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Multiple positive solutions for nonlinear coupled fractional Laplacian system with critical exponent

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Abstract

In this paper, we study the following critical system with fractional Laplacian:

$$\begin{cases} (-\Delta)^s u + \lambda_1 u = \mu_1 |u|^{2^*-2}u + \frac{\alpha\gamma}{2^*}|u|^{\alpha-2}u|v|^{\beta} & \text{in } \Omega, \\ (-\Delta)^s v + \lambda_2 v = \mu_2 |v|^{2^*-2}v + \frac{\beta\gamma}{2^*}|u|^{\alpha}|v|^{\beta-2}v & \text{in } \Omega, \\ u = v = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

where $(-\Delta)^s$ is the fractional Laplacian, $0 < s < 1$, $\mu_1, \mu_2 > 0$, $2^* = \frac{2N}{N-2s}$ is a fractional critical Sobolev exponent, $N > 2s$, $1 < \alpha, \beta < 2$, $\alpha + \beta = 2^*$, Ω is an open bounded set of $\mathbb{R}^N$ with Lipschitz boundary and $\lambda_1, \lambda_2 > -\lambda_{1,s}(\Omega)$, $\lambda_{1,s}(\Omega)$ is the first eigenvalue of the non-local operator $(-\Delta)^s$ with homogeneous Dirichlet boundary datum. By using the Nehari manifold, we prove the existence of a positive ground state solution of the system for all $\gamma > 0$. Via a perturbation argument and using the topological degree and a pseudo-gradient vector field, we show that this system has a positive higher energy solution. Then the asymptotic behaviors of the positive ground state solutions are analyzed when $\gamma \to 0$.

MSC: 35J50; 35B33; 35R11

Keywords: Fractional Laplacian; Critical exponent; Ground state solution; Higher energy solution

1 Introduction

The fractional Laplacian operator and fractional Sobolev space arise in a quite natural way in many different contexts, such as the thin obstacle problem, finance, phase transitions, anomalous diffusion, flame propagation and many others (see [1–4] and the references therein). In recent years, the corresponding non-local equation or systems involving fractional Laplacian with nonlinear terms have attracted the attention of many researchers, both for their interesting theoretical structure and their concrete applications (see [5–11] and the references therein).

There have been a lot of studies that consider a Laplacian equation or a Laplacian system (see [12–16] and the references therein). Compared to the Laplacian problem, the fractional Laplacian problem is non-local and more difficult to handle. For the following fractional Laplacian equation:

$$\begin{cases} (-\Delta)^s u = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \mathbb{R}^N \setminus \Omega. \end{cases} \tag{1}$$

Servadei and Valdinoci [17] showed that (1) has mountain pass type solution which is not identically zero. When $f(x,u) = \lambda u^q + u^{2^*-1}$, Barrios, Colorado, Servadei and Soria [18] obtained the existence and multiplicity solutions for system (1) under different conditions of λ.

For the following fractional Laplacian equation:

$$\begin{cases} (-\Delta)^s u = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2}$$

Caffarelli and Silvestre [19] studied an extension problem related to the fractional Laplacian in $\mathbb{R}^n$, which can transform the non-local problem into a local problem in $\mathbb{R}^{n+1}_+$. This method can be extended to bounded regions and is extensively used in recent articles. For example, when $f(x,u) = \lambda u^q + u^{\frac{n+s}{n-s}}$, Barrios, Colorado, de Pablo and Sánchez [5] proved the existence and multiplicity of solutions for equation (2) under suitable conditions of s and q. When $f(x,u) = |u|^{2^*-2}u + f(x)$ Colorado, de Pablo and Sánchez [6] proved the existence and the multiplicity of solutions for equation (2) under appropriate conditions on the size of f.

The following Brézis–Nirenberg problem for the fractional Laplacian:

$$\begin{cases} (-\Delta)^s u + \lambda_i u = |u|^{2^*-2}u & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

has been investigated by Servadei and Valdinoci [20, 21] and obtained a non-trivial solutions.

It is also natural to study the coupled system of equations. Li and Yang [22] considered the following subcritical case fractional Laplacian system:

$$\begin{cases} (-\Delta)^s u = \lambda |u|^{q-2}u + \frac{2\alpha}{\alpha+\beta}|u|^{\alpha-2}u|v|^{\beta} & \text{in } \Omega, \\ (-\Delta)^s v = \mu |v|^{q-2}v + \frac{2\beta}{\alpha+\beta}|u|^{\alpha}|v|^{\beta-2}v & \text{in } \Omega, \\ u = 0, \qquad v = 0 & \text{on } \mathbb{R}^N \setminus \Omega, \end{cases}$$

by using the Nehari manifold, fibering maps and the Lusternik–Schnirelmann category, they prove that the problem has at least $\operatorname{cat}(\Omega) + 1$ distinct positive solutions, where $\operatorname{cat}(\Omega)$ denotes the Lusternik–Schnirelmann category of Ω in itself. When the boundary conditions are replaced by $u = 0$, $v = 0$ on $\partial\Omega$, X. He, Squassina and Zou [23] using variational methods and a Nehari manifold decomposition proved that the system admits at least two positive solutions when the pair of parameters (λ, μ) belong to certain subset of $\mathbb{R}^2$.

We address the following critical system involving a fractional Laplacian:

$$\begin{cases} (-\Delta)^s u + \lambda_1 u = \mu_1 |u|^{2^*-2}u + \frac{\alpha\gamma}{2^*}|u|^{\alpha-2}u|v|^{\beta} & \text{in } \Omega, \\ (-\Delta)^s v + \lambda_2 v = \mu_2 |v|^{2^*-2}v + \frac{\beta\gamma}{2^*}|u|^{\alpha}|v|^{\beta-2}v & \text{in } \Omega, \\ u = v = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{3}$$

where $(-\Delta)^s$ is the fractional Laplacian, $0 < s < 1$, $\mu_1, \mu_2 > 0$, $2^* = \frac{2N}{N-2s}$ is a fractional critical Sobolev exponent, $N > 2s$, $1 < \alpha, \beta < 2$, $\alpha + \beta = 2^*$, Ω is an open bounded set of $\mathbb{R}^N$ with Lipschitz boundary and $\lambda_1, \lambda_2 > -\lambda_{1,s}(\Omega)$, $\lambda_{1,s}(\Omega)$ is the first eigenvalue of the non-local operator $(-\Delta)^s$ with homogeneous Dirichlet boundary datum.

