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Dynamics of blow-up solutions for the Schrödinger–Choquard equation

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Abstract

In this paper, we study the dynamics of blow-up solutions for the nonlinear Schrödinger–Choquard equation

$$i\psi_t + \Delta\psi = \lambda_1|\psi|^{p_1}\psi + \lambda_2(I_\alpha * |\psi|^{p_2})|\psi|^{p_2-2}\psi.$$

We first show existence of blow-up solutions and obtain a sharp threshold mass of global existence and blow-up for this equation with $\lambda_1 > 0$, $\lambda_2 < 0$, $0 < p_1 < \frac{4}{N}$ and $p_2 = 1 + \frac{2+\alpha}{N}$. Then we obtain some dynamical properties of blow-up solutions by the corresponding ground state of this equation with $\lambda_1 = 0$.

MSC: 35Q55; 35A15

Keywords: Nonlinear Schrödinger–Choquard equation; Blow-up solutions; The dynamical properties

1 Introduction

In this paper, we will investigate the blow-up solutions of the nonlinear Schrödinger–Choquard equation

$$\begin{cases} i\psi_t + \Delta\psi = \lambda_1|\psi|^{p_1}\psi + \lambda_2(I_\alpha * |\psi|^{p_2})|\psi|^{p_2-2}\psi, \\ \psi(0,x) = \psi_0(x), \end{cases} \tag{1.1}$$

where $\psi(t,x) : [0,T^*) \times \mathbb{R}^N \to \mathbb{C}$ is a complex valued function and $0 < T^* \le \infty$, $N \ge 3$, $\psi_0 \in H^1$, $0 < p_1 < \frac{4}{N-2}$, $1 + \frac{\alpha}{N} < p_2 < 1 + \frac{2+\alpha}{N-2}$, $\lambda_1, \lambda_2 \in \mathbb{R}$, $I_\alpha : \mathbb{R}^N \to \mathbb{R}$ is the Riesz potential defined by

$$I_\alpha(x) = \frac{\Gamma(\frac{N-\alpha}{2})}{\Gamma(\frac{\alpha}{2})\pi^{N/2}2^\alpha|x|^{N-\alpha}},$$

with $\max\{0, N-4\} < \alpha < N$ and Γ is the Gamma function.

Our main motivation for studying Eq. (1.1) is the loss of scaling invariance for this equation. When $p_2 > 0$, there exists a scaling transform for the nonlinear Choquard equation,

$$i\psi_t + \Delta\psi = \lambda_2\big(I_\alpha * |\psi|^{p_2}\big)|\psi|^{p_2-2}\psi, \tag{1.2}$$

which keeps it invariant. More precisely, the map

$$\psi(t,x) \mapsto \lambda^{-\frac{\alpha+2}{2p_2-2}} \psi\left(\frac{t}{\lambda^2}, \frac{x}{\lambda}\right) \tag{1.3}$$

maps a solution to (1.2) to another solution to (1.2). When $p_2 = 1 + \frac{2+\alpha}{N}$, the scaling transform (1.3) keeps the mass invariant. Thus, the nonlinearity $(I_\alpha * |\psi|^{p_2})|\psi|^{p_2-2}\psi$ is called L^2-critical.

When $\lambda_1 = 0$ and $p_2 = 2$, Eq. (1.1) simplifies to the Hartree equation. The Cauchy problem of (1.1) has been extensively investigated in [1–16]. The local well-posedness and global existence of (1.1) have been studied in [1]. Chen and Guo [3] studied the instability of standing waves. In the L^2-critical case, Miao et al. [10] studied the dynamical properties of the blow-up solutions. The soliton dynamics has been studied in [11].

When $\lambda_1 = 0$, $0 < \alpha < N$ and $1 + \frac{\alpha}{N} < p_2 < \frac{N+\alpha}{N-2}$, under the assumption that the local well-posedness holds for (1.1), Chen and Guo [3] derived the existence of blow-up solutions and the instability of standing waves. When $0 < \alpha < N$ and $1 + \frac{\alpha}{N} < p_2 < 1 + \frac{2+\alpha}{N}$, Squassina et al. in [17] studied the soliton dynamics of (1.1) under the assumption that the solution ψ of (1.1) is in $C([0,\infty), H^2) \cap C^1((0,\infty), L^2)$. In [18], Feng and Yuan systematically studied the Cauchy problem (1.1) for general $\max\{0, N-4\} < \alpha < N$ and $2 \leq p_2 < \frac{N+\alpha}{N-2}$. More precisely, they studied the local well-posedness, global existence, the existence of blow-up solutions and the dynamics of blow-up solutions. The sharp threshold of global existence and blow-up, the instability of standing wave of (1.1) with $\lambda_1 = 0$ and a harmonic potential have been investigated in [19].

