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Blow-up and delay for a parabolic–elliptic Keller–Segel system with a source term

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Abstract

In this paper, we are concerned with the parabolic–elliptic Keller–Segel system with a positive source term in a bounded domain in $\mathbb{R}^N$ ($N = 2, 3$), under homogeneous Dirichlet boundary condition, with time-dependent coefficients. Lower bounds for the blow-up time if the solutions blow up in finite time are derived under appropriate assumptions on data. Moreover, the exponential decay of the associated energies is also studied.

MSC: 35K37; 35B40; 92D40

Keywords: Parabolic–elliptic Keller–Segel system; Blow-up time; Lower bounds; Decay

1 Introduction

Let us consider the following parabolic–elliptic Keller–Segel system:

$$\begin{cases} u_t = \triangle u - k_1(t)\nabla\cdot(u^m\nabla v) + f(u), & x\in\Omega, t>0,\\ 0 = \triangle v - k_2(t)v + k_3(t)u, & x\in\Omega, t>0,\\ u|_{\partial\Omega} = 0, \qquad v|_{\partial\Omega} = 0, & t>0,\\ u(x,0) = u_0(x), & x\in\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded convex domain in $\mathbb{R}^N$ ($N = 2, 3$) with smooth boundary, $k_i(t)$ ($i = 1, 2, 3$) are positive and regular functions of t, $u_0(x)$ is a nonnegative function in Ω. Moreover, in the first equation, we assume that $f(u)$ is a nonnegative function and m is a positive constant.

The classical Keller–Segel system

$$\begin{cases} U_t = -\nabla(-\mu\nabla U + \chi U\nabla V), & x\in\Omega, t>0,\\ V_t = \nabla(D\nabla V) + gU - kV, & x\in\Omega, t>0, \end{cases} \tag{1.2}$$

was originally introduced in 1970 by Keller and Segel in [11], and it represents a fundamental model, which has great interest in biology, where Ω, denoting the capacity, is an open domain in $\mathbb{R}^N$ ($N \geq 1$), ∇ is the gradient operator , $U(x,t)$ denotes the cell density, and $V(x,t)$ represents the chemo-attractant. $\mu > 0$ is the amoeboid motility, $\chi > 0$ is the chemotactic sensitivity, $D > 0$ is the diffusion rate of cAMP, $g > 0$ is the rate of cAMP secretion per unit density of amoebae, $k > 0$ is the rate of degradation of cAMP in environment. The cross-diffusion term in the first equation reflects the assumption that individual cells partially adapt their motion so as to migrate toward increasing chemo-attractant.

In the last three decades, much attention has been devoted to studying the type of model (1.2) and its variations. See, for example, for system (1.2) with $\Omega = \mathbb{R}^N$, $\mu = \chi = D = 1$, $k = 0$. Kozono, Sugiyama, and Takada in [12] considered the problem whether there exists a finite-time self-similar solution of the backward type for the case of $N \geq 2$, and Sugiyama and Yahagi in [22] investigated the uniqueness and continuity of weak solutions with respect to the initial data for the Keller–Segel system of degenerate type. For more contribution along this line, we can see [3–6, 19–21], and the references therein.

In view of the biologically meaningful question whether or not cell populations spontaneously form aggregates, some studies focused on the issue whether solutions remain bounded or blow up (see [7–10, 16–18, 23, 26]).

Because practical experiences show how the specific parameters modeling chemotaxis phenomena are a general chemotaxis system, especially those influenced by logistic-type source (see [1, 2, 13–15, 25, 27]). In particular, Marras, Vernier-Piro, and Viglialoro in [15] considered the system

$$\begin{cases} u_t = \triangle u - k_1(t)\nabla \cdot (u^m \nabla v) + f(u), & x \in \Omega, t > 0, \\ v_t = k_2(t)\triangle v - k_3(t)v + k_4(t)u, & x \in \Omega, t > 0, \\ \frac{\partial u}{\partial n} + h(t)u = 0, \qquad \frac{\partial v}{\partial n} = 0, & x \in \partial\Omega, t > 0, \\ u(x,0) = u_0(x), \qquad v(x,0) = v_0(x), & x \in \Omega, \end{cases} \tag{1.3}$$

