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Positive solutions to fractional boundary-value problems with p-Laplacian on time scales

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Abstract

In this article, we consider the following boundary-value problem of nonlinear fractional differential equation with p-Laplacian operator:

$$D^{\alpha}(\phi_p(D^{\alpha}u(t))) = f(t,u(t)), \quad t \in [0,1]_T,$$
$$u(0) = u(\sigma(1)) = D^{\alpha}u(0) = D^{\alpha}u(\sigma(1)) = 0,$$

where $1 < \alpha \leq 2$ is a real number, the time scale T is a nonempty closed subset of $\mathbb{R}$. D^{α} is the conformable fractional derivative on time scales, $\phi_p(s) = |s|^{p-2}s$, $p > 1$, $\phi_p^{-1} = \phi_q$, $1/p + 1/q = 1$, and $f : [0,\sigma(1)] \times [0,+\infty) \to [0,+\infty)$ is continuous. By the use of the approach method and fixed-point theorems on cone, some existence and multiplicity results of positive solutions are acquired. Some examples are presented to illustrate the main results.

MSC: 34B15

Keywords: Conformable fractional derivative; Time scales; Fixed-point theorems on cone; p-Laplacian operator

1 Introduction

In this paper, the existence and multiplicity of positive solutions for the following fractional differential boundary-value problem on time scales is studied:

$$D^{\alpha}\big(\phi_p\big(D^{\alpha}u(t)\big)\big) = f\big(t,u(t)\big), \quad t \in [0,1]_T, \tag{1.1}$$

$$u(0) = u\big(\sigma(1)\big) = D^{\alpha}u(0) = D^{\alpha}u\big(\sigma(1)\big) = 0, \tag{1.2}$$

where $1 < \alpha \leq 2$, D^{α} is the conformable fractional derivative on time scales, $\phi_p(s) = |s|^{p-2}s$, $p > 1$, $\phi_p^{-1} = \phi_q$, $1/p + 1/q = 1$, and $f : [0,\sigma(1)] \times [0,+\infty) \to [0,+\infty)$ is continuous.

The existence of positive solutions for boundary-value problem on time scales has become the focus in recent years; for details, see [1–6]. Due to the wide applications, many researchers studied the existence of positive solutions for fractional derivatives boundary-value problem [7–21] and the references therein. Meanwhile, the boundary-value problem with p-Laplacian operator have also been discussed extensively in the literature; for example, see [4, 11, 22–27].

For $\alpha = 2$, problem (1.1), (1.2) is called a fourth order p-Laplacian boundary-value problem which has been studied in [4].

Dong et al. [22] investigated the boundary-value problem for a fractional differential equation with the p-Laplacian operator

$$D^\alpha\big(\phi_p\big(D^\alpha u(t)\big)\big) = f\big(t,u(t)\big), \quad 0 < t < 1, \tag{1.3}$$

$$u(0) = u(1) = D^\alpha u(0) = D^\alpha u(1) = 0, \tag{1.4}$$

where $1 < \alpha \leq 2$ is a real number, D^α is the conformable fractional derivative, $\phi_p(s) = |s|^{p-2}s$, $p > 1$, $\phi_p^{-1} = \phi_q$, $1/p + 1/q = 1$,$f : [0,1] \times [0,+\infty) \to [0,+\infty)$ is continuous. By the use of the fixed-point theorems on cone, some existence and multiplicity results of positive solutions are obtained.

Motivated by the work mentioned above, we investigate the existence and multiplicity of positive solutions for (1.3), (1.4) on time scales. The rest of this paper is organized as follows. In Sect. 2, we recall some concepts relative to the new conformable fractional calculus and give some lemmas with respect to the corresponding Green's function. In Sect. 3, we investigate the existence and multiplicity of positive solution for boundary-value problem (1.1), (1.2). In Sect. 4, we present some examples to illustrate our main results, respectively.

2 Preliminaries and lemmas

In this section, we introduced notations and definitions of conformable fractional derivative on time scales and some lemmas. Let T be a time scale and denote $[a,b]_T =: [a,b] \cap T$. These results can be found in the recent literature; see [2, 3, 6].

Definition 2.1 A time scale T is a nonempty closed subset of $\mathbb{R}$; assume that T has the topology that it inherits from the standard topology on $\mathbb{R}$. Define the forward and backward jump operators σ, ρ: $T \to T$ by

$$\sigma(t) = \inf\{\tau > t \mid \tau \in T\}, \qquad \rho(t) = \sup\{\tau < t \mid \tau \in T\}.$$

In this definition we put $\inf \emptyset = \sup T$, $\sup \emptyset = \inf T$. Set $\sigma^2(t) = \sigma(\sigma(t))$, $\rho^2(t) = \rho(\rho(t))$. The sets T^k and T_k which are derived from the time scale T are as follows:

$$T^k := \big\{t \in T : t \text{ is not maximal or } \rho(t) = t\big\},$$
$$T_k := \big\{t \in T : t \text{ is not minimal or } \sigma(t) = t\big\}.$$

Denote interval I on T by $I_T = I \cap T$.

Definition 2.2 If $f : T \to \mathbb{R}$ is a function and $t \in T_k$, then the delta derivative of f at the point t is defined to be the number $f^\Delta(t)$ (provided it exists) with the property that, for each $\epsilon > 0$, there is a neighborhood U of t such that

$$\big|f\big(\sigma(t)\big) - f(s) - f^\Delta(t)\big(\sigma(t) - s\big)\big| \leq \epsilon\big|\sigma(t) - s\big|$$

for all $s \in U$. The function f is called Δ-differentiable on T^k if $f^\Delta(t)$ exists for all $t \in T^k$.

Definition 2.3 ([3]) Let $\alpha \in (1,2]$ and $f : T \to \mathbb{R}$, $t \in T^k$. For $t > 0$, we define $D^\alpha f(t)$ to be the number (provided it exists) with the property that, given any $\varepsilon > 0$, there is a δ-neighborhood $V_t \subset T$ of $t, \delta > 0$, such that

$$\left|\left[f\bigl(\sigma(t)\bigr) - f(s)\right]t^{2-\alpha} - D^\alpha f(t)\left[\sigma(t) - s\right]\right| \le \epsilon\left|\sigma(t) - s\right|.$$

We call $D^\alpha f(t)$ the conformable fractional derivative of f of order α at t, and we define the conformable fractional derivative at 0 as $D^\alpha f(0) = \lim_{t\to 0^+} D^\alpha f(t)$.