The fractional Laplacian $(-\Delta)^s$ is defined by

$$-(-\Delta)^s u(x) = \frac{C(N,s)}{2} \int_{R^N} \frac{u(x+y) + u(x-y) - 2u(x)}{|y|^{N+2s}}\, dy, \quad x \in \mathbb{R}^N,$$

with

$$C(N,s) = \left(\int_{R^N} \frac{1 - \cos(\varsigma_1)}{|\varsigma|^{N+2s}}\, d\varsigma \right)^{-1} = 2^{2s}\pi^{-\frac{N}{2}} \frac{\Gamma(\frac{N+2s}{2})}{\Gamma(2-s)} s(1-s).$$

Guo, Luo and Zou [11], showed that when $\lambda_1, \lambda_2 \in (-\lambda_{1,s}(\Omega), 0)$, (3) has a positive ground state solution for all $\gamma > 0$. For more recent advances on this topic, see [24–26] and the references therein.

In [27], we have consider the following critical system:

$$\begin{cases} (-\Delta)^s u = \mu_1 |u|^{2^*-2}u + \frac{\alpha\gamma}{2^*}|u|^{\alpha-2}u|v|^{\beta} & \text{in } \mathbb{R}^n, \\ (-\Delta)^s v = \mu_2 |v|^{2^*-2}v + \frac{\beta\gamma}{2^*}|u|^{\alpha}|v|^{\beta-2}v & \text{in } \mathbb{R}^n, \\ u, v \in D_s(\mathbb{R}^n). \end{cases}$$

By using the Nehari manifold, under proper conditions, we establish the existence and nonexistence of a positive least energy solution of the above system.

In this paper, we study system (3) from another aspect to obtain the ground state solutions, higher energy solution and an analysis the asymptotic behaviors of the positive ground state solutions.

Let $D_s(\Omega)$ be Hilbert space as the completion of $C_c^\infty(\Omega)$ equipped with the norm

$$\|u\|^2_{D_s(\Omega)} = \frac{C(N,s)}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|y-x|^{N+2s}}\, dx\, dy.$$

Let

$$S_s = \inf_{u \in D_s(\mathbb{R}^N)\setminus\{0\}} \frac{\|u\|^2_{D_s(\mathbb{R}^N)}}{(\int_{R^N} |u|^{2^*}\, dx)^{\frac{2}{2^*}}} \tag{4}$$

be the sharp embedding constant of $D_s(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N)$ and S_s is attained (see [28]) in $\mathbb{R}^N$ by $\widetilde{u}_{\epsilon,y} = \kappa(\varepsilon^2 + |x-y|)^{-\frac{N-2s}{2}}$, where $\kappa \neq 0 \in \mathbb{R}$, $\varepsilon > 0$ and $y \in \mathbb{R}^N$. That is,

$$S_s = \frac{\|\widetilde{u}_{\epsilon,y}\|^2_{D_s(\mathbb{R}^N)}}{(\int_{\mathbb{R}^N} |\widetilde{u}_{\epsilon,y}|^{2^*}\, dx)^{\frac{2}{2^*}}}.$$

The energy functional associated with (3) is given by

$$
\begin{aligned}
E_\gamma(u,v) &= \frac{1}{2}\big\|(u,v)\big\|^2_{\mathcal{D}_s(\Omega)} + \frac{1}{2}\int_\Omega \big(\lambda_1 u^2 + \lambda_2 v^2\big)\,dx \\
&\quad - \frac{1}{2^*}\int_\Omega \big(\mu_1|u|^{2^*} + \mu_2|v|^{2^*} + \gamma|u|^\alpha|v|^\beta\big)\,dx,
\end{aligned}
$$

where $\mathcal{D}_s(\Omega) := D_s(\Omega)\times D_s(\Omega)$ is endowed with norm $\|(u,v)\|^2_{\mathcal{D}_s(\Omega)} = \|u\|^2_{D_s(\Omega)} + \|v\|^2_{D_s(\Omega)}$. Define the Nehari manifold

$$
\begin{aligned}
\mathbb{M} &= \bigg\{(u,v)\in\mathcal{D}_s(\Omega)\backslash\big\{(0,0)\big\} : \big\|(u,v)\big\|^2_{\mathcal{D}_s(\Omega)} + \int_\Omega\big(\lambda_1u^2+\lambda_2v^2\big)\,dx \\
&\quad = \int_\Omega\big(\mu_1|u|^{2^*} + \mu_2|v|^{2^*} + \gamma|u|^\alpha|v|^\beta\big)\,dx\bigg\}, \\
A_\gamma &:= \inf_{(u,v)\in\mathbb{M}} E_\gamma(u,v) = \inf_{(u,v)\in\mathbb{M}} \frac{s}{N}\bigg(\big\|(u,v)\big\|^2_{\mathcal{D}_s(\Omega)} + \int_\Omega\big(\lambda_1u^2+\lambda_2v^2\big)\,dx\bigg) \\
&\quad = \inf_{(u,v)\in\mathbb{M}} \frac{s}{N}\int_\Omega\big(\mu_1|u|^{2^*} + \mu_2|v|^{2^*} + \gamma|u|^\alpha|v|^\beta\big)\,dx.
\end{aligned}
$$

We say that (u,v) is a non-trivial solution of (3) if $u\not\equiv 0$, $v\not\equiv 0$ and (u,v) solves (3). Any non-trivial solution of (3) is in $\mathbb{M}$. Due to the fact that if we take $\varphi,\psi\in C_0^\infty(\Omega)$ with $\varphi,\psi\not\equiv 0$ and $\operatorname{supp}(\varphi)\cap\operatorname{supp}(\psi)=\emptyset$, then there exist $t_1,t_2>0$ such that $(t_1\varphi,t_2\psi)\in\mathbb{M}$, so $\mathbb{M}\neq\emptyset$.

Our main results are as follows.

Theorem 1.1

(i) *Assume* $-\lambda_{1,s}(\Omega) < \min\{\lambda_1,\lambda_2\} < 0$ *and* $N > 4s$. *Then system* (3) *has a positive ground state solution* $(u_\gamma,v_\gamma)\in\mathcal{D}_s(\Omega)$ *with* $E_\gamma(u_\gamma,v_\gamma) = A_\gamma$ *for all* $\gamma > 0$.