where $k_i(t)$ $(i = 1, 2, 3, 4)$ are positive and regular functions of t, $f(u)$ is a nonnegative function satisfying $f(u) \leq cu^2$, $c > 0$, and m is a positive constant. They showed that the lower bounds for blow-up time t^* to system (1.3) are obtained in both cases, the Neumann boundary condition (i.e., $h(t) = 0$) and the Robin boundary condition (i.e., $h(t) > 0$) for the three-dimensional case and provided $\frac{2}{3} < m < 1$ or for the two-dimensional case and provided $1 \leq m < 2$, respectively.

Homogeneous Dirichlet boundary condition for the chemotaxis system is prescribed by the disappearance of cell and chemo-attractant near the boundary.

Our aim in this paper is to investigate the lower bound for the blow-up time and decay criteria of associated energies to the parabolic–elliptic Keller–Segel system (1.1) under Dirichlet boundary conditions for the three-dimensional case and provided $\frac{2}{3} < m < 1$ or for the two-dimensional case and provided $1 \leq m < 2$, respectively. Let us point out that although the idea was used before for other problems, the adaptation of the procedure to our problem is not trivial at all. Due to the parabolic–elliptic Keller–Segel system (1.1) under Dirichlet boundary condition, we need more delicate estimates.

From biological point of view, solutions to system (1.1), representing the density and the chemo-attractant, must satisfy

$$u \geq 0, \qquad v \geq 0.$$

Thus it is reasonable to require throughout that the initial datum $u_0 \in C^0(\overline{\Omega})$ be nonnegative.

2 Lower bound for blow-up time

In this section, we give lower bounds for blow-up time to system (1.1) in the three-dimensional case and provided $\frac{2}{3} < m < 1$, and in the two-dimensional case and provided $1 \leq m < 2$, respectively.

2.1 Lower bound with $\Omega \subset \mathbb{R}^3$, $\frac{2}{3} < m < 1$

In this subsection, in order to obtain a lower bound for the blow-up time t^* of the solution (u,v) to system (1.1) with $N = 3$, we define the following auxiliary function:

$$\Phi(t) = \alpha(t) \int_\Omega u^2\,dx \tag{2.1}$$

with $\Phi_0 = \Phi(0) = \alpha(0) \int_\Omega u_0^2\,dx > 0$, and α is a suitable time-dependent positive function.

Definition 2.1 We say that (u,v) blows up in Φ-measure at time t^* if

$$\lim_{t\to t^*} \Phi(t) = \infty. \tag{2.2}$$

The main result in this subsection is given in the following theorem.

Theorem 2.1 *Let $\Omega \subset \mathbb{R}^3$ be a bounded convex domain with the origin inside. Suppose that $f(u)$ is a nonnegative function and satisfies*

$$f(u) \leq cu^2, \quad c > 0. \tag{2.3}$$

Moreover, let (u,v) be a classical solution to system (1.1), and (u,v) becomes unbounded in the Φ-measure at time $t = t^*$, with Φ defined in (2.1), then t^* satisfies the lower bound

$$t^* \geq \int_{\Phi(0)}^{+\infty} \frac{d\xi}{A_0\xi + A_1\xi^{\frac{m+2}{2}} + A_2\xi^{\frac{4-m}{4-3m}} + A_3\xi^{\frac{3}{2}} + A_4\xi^3}, \tag{2.4}$$