Lemma 2.1 ([3]) *Let $\alpha \in (1,2]$ and f be two times delta differentiable at $t \in T^k$. The following relation holds*: $D^\alpha f(t) = t^{2-\alpha} f^{\Delta\Delta}(t)$.

Definition 2.4 ([2]) A function $f : T \to \mathbb{R}$ is called regulated provided its right-sided limits exist (and are finite) at all right-dense points in T and its left-sided limits exist (and are finite) at all left-dense points in T.

Definition 2.5 ([2]) A function $f : T \to \mathbb{R}$ is called rd-continuous provided it is continuous at right-dense points in T and its left-sided limit exist (finite) at all left-dense points in T. The set of rd-continuous functions $f : T \to \mathbb{R}$ will be denoted by $C_{rd}(T,R)$.

Lemma 2.2 ([2]) *Assume $f : T \to \mathbb{R}$.*
(i) *If f is continuous, then f is rd-continuous.*
(ii) *If f is rd-continuous, then f is regulated.*

Definition 2.6 ([3]) Let $f : T \to \mathbb{R}$ be a regulated function and $1 < \alpha \le 2$. Then the α-fractional integral of f is defined by

$$I^\alpha f(t) = I^2\bigl(t^{\alpha-2} f(t)\bigr) = \int_0^t (t-s)s^{\alpha-2} f(s)\Delta s. \tag{2.1}$$

Lemma 2.3 *Let $t > 0$, $\alpha \in (1,2]$, and the function $f : [0,\infty)_T \to \mathbb{R}$ be rd-continuous, then $D^\alpha I^\alpha f(t) = f(t)$.*

Proof Since $f(t)$ is rd-continuous, then $f(t)$ is regulated, and $I^\alpha f(t)$ is twice times differentiable. In view of Lemma 2.1, one has

$$\begin{aligned} D^\alpha\bigl(I^\alpha f\bigr)(t) &= t^{2-\alpha}\left(\int_0^t (t-s)s^{\alpha-2} f(s)\Delta s\right)^{\Delta\Delta} \\ &= t^{2-\alpha} f(t) t^{\alpha-2} \\ &= f(t). \end{aligned}$$

The proof is complete. □

Lemma 2.4 (Mean value theorem [6]) *Let $a \ge 0$ and $f : T \to \mathbb{R}$ be a function continuous on $[a,b]_T$ which is conformable fractional differentiable of order with α on $[a,b]_T$. Then there exist $\xi, \tau \in [a,b]_T$ such that*

$$\xi^{\alpha-1} D^\alpha f(\xi) \le \frac{f(b) - f(a)}{b-a} \le \tau^{\alpha-1} D^\alpha f(\tau).$$

Lemma 2.5 *Let $\alpha \in (1,2]$, f be a α-differentiable function at $t > 0$, then $D^\alpha f(t) = 0$ for $t \in [0,1]_T$ if and only if $f(t) = a_0 + a_1 t$, where $a_k \in \mathbb{R}$, for $k = 0, 1$.*

Proof The sufficiency follows by the definition of the delta derivative on time scales.

Next, given $t_1, t_2 \in [0,1]_T$ with $t_1 < t_2$, by Lemma 2.4, there exists $\xi, \tau \in (t_1, t_2)_T$ such that

$$\xi^{\alpha-1} D^\alpha f(\xi) \le \frac{f^\Delta(t_2) - f^\Delta(t_1)}{t_2 - t_1} \le \tau^{\alpha-1} D^\alpha f(\tau).$$

By means of $D^\alpha f(\xi) = D^\alpha f(\tau) = 0$, we have $f^\Delta(t_2) = f^\Delta(t_1)$, with the arbitrariness of t_1, t_2, one has $f^\Delta(t)$ is a constant, so $f(t) = a_0 + a_1 t$, for $t \in [0,1]_T$. □

With Lemma 2.3 and Lemma 2.5, the following lemma is immediate.

Lemma 2.6 *Assume that $u \in C(0,+\infty)_T$ with a fractional derivative of order $\alpha \in (1,2]$. Then*

$$I^\alpha D^\alpha u(t) = u(t) + c_0 + c_1 t, \tag{2.2}$$

for some $c_k \in \mathbb{R}$, $k = 0, 1$.

We present below the Green's function and its properties.

Lemma 2.7 *Given $y \in C[0,\sigma(1)]_T$, the unique solution of*

$$D^\alpha u(t) + y(t) = 0, \quad t \in [0,1]_T, \tag{2.3}$$

$$u(0) = u\bigl(\sigma(1)\bigr) = 0, \tag{2.4}$$

is

$$u(t) = \int_0^{\sigma(1)} G(t,s) y(s) \Delta s,$$

where

$$G(t,s) = \frac{1}{\sigma(1)} \begin{cases} (\sigma(1) - t) s^{\alpha-1}, & \text{for } 0 \le s \le t; \\ t s^{\alpha-2}(\sigma(1) - s), & \text{for } t \le s \le 1. \end{cases} \tag{2.5}$$

Proof By the use of the Lemma 2.6, we can deduce from equation (2.3) an equivalent integral equation,

$$\begin{aligned} u(t) &= -I^\alpha y(t) + c_0 + c_1 t \\ &= -\int_0^t (t-s) s^{\alpha-2} y(s) \Delta s + c_0 + c_1 t, \end{aligned}$$

for some $c_0, c_1 \in \mathbb{R}$. By (2.4), there are

$$c_0 = 0, \qquad c_1 = \frac{1}{\sigma(1)} \int_0^{\sigma(1)} \bigl[\sigma(1) - s\bigr] s^{\alpha-2} y(s) \Delta s.$$

