Wang et al. *Boundary Value Problems* (2018) 2018:94
https://doi.org/10.1186/s13661-018-1012-0

Boundary Value Problems
a SpringerOpen Journal

RESEARCH **Open Access**

Infinitely many solutions for impulsive fractional boundary value problem with p-Laplacian

Yang Wang[1], Yansheng Liu[1*] and Yujun Cui[2]

*Correspondence:
ysliu@sdnu.edu.cn
[1]School of Mathematics and Statistics, Shandong Normal University, Jinan, P.R. China
Full list of author information is available at the end of the article

Abstract

This paper deals with the existence of infinitely many solutions for a class of impulsive fractional boundary value problems with p-Laplacian. Based on a variant fountain theorem, the existence of infinitely many nontrivial high or small energy solutions is obtained. In addition, two examples are worked out to illustrate the effectiveness of the main results.

MSC: 26A33; 34B15

Keywords: Fractional p-Laplacian; Variant fountain theorems; Impulsive effects; Infinitely many solutions

1 Introduction

Consider the following nonlinear impulsive fractional boundary value problem (BVP, for short):

$$\begin{cases} D_{T^-}^{\alpha}\Phi_p({}^cD_{0^+}^{\alpha}u(t)) + |u(t)|^{p-2}u(t) = f(t,u(t)), & t \in [0,T], t \neq t_j, \\ \Delta(D_{T^-}^{\alpha-1}\Phi_p({}^cD_{0^+}^{\alpha}u))(t_j) = I_j(u(t_j)), & j = 1,2,\ldots,m, \\ u(0) = u(T) = 0, \end{cases} \tag{1.1}$$

where $\alpha \in (\frac{1}{p}, 1]$, $p > 1$, $\Phi_p(s) = |s|^{p-2}s$, $D_{T^-}^{\alpha}$ represents the right Riemann–Liouville fractional derivative of order α and ${}^cD_{0^+}^{\alpha}$ represents the left Caputo fractional derivative of order α, $0 = t_0 < t_1 < \cdots < t_{m+1} = T$ and

$$\begin{aligned} \Delta\big(D_{T^-}^{\alpha-1}\Phi_p\big({}^cD_{0^+}^{\alpha}u\big)\big)(t_j) &= D_{T^-}^{\alpha-1}\Phi_p\big({}^cD_{0^+}^{\alpha}u\big)\big(t_j^+\big) - D_{T^-}^{\alpha-1}\Phi_p\big({}^cD_{0^+}^{\alpha}u\big)\big(t_j^-\big), \\ D_{T^-}^{\alpha-1}\Phi_p\big({}^cD_{0^+}^{\alpha}u\big)\big(t_j^+\big) &= \lim_{t\to t_j^+} D_{T^-}^{\alpha-1}\Phi_p\big({}^cD_{0^+}^{\alpha}u\big)(t), \\ D_{T^-}^{\alpha-1}\Phi_p\big({}^cD_{0^+}^{\alpha}u\big)\big(t_j^-\big) &= \lim_{t\to t_j^-} D_{T^-}^{\alpha-1}\Phi_p\big({}^cD_{0^+}^{\alpha}u\big)(t). \end{aligned}$$

$f : [0,T] \times \mathbb{R} \to \mathbb{R}$ and $I_j : \mathbb{R} \to \mathbb{R}$ are continuous.

Fractional differential equations have gained importance because of their numerous applications in various fields such as chemical physics, neural network model, signal processing and control, mechanics and engineering, fractal theory, and so on. For details, see [1–5] and the references therein. Recently, the existence and multiplicity of solutions for nonlinear fractional differential equations have been studied extensively by using the theory of coincidence degree, some fixed point theorems, upper-lower solution method, monotone iterative method, etc. [6–9]. It should be noted that the critical point theory and variational methods have proved to be a very effective approach in dealing with the existence and multiple solutions for fractional boundary value problems, see [10–17].

On the other hand, the impulsive differential equation is used to describe the dynamics of processes in which sudden, discontinuous jumps occur. It has numerous applications in many fields such as population dynamics, ecology, optimal control, economics, and so on. For details, see [18–21] and the references therein. Recently, many authors have studied the existence of solutions for impulsive fractional boundary value problem by using variational methods and critical point theory, see [22–30].

For example, Heidarkhani et al. [26] and [28] studied the following impulsive nonlinear fractional boundary value problem:

$$\begin{cases} D_{T^-}^{\alpha}({}^cD_{0^+}^{\alpha}u(t)) + a(t)u(t) = \lambda f(t,u(t)) + h(u(t)), \quad t \in [0,T], t \neq t_j, \\ \Delta(D_{T^-}^{\alpha-1}({}^cD_{0^+}^{\alpha}u))(t_j) = \mu I_j(u(t_j)), \quad j = 1,2,\dots,m, \\ u(0) = u(T) = 0. \end{cases} \tag{1.2}$$

Based on variational methods and critical point theory, they obtained the existence results of infinitely many classical solutions and three solutions for problem (1.2). In particular, Rodríguez-López and Tersian [22] established one and three solutions for problem (1.2) when $h(u(t)) \equiv 0$. In [25], Heidarkhani and Salari obtained the existence of two and three weak solutions for a class of nonlinear impulsive fractional systems by applying variational methods.

Furthermore, the p-Laplacian often occurs in non-Newtonian fluid theory, nonlinear elastic mechanics, and so on. So, the impulsive fractional boundary value problem with p-Laplacian is worth considering. For instance, in [31], basing on the mountain pass theorem and minimax methods, the existence of multiple solutions for BVP(1.1) is obtained.

To the best of our knowledge, there are fewer results on the existence and multiplicity of solutions for impulsive fractional boundary value problem with p-Laplacian. Inspired by the above references, we apply variant fountain theorems to study the existence of infinitely many small or high energy solutions for BVP (1.1). The main new features presented in this paper are as follows. Firstly, the main results of this paper are different from those in the aforementioned references, and extend the results obtained in [31]. Secondly, the main tool of this paper is variant fountain theorems, which is different from the aforementioned papers. Thirdly, the assumed conditions in this paper are easier to verify than those in [31]. Finally, two examples are worked out to demonstrate the effectiveness of our results.

For convenience, we list the following assumptions.

(H_1) $I_j(u)$ $(j = 1,2,\dots,m)$ are odd about u and satisfy $\int_0^u I_j(s)\,ds \geq 0$ for all $u \in \mathbb{R}$.

(H_2) There exist $b_j > 0$ and $\gamma_j \in (p-1,+\infty)$ such that

$$|I_j(u)| \leq b_j|u|^{\gamma_j}.$$

(H_3) There exist $b_j > 0$, $\mu > p$, and $\gamma_j \in (p-1, \mu-1)$ such that

$$\left|I_j(u)\right| \le b_j|u|^{\gamma_j}, \qquad I_j(u)u \le \mu \int_0^u I_j(s)\,ds.$$

(F_1) There exist $\eta \in (p-1,p)$ and $b(t) \in L^{\frac{p}{p-\eta}}[0,T]$ with $b(t) \ge 0$ such that

$$\left|f(t,u)\right| \le b(t)\left(1+|u|^{\eta-1}\right), \quad \forall (t,u) \in [0,T]\times\mathbb{R}.$$

(F_2) There exist $\sigma \in (p-1,\eta)$ and $d > 0$ such that

$$\lim_{|u|\to\infty} \frac{F(t,u)}{|u|^{\sigma}} > d, \quad \text{uniformly for } t \in [0,T], \text{ where } F(t,u) = \int_0^u f(t,s)\,ds.$$

(F_3) $\lim_{|u|\to 0} \frac{f(t,u)}{|u|^{p-1}} = 0$, uniformly for $t \in [0,T]$.
(F_4) $F(t,u) \ge 0$, $\forall (t,u) \in [0,T]\times\mathbb{R}$.
(F_5) $F(t,-u) = F(t,u)$, $\forall (t,u) \in [0,T]\times\mathbb{R}$.
(F_6) There exist constants $\theta_1 > 0$, $\theta_2 > 0$, and $q > p$ such that

$$\left|f(t,u)\right| \le \theta_1|u|^{p-1} + \theta_2|u|^{q-1} \quad \text{for all } t \in [0,T], u \in \mathbb{R}.$$

(F_7) There exists $\mu > p$ such that

$$-\mu F(t,u) + uf(t,u) \ge 0 \quad \text{for all } t \in [0,T], u \in \mathbb{R}.$$

Here are our main results.