where

$$\begin{aligned}
&A_0 = \frac{\alpha'}{\alpha}, \qquad A_1 = c_1 2^{\frac{3}{2}m-1} p_1^{\frac{3}{2}m} \alpha^{-\frac{m+2}{2}},\\
&A_2 = c_1 2^{\frac{3}{2}m-1} \frac{p_2^{\frac{3}{2}m}}{\varepsilon_0^{\frac{3m}{4-3m}}} \cdot \frac{4-3m}{4} \alpha^{\frac{m-4}{4-3m}}, \qquad A_3 = a_1\alpha^{-\frac{3}{2}}, \qquad A_4 = \frac{a_2}{\varepsilon_1^2}\alpha^{-3},\\
&c_1 = \frac{2\alpha k_1 k_3}{m+1}, \qquad a_1 = \alpha c(2p_1)^{\frac{3}{2}}, \qquad a_2 = 2^{-\frac{1}{2}}\alpha c p_2^{\frac{3}{2}},\\
&p_1 = \frac{3}{2\rho_0}, \qquad p_2 = \frac{d}{\rho_0} + 1, \qquad \rho_0 = \min_{\partial\Omega} x \cdot \nu > 0, \qquad d = \max_{\overline{\Omega}} |x|.
\end{aligned} \tag{2.5}$$

Among that ε_0, ε_1 present positive constants, ν denotes the unit normal vector directed outward on $\partial\Omega$.

Proof By using Hölder's inequality and the arithmetic inequality

$$a^r b^s \leq ra + sb, \quad a > 0, b > 0, s + r = 1, \tag{2.6}$$

we obtain

$$\int_{\Omega} u^{m+2}\,dx \le \left(\int_{\Omega} u^2\,dx\right)^{1-m}\left(\int_{\Omega} u^3\,dx\right)^{m}. \tag{2.7}$$

Next, we estimate the term $\int_{\Omega} u^3\,dx$ appearing in (2.7) by means of the following inequality (see Lemma A2 in [17]):

$$\left(\int_{\Omega} |u|^3\,dx\right)^{m} \le \left[p_1\int_{\Omega} u^2\,dx + p_2\left(\int_{\Omega} u^2\,dx\right)^{\frac{1}{2}}\left(\int_{\Omega} |\nabla u|^2\,dx\right)^{\frac{1}{2}}\right]^{\frac{3}{2}m} \tag{2.8}$$

with $p_1 = \frac{3}{2\rho_0}$, $p_2 = \frac{d}{\rho_0} + 1$, where $\rho_0 = \min_{\partial\Omega} x\cdot\nu > 0$, $d = \max_{\overline{\Omega}}|x|$, ν denotes the unit normal vector directed outward on $\partial\Omega$.

From (2.8) and the inequality

$$(a+b)^s \le 2^{s-1}\left(a^s + b^s\right), \quad a > 0, b > 0, s \ge 1, \tag{2.9}$$

we achieve

$$\begin{aligned}&\left(\int_{\Omega} |u|^3\,dx\right)^{m}\\ &\quad\le 2^{\frac{3}{2}m-1}\left[p_1^{\frac{3}{2}m}\left(\int_{\Omega} u^2\,dx\right)^{\frac{3}{2}m} + p_2^{\frac{3}{2}m}\left(\int_{\Omega} u^2\,dx\right)^{\frac{3}{4}m}\left(\int_{\Omega} |\nabla u|^2\,dx\right)^{\frac{3}{4}m}\right].\end{aligned} \tag{2.10}$$

Inserting this estimate (2.10) into (2.7), we have

$$\begin{aligned}\int_{\Omega} u^{m+2}\,dx \le{}& 2^{\frac{3}{2}m-1}p_1^{\frac{3}{2}m}\left(\int_{\Omega} u^2\,dx\right)^{\frac{m+2}{2}} + 2^{\frac{3}{2}m-1}\frac{p_2^{\frac{3}{2}m}}{\varepsilon_0^{\frac{3m}{4-3m}}}\cdot\frac{4-3m}{4}\left(\int_{\Omega} u^2\,dx\right)^{\frac{4-m}{4-3m}}\\ &+ 2^{\frac{3}{2}m-1}p_2^{\frac{3}{2}m}\cdot\frac{3}{4}m\varepsilon_0\int_{\Omega} |\nabla u|^2\,dx,\end{aligned} \tag{2.11}$$

where ε_0 is a positive constant.