However, in the above papers, the scale invariance plays an important role in the study of the dynamics of blow-up solutions to (1.2); see [7, 10, 12, 14, 18, 20, 21]. Because there exists no scale invariance for (1.1), the study of blow-up solutions to (1.1) is a very interesting problem. On the other hand, as far as we know, the existence of blow-up solutions to (1.1) with $\lambda_1 > 0$, $\lambda_2 < 0$, $0 < p_1 < \frac{4}{N}$ and $p_2 = 1 + \frac{2+\alpha}{N}$ has not been obtained yet. Hence, in this paper, we first show the existence of blow-up solutions and obtain the sharp threshold mass $\|u\|_{L^2}$ of global existence and blow-up for (1.1), where u is a ground state solution of the elliptic equation

$$-\Delta u + u - \left(I_\alpha * |u|^p\right)|u|^{p-2}u = 0. \tag{1.4}$$

Then, for overcoming the difficulty of the loss of scale invariance, we apply the ground state solution u of (1.4) to describe the dynamical properties of blow-up solutions to (1.1), including L^2-concentration, limiting profile and blow-up rates.

This paper is organized as follows: in Sect. 2, we recall some preliminaries. In Sect. 3, we firstly show the existence of blow-up solutions to (1.1) with $\lambda_1 = 1$, $\lambda_2 = -1$, $0 < p_1 < \frac{4}{N}$ and $p_2 = 1 + \frac{2+\alpha}{N}$, and then obtain the sharp threshold mass $\|u\|_{L^2}$ of global existence and blow-up. In Sect. 4, we will consider some dynamical properties of blow-up solutions to (1.1) with $\lambda_1 = 1$, $\lambda_2 = -1$, $0 < p_1 < \frac{4}{N}$ and $p_2 = 1 + \frac{2+\alpha}{N}$. Section 5 is a concluding section.

Notation In this paper, we use the following notations. We always denote u the ground state solution of (1.4). $\Sigma := \{\psi \in H^1, x\psi \in L^2\}$ is the energy space equipped with the norm $\|\psi\|_\Sigma := \|\psi\|_{H^1} + \|x\psi\|_{L^2}$.

2 Preliminaries

In order to study the blow-up solutions to (1.1), we firstly make the following assumption about the local well-posedness of (1.1).

Assumption 1 Let $\psi_0 \in H^1$, $N \geq 3$, $0 < p_1 < \frac{4}{N-2}$ and $1 + \frac{\alpha}{N} < p_2 < 1 + \frac{2+\alpha}{N-2}$. Then there exist $T^* > 0$ and a unique maximal solution $\psi \in C([0,T^*), H^1)$. In addition, if $T^* < \infty$, then $\|\psi(t)\|_{H^1} \to \infty$ as $t \uparrow T^*$. Moreover, the solution $\psi(t)$ satisfies

$$\big\|\psi(t)\big\|_{L^2} = \|\psi_0\|_{L^2}, \tag{2.1}$$

$$E\big(\psi(t)\big) = E(\psi_0), \tag{2.2}$$

for all $0 \leq t < T^*$, where $E(\psi(t))$ is defined by

$$\begin{aligned} E\big(\psi(t)\big) := {}& \frac{1}{2}\int_{\mathbb{R}^N} \big|\nabla\psi(t,x)\big|^2\,dx + \frac{\lambda_1}{p_1+2}\int_{\mathbb{R}^N} \big|\psi(t,x)\big|^{p_1+2}\,dx \\ &+ \frac{\lambda_2}{2p_2}\int_{\mathbb{R}^N} \big(I_\alpha * |\psi|^{p_2}\big)(t,x)\big|\psi(t,x)\big|^{p_2}\,dx. \end{aligned} \tag{2.3}$$

When $0 < p_1 < \frac{4}{N-2}$ and $2 \leq p_2 < 1 + \frac{2+\alpha}{N-2}$, this assumption can easily be proved by the Strichartz estimates and a fixed point argument; see [1, 18].

By the same argument as that in [1], one can easily derive the following lemma.

Lemma 2.1 ([1]) *Let $\psi_0 \in \Sigma := \{u \in H^1, xu \in L^2\}$. Assume that the solution $\psi(t)$ to* (1.1) *exists on the interval* $[0,T^*)$. *Then $\psi(t) \in \Sigma$ for all $t \in [0,T^*)$. Moreover, let $J(t) = \int_{\mathbb{R}^N} |x\psi(t,x)|^2\,dx$, then*

$$J'(t) = -4\,\mathrm{Im}\int_{\mathbb{R}^N} \psi(t,x)x\cdot\nabla\bar{\psi}(t,x)\,dx, \tag{2.4}$$

and

$$\begin{aligned} J''(t) = {}& 8\int_{\mathbb{R}^N} \big|\nabla\psi(t,x)\big|^2\,dx + \frac{4N\lambda_1 p_1}{p_1+2}\int_{\mathbb{R}^N} \big|\psi(t,x)\big|^{p_1+2}\,dx \\ &+ \lambda_2\frac{4p_2N - 4N - 4\alpha}{p_2}\int_{\mathbb{R}^N} \big(I_\alpha * |\psi|^{p_2}\big)(t,x)\big|\psi(t,x)\big|^{p_2}\,dx. \end{aligned} \tag{2.5}$$

As a direct result of this lemma, we have the following lemma.