Theorem 1.1 *Assume that (H_1)–(H_2) and (F_1)–(F_5) hold. Then BVP (1.1) possesses infinitely many small energy solutions $u_k \in E \setminus \{0\}$ satisfying*

$$\begin{aligned}&\frac{1}{p}\int_0^T \left(\left|{}^cD_{0^+}^{\alpha}u_k(t)\right|^p + \left|u_k(t)\right|^p\right)dt \\ &\quad + \sum_{j=1}^m \int_0^{u_k(t_j)} I_j(s)\,ds - \int_0^T F\big(t,u_k(t)\big)\,dt \to 0^- \quad \text{as } k \to \infty.\end{aligned}$$

Theorem 1.2 *Assume that (H_1), (H_3) and (F_4)–(F_7) hold. Then BVP (1.1) possesses infinitely many high energy solutions $u^k \in E \setminus \{0\}$ satisfying*

$$\begin{aligned}&\frac{1}{p}\int_0^T \left(\left|{}^cD_{0^+}^{\alpha}u^k(t)\right|^p + \left|u^k(t)\right|^p\right)dt \\ &\quad + \sum_{j=1}^m \int_0^{u^k(t_j)} I_j(s)\,ds - \int_0^T F\big(t,u^k(t)\big)\,dt \to \infty \quad \text{as } k \to \infty.\end{aligned}$$

The rest of this paper is organized as follows. Section 2 contains some preliminary results. In Sect. 3, we apply variant fountain theorems to prove the existence of infinitely many small or high energy solutions for BVP (1.1). In Sect. 4, two examples are presented to illustrate the main results.

2 Preliminaries

To obtain multiple solutions for BVP (1.1), it is necessary to introduce several definitions and preliminary lemmas which are used further in this paper.

Let $\mathrm{AC}[a,b]$ be the space of absolutely continuous functions on $[a,b]$.

Definition 2.1 ([10]) Let f be a function defined on $[a,b]$ and $0<\alpha\leq 1$. The left and right Riemann–Liouville fractional integrals of order α for the function f are defined by

$$D_{a^+}^{-\alpha}f(t)=\frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}f(s)\,ds,\quad t\in[a,b],$$
$$D_{b^-}^{-\alpha}f(t)=\frac{1}{\Gamma(\alpha)}\int_t^b (s-t)^{\alpha-1}f(s)\,ds,\quad t\in[a,b],$$

while the right-hand side is pointwise defined on $[a,b]$.

Definition 2.2 ([10]) Let $f\in\mathrm{AC}[a,b]$ and $0<\alpha\leq 1$. The left and right Riemann–Liouville fractional derivatives of order α for the function f are defined by

$$D_{a^+}^{\alpha}f(t)=\frac{d}{dt}D_{a^+}^{\alpha-1}f(t)=\frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_a^t (t-s)^{-\alpha}f(s)\,ds,\quad t\in[a,b],$$
$$D_{b^-}^{\alpha}f(t)=-\frac{d}{dt}D_{b^-}^{\alpha-1}f(t)=-\frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_t^b (s-t)^{-\alpha}f(s)\,ds,\quad t\in[a,b].$$

Definition 2.3 ([10]) Let $f\in\mathrm{AC}[a,b]$ and $0<\alpha\leq 1$. The left and right Caputo fractional derivatives of order α for the function f are defined by

$${}^cD_{a^+}^{\alpha}f(t)=D_{a^+}^{\alpha-1}f'(t)=\frac{1}{\Gamma(1-\alpha)}\int_a^t (t-s)^{-\alpha}f'(s)\,ds,\quad t\in[a,b],$$
$${}^cD_{b^-}^{\alpha}f(t)=-D_{b^-}^{\alpha-1}f'(t)=-\frac{1}{\Gamma(1-\alpha)}\int_t^b (s-t)^{-\alpha}f'(s)\,ds,\quad t\in[a,b].$$

In particular, when $\alpha=1$, we have ${}^cD_{a^+}^{1}f(t)=f'(t)$ and ${}^cD_{b^-}^{1}f(t)=-f'(t)$.

Lemma 2.4 ([32])

(1) *If* $u\in L^p[a,b]$, $v\in L^p[a,b]$, *and* $p\geq 1$, $q\geq 1$, $\frac{1}{p}+\frac{1}{q}\leq 1+\alpha$ *or* $p\neq 1$, $q\neq 1$, $\frac{1}{p}+\frac{1}{q}=1+\alpha$, *then*

$$\int_a^b \big(D_{a^+}^{-\alpha}u(t)\big)v(t)\,dt=\int_a^b u(t)\big(D_{b^-}^{-\alpha}v(t)\big)\,dt.$$

(2) *If* $0<\alpha\leq 1$, $u\in\mathrm{AC}[a,b]$, *and* $v\in L^p[a,b]$ $(1\leq p<\infty)$, *then*

$$\int_a^b u(t)\big({}^cD_{a^+}^{\alpha}v(t)\big)\,dt=D_{b^-}^{\alpha-1}u(t)v(t)\Big|_{t=a}^{t=b}+\int_a^b D_{b^-}^{\alpha}u(t)v(t)\,dt.$$

Denote

$$\|u\|_{L^p}=\left(\int_0^T \big|u(t)\big|^p\,dt\right)^{\frac{1}{p}},\qquad \|u\|_\infty=\max_{t\in[0,T]}\big|u(t)\big|.$$

Definition 2.5 Let $0 < \alpha \leq 1$, $1 < p < \infty$. The fractional derivative space $E_0^{\alpha,p}$ is defined by the closure of $C_0^\infty([0,T],\mathbb{R})$ with respect to the norm

$$\|u\|_{E_0^{\alpha,p}} = \left(\int_0^T \left(\left|{}^cD_{0^+}^\alpha u(t)\right|^p dt + \int_0^T \left|u(t)\right|^p\right) dt\right)^{\frac{1}{p}}, \quad \forall u \in E_0^{\alpha,p}. \tag{2.1}$$

Remark 2.6
(1) $E_0^{\alpha,p}$ is a reflexive and separable Banach space.
(2) For any $u \in E_0^{\alpha,p}$, we have $u \in L^p([0,T],\mathbb{R})$, ${}^cD_{a^+}^\alpha u \in L^p([0,T],\mathbb{R})$, and $u(0) = u(T) = 0$.