(ii) *Assume* $-\lambda_{1,s}(\Omega) < \min\{\lambda_1,\lambda_2\} < 0$, $N > 4s$ *and let* γ_n *be a sequence with* $\gamma_n \to 0$ *as* $n\to+\infty$. *Then, passing to a subsequence,* $(u_{\gamma_n},v_{\gamma_n})\to(\overline{u},\overline{v})$ *strongly in* $D_s(\Omega)\times D_s(\Omega)$ *as* $n\to+\infty$, *and one of the following conclusions holds:*

(1) $(\overline{u},0)$ *is a positive ground state solution of*

$$
\begin{cases}
(-\Delta)^s u + \lambda_1 u = \mu_1|u|^{2^*-2}u & \text{in } \Omega; \\
u = 0 & \text{on } \mathbb{R}^N\setminus\Omega.
\end{cases}
$$

(2) $(0,\overline{v})$ *is a positive ground state solution of*

$$
\begin{cases}
(-\Delta)^s v + \lambda_2 v = \mu_2|v|^{2^*-2}v & \text{in } \Omega, \\
v = 0 & \text{on } \mathbb{R}^N\setminus\Omega.
\end{cases}
$$

If

$$
\left(\frac{\mu_1}{\mu_2}\right)^{-\frac{N-2s}{2s}} < \frac{m_{\lambda_2}}{m_{\lambda_1}} \quad \text{implies that} \quad m_{\lambda_1,\mu_1} < m_{\lambda_2,\mu_2},
$$

then (1) *holds.*

If

$$\left(\frac{\mu_1}{\mu_2}\right)^{-\frac{N-2s}{2s}} > \frac{m_{\lambda_2}}{m_{\lambda_1}} \quad \textit{implies that} \quad m_{\lambda_1,\mu_1} > m_{\lambda_2,\mu_2},$$

then (2) *holds, where* m_{λ_i} *and* m_{λ_i,μ_i} *see Lemma* 2.1 *and Remark* 2.1 *in the next section.*

Theorem 1.2 *Assume* $-\lambda_{1,s}(\Omega) < \min\{\lambda_1, \lambda_2\} < 0$ *and* $N > 4s$, *then there exists a* $\gamma_0 > 0$ *such that, for* $|\gamma| < \gamma_0$, *system* (3) *has a positive higher energy solution* $(\widehat{u}_\gamma, \widehat{v}_\gamma)$ *with* $E_\gamma(\widehat{u}_\gamma, \widehat{v}_\gamma) > A_\gamma$.

Remark 1.1 Although the method in this paper to obtain the ground state solution is different from Z. Guo, S. Luo and W. Zou [11], we get similar result as Theorem 1.2 in [11].

Remark 1.2 In the proof of Theorem 1.1, we should point out that $1 < \alpha, \beta < 2$ is an essential condition.

Remark 1.3 In the proof of Theorem 1.1, we need $N > 4s$, due to $1 < \alpha, \beta < 2$ and $2 < \alpha + \beta = 2^* < 4$. For $2s < N < 4s$, the method in this paper does not work and it should be interesting to get a ground state solution.

Remark 1.4 It is easy to see that, for $\gamma > 0$ sufficiently small, the higher energy solutions in Theorem 1.2 are different from the ground state solutions in Theorem 1.1. That is system (3) has at least two positive solutions for $\lambda_1, \lambda_2 < 0$ and $\gamma > 0$ sufficiently small.

In order to prove Theorem 1.1, we use the classical mountain pass theorem, due to each equation in this system is critical exponent, so the embedding for $\mathcal{D}_s(\Omega) \hookrightarrow L^{2^*}(\Omega)$ is not compact embedding. Thus, we need estimate A_γ such that A_γ is strict less than $\min\{\mu_1^{-\frac{N-2s}{2s}} \frac{s}{N} S_s^{\frac{N}{2s}}, \mu_2^{-\frac{N-2s}{2s}} \frac{s}{N} S_s^{\frac{N}{2s}}\}$ (see Lemma 2.4). The main idea to prove Theorem 1.2 is to regard system (3) as a perturbation of system (24) by $\frac{\alpha\gamma}{2^*}|u|^{\alpha-2}u|v|^\beta$ and $\frac{\beta\gamma}{2^*}|u|^\alpha|v|^{\beta-2}v$, then use the topological degree and the pseudo-gradient vector field to show some lemmas that will be used to get another positive solution. The idea is originally from [29].

The paper is organized as follows. In Sect. 2, we introduce some preliminaries that will be used to prove theorems. In Sect. 3, we prove Theorem 1.1 and Theorem 1.2 will be proved in Sect. 4.

2 Some preliminaries

For the following fractional Brézis–Nirenberg problem:

$$\begin{cases} (-\Delta)^s u + \lambda_i u = \mu_i |u|^{2^*-2} u & \text{in } \Omega, \\ u = 0 & \text{on } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{5}$$

we define

$$J_{\lambda_i,\mu_i}(u) = \frac{1}{2}\|u\|^2_{D_s(\Omega)} + \frac{1}{2}\int_\Omega \lambda_i u^2\,dx - \frac{1}{2^*}\int_\Omega \mu_i |u|^{2^*}\,dx$$

and

$$m_{\lambda_i,\mu_i} = \inf_{u\in\mathbb{M}_i} J_{\lambda_i,\mu_i}(u),$$

where

$$\mathbb{M}_i = \left\{u \in D_s(\Omega)\backslash\{0\} : \|u\|^2_{D_s(\Omega)} + \int_\Omega \lambda_i u^2\,dx = \int_\Omega \mu_i|u|^{2^*}\,dx\right\}.$$

Lemma 2.1 (See [20]) *When* $\mu_i = 1$ *and assume* $-\lambda_{1,s}(\Omega) < \min\{\lambda_1,\lambda_2\} < 0$ *and* $N > 4s$, *then* (5) *has a non-trivial ground state solution such that*

$$J_i(u_{\lambda_i}) = m_{\lambda_i} < \frac{s}{N}S_s^{\frac{N}{2s}}, \quad i = 1,2. \tag{6}$$

Remark 2.1 By Lemma 2.1, it is easy to see, when $\mu_i = 1$, if u_{λ_i} is a non-trivial ground state solution of (5), then $u_{\lambda_i,\mu_i} = \mu_i^{-\frac{1}{2^*-2}}u_{\lambda_i}$ is a non-trivial ground state solution of (5) for $0 < \mu_i \neq 1$ and the energy of (5) satisfies

$$J_{\lambda_i,\mu_i}(u_{\lambda_i,\mu_i}) = m_{\lambda_i,\mu_i} < \mu_i^{-\frac{N-2s}{2s}}\frac{s}{N}S_s^{\frac{N}{2s}}. \tag{7}$$

In order to prove Theorem 1.1, we give the following lemmas.