Lemma 2.2 *If the solution $\psi(t)$ to* (1.1) *with $\psi_0 \in \Sigma$ blows up at the finite time T^*, then there exists $C > 0$ such that for all $t \in [0,T^*)$*

$$\int_{\mathbb{R}^N} |x|^2\big|\psi(t,x)\big|^2\,dx \leq C.$$

Next, we summarize some results about the ground state of (1.4), which is very important in the study of blow-up solutions to (1.1).

Lemma 2.3 ([17, 22]) *Let $\alpha \in (0,N)$ and $1+\frac{\alpha}{N} < p < 1+\frac{2+\alpha}{N-2}$. Then* (1.4) *admits a ground state solution u in H^1. Moreover, let u_1 and u_2 be two any ground state solutions of* (1.4), *then $\|u_1\|_{L^2} = \|u_2\|_{L^2}$.*

Finally, we recall a useful result which gives the best constant in a Gagliardo–Nirenberg type inequality; see [18].

Lemma 2.4 *The best constant in the Gagliardo–Nirenberg type inequality*

$$\int_{\mathbb{R}^N} \bigl(I_\alpha * |\psi|^p\bigr)|\psi|^p\,dx \le C_{\alpha,p}\biggl(\int_{\mathbb{R}^N} |\nabla\psi|^2\,dx\biggr)^{\frac{Np-N-\alpha}{2}} \biggl(\int_{\mathbb{R}^N} |\psi|^2\,dx\biggr)^{\frac{N+\alpha-Np+2p}{2}} \tag{2.6}$$

is

$$C_{\alpha,p} = \frac{2p}{2p-Np+N+\alpha}\biggl(\frac{2p-Np+N+\alpha}{Np-N-\alpha}\biggr)^{\frac{Np-N-\alpha}{2}} \|u\|_{L^2}^{2-2p}.$$

In particular, in the L^2-critical case, i.e., $p = 1+\frac{2+\alpha}{N}$, $C_{\alpha,p} = p\|u\|_{L^2}^{2-2p}$.

3 The sharp threshold mass of global existence and blow-up

From the local well-posedness of the nonlinear Schrödinger–Choquard equation, for small initial data ψ_0, the solution $\psi(t)$ to (1.1) exists globally, and the solution $\psi(t)$ may blow up for some large initial data. Therefore, whether there are some sharp thresholds of global existence and blow-up for (1.1) is a very interesting problem. In particular, the sharp thresholds of global existence and blow-up for nonlinear Schrödinger equations are pursued strongly (see [1, 2, 19, 23–25] and the references therein).

In the following, applying the inequality (2.6) and a scaling argument, we derive the existence of blow-up solutions to (1.1) and a sharp threshold of global existence and blow-up.

Theorem 3.1 *Let $\psi_0 \in H^1$, $\lambda_1 = 1$, $\lambda_2 = -1$, $0 < p_1 < \frac{4}{N}$ and $p_2 = 1+\frac{2+\alpha}{N}$. Then we have*:

(i) *If $\|\psi_0\|_{L^2} < \|u\|_{L^2}$, then the solution $\psi(t)$ to* (1.1) *exists globally.*

(ii) *Let $\psi_0 = c\rho^{\frac{N}{2}}u(\rho x)$ and $|x|\psi_0 \in L^2$, where $|c| \ge 1$, and $\rho > 0$ and satisfies*

$$\frac{2|c|^{p_1}\|u\|_{L^{p_1+2}}^{p_1+2}}{(p_1+2)(|c|^{2p_2-2}-1)\|\nabla u\|_{L^2}^2} < \rho^{2-\frac{N}{2}p_1}. \tag{3.1}$$

Then the solution $\psi(t)$ to (1.1) *blows up in finite time.*

Remark We see from Theorem 1.2 in [18] that the critical value about the initial data for global existence of (1.1) with $\lambda_1 = 0$ and (1.1) is the same.

Proof (i) Firstly, by (2.3) and (2.6), we have

$$\begin{aligned} E(\psi_0) &= E\bigl(\psi(t)\bigr) \\ &= \frac{1}{2}\int_{\mathbb{R}^N} \bigl|\nabla\psi(t,x)\bigr|^2\,dx - \frac{1}{2p_2}\int_{\mathbb{R}^N} \bigl(I_\alpha * |\psi|^{p_2}\bigr)(t,x)\bigl|\psi(t,x)\bigr|^{p_2}\,dx \end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{p_1+2}\int_{\mathbb{R}^N}\left|\psi(t,x)\right|^{p_1+2}dx \\
&\geq \left(\frac{1}{2}-\frac{\|\psi_0\|_{L^2}^{2p_2-2}}{2\|u\|_{L^2}^{2p_2-2}}\right)\left\|\nabla\psi(t)\right\|_{L^2}^2.
\end{aligned}
$$

It follows from $\|\psi_0\|_{L^2} < \|u\|_{L^2}$ and $E(\psi_0) = E(\psi(t))$ that there exists a constant C such that $\|\nabla\psi(t)\|_{L^2} \leq C$ for all $t > 0$. Therefore, the solution $\psi(t)$ to (1.1) exists globally.