Therefore, the unique solution of Problem (2.3), (2.4) is

$$\begin{aligned}
u(t) &= -\int_0^t (t-s)s^{\alpha-2}y(s)\Delta s + \frac{t}{\sigma(1)}\int_0^{\sigma(1)} \bigl(\sigma(1)-s\bigr)s^{\alpha-2}y(s)\Delta s \\
&= \frac{1}{\sigma(1)}\int_0^t \bigl[-\sigma(1)ts^{\alpha-2} + \sigma(1)s^{\alpha-1}\bigr]y(s)\Delta s \\
&\quad + \frac{1}{\sigma(1)}\int_0^{\sigma(1)} \bigl[t\sigma(1)s^{\alpha-2} - ts^{\alpha-2}\bigr]y(s)\Delta s \\
&= \frac{1}{\sigma(1)}\int_0^t \bigl(\sigma(1)-t\bigr)(s)^{\alpha-1}y(s)\Delta s \\
&\quad + \frac{1}{\sigma(1)}\int_t^{\sigma(1)} ts^{\alpha-2}\bigl(\sigma(1)-s\bigr)y(s)\Delta s \\
&= \int_0^{\sigma(1)} G(t,s)y(s)\Delta s.
\end{aligned}$$

The proof is complete. □

We point out here that (2.5) becomes the usual Green's function when $\alpha = 2$ on time scales.

Lemma 2.8 *Let* $y \in C[0,\sigma(1)]$ *and* $1 < \alpha \le 2$. *Then the problem*

$$D^\alpha\bigl(\phi_p\bigl(D^\alpha u(t)\bigr)\bigr) = y(t), \quad t \in [0,1]_T, \tag{2.6}$$

$$u(0) = u\bigl(\sigma(1)\bigr) = D^\alpha u(0) = D^\alpha u\bigl(\sigma(1)\bigr) = 0, \tag{2.7}$$

has a unique solution

$$u(t) = \int_0^{\sigma(1)} G(t,s)\phi_q\biggl(\int_0^{\sigma(1)} G(s,\tau)y(\tau)\Delta\tau\biggr)\Delta s. \tag{2.8}$$

Proof Applying operator I^α on both sides of (2.6), with Lemma 2.6,

$$\phi_p\bigl(D^\alpha u(t)\bigr) + C_0 + C_1 t = I^\alpha y(t).$$

So,

$$\begin{aligned}
\phi_p\bigl(D^\alpha u(t)\bigr) &= I^\alpha y(t) - C_0 - C_1 t \\
&= \int_0^t (t-\tau)\tau^{\alpha-2}y(\tau)\Delta\tau - C_0 - C_1 t
\end{aligned}$$

for some $C_0, C_1 \in \mathbb{R}$. By the boundary conditions $D^\alpha u(0) = D^\alpha u(\sigma(1)) = 0$, as a consequence we have

$$C_0 = 0, \qquad C_1 = \frac{1}{\sigma(1)}\int_0^{\sigma(1)} \bigl(\sigma(1)-\tau\bigr)\tau^{\alpha-2}y(\tau)\Delta\tau.$$

Therefore, the solution $u(t)$ of fractional differential equation boundary-value problem (2.6) and (2.7) satisfies

$$\begin{aligned}\phi_p\bigl(D^\alpha u(t)\bigr) &= \int_0^t (t-\tau)\tau^{\alpha-2}y(\tau)\Delta\tau \\ &\quad - \frac{1}{\sigma(1)}t\int_0^{\sigma(1)}\bigl(\sigma(1)-\tau\bigr)\tau^{\alpha-2}y(\tau)\Delta\tau \\ &= -\int_0^{\sigma(1)} G(t,\tau)y(\tau)\Delta\tau.\end{aligned}$$

Thus, the fractional differential equation boundary-value problem (2.6) and (2.7) is equivalent to the problem

$$\begin{aligned}&D^\alpha u(t) + \phi_q\biggl(\int_0^{\sigma(1)} G(t,\tau)y(\tau)\Delta\tau\biggr) = 0, \quad 0 < t < 1, \\ &u(0) = u\bigl(\sigma(1)\bigr) = 0.\end{aligned}$$

Lemma 2.7 implies that fractional differential equation boundary-value problem (2.6), (2.7) has a unique solution

$$u(t) = \int_0^{\sigma(1)} G(t,s)\phi_q\biggl(\int_0^{\sigma(1)} G(s,\tau)y(\tau)\Delta\tau\biggr)\Delta s.$$

The proof is complete. □

Lemma 2.9 *The function* $G(t,s)$ *defined by* (2.5) *satisfies*:

(i) $G(t,s) \ge 0$, *for* $t \in [0,\sigma(1)]$, $s \in [0,1]$, *and* $G(t,s) > 0$, *for* $t \in (0,\sigma(1))$, $s \in (0,1)$;

(ii) $G(t,s) \le G(s,s)$, *for* $t \in [0,\sigma(1)]$, $s \in [0,1]$;

(iii) $G(t,s) \ge \frac{\sigma(1)}{4}G(s,s)$, *for* $t \in [\frac{\sigma(1)}{4}, \frac{3\sigma(1)}{4}]$, $s \in [0,1]$.

Proof Observing the expression of $G(t,s)$, it is clear that $G(t,s) \ge 0$ for $t \in [0,\sigma(1)]$, $s \in [0,1]$, and $G(t,s) > 0$, for $t \in (0,\sigma(1))$, $s \in (0,1)$. Moreover, $G(t,s)$ is decreasing with respect to t for $s \le t$, and increasing for $t \le s$. By the fact

$$\frac{G(t,s)}{G(s,s)} = \begin{cases} \frac{t}{s}, & t \le s; \\ \frac{\sigma(1)-t}{\sigma(1)-s}, & s \le t.\end{cases}$$

We have

$$G(t,s) \le G(s,s),$$

for $t \in [0,\sigma(1)]$, $s \in [0,1]$. Furthermore, if $t \in [\frac{\sigma(1)}{4}, \frac{3\sigma(1)}{4}]$, $s \in [0,1]$, one has

$$\frac{G(t,s)}{G(s,s)} \ge \frac{\sigma(1)}{4},$$

which implies the desired results. □

Lemma 2.10 ([27]) *The following relations hold*:

(1) *If* $1 < q \le 2$, *then* $|\phi_q(u+v) - \phi_q(u)| \le 2^{2-q}|v|^{q-1}$ *for* $u, v \in \mathbb{R}$.