Then, differentiating $\Phi(t)$ and using the fact $u|_{\partial\Omega} = 0$, we have

$$\begin{aligned}\Phi'(t) &= 2\alpha\int_{\Omega} uu_t\,dx + \alpha'\int_{\Omega} u^2\,dx\\ &= 2\alpha\int_{\Omega} u\big(\triangle u - k_1(t)\nabla\cdot\left(u^m\nabla v\right) + f(u)\big)\,dx + \alpha'\int_{\Omega} u^2\,dx\\ &= 2\alpha\int_{\Omega} u\triangle u\,dx - 2\alpha k_1\int_{\Omega} u\nabla\cdot\left(u^m\nabla v\right)dx + 2\alpha\int_{\Omega} uf(u)\,dx + \alpha'\int_{\Omega} u^2\,dx\\ &= -2\alpha\int_{\Omega} |\nabla u|^2\,dx - \frac{2\alpha k_1}{m+1}\int_{\Omega} u^{m+1}\triangle v\,dx + 2\alpha\int_{\Omega} uf(u)\,dx + \alpha'\int_{\Omega} u^2\,dx\\ &= -2\alpha\int_{\Omega} |\nabla u|^2\,dx - \frac{2\alpha k_1k_2}{m+1}\int_{\Omega} u^{m+1}v\,dx + \frac{2\alpha k_1k_3}{m+1}\int_{\Omega} u^{m+2}\,dx\\ &\quad + 2\alpha\int_{\Omega} uf(u)\,dx + \alpha'\int_{\Omega} u^2\,dx.\end{aligned} \tag{2.12}$$

In the forth term (2.12), we use (2.3), (2.8), and (2.9), which leads to

$$2\alpha\int_\Omega uf(u)\,dx \le 2\alpha c\int_\Omega u^3\,dx \le a_1\left(\int_\Omega u^3\,dx\right)^{\frac{3}{2}} + \frac{a_2}{\varepsilon_1^2}\left(\int_\Omega u^3\,dx\right)^3 + 3a_2\varepsilon_1\int_\Omega |\nabla u|^2\,dx \tag{2.13}$$

with

$$a_1 = \alpha c(2p_1)^{\frac{3}{2}}, \qquad a_2 = 2^{-\frac{1}{2}}\alpha c p_2^{\frac{3}{2}}.$$

Here, we also used Hölder's inequality and Young's inequality with $\varepsilon_1 > 0$.

Inserting (2.11) and (2.13) into (2.12), we obtain

$$\begin{aligned}
\Phi'(t) \le{}& -2\alpha\int_\Omega |\nabla u|^2\,dx + c_1 2^{\frac{3}{2}m-1}p_1^{\frac{3}{2}m}\left(\int_\Omega u^2\,dx\right)^{\frac{m+2}{2}} \\
&+ c_1 2^{\frac{3}{2}m-1}\frac{p_2^{\frac{3}{2}m}}{\varepsilon_0^{\frac{3m}{4-3m}}}\cdot\frac{4-3m}{4}\left(\int_\Omega u^2\,dx\right)^{\frac{4-m}{4-3m}} + c_1 2^{\frac{3}{2}m-1}p_2^{\frac{3}{2}m}\cdot\frac{3}{4}m\varepsilon_0\int_\Omega |\nabla u|^2\,dx \\
&+ a_1\left(\int_\Omega u^3\,dx\right)^{\frac{3}{2}} + \frac{a_2}{\varepsilon_1^2}\left(\int_\Omega u^3\,dx\right)^3 + 3a_2\varepsilon_1\int_\Omega |\nabla u|^2\,dx + \alpha'\int_\Omega u^2\,dx \\
={}& \left(3a_2\varepsilon_1 + c_1 2^{\frac{3}{2}m-1}p_2^{\frac{3}{2}m}\cdot\frac{3}{4}m\varepsilon_0 - 2\alpha\right)\int_\Omega |\nabla u|^2\,dx \\
&+ c_1 2^{\frac{3}{2}m-1}p_1^{\frac{3}{2}m}\alpha^{-\frac{m+2}{2}}\left(\alpha\int_\Omega u^2\,dx\right)^{\frac{m+2}{2}} \\
&+ c_1 2^{\frac{3}{2}m-1}\frac{p_2^{\frac{3}{2}m}}{\varepsilon_0^{\frac{3m}{4-3m}}}\cdot\frac{4-3m}{4}\alpha^{\frac{m-4}{4-3m}}\left(\alpha\int_\Omega u^2\,dx\right)^{\frac{4-m}{4-3m}} + \frac{\alpha'}{\alpha}\left(\alpha\int_\Omega u^2\,dx\right) \\
&+ a_1\alpha^{-\frac{3}{2}}\left(\alpha\int_\Omega u^3\,dx\right)^{\frac{3}{2}} + \frac{a_2}{\varepsilon_1^2}\alpha^{-3}\left(\alpha\int_\Omega u^3\,dx\right)^3
\end{aligned} \tag{2.14}$$