Lemma 2.7 ([32]) *Let $0 < \alpha \leq 1$ and $1 < p < \infty$. For any $u \in E_0^{\alpha,p}$, we have*

$$\|u\|_{L^p} \leq \frac{T^\alpha}{\Gamma(\alpha+1)} \left\|{}^cD_{0^+}^\alpha u\right\|_{L^p}. \tag{2.2}$$

In addition, for $\frac{1}{p} < \alpha \leq 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, we have

$$\|u\|_\infty \leq \frac{T^{\alpha-\frac{1}{p}}}{\Gamma(\alpha)(\alpha q - q + 1)^{\frac{1}{q}}} \left\|{}^cD_{0^+}^\alpha u\right\|_{L^p}. \tag{2.3}$$

Remark 2.8 According to Lemma 2.7, it is easy to see that the norm of $E_0^{\alpha,p}$ defined in (2.1) is equivalent to the following norm:

$$\|u\|_{\alpha,p} = \left(\int_0^T \left|{}^cD_{0^+}^\alpha u(t)\right|^p dt\right)^{\frac{1}{p}}, \quad \forall u \in E_0^{\alpha,p}. \tag{2.4}$$

Lemma 2.9 ([32]) *Let $\frac{1}{p} < \alpha \leq 1$. If the sequence $\{u_k\}$ converges weakly to u in $E_0^{\alpha,p}$, i.e., $u_k \rightharpoonup u$, then $u_k \to u$ in $C[0,T]$, i.e., $\|u - u_k\|_\infty \to 0$ as $k \to \infty$.*

In the following, we denote $E = E_0^{\alpha,p}$, $\|u\| = \|u\|_{E_0^{\alpha,p}}$, $\|u\|_p = \|u\|_{L^p}$ for convenience.

Definition 2.10 A function

$$u \in \left\{u \in \mathrm{AC}[0,T] : \int_{t_j}^{t_{j+1}} \left(\left|{}^cD_{0^+}^\alpha u(t)\right|^p + \left|u(t)\right|^p\right) dt < \infty, j = 0, 1, \dots, m\right\}$$

is called a classical solution of BVP (1.1) if
(1) u satisfies (1.1).
(2) The limits $D_{T^-}^{\alpha-1}\Phi_p({}^cD_{0^+}^\alpha u)(t_j^+)$, $D_{T^-}^{\alpha-1}\Phi_p({}^cD_{0^+}^\alpha u)(t_j^-)$ exist.

Definition 2.11 A function $u \in E$ is a weak solution of BVP (1.1) if

$$\begin{aligned}&\int_0^T \left|{}^cD_{0^+}^\alpha u(t)\right|^{p-2}\left({}^cD_{0^+}^\alpha u(t)\right)\left({}^cD_{0^+}^\alpha v(t)\right) dt + \int_0^T \left|u(t)\right|^{p-2} u(t)v(t)\, dt \\ &\quad + \sum_{j=1}^m I_j\left(u(t_j)\right)v(t_j) - \int_0^T f\left(t, u(t)\right)v(t)\, dt = 0, \quad \forall u \in E.\end{aligned} \tag{2.5}$$

The energy functional $J : E \to \mathbb{R}$ associated with BVP (1.1) is defined by

$$J(u) = \frac{1}{p}\int_0^T \left(\left|{}^cD_{0^+}^\alpha u(t)\right|^p + \left|u(t)\right|^p\right) dt + \sum_{j=1}^m \int_0^{u(t_j)} I_j(s)\, ds - \int_0^T F\big(t, u(t)\big)\, dt, \quad \forall u \in E. \tag{2.6}$$

It is easy to see that $J \in C^1(E, \mathbb{R})$, and

$$\begin{aligned}\big(J'(u), v\big) ={}& \int_0^T \left|{}^cD_{0^+}^\alpha u(t)\right|^{p-2}\big({}^cD_{0^+}^\alpha u(t)\big)\big({}^cD_{0^+}^\alpha v(t)\big)\, dt + \int_0^T \left|u(t)\right|^{p-2} u(t)v(t)\, dt \\ &+ \sum_{j=1}^m I_j\big(u(t_j)\big)v(t_j) - \int_0^T f\big(t, u(t)\big)v(t)\, dt \\ ={}& 0, \quad \forall u \in E.\end{aligned} \tag{2.7}$$

Moreover, the critical points of J correspond to the weak solutions of BVP (1.1).

Lemma 2.12 ([31]) *If $u \in E$ is a weak solution of BVP (1.1), then u is a classical solution of BVP (1.1).*

To prove our main results, we need the following two variant fountain theorems in [33].

Let X be a Banach space with the norm $\|\cdot\|$ and $X = \overline{\bigoplus_{j\in\mathbb{N}} X_j}$ with $\dim X_j < \infty$ for each $j \in \mathbb{N}$. Set $W_k = \bigoplus_{j=1}^k X_j$, $Z_k = \overline{\bigoplus_{j=k}^\infty X_j}$, $B_k = \{u \in W_k : \|u\| \le \rho_k\}$, $S_k = \{u \in Z_k : \|u\| = r_k\}$, where $\rho_k > r_k > 0$.

Consider a family of C^1 functionals $J_\lambda : X \to \mathbb{R}$ defined by

$$J_\lambda(u) = A(u) - \lambda B(u), \quad \lambda \in [1, 2],$$

where $A, B : X \to \mathbb{R}$ are two functions.

Lemma 2.13 ([33]) *Assume that the functional J_λ defined above satisfies:*

- (B_1) *J_λ maps bounded sets into bounded sets uniformly for $\lambda \in [1, 2]$, and $J_\lambda(-u) = J_\lambda(u)$ for all $(\lambda, u) \in [1, 2] \times X$;*
- (B_2) *$B(u) \ge 0$ for all $u \in X$, $B(u) \to \infty$ as $\|u\| \to \infty$ on any finite dimensional subspace of X;*
- (B_3) *there exist $\rho_k > r_k > 0$ such that*

$$a_k(\lambda) = \inf_{u\in Z_k, \|u\|=\rho_k} J_\lambda(u) \ge 0, \qquad b_k(\lambda) = \max_{u\in W_k, \|u\|=r_k} J_\lambda(u) < 0, \quad \forall \lambda \in [1, 2]$$

and

$$d_k(\lambda) = \inf_{u\in Z_k, \|u\|\le\rho_k} J_\lambda(u) \to 0 \quad \text{as } k \to \infty \text{ uniformly for } \lambda \in [1, 2].$$

Then there exist $\lambda_n \to 1$, $u_n(\lambda_n) \in W_n$ such that

$$J'_{\lambda_n}|_{W_n}\big(u(\lambda_n)\big) = 0, \qquad J_{\lambda_n}\big(u(\lambda_n)\big) \to c_k \quad \text{as } n \to \infty,$$

where $c_k \in [d_k(2), b_k(1)]$. *In particular, if* $\{u(\lambda_n)\}$ *has a convergent subsequence for every* k, *then* J_1 *has infinitely many nontrivial critical points* $\{u_k\} \in X \setminus \{0\}$ *satisfying* $J_1(u_k) \to 0^-$ *as* $k \to \infty$.