(2) *If* $q > 2$, *then* $|\phi_q(u+v) - \phi_q(u)| \le (q-1)(|u|+|v|)^{q-2}|v|$ *for* $u, v \in \mathbb{R}$.

Lemma 2.11 ([28]) *Suppose E is a Banach space and* $T_n : E \to E$, $n = 3, 4, \dots$ *are completely continuous operators*, $T : E \to E$. *If* $\|T_n u - Tu\|$ *uniformly to zero when* $n \to \infty$ *for all bounded set* $\Omega \subseteq E$, *then* $T : E \to E$ *is completely continuous.*

Definition 2.7 The map θ is said to be a nonnegative continuous concave functional on a cone P of a Banach space E provided that $\theta : P \to [0, \infty)$ is continuous and

$$\theta\bigl(tx + (1-t)y\bigr) \ge t\theta(x) + (1-t)\theta(y)$$

for all $x, y \in P$ and $0 < t < 1$.

The following fixed-point theorems are useful in our proofs.

Lemma 2.12 ([29]) *Let E be a Banach space*, $P \subseteq E$ *be a cone, and* Ω_1, Ω_2 *be two bounded open balls of E centered at the origin with* $\overline{\Omega_1} \subset \Omega_2$. *Suppose that* $\mathcal{A} : P \cap (\overline{\Omega_2} \backslash \Omega_1) \to P$ *is a completely continuous operator such that either*

(i) $\|\mathcal{A}x\| \le \|x\|$, $x \in P \cap \partial\Omega_1$, *and* $\|\mathcal{A}x\| \ge \|x\|$, $x \in P \cap \partial\Omega_2$, *or*

(ii) $\|\mathcal{A}x\| \ge \|x\|$, $x \in P \cap \partial\Omega_1$, *and* $\|\mathcal{A}x\| \le \|x\|$, $x \in P \cap \partial\Omega_2$,

holds. Then $\mathcal{A}$ *has a fixed point in* $P \cap (\overline{\Omega_2} \backslash \Omega_1)$.

Lemma 2.13 ([30]) *Let P be a cone in a real Banach space E*, $P_c = \{x \in P \mid \|x\| \le c\}$, θ *be a nonnegative continuous concave functional on P such that* $\theta(x) \le \|x\|$, *for all* $x \in \overline{P_c}$, *and* $P(\theta, b, d) = \{x \in P \mid b \le \theta(x), \|x\| \le d\}$. *Suppose* $\mathcal{A} : \overline{P_c} \to \overline{P_c}$ *is a completely continuous and there exist constants* $0 < a < b < d \le c$ *such that*

(C1) $\{x \in P(\theta, b, d) \mid \theta(x) > b\}$ *is nonempty, and* $\theta(\mathcal{A}x) > b$, *for* $x \in P(\theta, b, d)$;

(C2) $\|\mathcal{A}x\| < a$, *for* $x \le a$;

(C3) $\theta(\mathcal{A}x) > b$, *for* $x \in P(\theta, b, c)$ *with* $\|\mathcal{A}x\| > d$.

Then $\mathcal{A}$ *has at least three fixed points* x_1, x_2, x_3 *with*

$$\|x_1\| < a, \qquad b < \theta(x_2), \qquad a < \|x_3\|, \qquad \theta(x_3) < b.$$

Remark 2.1 ([30]) If we have $d = c$, then condition (C1) of Lemma 2.13 implies condition (C3) of Lemma 2.13.

3 Existence results

Let $E = \{u : [0, \sigma(1)] \to \mathbb{R}\}$ be endowed with the ordering $u \le v$ if $u(t) \le v(t)$ for all $t \in [0, \sigma(1)]$, and the norm $\|u\| = \max_{0 \le t \le \sigma(1)} |u(t)|$. Define

$$P = \left\{ u \in E \mid u(t) \ge 0 \text{ on } \bigl[0, \sigma(1)\bigr], u(t) \ge \frac{\sigma(1)}{4}\|u\| \text{ for } t \in \left[\frac{\sigma(1)}{4}, \frac{3\sigma(1)}{4}\right] \right\}.$$

Given a function $f \in C([0,\sigma(1)] \times [0,\infty),[0,\infty))$, define $T, T_n : P \to E$ as

$$(Tu)(t) := \int_0^{\sigma(1)} G(t,s)\phi_q\left(\int_0^{\sigma(1)} G(s,\tau)f\big(\tau,u(\tau)\big)\Delta\tau\right)\Delta s,$$
$$(T_nu)(t) := \int_{\frac{1}{n}}^{\sigma(1)} G(t,s)\phi_q\left(\int_{\frac{1}{n}}^{\sigma(1)} G(s,\tau)f\big(\tau,u(\tau)\big)\Delta\tau\right)\Delta s, \quad n = 3,4,\ldots.$$

Lemma 3.1 *$T : P \to P$ is completely continuous.*

Proof Firstly, take the constant in the second member to be independent on n, Hence, we show that $T_n : P \to P$ are completely continuous for $n = 3,4,\ldots$. Given $u \in P$, with Lemma 2.9 and the nonnegativity of $f(t,u)$, one has

$$\begin{aligned}(T_nu)(t) &= \int_{\frac{1}{n}}^{\sigma(1)} G(t,s)\phi_q\left(\int_{\frac{1}{n}}^{\sigma(1)} G(s,\tau)f\big(\tau,u(\tau)\big)\Delta\tau\right)\Delta s\\ &\le \int_{\frac{1}{n}}^{\sigma(1)} G(s,s)\phi_q\left(\int_{\frac{1}{n}}^{\sigma(1)} G(s,\tau)f\big(\tau,u(\tau)\big)\Delta\tau\right)\Delta s,\end{aligned}$$

so

$$\|T_nu\| \le \int_{\frac{1}{n}}^{\sigma(1)} G(s,s)\phi_q\left(\int_{\frac{1}{n}}^{\sigma(1)} G(s,\tau)f\big(\tau,u(\tau)\big)\Delta\tau\right)\Delta s.$$