Lemma 2.2 *Define* $\widehat{A_\gamma} := \inf_{\sigma\in\Gamma}\max_{t\in[0,1]} E_\gamma(\sigma(t))$, *then there exist a sequence* $\{(u_n,v_n)\} \subset \mathcal{D}_s(\Omega)$ *such that*

$$E_\gamma(u_n,v_n) \to \widehat{A_\gamma} \quad \text{and} \quad E'_\gamma(u_n,v_n) \to 0 \quad \text{as } n \to +\infty, \tag{8}$$

where

$$\Gamma = \bigl\{\sigma \in \mathcal{C}\bigl([0,1],\mathcal{D}_s(\Omega)\bigr) : \sigma(0) = (0,0), \sigma(1) = (u_0,v_0)\bigr\}.$$

Proof We first claim that E_γ possesses a mountain pass geometry around $(0,0)$;

(1) there exist $\alpha,\rho > 0$, such that $E_\gamma(u,v) > \alpha$ for all $\|(u,v)\|_{\mathcal{D}_s(\Omega)} = \rho$;

(2) there exist $(u_0,v_0) \in \mathcal{D}_s(\Omega)$ such that $\|(u_0,v_0)\|_{\mathcal{D}_s(\Omega)} > \rho$ and $E_\gamma(u_0,v_0) < 0$.

Since $\lambda_1,\lambda_2 > -\lambda_{1,s}(\Omega)$ and the Sobolev embedding theorem $D_s(\Omega) \hookrightarrow L^2(\Omega)$, it is easy to see $\|\cdot\|_{\lambda_i}$, $i = 1,2$, are equivalent to $\|\cdot\|_{D_s(\Omega)}$, where $\|u\|_{\lambda_i} = (\|u\|^2_{D_s(\Omega)} + \int_\Omega \lambda_i u^2\,dx)^{\frac{1}{2}}$. On the one hand, by the Hölder inequality and the Young inequality, we have

$$\int_\Omega |u|^\alpha|v|^\beta\,dx \le \frac{\alpha}{2^*}\int_\Omega |u|^{2^*}\,dx + \frac{\beta}{2^*}\int_\Omega |v|^{2^*}\,dx.$$

Hence

$$\begin{aligned} E_\gamma(u,v) &= \frac{1}{2}\bigl\|(u,v)\bigr\|^2_{\mathcal{D}_s(\Omega)} + \frac{1}{2}\int_\Omega\bigl(\lambda_1 u^2 + \lambda_2 v^2\bigr)\,dx \\ &\quad - \frac{1}{2^*}\int_\Omega\bigl(\mu_1|u|^{2^*} + \mu_2|v|^{2^*} + \gamma|u|^\alpha|v|^\beta\bigr)\,dx \end{aligned}$$

$$\begin{aligned}&\geq \frac{1}{2}\left\|(u,v)\right\|^2_{\mathcal{D}_s(\Omega)}+\frac{1}{2}\int_\Omega\left(\lambda_1u^2+\lambda_2v^2\right)dx\\&\quad-\frac{1}{2^*}\int_\Omega\left((\mu_1+\alpha\gamma)|u|^{2^*}+(\mu_2+\beta\gamma)|v|^{2^*}\right)dx\\&\geq C_1\left\|(u,v)\right\|^2_{\mathcal{D}_s(\Omega)}-C_2\left\|(u,v)\right\|^{2^*}_{\mathcal{D}_s(\Omega)}.\end{aligned}$$

Choose $\rho>0$ sufficiently small, if $\|(u,v)\|^2_{\mathcal{D}_s(\Omega)}=\rho$, then

$$E_\gamma(u,v)\geq C_1\left\|(u,v)\right\|^2_{\mathcal{D}_s(\Omega)}-C_2\left\|(u,v)\right\|^{2^*}_{\mathcal{D}_s(\Omega)}>\frac{1}{4}C_1\rho^2>0.$$

On the other hand, we can choose $\varphi,\psi\in\mathcal{C}_0^\infty(\Omega)$ with $\varphi,\psi\not\equiv 0$ and $\operatorname{suup}(\varphi)\cap\operatorname{suup}(\psi)=\emptyset$, then there exists $t_0>0$ such that $E_\gamma(t_0\varphi,t_0\psi)<0$ and $\|(t_0\varphi,t_0\psi)\|^2_{\mathcal{D}_s(\Omega)}>\rho$. Then we can take $(u_0,v_0)=(t_0\varphi,t_0\psi)$.

By the mountain pass theorem, for the constant $0<\widehat{A_\gamma}:=\inf_{\sigma\in\Gamma}\max_{t\in[0,1]}E_\gamma(\sigma(t))$, there exists a $(PS)_{\widehat{A_\gamma}}$ sequence $\{(u_n,v_n)\}\subset\mathcal{D}_s(\Omega)$, that is,

$$E_\gamma(u_n,v_n)\to\widehat{A_\gamma}\quad\text{and}\quad E'_\gamma(u_n,v_n)\to 0\quad\text{as } n\to+\infty,$$

where

$$\Gamma=\left\{\sigma\in\mathcal{C}\left([0,1],\mathcal{D}_s(\Omega)\right):\sigma(0)=(0,0),\sigma(1)=(u_0,v_0)\right\}.$$ □

Lemma 2.3 $\widehat{A_\gamma}=\inf_{\mathcal{D}_s(\Omega)\setminus\{(0,0)\}}\max_{t>0}E_\gamma(tu,tv)=A_\gamma$.

Proof For any $(u,v)\in\mathcal{D}_s(\Omega)$ with $(u,v)\neq(0,0)$, there exists a unique $t_{\gamma,u,v}>0$ such that

$$\begin{aligned}\max_{t>0}E_\gamma(tu,tv)&=E_\gamma(t_{\gamma,u,v}u,t_{\gamma,u,v}v)\\&=\frac{s}{N}t^2_{\gamma,u,v}\left(\left\|(u,v)\right\|^2_{\mathcal{D}_s(\Omega)}+\int_\Omega\left(\lambda_1u^2+\lambda_2v^2\right)dx\right)\\&=\frac{s}{N}t^{2^*}_{\gamma,u,v}\int_\Omega\left(\mu_1|u|^{2^*}+\mu_2|v|^{2^*}+\gamma|u|^\alpha|v|^\beta\right)dx,\end{aligned}$$

where $t_{\gamma,u,v}>0$ satisfies

$$t^{2^*-2}_{\gamma,u,v}=\frac{\|(u,v)\|^2_{\mathcal{D}_s(\Omega)}+\int_\Omega(\lambda_1u^2+\lambda_2v^2)\,dx}{\int_\Omega(\mu_1|u|^{2^*}+\mu_2|v|^{2^*}+\gamma|u|^\alpha|v|^\beta)\,dx},\tag{9}$$

which implies that $(t_{\gamma,u,v}u,t_{\gamma,u,v}v)\in\mathbb{M}$. Combining this with $\max_{t>0}E_\gamma(tu,tv)=E_\gamma(t_{\gamma,u,v}u,t_{\gamma,u,v}v)$, by the definition of A_γ and $\widehat{A_\gamma}$, we can deduce that

$$\widehat{A_\gamma}=\inf_{\mathcal{D}_s(\Omega)\setminus\{(0,0)\}}\max_{t>0}E_\gamma(tu,tv)=A_\gamma.$$ □