with $c_1 = \frac{2\alpha k_1 k_3}{m+1}$.

To simplify the right-hand side of (2.14), we choose appropriate constants ε_0 and ε_1 such that

$$3a_2\varepsilon_1 + c_1 2^{\frac{3}{2}m-1}p_2^{\frac{3}{2}m}\cdot\frac{3}{4}m\varepsilon_0 - 2\alpha = 0.$$

Hence, we can estimate (2.14) as

$$\Phi'(t) \le A_0\Phi + A_1\Phi^{\frac{m+2}{2}} + A_2\Phi^{\frac{4-m}{4-3m}} + A_3\Phi^{\frac{3}{2}} + A_4\Phi^3, \tag{2.15}$$

where

$$A_0 = \frac{\alpha'}{\alpha}, \qquad A_1 = c_1 2^{\frac{3}{2}m-1}p_1^{\frac{3}{2}m}\alpha^{-\frac{m+2}{2}}, \qquad A_2 = c_1 2^{\frac{3}{2}m-1}\frac{p_2^{\frac{3}{2}m}}{\varepsilon_0^{\frac{3m}{4-3m}}}\cdot\frac{4-3m}{4}\alpha^{\frac{m-4}{4-3m}},$$

$$A_3 = a_1\alpha^{-\frac{3}{2}}, \qquad A_4 = \frac{a_2}{\varepsilon_1^2}\alpha^{-3}.$$

Now, integrating (2.15) over $(0,t)$, we have

$$t \geq \int_{\Phi(0)}^{\Phi(t)} \frac{d\xi}{A_0\xi + A_1\xi^{\frac{m+2}{2}} + A_2\xi^{\frac{4-m}{4-3m}} + A_3\xi^{\frac{3}{2}} + A_4\xi^3}.$$

Thus the proof of Theorem 2.1 is completed. □

2.2 Lower bound with $\Omega \subset \mathbb{R}^2$, $1 \leq m < 2$

In this subsection, we consider system (1.1) in the case $\Omega \subset \mathbb{R}^2$, we have the following main result.

Theorem 2.2 *Let (u,v) be a classical solution of system (1.1) in a convex region $\Omega \subset \mathbb{R}^2$ with smooth boundary and $1 \leq m < 2$. Suppose that $f(u)$ is a nonnegative function and satisfies (2.3). If (u,v) blows up in the Φ-measure at time t^*, with Φ defined in (2.1), then t^* satisfies the lower bound*

$$t^* \geq \int_{\Phi(0)}^{+\infty} \frac{d\xi}{\overline{A}_0\xi + \overline{A}_1\xi^{\frac{m+2}{2}} + \overline{A}_2\xi^{\frac{2}{2-m}} + \overline{A}_3\xi^{\frac{3}{2}} + \overline{A}_4\xi^3}, \tag{2.16}$$