Lemma 2.14 ([33]) *Assume that the functional* J_λ *defined above satisfies*

(A_1) J_λ *maps bounded sets into bounded sets uniformly for* $\lambda \in [1,2]$, *and* $J_\lambda(-u) = J_\lambda(u)$ *for all* $(\lambda, u) \in [1,2] \times X$;

(A_2) $B(u) \geq 0$ *for all* $u \in X$, $A(u) \to \infty$ *or* $B(u) \to \infty$ *as* $\|u\| \to \infty$; *or*

(A_3) $B(u) \leq 0$ *for all* $u \in X$, $B(u) \to -\infty$ *as* $\|u\| \to \infty$;

(A_4) *there exist* $\rho_k > r_k > 0$ *such that*

$$b_k(\lambda) = \inf_{u \in Z_k, \|u\| = r_k} J_\lambda(u) > a_k(\lambda) = \max_{u \in W_k, \|u\| = \rho_k} J_\lambda(u), \quad \forall \lambda \in [1,2].$$

Then

$$b_k(\lambda) \leq c_k(\lambda) = \inf_{\gamma \in \Gamma_k} \max_{u \in B_k} J_\lambda\big(\gamma(u)\big), \quad \forall \lambda \in [1,2],$$

where $\Gamma_k = \{\gamma \in C(B_k, X) : \gamma \text{ is odd}, \gamma|_{\partial B_k} = \mathrm{id}\}$. *Moreover, for almost every* $\lambda \in [1,2]$, *there exists a sequence* $\{u_n^k(\lambda)\}$ *such that*

$$\sup_n \big\|u_n^k(\lambda)\big\| < \infty, \qquad J'_\lambda\big(u_n^k(\lambda)\big) \to 0 \quad \text{and} \quad J_\lambda\big(u_n^k(\lambda)\big) \to c_k(\lambda) \quad \text{as } n \to \infty.$$

As E is a separable and reflexive Banach space, then there exist $\{e_j\}_{j=1}^\infty \subset E$ and $\{e_j^*\}_{j=1}^\infty \subset E^*$ such that

$$E = \overline{\operatorname{span}\{e_j\}}, \qquad E^* = \overline{\operatorname{span}\{e_j^*\}},$$
$$\big(e_i^*, e_i\big) = 1, \qquad \big(e_j^*, e_i\big) = 0 \quad (i \neq j).$$

Define $X_j = \operatorname{span}\{e_j\}$, $W_k = \bigoplus_{j=1}^k X_j$, $Z_k = \overline{\bigoplus_{j=k}^\infty X_j}$. In order to apply Lemma 2.13 and Lemma 2.14 to prove the existence of infinitely many solutions of BVP (1.1), we define A, B, and J_λ on a fractional derivative space E by

$$A(u) = \frac{1}{p}\|u\|^p + \sum_{j=1}^m \int_0^{u(t_j)} I_j(s)\,ds, \qquad B(u) = \int_0^T F\big(t, u(t)\big)\,dt,$$

and

$$\begin{aligned} J_\lambda(u) &= A(u) - \lambda B(u) \\ &= \frac{1}{p}\|u\|^p + \sum_{j=1}^m \int_0^{u(t_j)} I_j(s)\,ds - \lambda \int_0^T F\big(t, u(t)\big)\,dt, \quad \forall u \in E, \lambda \in [1,2]. \end{aligned}$$

3 Proof of the main results

In order to complete the proof of our main results, it is necessary to give the following two lemmas. Because of using similar arguments to the proofs of Lemma 3.2 and Lemma 3.5 in [15], we omit the proving processes for convenience.

Lemma 3.1 *Let H be any finite dimensional subspace of E. Then there exists a constant $\varepsilon_0 > 0$ such that*

$$\operatorname{meas}\bigl\{t \in [0,T] : \bigl|u(t)\bigr| \geq \varepsilon_0 \|u\|\bigr\} \geq \varepsilon_0, \quad \forall u \in H \setminus \{0\}. \tag{3.1}$$

Lemma 3.2 *Let $\alpha_r(k) = \sup_{u \in Z_k, \|u\|=1} \|u\|_r$ with $r \geq p$. Then $\alpha_r(k) \to 0$ as $k \to \infty$.*

Now we are ready to prove Theorem 1.1 and Theorem 1.2.

Proof of Theorem 1.1 By (F_1) and (F_3), for $\forall \varepsilon > 0$, there exists $\delta_\varepsilon > 0$ such that

$$\bigl|F(t,u)\bigr| \leq \varepsilon |u|^p + \delta_\varepsilon b(t)|u|^\eta. \tag{3.2}$$

Combining (3.2), (H_2), and Lemma 2.7, it is easily seen that J_λ maps bounded sets into bounded sets uniformly for $\lambda \in [1,2]$. It follows from (H_1) and (F_5) that $J_\lambda(-u) = J_\lambda(u)$ for all $(\lambda, u) \in [1,2] \times E$. Thus, condition (B_1) holds.

Next, we verify condition (B_2).

According to (F_4), $B(u) \geq 0$ is obvious. By (F_2), there exists $M > 0$ such that

$$F(t,u) \geq d|u|^\sigma \quad \text{for all } |u| > M. \tag{3.3}$$

Assume that H is a finite dimensional subspace of E. According to Lemma 3.1, there exists $\varepsilon_0 > 0$ such that (3.1) holds. Then

$$\operatorname{meas}(D_u) \geq \varepsilon_0, \quad \forall u \in H \setminus \{0\},$$

where $D_u = \{t \in [0,T] : |u(t)| \geq \varepsilon_0 \|u\|\}$. Hence, for any $u \in H$ with $\|u\| \geq \frac{M}{\varepsilon_0}$, by (3.3), we get

$$\begin{aligned} B(u) &= \int_0^T F\bigl(t,u(t)\bigr)\,dt \\ &\geq \int_{D_u} d\bigl|u(t)\bigr|^\sigma\,dt \\ &\geq d\varepsilon_0^\sigma \|u\|^\sigma \operatorname{meas}(D_u) \\ &\geq d\varepsilon_0^{1+\sigma} \|u\|^\sigma. \end{aligned}$$

This means that $B(u) \to \infty$ as $\|u\| \to \infty$ on any finite dimensional subspace. Hence, condition (B_2) holds.

In the end, we claim that condition (B_3) holds.

For $u \in Z_k$, by (3.2), Hölder's inequality, and (H_1), we have

$$\begin{aligned} J_\lambda(u) &\geq \frac{1}{p}\|u\|^p - \int_0^T \bigl(\varepsilon|u|^p + \delta_\varepsilon b(t)|u|^\eta\bigr)\,dt \\ &\geq \frac{1}{2p}\|u\|^p - \delta\bigl\|b(t)\bigr\|_{\frac{p}{p-\eta}} \|u\|_p^\eta, \end{aligned}$$

where we choose $\varepsilon = \frac{1}{2p}$. According to the definition of $\alpha_r(k)$ in Lemma 3.2, we have

$$\|u\|_p^\eta \le \alpha_p^\eta(k)\|u\|^\eta, \quad \forall u \in Z_k.$$

Hence,

$$J_\lambda(u) \ge \frac{1}{2p}\|u\|^p - \delta\big\|b(t)\big\|_{\frac{p}{p-\eta}}\alpha_p^\eta(k)\|u\|^\eta.$$

Choose $\rho_k = (4p\delta\|b(t)\|_{\frac{p}{p-\eta}}\alpha_p^\eta(k))^{\frac{1}{p-\eta}}$. Then $\rho_k \to 0^+$ as $k \to \infty$. Therefore,

$$a_k(\lambda) = \inf_{u\in Z_k,\|u\|=\rho_k} J_\lambda(u) \ge \frac{1}{4p}\rho_k^p > 0.$$

In addition, for $\lambda \in [1,2]$ and $u \in Z_k$ with $\|u\| \le \rho_k$, we have

$$J_\lambda(u) \ge -\delta\big\|b(t)\big\|_{\frac{p}{p-\eta}}\alpha_p^\eta(k)\rho_k^\eta \to 0^+, \quad k \to \infty.$$

So,

$$d_k(\lambda) = \inf_{u\in Z_k,\|u\|\le\rho_k} J_\lambda(u) \to 0 \quad \text{as } k \to \infty.$$