Lemma 2.4 $A_\gamma<\min\{m_{\lambda_1,\mu_1},m_{\lambda_2,\mu_2}\}<\min\{\mu_1^{-\frac{N-2s}{2s}}\frac{s}{N}S_s^{\frac{N}{2s}},\mu_2^{-\frac{N-2s}{2s}}\frac{s}{N}S_s^{\frac{N}{2s}}\}$.

Proof By Remark 2.1, we obtain $\min\{m_{\lambda_1,\mu_1},m_{\lambda_2,\mu_2}\}<\min\{\mu_1^{-\frac{N-2s}{2s}}\frac{s}{N}S_s^{\frac{N}{2s}},\mu_2^{-\frac{N-2s}{2s}}\frac{s}{N}S_s^{\frac{N}{2s}}\}$.

Next we prove that $A_\gamma<m_{\lambda_1,\mu_1}$ and $A_\gamma<m_{\lambda_2,\mu_2}$.

Define a function $H : \mathbb{R}^2 \to \mathbb{R}$ by

$$H(t,\tau) = \Psi(tu_{\lambda_1,\mu_1}, t\tau v_{\lambda_2,\mu_2}),$$

where

$$\Psi(u,v) = \|(u,v)\|^2_{\mathcal{D}_s(\Omega)} + \int_\Omega (\lambda_1 u^2 + \lambda_2 v^2)\,dx - \int_\Omega (\mu_1|u|^{2^*} + \mu_2|v|^{2^*} + \gamma|u|^\alpha|v|^\beta)\,dx.$$

Since $H(1,0) = 0$ and $H_t(1,0) \neq 0$, by the implicit function theorem, there exist $\delta > 0$ and a function $t(\tau) \in C^1(-\delta,\delta)$ such that

$$t(0) = 1, \qquad t'(\tau) = -\frac{H_\tau(t,\tau)}{H_t(t,\tau)} \quad \text{and} \quad H\bigl(t(\tau),\tau\bigr) = 0, \quad \forall \tau \in (-\delta,\delta),$$

which implies that

$$\bigl(t(\tau)u_{\lambda_1,\mu_1}, t(\tau)\tau v_{\lambda_2,\mu_2}\bigr) \in \mathbb{M}, \quad \forall \tau \in (-\delta,\delta).$$

Since $1 < \beta < 2$, by direct calculation, we have

$$\lim_{\tau\to 0} \frac{t'(\tau)}{|\tau|^{\beta-2}\tau} = \frac{-\beta\gamma \int_\Omega u_{\lambda_1,\mu_1} v_{\lambda_2,\mu_2}\,dx}{(2^*-2)\int_\Omega \mu_1 |u_{\lambda_1,\mu_1}|^{2^*}\,dx} < 0.$$

That is,

$$t'(\tau) = \frac{-\beta\gamma \int_\Omega u_{\lambda_1,\mu_1} v_{\lambda_2,\mu_2}\,dx}{(2^*-2)\int_\Omega |u_{\lambda_1,\mu_1}|^{2^*}\,dx} |\tau|^{\beta-2}\tau\bigl(1+o(1)\bigr) \quad \text{as } \tau \to 0.$$

So

$$t(\tau) = 1 - \frac{\gamma \int_\Omega u_{\lambda_1,\mu_1} v_{\lambda_2,\mu_2}\,dx}{(2^*-2)\int_\Omega \mu_1|u_{\lambda_1,\mu_1}|^{2^*}\,dx} |\tau|^{\beta}\bigl(1+o(1)\bigr) \quad \text{as } \tau \to 0.$$

Consequently, we have

$$t^{2^*}(\tau) = 1 - \frac{2^*\gamma \int_\Omega u_{\lambda_1,\mu_1} v_{\lambda_2,\mu_2}\,dx}{(2^*-2)\int_\Omega \mu_1|u_{\lambda_1,\mu_1}|^{2^*}\,dx} |\tau|^{\beta}\bigl(1+o(1)\bigr) \quad \text{as } \tau \to 0.$$

Thus

$$\begin{aligned}
A_\gamma &\le E_\gamma\bigl(t(\tau)u_{\lambda_1,\mu_1}, t(\tau)\tau v_{\lambda_2,\mu_2}\bigr) - \frac{1}{2}\Psi(tu_{\lambda_1,\mu_1}, t\tau v_{\lambda_2,\mu_2}) \\
&\le \frac{s}{N} t^{2^*}(\tau) \int_\Omega \bigl(\mu_1|u_{\lambda_1,\mu_1}|^{2^*} + \tau^{2^*}\mu_2|v_{\lambda_2,\mu_2}|^{2^*} + \gamma\tau^\beta|u_{\lambda_1,\mu_1}|^\alpha|v_{\lambda_2,\mu_2}|^\beta\bigr)\,dx \\
&\le \frac{s}{N}\int_\Omega \mu_1|u_{\lambda_1,\mu_1}|^{2^*}\,dx - |\tau|^\beta \frac{\beta\gamma}{2^*}\int_\Omega |u_{\lambda_1,\mu_1}|^\alpha|v_{\lambda_2,\mu_2}|^\beta\,dx + o\bigl(|\tau|^\beta\bigr) \\
&< \frac{s}{N}\int_\Omega \mu_1|u_{\lambda_1,\mu_1}|^{2^*}\,dx = m_{\lambda_1,\mu_1}, \quad \text{as } |\tau| > 0 \text{ small enough.}
\end{aligned}$$

Hence $A_\gamma < m_{\lambda_1,\mu_1}$. Similarly, by the same arguments, we have $A_\gamma < m_{\lambda_2,\mu_2}$.

This completes the proof of Lemma 2.4. □

3 Proof of Theorem 1.1

In this section, we prove the first part of Theorem 1.1 by two steps. We prove the existence of ground state solutions for system (3) in step one, then we claim there exist a positive ground state solutions. Finally, we prove the second part of Theorem 1.1.