For $u \in P$,

$$\begin{aligned}\min_{\frac{\sigma(1)}{4}\le t\le\frac{3\sigma(1)}{4}}(T_nu)(t) &= \min_{\frac{\sigma(1)}{4}\le t\le\frac{3\sigma(1)}{4}}\int_{\frac{1}{n}}^{\sigma(1)} G(t,s)\phi_q\left(\int_{\frac{1}{n}}^{\sigma(1)} G(s,\tau)f\big(\tau,u(\tau)\big)\Delta\tau\right)\Delta s\\ &\ge \frac{\sigma(1)}{4}\int_{\frac{1}{n}}^{\sigma(1)} G(s,s)\phi_q\left(\int_{\frac{1}{n}}^{\sigma(1)} G(s,\tau)f\big(\tau,u(\tau)\big)\Delta\tau\right)\Delta s.\end{aligned}$$

It follows that

$$\min_{\frac{\sigma(1)}{4}\le t\le\frac{3\sigma(1)}{4}}(T_nu)(t) \ge \frac{\sigma(1)}{4}\|T_nu\|.$$

Hence, $T_nu \in P$, and so $T_n : P \to P$. Let $\Omega \subset P$ be bounded, i.e., there exists a positive constant $M > 0$ such that $\|u\| \le M$ for all $u \in \Omega$. Let

$$L = \max_{0\le t\le\sigma(1),0\le u\le M}\big|f(t,u)\big| + 1, \qquad H = \int_0^{\sigma(1)} G(s,s)\Delta s + 1,$$

then, for $u \in \Omega$, we have

$$\begin{aligned}\big|(T_nu)(t)\big| &= \int_{\frac{1}{n}}^{\sigma(1)} G(t,s)\phi_q\left(\int_{\frac{1}{n}}^{\sigma(1)} G(s,\tau)f\big(\tau,u(\tau)\big)\Delta\tau\right)\Delta s\\ &\le L^{q-1}H^q < +\infty.\end{aligned}$$

Hence, $T_n(\Omega)$ is bounded for $n = 3,4,\ldots$.

On the other hand, given $\epsilon > 0$, let

$$\delta = \frac{\epsilon}{2\sigma(1)L^{q-1}H^{q-1}},$$

then, for each $u \in \Omega$, $t_1, t_2 \in [0, \sigma(1)]$, $t_1 \leq t_2$, and $t_2 - t_1 < \delta$, one has

$$\left|(T_n u)(t_2) - (T_n u)(t_1)\right| < \epsilon.$$

That is to say $T_n(\Omega)$ has equicontinuity. In fact, we consider three situations.

(1) $0 < t_1 \leq t_2 < \frac{1}{n}$.

$$\begin{aligned}
&\left|(T_n u)(t_2) - (T_n u)(t_1)\right| \\
&\quad = \left| \int_{\frac{1}{n}}^{\sigma(1)} G(t_2,s)\phi_q\left(\int_{\frac{1}{n}}^{\sigma(1)} G(s,\tau)f\bigl(\tau,u(\tau)\bigr)\Delta\tau\right)\Delta s \right. \\
&\qquad \left. - \int_{\frac{1}{n}}^{\sigma(1)} G(t_1,s)\phi_q\left(\int_{\frac{1}{n}}^{\sigma(1)} G(s,\tau)f\bigl(\tau,u(\tau)\bigr)\Delta\tau\right)\Delta s \right| \\
&\quad \leq L^{q-1}H^{q-1}\int_{\frac{1}{n}}^{\sigma(1)} \left|G(t_2,s) - G(t_1,s)\right|\Delta s \\
&\quad = \frac{1}{\sigma(1)}L^{q-1}H^{q-1}\int_{\frac{1}{n}}^{\sigma(1)} (t_2 - t_1)s^{\alpha-2}\bigl(\sigma(1) - s\bigr)\Delta s \\
&\quad \leq \frac{1}{\sigma(1)}L^{q-1}H^{q-1}(t_2 - t_1)\int_0^{\sigma(1)} \bigl[\sigma(1) - s\bigr]\Delta s \\
&\quad \leq L^{q-1}H^{q-1}(t_2 - t_1)\sigma(1) \\
&\quad < \epsilon.
\end{aligned}$$

(2) $0 < t_1 \leq \frac{1}{n} \leq t_2 < 1$.

$$\begin{aligned}
&\left|(T_n u)(t_2) - (T_n u)(t_1)\right| \\
&\quad \leq L^{q-1}H^{q-1}\left(\int_{\frac{1}{n}}^{t_2} \left|G(t_2,s) - G(t_1,s)\right|\Delta s + \int_{t_2}^{\sigma(1)} \left|G(t_2,s) - G(t_1,s)\right|\Delta s\right) \\
&\quad \leq L^{q-1}H^{q-1}\frac{1}{\sigma(1)}\left(\int_{\frac{1}{n}}^{t_2} \bigl[(t_1 - s)s^{\alpha-2}\sigma(1) + (t_2 - t_1)s^{\alpha-1}\bigr]\Delta s \right. \\
&\qquad \left. + \int_{t_2}^{\sigma(1)} (t_2 - t_1)s^{\alpha-2}\bigl(\sigma(1) - s\bigr)\Delta s\right) \\
&\quad \leq \frac{1}{\sigma(1)}L^{q-1}H^{q-1}(t_2 - t_1)\int_0^{\sigma(1)} \bigl(s^{\alpha-2}\sigma(1) + s^{\alpha-1}\bigr)\Delta s \\
&\quad \leq \frac{1}{\sigma(1)}L^{q-1}H^{q-1}(t_2 - t_1)\int_0^{\sigma(1)} \bigl[\sigma(1) + s\bigr]\Delta s \\
&\quad \leq 2\frac{1}{\sigma(1)}L^{q-1}H^{q-1}(t_2 - t_1)\int_0^{\sigma(1)} \sigma(1)\Delta s \\
&\quad \leq 2L^{q-1}H^{q-1}\sigma(1)(t_2 - t_1) \\
&\quad < \epsilon.
\end{aligned}$$

(3) $\frac{1}{n} < t_1 \leq t_2 < 1$.