where

$$\overline{A}_0 = \frac{\alpha'}{\alpha}, \qquad \overline{A}_1 = c_1\frac{2^{\frac{3m}{2}-1}p_1^m}{3^m}\alpha^{-\frac{m+2}{2}}, \qquad \overline{A}_2 = c_1 2^{\frac{m}{2}-1}p_2^m\frac{2-m}{2}\varepsilon_2^{\frac{m}{m-2}}\alpha^{\frac{m}{m-2}},$$

$$\overline{A}_3 = a_1\alpha^{-\frac{3}{2}}, \qquad \overline{A}_4 = \frac{a_2}{\varepsilon_1^2}\alpha^{-3}$$

with c_1, a_1, a_2, p_1, p_2, ρ_0, d defined in (2.5). Among that, ε_1, ε_2 present positive constants, ν denotes the unit normal vector directed outward on $\partial\Omega$.

Proof By using Hölder's inequality, we obtain

$$\int_\Omega u^{m+2}\,dx \leq \left(\int_\Omega u^2\,dx\right)^{1-\frac{m}{2}}\left(\int_\Omega u^4\,dx\right)^{\frac{m}{2}}. \tag{2.17}$$

In order to estimate the term $\int_\Omega u^4\,dx$, we use the following inequality (see (3.2) and (3.4) in [17]):

$$\left(\int_\Omega |u|^4\,dx\right)^{\frac{1}{2}} \leq \frac{\sqrt{2}}{2}\left[\frac{1}{\rho_0}\int_\Omega u^2\,dx + \left(1+\frac{d}{\rho_0}\right)\left(\int_\Omega u^2\,dx\int_\Omega |\nabla u|^2\,dx\right)^{\frac{1}{2}}\right] \tag{2.18}$$

with ρ_0, d defined in (2.5).

Inserting (2.18) into (2.17), we obtain

$$\int_\Omega u^{m+2}\,dx \leq 2^{-\frac{m}{2}}\left(\int_\Omega u^2\,dx\right)^{1-\frac{m}{2}} \times \left[\frac{1}{\rho_0}\int_\Omega u^2\,dx + \left(1+\frac{d}{\rho_0}\right)\left(\int_\Omega u^2\,dx\int_\Omega |\nabla u|^2\,dx\right)^{\frac{1}{2}}\right]^m$$

$$
\begin{aligned}
&\le 2^{\frac{m}{2}-1}\frac{1}{\rho_0^m}\left(\int_\Omega u^2\,dx\right)^{1+\frac{m}{2}} + 2^{\frac{m}{2}-1}\left(1+\frac{d}{\rho_0}\right)^m\left(\int_\Omega u^2\,dx\int_\Omega |\nabla u|^2\,dx\right)^{\frac{m}{2}} \\
&\le \frac{2^{\frac{3m}{2}-1}p_1^m}{3^m}\left(\int_\Omega u^2\,dx\right)^{1+\frac{m}{2}} + 2^{\frac{m}{2}-1}p_2^m\frac{m}{2}\varepsilon_2\int_\Omega |\nabla u|^2\,dx \\
&\quad + 2^{\frac{m}{2}-1}p_2^m\frac{2-m}{2}\varepsilon_2^{\frac{m}{m-2}}\left(\int_\Omega u^2\,dx\right)^{\frac{2}{2-m}}. \qquad (2.19)
\end{aligned}
$$

Starting from (2.12), if we apply (2.13) and (2.19), we obtain

$$
\begin{aligned}
\Phi'(t) &\le \left(3a_2\varepsilon_1 + c_1 2^{\frac{m}{2}-1}p_2^m\cdot\frac{m}{2}\varepsilon_2 - 2\alpha\right)\int_\Omega |\nabla u|^2\,dx + c_1\frac{2^{\frac{3m}{2}-1}p_1^m}{3^m}\left(\int_\Omega u^2\,dx\right)^{1+\frac{m}{2}} \\
&\quad + c_1 2^{\frac{m}{2}-1}p_2^m\frac{2-m}{2}\varepsilon_2^{\frac{m}{m-2}}\left(\int_\Omega u^2\,dx\right)^{\frac{2}{2-m}} + \frac{\alpha'}{\alpha}\left(\alpha\int_\Omega u^2\,dx\right) \\
&\quad + a_1\alpha^{-\frac{3}{2}}\left(\alpha\int_\Omega u^3\,dx\right)^{\frac{3}{2}} + \frac{a_2}{\varepsilon_1^2}\alpha^{-3}\left(\alpha\int_\Omega u^3\,dx\right)^3. \qquad (2.20)
\end{aligned}
$$