(ii) Since $|x|\psi_0 \in L^2$, $J(t) = \int_{\mathbb{R}^N} |x\psi(t,x)|^2\,dx$ is well defined. We deduce from Lemma 2.1 that

$$
J''(t) = 16E(\psi_0) - \frac{16-4Np_1}{p_1+2}\int_{\mathbb{R}^N}\left|\psi(t,x)\right|^{p_1+2}dx. \tag{3.2}
$$

Since $\psi_0(x) = c\rho^{\frac{N}{2}}u(\rho x)$ and the Pohožaev identity of (1.4), i.e., $\frac{1}{2}\int_{\mathbb{R}^N}|\nabla u(x)|^2\,dx = \frac{1}{2p_2}\int_{\mathbb{R}^N}(I_\alpha * |u|^{p_2})(x)|u(x)|^{p_2}\,dx$ (see [18]), it follows that

$$
\begin{aligned}
E(\psi_0) &= \frac{|c|^2\rho^2}{2}\int_{\mathbb{R}^N}\left|\nabla u(x)\right|^2dx - \frac{|c|^{2p_2}\rho^2}{2p_2}\int_{\mathbb{R}^N}\left(I_\alpha * |u|^{p_2}\right)(x)\left|u(x)\right|^{p_2}dx \\
&\quad + \frac{|c|^{p_1+2}\rho^{\frac{N}{2}p_1}}{p_1+2}\int_{\mathbb{R}^N}\left|u(x)\right|^{p_1+2}dx \\
&= -\frac{|c|^2\rho^2}{2}\left(|c|^{2p_2-2}-1\right)\|\nabla u\|_{L^2}^2 + \frac{|c|^{p_1+2}\rho^{\frac{N}{2}p_1}}{p_1+2}\int_{\mathbb{R}^N}\left|u(x)\right|^{p_1+2}dx.
\end{aligned}
$$

Thus, it follows from (3.1) that $E(\psi_0) < 0$. We deduce from (3.2) that $J''(t) < 16E(\psi_0) < 0$. By a standard argument, the solution $\psi(t)$ to (1.1) with $\psi_0 = c\rho^{\frac{N}{2}}u(\rho x)$ blows up in finite time. □

4 Dynamics of blow-up solutions in the L^2-critical case

In this section, we study the dynamical properties of blow-up solutions for (1.1) with $\lambda_1 = 1$, $\lambda_2 = -1$, $0 < p_1 < \frac{4}{N}$ and $p_2 = 1 + \frac{2+\alpha}{N}$. For this purpose, we firstly recall a refined compactness lemma which has been proved in [18] by the inequality (2.6) and the profile decomposition theory.

Lemma 4.1 *Let $p_2 = 1 + \frac{2+\alpha}{N}$. If $\{\psi_n\}_{n=1}^\infty$ is a bounded sequence in H^1 and satisfies*

$$
\limsup_{n\to\infty}\|\nabla\psi_n\|_{L^2}^2 \leq M, \qquad \limsup_{n\to\infty}\int_{\mathbb{R}^N}\left(I_\alpha * |\psi_n|^{p_2}\right)|u_n|^{p_2}\,dx \geq m.
$$

Then there exists $\{x_n\}_{n=1}^\infty \subset \mathbb{R}^N$, such that, up to a subsequence,

$$
\psi_n(\cdot + x_n) \rightharpoonup \Psi
$$

with $\|\Psi\|_{L^2} \geq (\frac{m}{p_2 M})^{\frac{1}{2p_2-2}}\|u\|_{L^2}$.

Theorem 4.2 (L^2-concentration) *Assume that $\psi_0 \in H^1$, $\lambda_1 = 1$, $\lambda_2 = -1$, $0 < p_1 < \frac{4}{N}$ and $p_2 = 1 + \frac{2+\alpha}{N}$. Let the solution $\psi(t)$ to (1.1) blow up at the finite time T^*. If $a(t) : [0, T^*) \mapsto \mathbb{R}$*

is a real-valued function and $a(t)\|\nabla\psi(t)\|_{L^2} \to \infty$ *as* $t \to T^*$. *Then there exists* $x(t) \in \mathbb{R}^N$ *such that*

$$\liminf_{t\to T^*} \int_{|x-x(t)|\le a(t)} \left|\psi(t,x)\right|^2 dx \ge \int_{\mathbb{R}^N} \left|u(x)\right|^2 dx. \tag{4.1}$$