$$
\begin{aligned}
&\left|(T_n u)(t_2) - (T_n u)(t_1)\right| \\
&\leq L^{q-1}H^{q-1}\left(\int_{\frac{1}{n}}^{t_1}\left|G(t_2,s) - G(t_1,s)\right|\Delta s + \int_{t_1}^{t_2}\left|G(t_2,s) - G(t_1,s)\right|\Delta s \right.\\
&\quad \left. + \int_{t_2}^{\sigma(1)}\left|G(t_2,s) - G(t_1,s)\right|\Delta s\right) \\
&= \frac{1}{\sigma(1)}L^{q-1}H^{q-1}\left[\int_{t_1}^{t_2}\left(\left(\sigma(1) - t_2\right)s^{\alpha-1} - t_1 s^{\alpha-2}\left(\sigma(1) - s\right)\right)\Delta s \right.\\
&\quad \left. + \int_{\frac{1}{n}}^{t_1}(t_2 - t_1)s^{\alpha-1}\Delta s + \int_{t_2}^{\sigma(1)}\left((t_2 - t_1)s^{\alpha-2}\sigma(1) - s^{\alpha-1}\right)\Delta s\right] \\
&\leq \frac{1}{\sigma(1)}L^{q-1}H^{q-1}(t_2 - t_1)\int_0^{\sigma(1)}\left(s^{\alpha-2}\sigma(1) + s^{\alpha-1}\right)\Delta s \\
&\leq \frac{1}{\sigma(1)}L^{q-1}H^{q-1}(t_2 - t_1)\int_0^{\sigma(1)}\left[\sigma(1) + s\right]\Delta s \\
&< \epsilon.
\end{aligned}
$$

By the means of the Arzela–Ascoli theorem, we see that $T_n : P \to P$ are completely continuous operators.

Secondly, it is clear that $T : P \to P$. We prove that $T_n : P \to P$ have uniform convergence to T and $T : P \to P$ is completely continuous too.

With the use of Lemma 2.10,

$$\phi_q(A + B) < \phi_q(A) + 2\phi_q(B) + (q-1)(A + B)^{q-2}B.$$

Given $\epsilon > 0$, let

$$K = \left(\frac{\sigma(1)\cdot(2L^{q-1}H + qL^{q-1}H^{q-1})}{\epsilon}\right),$$

then $\|T_n u - Tu\| < \epsilon$, for all $n > N$. In fact,

$$
\begin{aligned}
&\|T_n u - Tu\| \\
&= \max_{0\leq t\leq\sigma(1)}\left|(T_n u)(t) - (Tu)(t)\right| \\
&= \max_{0\leq t\leq\sigma(1)}\left|\int_0^{\sigma(1)}G(t,s)\phi_q\left(\int_0^{\sigma(1)}G(s,\tau)f\left(\tau,u(\tau)\right)\Delta\tau\right)\Delta s \right.\\
&\quad \left. - \int_{\frac{1}{n}}^{\sigma(1)}G(t,s)\phi_q\left(\int_{\frac{1}{n}}^{\sigma(1)}G(s,\tau)f\left(\tau,u(\tau)\right)\Delta\tau\right)\Delta s\right| \\
&< \max_{0\leq t\leq\sigma(1)}\int_0^{\sigma(1)}G(t,s)\left[\phi_q\left(\int_{\frac{1}{n}}^{\sigma(1)}G(s,\tau)f\left(\tau,u(\tau)\right)\Delta\tau\right)\right.\\
&\quad + 2\phi_q\left(\int_0^{\frac{1}{n}}G(s,\tau)f\left(\tau,u(\tau)\right)\Delta\tau\right)
\end{aligned}
$$

$$
\begin{aligned}
&\quad + (q-1)\left(\int_0^{\sigma(1)} G(s,\tau)f\big(\tau,u(\tau)\big)\Delta\tau\right)^{q-2}\int_0^{\frac{1}{n}} G(s,\tau)f\big(\tau,u(\tau)\big)\Delta\tau\Bigg]\Delta s \\
&\quad - \int_{\frac{1}{n}}^{\sigma(1)} G(t,s)\phi_q\left(\int_{\frac{1}{n}}^{\sigma(1)} G(s,\tau)f\big(s,u(s)\big)\Delta\tau\right)\Delta s \\
&\leq \max_{0\leq t\leq\sigma(1)}\int_0^{\frac{1}{n}} G(t,s)\phi_q\left(\int_{\frac{1}{n}}^{\sigma(1)} G(s,\tau)f\big(\tau,u(\tau)\big)\Delta\tau\right)\Delta s \\
&\quad + 2L^{q-1}\int_0^{\sigma(1)} G(s,s)\Delta s\phi_q\left(\int_0^{\frac{1}{n}} G(\tau,\tau)\Delta\tau\right) \\
&\quad + (q-1)L^{q-1}\int_0^{\sigma(1)} G(s,s)\Delta s\left(\int_0^{\sigma(1)} G(\tau,\tau)\Delta\tau\right)^{q-2}\int_0^{\frac{1}{n}} G(\tau,\tau)\Delta\tau \\
&\leq \sigma(1)\cdot\left(L^{q-1}H^{q-1} + 2L^{q-1}H + (q-1)L^{q-1}H^{q-1}\right)\cdot\left(\frac{1}{n}\right) \\
&< \epsilon.
\end{aligned}
$$

By the use of Lemma 2.11, $T: P \to P$ is completely continuous. □

We take into account that the Green's function satisfy $G(t,s) \geq 0$ for $t \in [0,\sigma(1)]$, $s \in [0,1]$, and $G(t,s) > 0$, for $t \in (0,\sigma(1))$, $s \in (0,1)$. The following constants are well defined:

$$
\begin{aligned}
M &= \left(\int_0^{\sigma(1)} G(s,s)\Delta s\phi_q\left(\int_0^{\sigma(1)} G(\tau,\tau)\Delta\tau\right)\right)^{-1}, \\
N &= \left(\int_{\frac{\sigma(1)}{4}}^{\frac{3\sigma(1)}{4}} \frac{\sigma(1)}{4}G(s,s)\Delta s\phi_q\left(\int_{\frac{\sigma(1)}{4}}^{\frac{3\sigma(1)}{4}} \frac{\sigma(1)}{4}G(\tau,\tau)\Delta\tau\right)\right)^{-1}.
\end{aligned}
$$

Theorem 3.1 *Let $f \in C([0,\sigma(1)] \times [0,\infty), [0,\infty))$. Assume that there exist two different positive constants r_2, r_1, and $r_2 \neq r_1$ such that*

(H1) *$f(t,u) \leq \phi_p(Mr_1)$, for $(t,u) \in [0,\sigma(1)] \times [0,r_1]$;*

(H2) *$f(t,u) \geq \phi_p(Nr_2)$, for $(t,u) \in [\frac{\sigma(1)}{4}, \frac{3\sigma(1)}{4}] \times [\frac{\sigma(1)}{4}r_2, r_2]$.*

Then Problem (1.1), (1.2) *has at least one positive solution u such that* $\min\{r_2,r_1\} \leq \|u\| \leq \max\{r_2,r_1\}$.