Now, choosing appropriate constants ε_1 and ε_2 such that

$$
3a_2\varepsilon_1 + c_1 2^{\frac{m}{2}-1}p_2^m\cdot\frac{m}{2}\varepsilon_2 - 2\alpha = 0,
$$

we get

$$
\Phi'(t) \le A_0\Phi + A_1\Phi^{\frac{m+2}{2}} + A_2\Phi^{\frac{2}{2-m}} + A_3\Phi^{\frac{3}{2}} + A_4\Phi^3, \qquad (2.21)
$$

where

$$
\begin{aligned}
&\overline{A}_0 = \frac{\alpha'}{\alpha}, \qquad \overline{A}_1 = c_1\frac{2^{\frac{3m}{2}-1}p_1^m}{3^m}\alpha^{-\frac{m+2}{2}}, \qquad \overline{A}_2 = c_1 2^{\frac{m}{2}-1}p_2^m\frac{2-m}{2}\varepsilon_2^{\frac{m}{m-2}}\alpha^{\frac{m}{m-2}}, \\
&\overline{A}_3 = a_1\alpha^{-\frac{3}{2}}, \qquad \overline{A}_4 = \frac{a_2}{\varepsilon_1^2}\alpha^{-3}.
\end{aligned}
$$

Now, integrating (2.21) over $(0,t)$, we have

$$
t \ge \int_{\Phi(0)}^{\Phi(t)} \frac{d\xi}{\overline{A}_0\xi + \overline{A}_1\xi^{\frac{m+2}{2}} + \overline{A}_2\xi^{\frac{2}{2-m}} + \overline{A}_3\xi^{\frac{3}{2}} + \overline{A}_4\xi^3}.
$$

Thus the proof of Theorem 2.2 is completed. □

3 Exponential decay for the associated energies

In this section, we focus on the exponential decay of the associated energies for system (1.1). We only consider the case when $\Omega \subset \mathbb{R}^3$, the case when $\Omega \subset \mathbb{R}^2$ being completely similar. In order to state our main result, we need the following condition:

$$
-D_1\lambda_1 + D_2\lambda_1^{\frac{3}{4}}\Phi(0)^{\frac{1}{2}} + D_3\Phi(0)^{\frac{1}{2}} + D_4 < 0, \qquad (3.1)
$$

where

$$D_1 = 2\alpha, \qquad D_2 = 2^{\frac{1}{2}}(c_1 m + 2\alpha)p_2^{\frac{3}{2}}\alpha^{-\frac{3}{4}}, \qquad D_3 = 2^{\frac{1}{2}}(c_1 m + 2\alpha)p_1^{\frac{3}{2}}\alpha^{-\frac{3}{42}},$$
$$D_4 = \frac{c_1(1-m)+\alpha'}{\alpha}, \qquad \Phi(0) = \alpha(0)\int_\Omega u_0(x)^2\,dx \tag{3.2}$$

and λ_1 is the first eigenvalue for the boundary value problem as follows:

$$\begin{cases} \triangle\varphi + \lambda\varphi = 0, & x \in \Omega, \\ \varphi|_{\partial\Omega} = 0, & \\ \varphi > 0, & x \in \Omega. \end{cases}$$

Remark 3.1 Let us note that upon appropriate choices of D_1, D_2, D_3, D_4, and $\Phi(0)$, one can obtain condition (3.1).

We are now ready to state our main result about the decay of energies for system (1.1).

Theorem 3.2 *Assume that* $\Phi(0) > 0$, $2k_2 > k_3$ *and condition* (3.1) *is satisfied. Then the solution decays exponentially to zero in* $L^2(\Omega)$.

