0,\delta)}\left(|\nabla\varphi_j|^p + \lambda V(x)|\varphi_j|^p\right)\mathrm{d}x - \frac{\tau_j^q}{q}\lambda\int_{B(x_0,\delta)} P(x)|\varphi_j|^q\,\mathrm{d}x\right]\\ &\leq \frac{q-p}{pq}\sum_{j=1}^n \frac{(\int_{B(x_0,\delta)}|\nabla\varphi_j|^p\,\mathrm{d}x)^{\frac{q}{q-p}}}{\lambda^{\frac{p}{q-p}}P_m^{\frac{p}{q-p}}(\int_{B(x_0,\delta)}|\varphi_j|^q\,\mathrm{d}x)^{\frac{p}{q-p}}}.\end{aligned}$$

Consequently, there exists a $\Lambda_n > 0$ such that $\sup_{u\in E_n} J_\lambda(u) < c^*$ for $\lambda > \Lambda_n$. Similar to the proof of Theorem 1.1, Problem (1.5) has at least n pairs of nonnegative nontrivial solutions. The proof of Theorem 1.4 is complete. □

4 Conclusion

In this paper, we study the following quasilinear Neumann problem with critical Sobolev exponent:

$$\begin{cases} -\varepsilon^p\Delta_p u + V(x)|u|^{p-2}u = Q(x)|u|^{p^*-2}u + P(x)|u|^{q-2}u, & x\in\Omega,\\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = 0, & x\in\partial\Omega,\end{cases}$$

where the weight functions $V(x)$ is continuous in Ω and $Q(x)$, $P(x)$ are continuous on $\overline{\Omega}$. Due to the lack of compactness of the embedding of $W^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$ and the fact that the weight function $V(x)$ may be unbounded close to the boundary $\partial\Omega$, some classical methods may not directly be applied to our problem. We introduce a suitable weighted Sobolev space and add restrictions on the weight functions $Q(x)$ and $P(x)$ to prove the corresponding functional of problem satisfies $(PS)_c$-condition in a suitable range by the Lions concentration-compactness principle, then apply the mountain pass lemma, the existence and multiplicity of nontrivial solutions are obtained.

Acknowledgements
The authors would like to thank the editor and the referees for their valuable comments and suggestions, which improved the quality of our manuscript.

Funding
This work is supported by the National Natural Science Foundation of China (11601122) and by DR Fund of Henan University of Technology (150152).

Abbreviations
Not applicable

Availability of data and materials
Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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Received: 18 March 2018 Accepted: 18 April 2018 Published online: 27 April 2018

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