Proof Set

$$\rho_n := \|\nabla u\|_{L^2} / \left\|\nabla\psi(t_n)\right\|_{L^2} \quad \text{and} \quad v_n(x) := \rho_n^{\frac{N}{2}} \psi(t_n, \rho_n x),$$

where $\{t_n\}_{n=1}^{\infty} \subseteq [0, T^*)$ and $t_n \to T^*$ as $n \to \infty$. Then the sequence $\{v_n\}$ satisfies

$$\begin{aligned} &\|v_n\|_{L^2} = \left\|\psi(t_n)\right\|_{L^2} = \|\psi_0\|_{L^2}, \\ &\|\nabla v_n\|_{L^2} = \rho_n \left\|\nabla\psi(t_n)\right\|_{L^2} = \|\nabla u\|_{L^2}. \end{aligned} \tag{4.2}$$

It follows from (2.3) that

$$\begin{aligned} H(v_n) &:= \frac{1}{2}\int_{\mathbb{R}^N} \left|\nabla v_n(x)\right|^2 dx - \frac{1}{2p_2}\int_{\mathbb{R}^N} \left(I_\alpha * |v_n|^{p_2}\right)(x)\left|v_n(x)\right|^{p_2} dx \\ &= \rho_n^2 \left(\frac{1}{2}\int_{\mathbb{R}^N} \left|\nabla\psi(t_n,x)\right|^2 dx - \frac{1}{2p_2}\int_{\mathbb{R}^N} \left(I_\alpha * \left|\psi(t_n)\right|^{p_2}\right)(x)\left|\psi(t_n,x)\right|^{p_2} dx \right) \\ &= \rho_n^2 \left(E(\psi_0) - \frac{1}{p_1+2}\int_{\mathbb{R}^N} \left|\psi(t_n,x)\right|^{p_1+2} dx \right). \end{aligned} \tag{4.3}$$

Hence, by the Gagliardo–Nirenberg inequality

$$\int_{\mathbb{R}^N} \left|\psi(x)\right|^{p_1+2} dx \le C\|\psi\|_{L^2}^{p_1+2-\frac{Np_1}{2}} \|\nabla\psi\|_{L^2}^{\frac{Np_1}{2}},$$

and $0 < p_1 < \frac{4}{N}$, it follows that

$$\begin{aligned} \left|H(v_n)\right| &\le \rho_n^2 \left(\left|E(\psi_0)\right| + \frac{1}{p_1+2}\int_{\mathbb{R}^N} \left|\psi(t_n,x)\right|^{p_1+2} dx \right) \\ &\le \frac{|E(\psi_0)|\|\nabla u\|_{L^2}^2}{\|\nabla\psi(t_n)\|_{L^2}^2} + C\frac{\|\nabla\psi\|_{L^2}^2 \|\nabla\psi(t_n)\|_{L^2}^{\frac{Np_1}{2}}}{\|\nabla\psi(t_n)\|_{L^2}^2} \to 0 \quad \text{as } n \to \infty. \end{aligned} \tag{4.4}$$

This yields $\int_{\mathbb{R}^N} (I_\alpha * |v_n|^{p_2})|v_n|^{p_2} dx \to p_2\|\nabla u\|_{L^2}^2$.

Set $m = p_2\|\nabla u\|_{L^2}^2$ and $M = \|\nabla u\|_{L^2}^2$. Then we deduce from Lemma 4.1 that there exist $V \in H^1$ and $\{x_n\}_{n=1}^{\infty} \subset \mathbb{R}^N$ such that, up to a subsequence,

$$v_n(\cdot + x_n) = \rho_n^{N/2}\psi\left(t_n, \rho_n(\cdot + x_n)\right) \rightharpoonup V \quad \text{weakly in } H^1 \tag{4.5}$$

with

$$\|V\|_{L^2} \ge \|u\|_{L^2}. \tag{4.6}$$

Therefore, we have

$$\begin{aligned}\liminf_{n\to\infty}\int_{|x|\le r}\left|v_n(t_n,x+x_n)\right|^2dx&=\liminf_{n\to\infty}\int_{|x|\le r}\rho_n^N\left|\psi\left(t_n,\rho_n(x+x_n)\right)\right|^2dx\\&\ge\int_{|x|\le r}\left|V(x)\right|^2dx,\quad\text{for every } r>0.\end{aligned}\tag{4.7}$$

From the assumption on $a(t)$, we have

$$\frac{a(t_n)}{\rho_n}=\frac{a(t_n)\|\nabla\psi(t_n)\|_{L^2}}{\|\nabla u\|_{L^2}}\to\infty,\quad\text{as } n\to\infty.$$