Proof Let (u, v) be the unique solution of (1.1), we put

$$\Phi(t) = \alpha(t)\int_\Omega u^2\,dx, \Psi(t) = \frac{(2k_2 - k_3)\alpha(t)}{k_3}\int_\Omega v^2\,dx.$$

Following the same idea as in the proof of Theorem 2.1, we have

$$\Phi'(t) \le \left(\int_\Omega |\nabla u|^2\,dx\right)^{\frac{3}{4}}\left(-D_1\left(\int_\Omega |\nabla u|^2\,dx\right)^{\frac{1}{4}} + D_2\Phi(t)^{\frac{3}{4}}\right) + D_3\Phi(t)^{\frac{3}{2}} + D_4\Phi(t) \tag{3.3}$$

with D_1, D_2, D_3, D_4 defined in (3.2).

Inserting the result

$$\int_\Omega |\nabla u|^2\,dx \ge \lambda_1\int_\Omega u^2\,dx,$$

we obtain

$$\Phi'(t) \le \left(\int_\Omega |\nabla u|^2\,dx\right)^{\frac{3}{4}}\left(-D_1\lambda_1^{\frac{1}{4}}\Phi(t)^{\frac{1}{4}} + D_2\Phi(t)^{\frac{3}{4}}\right) + D_3\Phi(t)^{\frac{3}{2}} + D_4\Phi(t). \tag{3.4}$$

We claim, by an argument similar to that of Wang, Wang, and Zhou in [24], that

$$\Phi'(t) < 0 \tag{3.5}$$

and

$$\Phi'(t) \le \Phi(t)\left(-D_1\lambda_1^{\frac{1}{4}}\Phi(t)^{\frac{1}{4}} + D_2\Phi(t)^{\frac{3}{4}}\right) + D_3\Phi(t)^{\frac{3}{2}} + D_4\Phi(t). \tag{3.6}$$

By (3.5) and (3.6), there exists a positive constant γ such that

$$D_2\Phi(t)^{\frac{3}{4}} + D_3\Phi(t)^{\frac{3}{2}} < D_2\Phi(0)^{\frac{3}{4}} + D_3\Phi(0)^{\frac{3}{2}} < D_1\lambda_1 - D_4 - \gamma.$$

Thus, one has

$$\Phi'(t) \le -\gamma\Phi(t),$$

which yields

$$\Phi(t) \le \Phi(0)\exp(-\gamma t). \tag{3.7}$$

This proves that u decays exponentially to zero in $L^2(\Omega)$.

Next, we study the decay behavior of $\Psi(t)$. Multiplying both sides of the second equation in (1.1) by v and then integrating over Ω, we obtain

$$\int_\Omega v\triangle v\,dx = k_2\int_\Omega v^2\,dx - k_3\int_\Omega uv\,dx.$$

Namely,

$$-\int_\Omega |\nabla v|^2\,dx = k_2\int_\Omega v^2\,dx - k_3\int_\Omega uv\,dx.$$

By Hölder's inequality and the arithmetic inequality (2.6), we have

$$k_2\int_\Omega v^2\,dx + \int_\Omega |\nabla v|^2\,dx = k_3\int_\Omega uv\,dx \le \frac{k_3}{2}\int_\Omega u^2\,dx + \frac{k_3}{2}\int_\Omega v^2\,dx.$$

Thus,

$$\left(k_2 - \frac{k_3}{2}\right)\int_\Omega v^2\,dx \le \frac{k_3}{2}\int_\Omega u^2\,dx,$$

which together with (3.7) yields

$$\Psi(t) = \frac{(2k_2 - k_3)\alpha(t)}{k_3}\int_\Omega v^2\,dx \le \alpha(t)\int_\Omega u^2\,dx = \Phi(t) \le \Phi(0)\exp(-\gamma t),$$

as desired. This completes the proof. □

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Authors' contributions
YJ completed the main study and wrote the manuscript, WZ checked the proofs process and verified the calculation. Moreover, all the authors read and approved the last version of the manuscript.

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