By (F_1)–(F_3), we have

$$F(t,u) \ge d|u|^\sigma - \varepsilon|u|^p - \delta_\varepsilon b(t)|u|^\eta. \tag{3.4}$$

If $u \in W_k$, by the equivalence of any norm in a finite dimensional space, (3.4), (H_2), Lemma 2.7, and Hölder's inequality, we get

$$\begin{aligned} J_\lambda(u) &= \frac{1}{p}\|u\|^p + \sum_{j=1}^m \int_0^{u(t_j)} I_j(s)\,ds - \lambda\int_0^T F\big(t,u(t)\big)\,dt \\ &\le \frac{1}{p}\|u\|^p + \sum_{j=1}^m \frac{b_j}{\gamma_j+1}\big|u(t_j)\big|^{\gamma_j+1} - d\int_0^T |u|^\sigma\,dt + \varepsilon\int_0^T |u|^p\,dt + \delta_\varepsilon\int_0^T b(t)|u|^\eta\,dt \\ &\le \|u\|^p + M\|u\|^{\gamma_j+1} - \delta_1\|u\|^\sigma + \delta\big\|b(t)\big\|_{\frac{p}{p-\eta}}\|u\|^\eta. \end{aligned}$$

Choose $r_k > 0$ small enough and $r_k < \rho_k$ such that

$$b_k(\lambda) = \max_{u\in W_k,\|u\|=r_k} J_\lambda(u) < 0.$$

This guarantees that condition (B_3) holds.

Consequently, by Lemma 2.13, for every $k \in \mathbb{N}$, there exist $\lambda_n \to 1$, $u_n(\lambda_n) \in W_n$ such that

$$J'_{\lambda_n}|_{W_n}\big(u(\lambda_n)\big) = 0, \qquad J_{\lambda_n}\big(u(\lambda_n)\big) \to c_k \in \big[d_k(2), b_k(1)\big] \quad \text{as } n \to \infty.$$

For simplicity, denote $u(\lambda_n)$ by u_n. Now we show that $\{u_n\}$ has a strong convergent subsequence for every $k \in \mathbb{N}$. In fact, by (3.2), (H_1), Hölder's inequality, and Lemma 2.7, we get

$$\begin{aligned}\|u_n\|^p &= pJ_{\lambda_n}(u_n) - p\sum_{j=1}^{m}\int_0^{u_n(t_j)} I_j(s)\,ds + p\lambda_n\int_0^T F(t,u_n)\,dt \\ &\le pc_k + p\lambda_n\delta_2\|u_n\|^p + p\lambda_n\delta\big\|b(t)\big\|_{\frac{p}{p-\eta}}\|u_n\|^\eta.\end{aligned}$$

This means that $\{u_n\}$ is bounded in E. Without loss of generality, we may assume $u_n \rightharpoonup u$ in E. Since $\{e_j\}$ is a completely orthonormal basis of E, $W_n = L(e_1,e_2,\dots,e_n)$, $u = \sum_{j=1}^{\infty}(e_j,u)e_j$. Let $P_n : E \mapsto W_n$ be the orthogonal projection operator. We know that $P_n u = \sum_{j=1}^{n}(e_j,u)e_j$ and $P_n u \to u$ in E as $n\to\infty$. Therefore $u_n - P_n u \rightharpoonup 0$ in E as $n\to\infty$. Moreover, it follows from $J_1'(u) \in E^*$ that

$$\big(J_1'(u), u_n - P_n u\big) \to 0, \quad n\to\infty. \tag{3.5}$$

Also, since $J_1' \in C(E \to E^*)$ and $P_n u \to u$ in E, we have

$$\big(J_1'(P_n u) - J_1'(u), u_n - P_n u\big) \to 0, \quad n\to\infty. \tag{3.6}$$

Therefore, by (3.5) and (3.6), we get

$$\big(J_1'(P_n u), u_n - P_n u\big) \to 0, \quad n\to\infty. \tag{3.7}$$

Note that $P_n u_n = u_n$ and $(J_{\lambda_n}'(P_n u_n), P_n(u_n - u)) = 0$ since $u_n \in W_n$ and $J_{\lambda_n}'|_{W_n}(u_n) = 0$. By the continuity of f, I_j, and Lemma 2.9, it is easily seen that

$$\begin{aligned}&\lambda_n\int_0^T f(t,u_n)(u_n - P_n u)\,dt \to 0, \quad n\to\infty. \\ &\int_0^T f(t,P_n u)(u_n - P_n u)\,dt \to 0, \quad n\to\infty. \\ &\sum_{j=1}^{m}\big[I_j\big(u_n(t_j)\big) - I_j\big(P_n u(t_j)\big)\big]\big(u_n(t_j) - P_n u(t_j)\big) \to 0, \quad n\to\infty.\end{aligned}$$

Set

$$\begin{aligned}\psi_1 &= \int_0^T \big[\big|{}^cD_{0^+}^\alpha u_n(t)\big|^{p-2}\big({}^cD_{0^+}^\alpha u_n(t)\big) - \big|{}^cD_{0^+}^\alpha P_n u(t)\big|^{p-2}\big({}^cD_{0^+}^\alpha P_n u(t)\big)\big] \\ &\quad\times\big[{}^cD_{0^+}^\alpha u_n(t) - {}^cD_{0^+}^\alpha P_n u(t)\big]\,dt, \\ \psi_2 &= \int_0^T \big(\big|u_n(t)\big|^{p-2}u_n - \big|P_n u(t)\big|^{p-2}P_n u(t)\big)\big(u_n(t) - P_n u(t)\big)\,dt.\end{aligned}$$

Therefore,

$$\begin{aligned}\psi_1+\psi_2 &= \left(J'_{\lambda_n}(u_n),u_n-P_nu\right)-\left(J'_1(P_nu),u_n-P_nu\right)\\&\quad-\sum_{j=1}^{m}\left[I_j\left(u_n(t_j)\right)-I_j\left(P_nu(t_j)\right)\right]\left(u_n(t_j)-P_nu(t_j)\right)\\&\quad+\lambda_n\int_0^T f(t,u_n)(u_n-P_nu)\,dt-\int_0^T f(t,P_nu)(u_n-P_nu)\,dt\\&\to 0.\end{aligned}$$

In what follows, we prove $\|u_n-P_nu\|\to 0$ in two cases.

Case 1: $p\ge 2$.

According to the following inequality (see [34], Lemma 4.2)

$$\left(|x|^{p-2}x-|y|^{p-2}y\right)(x-y)\ge\omega|x-y|^p,$$

there exist $\omega_1>0$, $\omega_2>0$ such that

$$\psi_1\ge\omega_1\int_0^T\left|{}^cD_{0^+}^{\alpha}u_n(t)-{}^cD_{0^+}^{\alpha}P_nu(t)\right|^p dt, \tag{3.8}$$

$$\psi_2\ge\omega_2\int_0^T\left|u_n(t)-P_nu(t)\right|^p dt. \tag{3.9}$$

Combining (3.8) and (3.9), we get

$$\psi_1+\psi_2\ge M_1\|u_n-P_nu\|^p,$$

where $M_1=\min\{\omega_1,\omega_2\}$. Thus, $\|u_n-P_nu\|\to 0$.

Case 2: $1<p<2$.