Then $r\rho_n < a(t_n)$ for sufficiently large n. Therefore, it follows from (4.5) that

$$\begin{aligned}&\liminf_{n\to\infty}\sup_{y\in\mathbb{R}^N}\int_{|x-y|\le a(t_n)}\left|\psi(t_n,x)\right|^2dx\\&\quad\ge\liminf_{n\to\infty}\sup_{y\in\mathbb{R}^N}\int_{|x-y|\le r\rho_n}\left|\psi(t_n,x)\right|^2dx\\&\quad\ge\liminf_{n\to\infty}\int_{|x-x_n|\le r\rho_n}\left|\psi(t_n,x)\right|^2dx\\&\quad=\liminf_{n\to\infty}\int_{|x|\le r}\rho_n^N\left|\psi\left(t_n,\rho_n(x+x_n)\right)\right|^2dx.\end{aligned}$$

This and (4.7) imply that

$$\liminf_{n\to\infty}\sup_{y\in\mathbb{R}^N}\int_{|x-y|\le a(t_n)}\left|\psi(t_n,x)\right|^2dx\ge\int_{\mathbb{R}^N}\left|V(x)\right|^2dx\ge\int_{\mathbb{R}^N}\left|u(x)\right|^2dx.$$

Since the sequence $\{t_n\}_{n=1}^{\infty}$ is arbitrary, it follows that

$$\liminf_{t\to T^*}\sup_{y\in\mathbb{R}^N}\int_{|x-y|\le a(t)}\left|\psi(t,x)\right|^2dx\ge\int_{\mathbb{R}^N}\left|u(x)\right|^2dx.\tag{4.8}$$

Furthermore, for every $t\in[0,T^*)$, the function $y\mapsto h(y)=\int_{|x-y|\le a(t)}|\psi(t,x)|^2\,dx$ is continuous and $h(y)\to 0$ as $|y|\to\infty$. Hence, there is $x(t)\in\mathbb{R}^N$ such that

$$\sup_{y\in\mathbb{R}^N}\int_{|x-y|\le a(t)}\left|\psi(t,x)\right|^2dx=\int_{|x-x(t)|\le a(t)}\left|\psi(t,x)\right|^2dx,$$

which, together with (4.8), implies (4.1). □

In the following, we will study some properties of blow-up solutions to (1.1) with $\|\psi_0\|_{L^2}=\|u\|_{L^2}$. When $p=2$ or $\alpha=2$, the uniqueness of the ground state of (1.4) plays an important role in the characterization of blow-up solutions to (1.2) in [7, 10]. However, the uniqueness of ground states of (1.4) with $0<\alpha<N$ and $1+\frac{\alpha}{N}<p_2<\frac{N+\alpha}{N-2}$ is not known, we cannot apply the method in [7, 10] to study the dynamics of the blow-up solutions.

Theorem 4.3 *Assume that $\psi_0\in\Sigma$, $\lambda_1=1$, $\lambda_2=-1$, $0<p_1<\frac{4}{N}$ and $p_2=1+\frac{2+\alpha}{N}$. Let the solution $\psi(t)$ to* (1.1) *blow up at the finite time T^* and $\|\psi_0\|_{L^2}=\|u\|_{L^2}$. Then there exists*

$x_0 \in \mathbb{R}^N$ *such that*

$$\left|\psi(t,x)\right|^2 \to \|u\|_{L^2}^2 \delta_{x_0} \quad \textit{as } t \to T^* \tag{4.9}$$

in the sense of a distribution.

Proof Firstly, it follows from Theorem 4.2 that for all $r > 0$

$$\liminf_{t\to T^*} \int_{|x-x(t)|<r} \left|\psi(t,x)\right|^2 dx \geq \|u\|_{L^2}^2. \tag{4.10}$$

This and (2.1) yield for all $r > 0$

$$\|u\|_{L^2}^2 = \|\psi_0\|_{L^2}^2 = \left\|\psi(t)\right\|_{L^2}^2 \geq \liminf_{t\to T^*} \int_{|x-x(t)|<r} \left|\psi(t,x)\right|^2 dx \geq \|u\|_{L^2}^2.$$

This implies

$$\left|\psi\big(t,x+x(t)\big)\right|^2 \to \|u\|_{L^2}^2 \delta_{x=0} \quad \text{as } t \to T^*. \tag{4.11}$$

On the other hand, it follows from the inequality (2.6) and (4.3) that for any $\varepsilon > 0$ and any real-valued function θ

$$\begin{aligned} H\big(e^{\pm i\epsilon\theta}\psi(t)\big) &= \frac{\epsilon^2}{2}\int_{\mathbb{R}^N} \left|\psi(t,x)\right|^2 \left|\nabla\theta(x)\right|^2 dx \\ &\quad \mp \epsilon \operatorname{Im} \int_{\mathbb{R}^N} \bar{\psi}(t,x)\nabla\psi(t,x)\cdot\nabla\theta(x)\,dx + H\big(\psi(t)\big) \\ &\geq \frac{1}{2}\int_{\mathbb{R}^N} \left|\nabla\big(e^{\pm i\epsilon\theta}\psi(t,x)\big)\right|^2 dx \left(1 - \frac{\|\psi_0\|_{L^2}^{2p_2-2}}{\|u\|_{L^2}^{2p_2-2}}\right) = 0. \end{aligned}$$