Proof Proof of the first part of Theorem 1.1.

Step one. Prove the existence of ground state solutions for system (3).

By (8) and Lemma 2.3, there exists $\{(u_n, v_n)\} \subset \mathcal{D}_s(\Omega)$ such that

$$\lim_{n\to+\infty} E_\gamma(u_n, v_n) = A_\gamma, \qquad \lim_{n\to+\infty} E'_\gamma(u_n, v_n) = 0.$$

Next, we claim $\{(u_n, v_n)\}$ is bounded in $\mathcal{D}_s(\Omega)$.

Let $z_n = \{(u_n, v_n)\}$, assuming by contradiction that $\|z_n\| := \|z_n\|_{\mathcal{D}_s(\Omega)} \to +\infty$ as $n \to +\infty$. Put

$$\widetilde{z_n} = (\widetilde{u_n}, \widetilde{v_n}) = \frac{z_n}{\|z_n\|} = \left(\frac{u_n}{\|z_n\|}, \frac{v_n}{\|z_n\|}\right).$$

Since $\{z_n\}$ is a $(PS)_{A_\gamma}$ sequence for E_γ and $\|z_n\| \to +\infty$ as $n \to +\infty$, we have

$$\begin{aligned} &\frac{\|z_n\|^2}{2}\left\|(\widetilde{u_n}, \widetilde{v_n})\right\|^2_{\mathcal{D}_s(\Omega)} + \frac{\|z_n\|^2}{2}\int_\Omega \left(\lambda_1|\widetilde{u_n}|^2 + \lambda_2|\widetilde{v_n}|^2\right) dx \\ &\quad - \frac{\|z_n\|^{2^*}}{2^*}\int_\Omega \left(\mu_1|\widetilde{u_n}|^{2^*} + \mu_2|\widetilde{v_n}|^{2^*} + \gamma|\widetilde{u_n}|^\alpha|\widetilde{v_n}|^\beta\right) dx = A_\gamma + o_n(1), \end{aligned} \tag{10}$$

$$\begin{aligned} &\|z_n\|^2\left\|(\widetilde{u_n}, \widetilde{v_n})\right\|^2_{\mathcal{D}_s(\Omega)} + \|z_n\|^2\int_\Omega \left(\lambda_1|\widetilde{u_n}|^2 + \lambda_2|\widetilde{v_n}|^2\right) dx \\ &\quad - \|z_n\|^{2^*}\int_\Omega \left(\mu_1|\widetilde{u_n}|^{2^*} + \mu_2|\widetilde{v_n}|^{2^*} + \gamma|\widetilde{u_n}|^\alpha|\widetilde{v_n}|^\beta\right) dx = o_n(1). \end{aligned} \tag{11}$$

Combining (10) with (11), we obtain

$$\frac{s}{N}\|z_n\|^{2^*-2}\int_\Omega \left(\mu_1|\widetilde{u_n}|^{2^*} + \mu_2|\widetilde{v_n}|^{2^*} + \gamma|\widetilde{u_n}|^\alpha|\widetilde{v_n}|^\beta\right) dx = o_n(1),$$

as $n \to +\infty$, we have a contradiction. Consequently, $\{(u_n, v_n)\}$ is bounded in $\mathcal{D}_s(\Omega)$. Thus, by the Sobolev embedding theorem, there exist $(u, v) \in \mathcal{D}_s(\Omega)$ such that

$$\begin{cases} (u_n, v_n) \rightharpoonup (u, v), & \text{weakly in } \mathcal{D}_s(\Omega), \\ (u_n, v_n) \to (u, v), & \text{strongly in } L^p(\Omega) \times L^p(\Omega), \text{for } 2 \le p < 2^*, \\ (u_n, v_n) \to (u, v), & \text{a.e. } \Omega. \end{cases} \tag{12}$$

Consequently, we have $E'_\gamma(u, v) = 0$. Set $w_n = u_n - u$ and $\sigma_n = v_n - v$. Then, by the Brézis–Lieb lemma [30],

$$\begin{aligned} \|u_n\|^{2^*}_{2^*} &= \|u\|^{2^*}_{2^*} + \|w_n\|^{2^*}_{2^*} + o_n(1), \\ \|v_n\|^{2^*}_{2^*} &= \|v\|^{2^*}_{2^*} + \|\sigma_n\|^{2^*}_{2^*} + o_n(1). \end{aligned} \tag{13}$$

By Lemma 2.1 in [31], we also have

$$\int_\Omega |w_n|^\alpha |\sigma_n|^\beta \, dx = \int_\Omega |u_n|^\alpha |v_n|^\beta \, dx - \int_\Omega |u|^\alpha |v|^\beta \, dx + o_n(1). \tag{14}$$

We have

$$\begin{aligned} \|w_n\|^2_{D_s(\Omega)} &= \|u_n\|^2_{D_s(\Omega)} - \|u\|^2_{D_s(\Omega)} + o_n(1), \\ \|\sigma_n\|^2_{D_s(\Omega)} &= \|v_n\|^2_{D_s(\Omega)} - \|v\|^2_{D_s(\Omega)} + o_n(1), \end{aligned} \tag{15}$$

and $E'_\gamma(u_n, v_n) \to 0$ as $n \to +\infty$. Combining this with (13), (14) and (15), we obtain

$$\|(w_n, \sigma_n)\|^2_{D_s(\Omega)} = \int_\Omega \left(\mu_1 |w_n|^{2^*} + \mu_2 |\sigma_n|^{2^*} + \gamma |w_n|^\alpha |\sigma_n|^\beta\right) dx + o_n(1) \tag{16}$$

and

$$\begin{aligned} E_\gamma(u_n, v_n) = E_\gamma(u, v) &+ \frac{1}{2} \|(w_n, \sigma_n)\|^2_{\mathcal{D}_s(\Omega)} \\ &- \frac{1}{2^*} \int_\Omega \left(\mu_1 |w_n|^{2^*} + \mu_2 |\sigma_n|^{2^*} + \gamma |w_n|^\alpha |\sigma_n|^\beta\right) dx + o_n(1). \end{aligned} \tag{17}$$

By (16) and (17), we have

$$E_\gamma(u_n, v_n) = E_\gamma(u, v) + \frac{s}{N} \|(w_n, \sigma_n)\|^2_{\mathcal{D}_s(\Omega)} + o_n(1). \tag{18}$$

Next, we prove that $(u_n, v_n) \to (u, v)$ strongly in $\mathcal{D}_s(\Omega)$. Let

$$\lim_{n\to+\infty} \|(w_n, 0)\|^2_{\mathcal{D}_s(\Omega)} = l_1, \qquad \lim_{n\to+\infty} \|(0, \sigma_n)\|^2_{\mathcal{D}_s(\Omega)} = l_2, \qquad \lim_{n\to+\infty} \|(w_n, \sigma_n)\|^2_{\mathcal{D}_s(\Omega)} = l,$$

if $l = 0$, then we have proved $(u_n, v_n) \to (u, v)$ strongly in $\mathcal{D}_s(\Omega)$, if $l > 0$ then

$$\lim_{n\to+\infty} \|(w_n, \sigma_n)\|^2_{\mathcal{D}_s(\Omega)} = l_1 + l_2 \geq \max\{l_1, l_2\}.$$

Case one $l_1 = 0$ or $l_2 = 0$.