According to the following inequality (see [34], Lemma 4.2)

$$\left(\left(|x|^{p-2}x-|y|^{p-2}y\right)(x-y)\right)^{\frac{p}{2}}\left(|x|^p+|y|^p\right)^{\frac{2-p}{2}}\ge\omega|x-y|^p,$$

there exist positive numbers ω_3 and ω_4 such that

$$\psi_1\ge\omega_3\int_0^T\frac{|{}^cD_{0^+}^{\alpha}u_n(t)-{}^cD_{0^+}^{\alpha}P_nu(t)|^2}{(|{}^cD_{0^+}^{\alpha}u_n(t)|+|{}^cD_{0^+}^{\alpha}P_nu(t)|)^{2-p}}\,dt, \tag{3.10}$$

$$\psi_2\ge\omega_4\int_0^T\frac{|u_n(t)-P_nu(t)|^2}{(|u_n(t)|+|P_nu(t)|)^{2-p}}\,dt. \tag{3.11}$$

By Hölder's inequality, we get

$$\begin{aligned}&\int_0^T\left|u_n(t)-P_nu(t)\right|^p dt\\&\quad\le\left(\int_0^T\frac{|u_n(t)-P_nu(t)|^2}{(|u_n(t)|+|P_nu(t)|)^{2-p}}\,dt\right)^{\frac{p}{2}}\left(\int_0^T\left(\left|u_n(t)\right|+\left|P_nu(t)\right|\right)^p dt\right)^{\frac{2-p}{2}}\\&\quad\le M_2\left(\|u_n\|_p^p+\|P_nu\|_p^p\right)^{\frac{2-p}{2}}\left(\int_0^T\frac{|u_n(t)-P_nu(t)|^2}{(|u_n(t)|+|P_nu(t)|)^{2-p}}\,dt\right)^{\frac{p}{2}},\end{aligned}$$

where $M_2 = 2^{\frac{(p-1)(2-p)}{2}}$. Therefore,

$$\int_0^T \frac{|u_n(t) - P_n u(t)|^2}{(|u_n(t)| + |P_n u(t)|)^{2-p}}\,dt \\ \geq M_2^{-\frac{2}{p}} \left(\int_0^T \left|u_n(t) - P_n u(t)\right|^p dt\right)^{\frac{2}{p}} \left(\|u_n\|_p^p + \|P_n u\|_p^p\right)^{\frac{p-2}{p}}. \tag{3.12}$$

Similarly, we have

$$\int_0^T \frac{|{}^cD_{0^+}^\alpha u_n(t) - {}^cD_{0^+}^\alpha P_n u(t)|^2}{(|{}^cD_{0^+}^\alpha u_n(t)| + |{}^cD_{0^+}^\alpha P_n u(t)|)^{2-p}}\,dt \\ \geq M_2^{-\frac{2}{p}} \left(\int_0^T \left|{}^cD_{0^+}^\alpha u_n(t) - {}^cD_{0^+}^\alpha P_n u(t)\right|^p dt\right)^{\frac{2}{p}} \\ \cdot \left(\left\|{}^cD_{0^+}^\alpha u_n\right\|_p^p + \left\|{}^cD_{0^+}^\alpha P_n u\right\|_p^p\right)^{\frac{p-2}{p}}. \tag{3.13}$$

Combining (3.10)–(3.13), we get

$$\psi_1 + \psi_2 \geq M_{3n}\|u_n - P_n u\|^2,$$

where $M_{3n} = M_2^{-\frac{2}{p-1}} \min\{\omega_3(\|{}^cD_{0^+}^\alpha u_n\|_p^p + \|{}^cD_{0^+}^\alpha P_n u\|_p^p)^{\frac{p-2}{p}}, \omega_4(\|u_n\|_p^p + \|P_n u\|_p^p)^{\frac{p-2}{p}}\}$.

If $\{u_n\}$ has a subsequence (still relabeled $\{u_n\}$ for convenience) such that $\|u_n\| \to 0$ as $n \to \infty$, it is easy to see that $\|u_n - P_n u\| \to 0$. On the other hand, if $\inf_{n\geq 1}\|u_n\| > 0$, by the boundedness of $\{u_n\}$ in E, there exists $M_3 > 0$ such that $M_{3n} \geq M_3 > 0$. Then $\|u_n - P_n u\| \to 0$.

So, $u_n - P_n u \to 0$ in E as $n \to \infty$, which means that $u_n \to u$ in E as $n \to \infty$. By Lemma 2.13, we know that $J = J_1$ has infinitely many nontrivial critical points u_k. Consequently, BVP (1.1) has infinitely many small energy solutions. □

Proof of Theorem 1.2 For any $\varepsilon > 0$, it follows from (F_6) that there exist positive numbers θ_3 and θ_4 such that

$$\left|F(t,u)\right| \leq \theta_3|u|^p + \theta_4|u|^q. \tag{3.14}$$

Combining (3.14), (H_3), and Lemma 2.7, it is easily seen that J_λ maps bounded sets into bounded sets uniformly for $\lambda \in [1,2]$. By (H_1) and (F_5), $J_\lambda(-u) = J_\lambda(u)$ for all $(\lambda,u) \in [1,2] \times E$. Thus, condition (A_1) holds. Assumption (F_4) means that $B(u) \geq 0$. Condition (A_2) holds for the fact that $A(u) \geq \frac{1}{p}\|u\|^p \to \infty$ as $n \to \infty$ and $B(u) \geq 0$.

In what follows, we verify condition (A_4). For this sake, we need to prove that there exist two sequences $\rho_k > r_k > 0$ such that

$$b_k(\lambda) = \inf_{u\in Z_k,\|u\|=r_k} J_\lambda(u) > 0, \quad \forall\lambda \in [1,2], \tag{3.15}$$

$$a_k(\lambda) = \max_{u\in W_k,\|u\|=\rho_k} J_\lambda(u) < 0, \quad \forall\lambda \in [1,2]. \tag{3.16}$$

First, we prove that (3.15) is true.

For $u \in Z_k$, by (3.14), (H_1), and the definition of $\alpha_r(k)$ in Lemma 3.2, we have

$$\begin{aligned} J_\lambda(u) &\geq \frac{1}{p}\|u\|^p - 2\theta_3\|u\|_p^p - 2\theta_4\|u\|_q^q \\ &\geq \frac{1}{p}\|u\|^p - 2\theta_3\alpha_p^p(k)\|u\|^p - 2\theta_4\alpha_q^q(k)\|u\|^q. \end{aligned}$$

Choose $r_k = \frac{1}{\alpha_p(k)+\alpha_q(k)}$. Then $r_k \to \infty$ as $k \to \infty$. For any $u \in Z_k$ with $\|u\| = r_k$, we know

$$\begin{aligned} J_\lambda(u) &\geq \frac{1}{p}\|u\|^p - 2\theta_3\frac{\alpha_p^p(k)}{|\alpha_p(k)+\alpha_q(k)|^p} - 2\theta_4\frac{\alpha_q^q(k)}{|\alpha_p(k)+\alpha_q(k)|^q} \\ &\geq \frac{1}{p}r_k^p - 2\theta_3 - 2\theta_4 \\ &> 0. \end{aligned}$$

Therefore,

$$b_k(\lambda) = \inf_{u\in Z_k,\|u\|=r_k} J_\lambda(u) > 0, \quad \forall\lambda \in [1,2].$$

Next, we prove that (3.16) is true.