Proof By Lemma 3.1, $T: P \to P$ is completely continuous. Without loss of generality, suppose $0 < r_1 < r_2$, and let

$$
\Omega_1 := \big\{u \in P \mid \|u\| < r_1\big\}, \qquad \Omega_2 := \big\{u \in P \mid \|u\| < r_2\big\}.
$$

For $u \in \partial\Omega_1$, we have $0 \leq u(t) \leq r_1$ for all $t \in [0,\sigma(1)]$. It follows from (H1) that

$$
\begin{aligned}
\|Tu\| &= \max_{0\leq t\leq\sigma(1)}\left|\int_0^{\sigma(1)} G(t,s)\Phi_q\left(\int_0^{\sigma(1)} G(s,\tau)f\big(\tau,u(\tau)\big)\Delta\tau\right)\Delta s\right| \\
&\leq Mr_1\int_0^{\sigma(1)} G(s,s)\Delta s\Phi_q\left(\int_0^{\sigma(1)} G(\tau,\tau)\Delta\tau\right) = r_1 = \|u\|.
\end{aligned}
$$

So,

$$
\|Tu\| \leq \|u\|, \quad \text{for } u \in \partial\Omega_1.
$$

For $u \in \partial\Omega_2$, by the definition of P, we have

$$\min_{\frac{\sigma(1)}{4} \le t \le \frac{3\sigma(1)}{4}} u(t) \ge \frac{\sigma(1)}{4}\|u\| = \frac{\sigma(1)}{4} r_2.$$

By assumption (H2), for $t \in [\frac{\sigma(1)}{4}, \frac{3\sigma(1)}{4}]$, we have

$$\begin{aligned}
(Tu)(t) &= \int_0^{\sigma(1)} G(t_0,s)\phi_q\left(\int_0^{\sigma(1)} G(s,\tau)f\big(\tau,u(\tau)\big)\Delta\tau\right)\Delta s \\
&\ge \int_0^{\sigma(1)} \frac{\sigma(1)}{4} G(s,s)\phi_q\left(\int_{\frac{\sigma(1)}{4}}^{\frac{3\sigma(1)}{4}} \frac{\sigma(1)}{4} G(\tau,\tau)f\big(\tau,u(\tau)\big)\Delta\tau\right)\Delta s \\
&\ge Nr_2 \int_{\frac{\sigma(1)}{4}}^{\frac{3\sigma(1)}{4}} \frac{\sigma(1)}{4} G(s,s)\phi_q\left(\int_{\frac{\sigma(1)}{4}}^{\frac{3\sigma(1)}{4}} \frac{\sigma(1)}{4} G(\tau,\tau)\Delta\tau\right)\Delta s \\
&= r_2 = \|u\|.
\end{aligned}$$

So,

$$\|Tu\| \ge \|u\|, \quad \text{for } u \in \partial\Omega_2.$$

Therefore, by Lemma 2.12, we complete the proof. □

Theorem 3.2 *Suppose* $f \in C([0,\sigma(1)] \times [0,\infty), [0,\infty))$ *and there exist constants* $0 < a < b < c$ *such that the following assumptions hold*:

(A1) $f(t,u) \le \phi_p(Ma)$, *for* $(t,u) \in [0,\sigma(1)] \times [0,a]$;

(A2) $f(t,u) \ge \phi_p(Nb)$, *for* $(t,u) \in [\frac{\sigma(1)}{4}, \frac{3\sigma(1)}{4}] \times [b,c]$;

(A3) $f(t,u) \le \phi_p(Mc)$, *for* $(t,u) \in [0,\sigma(1)] \times [0,c]$.

Then the boundary-value problem (1.1), (1.2) *has at least three positive solutions* u_1, u_2, u_3 *with*

$$\max_{0\le t\le\sigma(1)} |u_1(t)| < a, \qquad b < \min_{\frac{\sigma(1)}{4} \le t \le \frac{3\sigma(1)}{4}} |u_2(t)| < \max_{0\le t\le\sigma(1)} |u_2(t)| \le c,$$

$$a < \max_{0\le t\le\sigma(1)} |u_3(t)| \le c, \qquad \min_{\frac{\sigma(1)}{4} \le t \le \frac{3\sigma(1)}{4}} |u_3(t)| < b.$$

Proof We show that all the conditions of Lemma 2.13 are satisfied. If $u \in \overline{P}_c$, then $\|u\| \le c$. Assumption (A3) implies $f(t,u(t)) \le \phi_p(Mc)$ for $0 \le t \le \sigma(1)$, consequently,

$$\begin{aligned}
\|Tu\| &= \max_{0\le t\le\sigma(1)} \left| \int_0^{\sigma(1)} G(t,s)\phi_q\left(\int_0^{\sigma(1)} G(s,\tau)f\big(\tau,u(\tau)\big)\Delta\tau\right)\Delta s \right| \\
&\le \int_0^{\sigma(1)} G(s,s)\phi_q\left(\int_0^{\sigma(1)} G(\tau,\tau)f\big(\tau,u(\tau)\big)\Delta\tau\right)\Delta s \\
&\le Mc \int_0^{\sigma(1)} G(s,s)\Delta s\phi_q\left(\int_0^{\sigma(1)} G(\tau,\tau)\Delta\tau\right) \le c.
\end{aligned}$$

Hence, $T : \overline{P}_c \to \overline{P}_c$. Similarly, if $u \in \overline{P}_a$, then assumption (A1) yields $f(t,u(t)) \le \phi_p(Ma)$, $0 \le t \le \sigma(1)$. Therefore, condition (C2) of Lemma 2.13 is satisfied.