This implies that

$$\begin{aligned} &\left|\mp \operatorname{Im} \int_{\mathbb{R}^N} \bar{\psi}(t,x)\nabla\psi(t,x)\cdot\nabla\theta(x)\,dx\right| \\ &\quad \leq \left(2H\big(\psi(t)\big)\int_{\mathbb{R}^N} \left|\psi(t,x)\right|^2 \left|\nabla\theta(x)\right|^2 dx\right)^{1/2}. \end{aligned} \tag{4.12}$$

Therefore, this and $H(\psi(t)) \leq E(\psi(t)) = E(\psi_0)$ yield

$$\begin{aligned} \left|\frac{d}{dt}\int_{\mathbb{R}^N} \left|\psi(t,x)\right|^2 x_j\,dx\right| &\leq C\left|\int_{\mathbb{R}^N} \bar{\psi}(t,x)\partial_j\psi(t,x)\,dx\right| \\ &\leq C\left|\int_{\mathbb{R}^N} \bar{\psi}(t,x)\nabla\psi(t,x)\nabla x_j\,dx\right| \\ &\leq C\left(2H\big(\psi(t)\big)\int_{\mathbb{R}^N} \left|\psi(t,x)\right|^2 |\nabla x_j|^2\,dx\right)^{1/2} \leq C, \end{aligned}$$

for every $j = 1, 2, \ldots, N$. This implies

$$\left|\int_{\mathbb{R}^N} \left|\psi(t_m,x)\right|^2 x_j\,dx - \int_{\mathbb{R}^N} \left|\psi(t_k,x)\right|^2 x_j\,dx\right| \leq C|t_m - t_k| \to 0 \quad \text{as } m,k \to \infty,$$

for every $j = 1, 2, \dots, N$, where $\{t_m\}_{m=1}^{\infty}, \{t_k\}_{k=1}^{\infty} \subseteq (0, T^*)$ and $\lim_{m\to\infty} t_m = \lim_{k\to\infty} t_k = T^*$. Thus, we have

$$\lim_{t\to T^*} \int_{\mathbb{R}^N} \left|\psi(t,x)\right|^2 x_j \, dx \quad \text{exists,}$$

for every $j = 1, 2, \dots, N$. Set

$$x_0 = \lim_{t\to T^*} \int_{\mathbb{R}^N} \left|\psi(t,x)\right|^2 x \, dx / \|u\|_{L^2}^2, \tag{4.13}$$

it follows that

$$\lim_{t\to T^*} \int_{\mathbb{R}^N} \left|\psi(t,x)\right|^2 x \, dx = \|u\|_{L^2}^2 x_0. \tag{4.14}$$

In addition, we deduce from Lemma 2.2 and (4.11) that

$$\begin{aligned}
&\int_{\mathbb{R}^N} |x|^2 \left|\psi\big(t, x + x(t)\big)\right|^2 dx \\
&\quad\le C \int_{\mathbb{R}^N} \left|x + x(t)\right|^2 \left|\psi\big(t, x + x(t)\big)\right|^2 dx + C\left|x(t)\right|^2 \int_{\mathbb{R}^N} \left|\psi\big(t, x + x(t)\big)\right|^2 dx \\
&\quad\le C + C\left|x(t)\right|^2 \|\psi_0\|_{L^2}^2 \\
&\quad\le C + C \limsup_{t\to T^*} \int_{|x|<1} \left|x + x(t)\right|^2 \left|\psi\big(t, x + x(t)\big)\right|^2 dx \\
&\quad\le C + C \int_{\mathbb{R}^N} |x|^2 \left|\psi(t,x)\right|^2 dx \le C.
\end{aligned} \tag{4.15}$$

This implies

$$\limsup_{t\to T^*} \left|x(t)\right| \le \frac{\sqrt{C}}{\|\psi_0\|_{L^2}} \tag{4.16}$$

and

$$\limsup_{t\to T^*} \int_{\mathbb{R}^N} |x|^2 \left|\psi\big(t, x + x(t)\big)\right|^2 dx \le C.$$

Thus, for any $\varepsilon > 0$, there is R_0 such that

$$\limsup_{t\to T^*} \left| \int_{|x|\ge R_0} x \left|\psi\big(t, x + x(t)\big)\right|^2 dx \right| \le \frac{C}{R_0} < \frac{\varepsilon}{2}.$$

We see from (4.11) that

$$\begin{aligned}
&\limsup_{t\to T^*} \left| \int_{\mathbb{R}^N} \left|\psi(t,x)\right|^2 x \, dx - x(t) \|u\|_{L^2}^2 \right| \\
&\quad= \limsup_{t\to T^*} \left| \int_{\mathbb{R}^N} \left|\psi(t,x)\right|^2 \big(x - x(t)\big) \, dx \right| \\
&\quad\le \limsup_{t\to T^*} \left| \int_{|x|\le R_0} \left|\psi\big(t, x + x(t)\big)\right|^2 x \, dx \right| + \frac{\varepsilon}{2} \le \varepsilon.
\end{aligned} \tag{4.17}$$

This and (4.14) imply that $\lim_{t\to T^*} x(t) = x_0$. Thus, it follows from (4.11) that

$$\left|\psi(t,x)\right|^2 \to \|u\|_{L^2}^2 \delta_{x=x_0} \quad \text{as } t \to T^*$$

in the sense of distribution. □

Finally, we study the blow-up rate of blow-up solutions to (1.1) with $\|\psi_0\|_{L^2} = \|u\|_{L^2}$.