By (F_7), there exists $\delta_3 > 0$ such that

$$F(t,u) \geq \delta_3|u|^\mu, \quad \text{for all } t \in [0,T], u \in \mathbb{R}. \tag{3.17}$$

According to (3.17), (H_3), and Lemma 2.7, we have

$$\begin{aligned} J_\lambda(u) &= \frac{1}{p}\|u\|^p + \sum_{j=1}^m \int_0^{u(t_j)} I_j(s)\,ds - \lambda\int_0^T F\bigl(t,u(t)\bigr)\,dt \\ &\leq \frac{1}{p}\|u\|^p + \sum_{j=1}^m \frac{b_j}{\gamma_j+1}\bigl|u(t_j)\bigr|^{\gamma_j+1} - \delta_3\int_0^T |u|^\mu\,dt \\ &\leq \frac{1}{p}\|u\|^p + \delta_4\|u\|^{\gamma_j+1} - \delta_5\|u\|^\mu. \end{aligned}$$

Hence, one can take $\rho_k > \gamma_k$ large enough such that

$$a_k(\lambda) = \max_{u\in W_k,\|u\|=\rho_k} J_\lambda(u) < 0.$$

Until now, all the conditions of Lemma 2.14 hold. Hence, for $\lambda \in [1,2]$, there exists a sequence $\{u_n^k(\lambda)\}_{n=1}^\infty$ such that

$$\begin{aligned} &\sup_n\bigl\|u_n^k(\lambda)\bigr\| < \infty, \\ &J'_\lambda\bigl(u_n^k(\lambda)\bigr) \to 0, \qquad J_\lambda\bigl(u_n^k(\lambda)\bigr) \to c_k(\lambda) = \inf_{\gamma\in\Gamma_k}\max_{u\in B_k} J_\lambda\bigl(\gamma(u)\bigr), \quad n \to \infty. \end{aligned} \tag{3.18}$$

Furthermore,

$$c_k(\lambda) \geq b_k(\lambda) \geq \frac{1}{p} r_k^p - 2\theta_3 - 2\theta_4 := \overline{b_k} \to \infty, \quad k \to \infty,$$

$$c_k(\lambda) \leq \max_{u \in B_k} J_1(u) := \overline{c_k}.$$

Thus,

$$\overline{b_k} \leq c_k(\lambda) \leq \overline{c_k}, \quad \lambda \in [1,2]. \tag{3.19}$$

Choose a sequence $\lambda_m \to 1$ such that (3.18) holds. Using similar arguments of the proof of Theorem 1.1, we can show that $\{u_n^k(\lambda_m)\}_{n=1}^{\infty}$ possesses a strong convergent subsequence. Thus, we suppose that $u_n^k(\lambda_m) \to u^k(\lambda_m)$ in E as $n \to \infty$. By (3.18) and (3.19), we can get

$$J'_{\lambda_m}\bigl(u^k(\lambda_m)\bigr) = 0, \qquad J_{\lambda_m}\bigl(u^k(\lambda_m)\bigr) \in [\overline{b_k}, \overline{c_k}] \quad \text{for } k \geq k_1.$$

In the following we prove that $\{u^k(\lambda_m)\}_{m=1}^{\infty}$ is bounded.

From (H_3) and (F_6), we have

$$\begin{aligned}
&\mu J_{\lambda_m}\bigl(u^k(\lambda_m)\bigr) - \bigl(J'_{\lambda_m}\bigl(u^k(\lambda_m)\bigr), u^k(\lambda_m)\bigr) \\
&\quad = \left(\frac{\mu}{p} - 1\right)\bigl\|u^k(\lambda_m)\bigr\|^p + \sum_{j=1}^{m}\left(\mu \int_0^{u^k(\lambda_m)(t_j)} I_j(s)\,ds - I_j\bigl(u^k(\lambda_m)(t_j)\bigr)u^k(\lambda_m)(t_j)\right) \\
&\qquad + \lambda_m \int_0^T \bigl[f\bigl(t, u^k(\lambda_m)\bigr)u^k(\lambda_m) - \mu F\bigl(t, u^k(\lambda_m)\bigr)\bigr]\,dt \\
&\quad \geq \left(\frac{\mu}{p} - 1\right)\bigl\|u^k(\lambda_m)\bigr\|^p.
\end{aligned}$$

Therefore, $\{u^k(\lambda_m)\}_{m=1}^{\infty}$ is bounded in E. Similar arguments of the proof of Theorem 1.1 show that $u^k(\lambda_m) \to u^k$ in E as $m \to \infty$ $(k \geq k_1)$. Then u^k is a critical point of $J = J_1$ with $I(u^k) \in [\overline{b_k}, \overline{c_k}]$. According to $\overline{b_k} \to \infty$ as $k \to \infty$, we know that BVP (1.1) has infinitely many nontrivial high energy solutions. □

4 Examples

In this section, two examples are given to illustrate our results.

Example 4.1 Consider the following nonlinear impulsive fractional boundary value problem:

$$\begin{cases} D_{T^-}^{0.8}\Phi_4({}^cD_{0^+}^{0.8}u(t)) + |u(t)|u(t) = f(t,u), & t \in [0,T], t \neq t_1, \\ \Delta(D_{T^-}^{-0.2}\Phi_4({}^cD_{0^+}^{0.8}u))(t_1) = u^5(t_1), \\ u(0) = u(T) = 0, \end{cases} \tag{4.1}$$

where

$$f(t,u) = \begin{cases} \frac{7}{2}e^t|u|^4, & |u| \leq 1, \\ \frac{7}{2}e^t|u|^{\frac{5}{2}}, & |u| > 1. \end{cases}$$

Choose $p = 4$, $\alpha = 0.8 \in (\frac{1}{4}, 1]$, and $I_1(u) = u^5(t_1)$. It is easy to show that assumption (H_1) holds. Take $b_1 = 2$, $\gamma_1 = 5 \in (3, +\infty)$. From this we can see that assumption (H_2) holds.

Moreover,

$$\bigl|f(t,u)\bigr| \leq 4e^t\bigl(1 + |u|^{\frac{5}{2}}\bigr).$$

Choose $\eta = \frac{7}{2} \in (3,4)$ and $b(t) = 4e^t$. This means that assumption (F_1) is satisfied.

Take $\sigma = \frac{13}{4} \in (3, \frac{7}{2})$. By a simple calculation, one has $F(t,u) = e^t|u|^{\frac{7}{2}}$ and $\lim_{|u|\to\infty} \frac{e^t|u|^{\frac{7}{2}}}{|u|^{\frac{13}{4}}} \to \infty$. Therefore, assumption (F_2) holds.

In addition, $\lim_{|u|\to 0} \frac{\frac{7}{2}e^t|u|^4}{|u|^3} = 0$ implies that assumption (F_3) holds.

Finally, it is easy to see that (F_4) and (F_5) hold. Consequently, BVP (4.1) has infinitely many small energy solutions by Theorem 1.1.

Example 4.2 Consider the following nonlinear fractional impulsive boundary value problem:

$$\begin{cases} D_{T^-}^{0.8}\Phi_{\frac{5}{2}}({}^cD_{0^+}^{0.8}u(t)) + |u(t)|^{\frac{1}{2}}u(t) = 5|u|^4\ln(|u|+1) + \frac{|u|^5}{|u|+1}, & t \in [0,T], t \neq t_1, \\ \Delta(D_{T^-}^{-0.2}\Phi_{\frac{5}{2}}({}^cD_{0^+}^{0.8}u))(t_1) = u^3(t_1), \\ u(0) = u(T) = 0. \end{cases} \tag{4.2}$$

First, choose $p = \frac{5}{2}$, $\alpha = 0.8 \in (\frac{2}{5}, 1]$, and $I_1(u) = u^3(t_1)$. From this one can see that assumption (H_1) holds. Taking $b_1 = 2$, $\mu = 5 > \frac{5}{2}$, and $\gamma_1 = 3 \in (\frac{3}{2}, 4)$ means that assumption (H_3) holds.