Choose

$$u(t) = \frac{b+c}{2}, \quad 0 \le t \le \sigma(1).$$

Then $u(t) \in P(\theta, b, c)$, $\theta(u) = \theta(\frac{b+c}{2}) > b$, consequently,

$$\{u \in P(\theta, b, c) \mid \theta(u) > b\} \neq \emptyset.$$

Hence, if $u \in P(\theta, b, c)$, then $b \le u(t) \le c$ for $\frac{\sigma(1)}{4} \le t \le \frac{3\sigma(1)}{4}$. From assumption (A3), we have $f(t, u(t)) \ge \phi_p(Nb)$ for $\frac{\sigma(1)}{4} \le t \le \frac{3\sigma(1)}{4}$. So

$$\begin{aligned}
\theta(Tu) &= \min_{\frac{\sigma(1)}{4} \le t \le \frac{3\sigma(1)}{4}} \int_0^{\sigma(1)} G(t,s)\Phi_q\left(\int_0^{\sigma(1)} G(s,\tau) f\big(\tau, u(\tau)\big)\Delta\tau\right)\Delta s \\
&> Nb \int_{\frac{\sigma(1)}{4}}^{\frac{3\sigma(1)}{4}} \frac{\sigma(1)}{4} G(s,s)\phi_q\left(\int_{\frac{\sigma(1)}{4}}^{\frac{3\sigma(1)}{4}} \frac{\sigma(1)}{4} G(\tau,\tau)\Delta\tau\right)\Delta s \\
&= b,
\end{aligned}$$

i.e.,

$$\theta(Tu) > b, \quad \text{for all } u \in P(\theta, b, c).$$

This shows that condition (C1) of Lemma 2.13 is satisfied.

By Lemma 2.13 and Remark 2.1, Problem (1.1), (1.2) has at least three positive solutions u_1, u_2, u_3, satisfying

$$\begin{aligned}
&\max_{0 \le t \le \sigma(1)} \big|u_1(t)\big| < a, \qquad b < \min_{\frac{\sigma(1)}{4} \le t \le \frac{3\sigma(1)}{4}} \big|u_2(t)\big|, \\
&a < \max_{0 \le t \le \sigma(1)} \big|u_3(t)\big|, \qquad \min_{\frac{\sigma(1)}{4} \le t \le \frac{3\sigma(1)}{4}} \big|u_3(t)\big| < b.
\end{aligned}$$

The proof is complete. □

4 Examples

Example 4.1 Let $T = \mathbb{R}$, $\alpha = \frac{3}{2}$, $p = 3$, consider the following fractional differential equation boundary-value problem:

$$D^{\frac{3}{2}}\big(\phi_3\big(D^{\frac{3}{2}} u(t)\big)\big) = 1 + t + \sin u, \quad t \in [0,1]_T, \tag{4.1}$$

$$u(0) = u(1) = D^{\frac{3}{2}} u(0) = D^{\frac{3}{2}} u(1) = 0. \tag{4.2}$$

By a simple computation, we obtain $M = 3.75$, $N \approx 5.987$. Choose $r_1 = 1$, $r_2 = \frac{1}{5}$, then

$$\begin{aligned}
&f(t,u) = 1 + t + \sin u \le 3.5 < \phi_p(Mr_1) = 3.75, \quad \text{for } (t,u) \in [0,1] \times [0,1], \\
&f(t,u) = 1 + t + \sin u \ge 2 > \phi_p(Nr_2) \approx 1.1974, \quad \text{for } (t,u) \in \left[\frac{1}{4}, \frac{3}{4}\right] \times \left[\frac{1}{20}, \frac{1}{5}\right].
\end{aligned}$$

With the use of Theorem 3.1, the fractional differential equation boundary-value problem (4.1) and (4.2) has at least one positive solution u such that $\frac{1}{5} \leq \|u\| \leq 1$.

Example 4.2 Let $T = \mathbb{R}$, consider the following fractional differential equation boundary-value problem:

$$D^{\frac{3}{2}}\big(\phi_3\big(D^{\frac{3}{2}}u(t)\big)\big) = f(t,u), \quad t \in [0,1]_T, \tag{4.3}$$

$$u(0) = u(1) = D^{\frac{3}{2}}u(0) = D^{\frac{3}{2}}u(1) = 0, \tag{4.4}$$

where

$$f(t,u) = \begin{cases} 2u + \frac{1}{10}t, & u \leq 1; \\ 6 + 7(u-1)^2 + \frac{1}{10}t, & u \geq 1. \end{cases}$$

We obtain $M = 3.75$, $N \approx 5.987$. Choose $a = 0.1$, $b = 1$, $c = 4$, then

$$f(t,u) = 2u + \frac{1}{10}t < 0.3 < \phi_p(Ma) = 0.375, \quad \text{for } (t,u) \in [0,1] \times [0,0.1],$$

$$f(t,u) = 6 + 7(u-1)^2 + \frac{t}{10} \geq 2 > \phi_p\left(\frac{Nb}{4}\right) \approx 1.497, \quad (t,u) \in \left[\frac{1}{4}, \frac{3}{4}\right] \times [1,4],$$

$$f(t,u) = 6 + 7(u-1)^2 + \frac{t}{10} \leq 9.1 < \phi_p(Mc) = 15, \quad (t,u) \in [0,1] \times [0,4].$$

With the use of Theorem 3.2, the fractional differential equation boundary-value problem (4.3) and (4.4) has at least three positive solutions u_1, u_2 and u_3 with

$$\max_{0 \leq t \leq 1}\big|u_1(t)\big| < 0.1, \qquad 1 < \min_{\frac{1}{4} \leq t \leq \frac{3}{4}}\big|u_2(t)\big| < \max_{0 \leq t \leq 1}\big|u_2(t)\big| \leq 4,$$

$$0.1 < \max_{0 \leq t \leq 1}\big|u_3(t)\big| \leq 4, \qquad \min_{\frac{1}{4} \leq t \leq \frac{3}{4}}\big|u_3(t)\big| < 1.$$

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