Theorem 4.4 *Assume that $\psi_0 \in \Sigma$, $\lambda_1 = 1$, $\lambda_2 = -1$, $0 < p_1 < \frac{4}{N}$ and $p_2 = 1 + \frac{2+\alpha}{N}$. Let the solution $\psi(t)$ to* (1.1) *blow up at the finite time T^* and $\|\psi_0\|_{L^2} = \|u\|_{L^2}$. Then there exists a constant $C > 0$ such that for all $t \in [0, T^*)$*

$$\left\|\nabla\psi(t)\right\|_{L^2} \geq \frac{C}{T^* - t}. \tag{4.18}$$

Proof Let $g \in C_0^\infty(\mathbb{R}^N)$ be a nonnegative radial function satisfying

$$g(x) = g\left(|x|\right) = |x|^2, \quad \text{if } |x| < 1 \quad \text{and} \quad \left|\nabla g(x)\right|^2 \leq Cg(x).$$

For $A > 0$, we define $g_A(x) = A^2 g(\frac{x}{A})$ and $h_A(t) = \int_{\mathbb{R}^N} g_A(x - x_0)|\psi(t,x)|^2\,dx$ with x_0 defined by (4.13).

It follows from (4.12) and $H(\psi(t)) \leq E(\psi(t)) = E(\psi_0)$ that for every $t \in [0, T^*)$

$$\begin{aligned}
\left|\frac{d}{dt}h_A(t)\right| &\leq C\left|\int_{\mathbb{R}^N} \bar{\psi}(t,x)\nabla\psi(t,x)\nabla g_A(x - x_0)\,dx\right| \\
&\leq 2\sqrt{H\left(\psi(t)\right)}\left(\int_{\mathbb{R}^N} \left|\psi(t,x)\right|^2\left|\nabla g_A(x - x_0)\right|^2 dx\right)^{1/2} \\
&\leq 2\sqrt{E(\psi_0)}\left(\int_{\mathbb{R}^N} \left|\psi(t,x)\right|^2\left|g_A(x - x_0)\right| dx\right)^{1/2} \\
&\leq C\sqrt{h_A(t)}.
\end{aligned} \tag{4.19}$$

This implies that there is a constant C such that $|\frac{d}{dt}\sqrt{h_A(t)}| \leq C$. Integrating on both sides with respect to time t on $[t_1, t]$, we have

$$\left|\sqrt{h_A(t)} - \sqrt{h_A(t_1)}\right| \leq C|t - t_1|. \tag{4.20}$$

On the other hand, from (4.9), we have

$$h_A(t_1) \to \|Q\|_{L^2} g_A(0) = 0 \quad \text{as } t_1 \to T^*.$$

Thus, let $t_1 \to T^*$ in (4.20), we have $h_A(t) \leq C(T^* - t)^2$. Now fix $t \in [0, T^*)$, it follows that

$$\lim_{A\to\infty} h_A(t) = \int_{\mathbb{R}^N} |x - x_0|^2\left|\psi(t,x)\right|^2 dx \leq C\left(T^* - t\right)^2.$$

Thus, we deduce from the uncertainty principle that

$$\left\|\nabla\psi(t)\right\|_{L^2} \geq \frac{\int_{\mathbb{R}^N} |\psi(x)|^2\,dx}{(\int_{\mathbb{R}^N} |x - x_0|^2|\psi(x)|^2\,dx)^{1/2}} \geq \frac{C}{T^* - t}, \quad \forall t \in [0, T^*).$$

This completes the proof. □

5 Conclusions

In this paper, we study the dynamics of blow-up solutions for the nonlinear Schrödinger–Choquard equation (1.1) with $0 < p_1 < \frac{4}{N}$ and $p_2 = 1 + \frac{2+\alpha}{N}$. In the previous papers, the scale invariance played an important role in the study of the dynamics of blow-up solutions to nonlinear Schrödinger equations. Because there exists no scale invariance for Eq. (1.1), the study of blow-up solutions to (1.1) is an interesting problem. We must overcome the difficulty brought about by the loss of scale invariance. For (1.1), we find that the ground state solution u to (1.4) exactly describes the sharp threshold mass of global existence and blow-up, the dynamical properties of blow-up solutions, including L^2-concentration, limiting profile and blow-up rates.

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