Next, a simple calculation shows that

$$F(t,u) = |u|^5\ln\bigl(|u|+1\bigr), \qquad -5F(t,u) + uf(t,u) = \frac{|u|^5}{|u|+1} \geq 0.$$

Hence, assumption (F_7) holds.

Finally, it is easy to show that (F_4)–(F_6) hold. Consequently, BVP (4.2) has infinitely many high energy solutions by Theorem 1.2.

Acknowledgements
The authors wish to thank anonymous referees for their valuable suggestions.

Funding
This work was supported by the National Natural Science Foundation of China (No. 11671237), and by Shandong Provincial Natural Science Foundation of China (No. ZR2013AM005).

List of abbreviations
BVP, boundary value problem; $\mathbf{AC}[a,b]$, the space of absolutely continuous functions on $[a,b]$.

Availability of data and materials
Not applicable.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Author details
[1]School of Mathematics and Statistics, Shandong Normal University, Jinan, P.R. China. [2]Department of Mathematics, Shandong University of Science and Technology, Qingdao, P.R. China.

Publisher's Note

Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 28 February 2018 Accepted: 4 June 2018 Published online: 15 June 2018

References

1. Droniou, J., Imbert, C.: Fractal first order partial differential equation. Arch. Ration. Mech. Anal. **182**, 299–331 (2006)
2. Mathieu, B., Melchior, P., Outsaloup, A., Ceyral, C.: Fractional differentiation for edge detection. Signal Process. **83**, 2421–2432 (2003)
3. Kirchner, J.W., Feng, X., Neal, C.: Fractal streamchemistry and its implications for contaminant transport in catchments. Nature **403**, 524–526 (2000)
4. Hilfer, R.: Applications of Fractional Calculus in Physics. World Scientific, Singapore (2000)
5. Carpinteri, A., Mainaardi, F.: Fractals and Fractional Calculus in Continuum Mechanics. Springer, Berlin (1997)
6. Cui, Y.J.: Uniqueness of solution for boundary value problems for fractional differential equations. Appl. Math. Lett. **51**, 48–54 (2016)
7. Zhang, K.M.: On a sign-changing solution for some fractional differential equations. Bound. Value Probl. **2017**, Article ID 59 (2017)
8. Zhao, D.L., Liu, Y.S.: Positive solutions for a class of fractional differential coupled system with integral boundary value conditions. J. Nonlinear Sci. Appl. **9**, 2922–2942 (2016)
9. Liu, Y.S.: Bifurcation techniques for a class of boundary value problems of fractional impulsive differential equations. J. Nonlinear Sci. Appl. **8**, 340–353 (2015)
10. Jiao, F., Zhou, Y.: Existence results for fractional boundary value problem via critical point theory. Int. J. Bifurc. Chaos **22**, Article ID 1250086 (2012)
11. Heidarkhani, S.: Infinitely many solutions for nonlinear perturbed fractional boundary value problems. An. Univ. Craiova, Ser. Mat. Inform., **41**, 88–103 (2014)
12. Heidarkhani, S., Zhou, Y., Caristi, G., Afrouzi, G.A., Moradi, S.: Existence results for fractional differential systems through a local minimization principle. Comput. Math. Appl. (2016). https://doi.org/10.1016/j.camwa.2016.04.012
13. Zhao, Y.L., Chen, H.B., Qin, B.: Multiple solutions for a coupled system of nonlinear fractional differential equations via variational methods. Appl. Math. Comput. **257**, 417–427 (2015)
14. Chen, J., Tang, X.H.: Infinitely many solutions for a class of fractional boundary value problem. Bull. Malays. Math. Sci. Soc. **36**, 1083–1097 (2013)
15. Chai, G.Q.: Infinitely many solutions for nonlinear fractional boundary value problems via variational methods. Adv. Differ. Equ. **2016**, Article ID 213 (2016)
16. Sun, H.R., Zhang, Q.G.: Existence of solutions for a fractional boundary value problem via the mountain pass method and an iterative technique. Comput. Math. Appl. **64**, 3436–3443 (2012)
17. Zhang, X.G., Liu, L.S., Wu, Y.H., Wiwatanapataphee, B.: Nontrivial solutions for fractional advection dispersion equation in anomalous diffusion. Appl. Math. Lett. **66**, 1–8 (2017)
18. Heidarkhani, S., Ferrara, M., Salari, A.: Infinitely many periodic solutions for a class of perturbed second-order differential equations with impulses. Acta Appl. Math. **139**, 81–94 (2015)
19. Bainov, D., Simeonov, P.: Systems with Impulse Effect. Ellis Horwood Series: Mathematics and Its Applications. Ellis Horwood, Chichester (1989)
20. Samoilenko, A.M., Perestyuk, N.A.: Impulsive Differential Equations. World Scientific, Singapore (1995)
21. Liu, X., Willms, A.: Impulsive controllability of linear dynamical systems with applications to maneuvers of spacecraft. Math. Probl. Eng. **2**, 277–299 (1996)
22. Rodríguez-López, R., Tersian, S.: Multiple solutions to boundary value problem for impulsive fractional differential equations. Fract. Calc. Appl. Anal. **17**(4), 1016–1038 (2014)
23. Heidarkhani, S., Zhao, Y.L., Caristi, G., Afrouzi, G.A., Moradi, S.: Infinitely many solutions for perturbed impulsive fractional differential systems. Appl. Anal. **96**(8), 1401–1424 (2017)
24. D'Agui, G., Di Bella, B., Tersian, S.: Multiplicity results for superlinear boundary value problems with impulsive effects. Math. Methods Appl. Sci. **39**, 1060–1068 (2016)
25. Heidarkhani, S., Salari, A.: Nontrivial solutions for impulsive fractional differential systems through variational methods. Comput. Math. Appl. (2016). https://doi.org/10.1016/j.camwa.2016.04.016
26. Heidarkhani, S., Salari, A., Caristi, G.: Infinitely many solutions for impulsive nonlinear fractional boundary value problems. Adv. Differ. Equ. **2016**, Article ID 196 (2016)
27. Heidarkhani, S., Cabada, A., Afrouzi, G.A., Moradi, S., Carist, G.: A variational approach to perturbed impulsive fractional differential equations. J. Comput. Appl. Math. **341**, 42–60 (2018)
28. Heidarkhani, S., Ferrara, M., Caristi, G., Salari, A.: Existence of three solutions for impulsive nonlinear fractional boundary value problems. Opusc. Math. **37**, 281–301 (2017)
29. Zhao, Y.L., Zhao, Y.L.: Nontrivial solutions for a class of perturbed fractional differential systems with impulsive effects. Bound. Value Probl. **2016**, Article ID 129 (2016)
30. Zhao, Y.L., Chen, H.B., Xu, C.: Nontrivial solutions for impulsive fractional differential equations via Morse theory. Appl. Math. Comput. **307**, 170–179 (2017)
31. Zhao, Y.L., Tang, L.: Multiplicity results for impulsive fractional differential equations with p-Laplacian via variational methods. Bound. Value Probl. **2017**, Article ID 123 (2017)
32. Zhou, Y.: Basic Theory of Fractional Differential Equations. World Scientific, Singapore (2014)
33. Zou, W.: Variant fountain theorems and their applications. Manuscr. Math. **104**, 343–358 (2001)
34. Jia, M., Liu, X.: Multiplicity of solutions for integral boundary value problems of fractional differential equations with upper and lower solutions. Appl. Math. Comput. **232**, 313–323 (2014)