If $l_2 = 0$, then (16) turns to

$$\|w_n\|^2_{D_s(\Omega)} = \int_\Omega \mu_1 |w_n|^{2^*} \, dx + o_n(1). \tag{19}$$

By the Sobolev embedding $D_s(\Omega) \hookrightarrow L^{2^*}(\Omega)$, we have $\|w_n\|^2_{D_s(\Omega)} \geq S_s(\int_\Omega |w_n|^{2^*} dx)^{\frac{2}{2^*}}$, hence $\|w_n\|^2_{D_s(\Omega)} \geq \mu_1^{-\frac{2}{2^*}} S_s(\int_\Omega \mu_1 |w_n|^{2^*} dx)^{\frac{2}{2^*}}$ combining this with (19), we can deduce that

$$l_1 \geq \mu_1^{-\frac{N-2s}{2s}} S_s^{\frac{N}{2s}}.$$

Similarly, if $l_1 = 0$, we have

$$l_2 \geq \mu_2^{-\frac{N-2s}{2s}} S_s^{\frac{N}{2s}}.$$

Since $E_\gamma(u,v) \geq 0$, let $n \to +\infty$ in (18), we obtain

$$A_\gamma \geq \max\left\{\mu_1^{-\frac{N-2s}{2s}} \frac{s}{N} S_s^{\frac{N}{2s}}, \mu_2^{-\frac{N-2s}{2s}} \frac{s}{N} S_s^{\frac{N}{2s}}\right\}.$$

This contradicts Lemma 2.4. Thus $E_\gamma(u,v) = A_\gamma$ and $E'_\gamma(u,v) = 0$. That is (u,v) is a non-trivial solution of system (3).

Case two $l_1 \neq 0$, $l_2 \neq 0$ and $l > 0$, we prove system (3) has a ground state solution.

For case two, in order to obtain a ground state solution for (3), we borrow some ideas from [11]. First we give the following lemma.

Lemma 3.1 (A result in [11]) *Define*

$$\widetilde{S}_s = \inf_{(u,v)\in\mathcal{D}_s(\mathbb{R}^N)\setminus\{(0,0)\}} \frac{\|(u,v)\|^2_{\mathcal{D}_s(\Omega)}}{(\int_{\mathbb{R}^N}(\mu_1|u|^{2^*} + \mu_2|v|^{2^*} + \gamma|u|^\alpha|v|^\beta)\,dx)^{\frac{2}{2^*}}},$$

$$\widetilde{S}_{s,\lambda_1,\lambda_2} := \inf_{(u,v)\in\mathcal{D}_s(\Omega)\setminus\{(0,0)\}} \frac{\|(u,v)\|^2_{\mathcal{D}_s(\Omega)} + \int_\Omega(\lambda_1u^2 + \lambda_2v^2)\,dx}{(\int_\Omega(\mu_1|u|^{2^*} + \mu_2|v|^{2^*} + \gamma|u|^\alpha|v|^\beta)\,dx)^{\frac{2}{2^*}}}.$$

Then

$$\widetilde{S}_{s,\lambda_1,\lambda_2} < \widetilde{S}_s.$$

Let $\{(u_n,u_n)\}$ be a minimizing sequence for $S_{s,\lambda_1,\lambda_2}$ normalized by

$$\int_\Omega\left(\mu_1|u_n|^{2^*} + \mu_2|v_n|^{2^*} + \gamma|u_n|^\alpha|v_n|^\beta\right)dx = 1, \tag{20}$$

that is,

$$\left\|(u_n,v_n)\right\|^2_{\mathcal{D}_s(\Omega)} + \int_\Omega\left(\lambda_1u_n^2 + \lambda_2v_n^2\right)dx = \widetilde{S}_{s,\lambda_1,\lambda_2} + o_n(1). \tag{21}$$

Since $\{u_n\}$ and $\{v_n\}$ are bounded in $D_s(\Omega)$, (12) holds and

$$\int_\Omega\left(\mu_1|u|^{2^*} + \mu_2|v|^{2^*} + \gamma|u|^\alpha|v|^\beta\right)dx \leq 1.$$

By (21), we have

$$\widetilde{S}_{s,\lambda_1,\lambda_2} - \int_\Omega\left(\lambda_1u_n^2 + \lambda_2v_n^2\right)dx + o_n(1) \geq \left\|(u_n,v_n)\right\|^2_{\mathcal{D}_s(\Omega)} \geq \widetilde{S}_s.$$

By (12) and Lemma 3.1, we have

$$-\int_\Omega\left(\lambda_1u^2 + \lambda_2v^2\right)dx \geq \widetilde{S}_s - \widetilde{S}_{s,\lambda_1,\lambda_2} > 0,$$

which implies that $(u,v) \not\equiv (0,0)$. By (15) and (21), we obtain

$$\widetilde{S}_{s,\lambda_1,\lambda_2} = \left\|(w_n,\sigma_n)\right\|^2_{D_s(\Omega)} + \left\|(u,v)\right\|^2_{D_s(\Omega)} + \int_\Omega\left(\lambda_1u^2 + \lambda_2v^2\right)dx + o_n(1). \tag{22}$$

Combining (13), (14) with (20), we have

$$
\begin{aligned}
1 = {} & \int_\Omega \left(\mu_1|u|^{2^*} + \mu_2|v|^{2^*} + \gamma|u|^\alpha|v|^\beta\right)dx \\
& + \int_\Omega \left(\mu_1|w_n|^{2^*} + \mu_2|\sigma_n|^{2^*} + \gamma|w_n|^\alpha|\sigma_n|^\beta\right)dx + o_n(1).
\end{aligned}
$$

Since

$$
\int_\Omega \left(\mu_1|u|^{2^*} + \mu_2|v|^{2^*} + \gamma|u|^\alpha|v|^\beta\right)dx \le 1
$$

and

$$
\int_\Omega \left(\mu_1|w_n|^{2^*} + \mu_2|\sigma_n|^{2^*} + \gamma|w_n|^\alpha|\sigma_n|^\beta\right)dx \le 1,
$$

we have

$$
\begin{aligned}
1 \le {} & \left(\int_\Omega \left(\mu_1|u|^{2^*} + \mu_2|v|^{2^*} + \gamma|u|^\alpha|v|^\beta\right)dx\right)^{\frac{2}{2^*}} \\
& + \left(\int_\Omega \left(\mu_1|w_n|^{2^*} + \mu_2|\sigma_n|^{2^*} + \gamma|w_n|^\alpha|\sigma_n|^\beta\right)dx\right)^{\frac{2}{2^*}} \\
\le {} & \left(\int_\Omega \left(\mu_1|u|^{2^*} + \mu_2|v|^{2^*} + \gamma|u|^\alpha|v|^\beta\right)dx\right)^{\frac{2}{2^*}} \\
& + \frac{1}{\widetilde{S}_s}\left\|(w_n,\sigma_n)\right\|^2_{\mathcal{D}_s(\Omega)} + o_n(1).
\end{aligned}
\tag{23}
$$

Combining (23), (22), Lemma 3.1 with $\widetilde{S}_{s,\lambda_1,\lambda_2} > 0$, we have

$$
\begin{aligned}
& \left\|(u,v)\right\|^2_{D_s(\Omega)} + \int_\Omega \left(\lambda_1 u^2 + \lambda_2 v^2\right)dx \\
& \quad \le \widetilde{S}_{s,\lambda_1,\lambda_2}\left(\int_\Omega \left(\mu_1|u|^{2^*} + \mu_2|v|^{2^*} + \gamma|u|^\alpha|v|^\beta\right)dx\right)^{\frac{2}{2^*}} \\
& \qquad + \left(\frac{\widetilde{S}_{s,\lambda_1,\lambda_2}}{\widetilde{S}_s} - 1\right)\left\|(w_n,\sigma_n)\right\|^2_{\mathcal{D}_s(\Omega)} + o_n(1) \\
& \quad \le \widetilde{S}_{s,\lambda_1,\lambda_2}\left(\int_\Omega \left(\mu_1|u|^{2^*} + \mu_2|v|^{2^*} + \gamma|u|^\alpha|v|^\beta\right)dx\right)^{\frac{2}{2^*}} + o_n,
\end{aligned}
$$

which implies that

$$
\frac{\|(u,v)\|^2_{D_s(\Omega)} + \int_\Omega(\lambda_1 u^2 + \lambda_2 v^2)\,dx}{(\int_\Omega(\mu_1|u|^{2^*} + \mu_2|v|^{2^*} + \gamma|u|^\alpha|v|^\beta)\,dx)^{\frac{2}{2^*}}} \le \widetilde{S}_{s,\lambda_1,\lambda_2}.
$$

Therefore, $\widetilde{S}_{s,\lambda_1,\lambda_2}$ is attained by (u,v). Thus, system (3) has a ground state solution.

Combining case one with case two, we prove that system (3) has a ground state solution.

Step two. We claim that there exists a positive ground state solution. Since

$$\begin{aligned}&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dx\,dy-\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{||u(x)|-|u(y)||^2}{|x-y|^{N+2s}}\,dx\,dy\\&\quad=2\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)||u(y)|-u(x)u(y)}{|x-y|^{N+2s}}\,dx\,dy\geq 0,\end{aligned}$$

we have

$$|||u|||_{D_s(\Omega)}\leq\|u\|_{D_s(\Omega)}.$$

Then, for the minimizing sequence $(u_n,v_n)\in\mathbb{M}$, we have

$$\begin{aligned}&|||u_n|||^2_{D_s(\Omega)}+|||v_n|||^2_{D_s(\Omega)}+\int_\Omega\left(\lambda_1|u_n|^2+\lambda_2|v_n|^2\right)dx\\&\quad\leq\|u_n\|^2_{D_s(\Omega)}+\|v_n\|^2_{D_s(\Omega)}+\int_\Omega\left(\lambda_1u_n^2+\lambda_2v_n^2\right)dx\\&\quad=\int_\Omega\left(\mu_1|u_n|^{2^*}+\mu_2|v_n|^{2^*}+\gamma|u_n|^\alpha|v_n|^\beta\right)dx,\end{aligned}$$

this implies that there exists $t_n\in(0,1]$ such that $(t_n|u_n|,t_n|v_n|)\in\mathbb{M}$. Hence, we can choose a minimizing sequence $(\overline{u}_n,\overline{v}_n)=(t_n|u_n|,t_n|v_n|)$ and the weak limit $(\overline{u},\overline{v})$ is nonnegative. By the strong maximum principle for the fractional Laplacian (see Proposition 2.17 in [4]), we have $\overline{u}$ and $\overline{v}$ are both positive.

Proof of the second part of Theorem 1.1

Let γ_n be a sequence with $\gamma_n\to 0$ as $n\to+\infty$. $\{(u_{\gamma_n},v_{\gamma_n})\}$ is bounded in $D_s(\Omega)\times D_s(\Omega)$, then there exists a subsequence, still denoted by $\{(u_{\gamma_n},v_{\gamma_n})\}$, such that $(u_{\gamma_n},v_{\gamma_n})\rightharpoonup(\overline{u},\overline{v})$ weakly in $D_s(\Omega)\times D_s(\Omega)$. Then $(\overline{u},\overline{v})$ satisfies

$$\begin{cases}(-\Delta)^s\overline{u}+\lambda_1\overline{u}=\mu_1|\overline{u}|^{2^*-2}\overline{u} & \text{in }\Omega,\\(-\Delta)^s\overline{v}+\lambda_2\overline{v}=\mu_2|\overline{v}|^{2^*-2}\overline{v} & \text{in }\Omega,\\\overline{u}=\overline{v}=0 & \text{in }\mathbb{R}^N\setminus\Omega.\end{cases}\tag{24}$$

Since $E_0'(u_{\gamma_n},v_{\gamma_n})\to 0$ and $\lim_{n\to+\infty}E_0(u_{\gamma_n},v_{\gamma_n})=\lim_{n\to+\infty}E_{\gamma_n}(u_{\gamma_n},v_{\gamma_n})$, we have

$$E_0'(\overline{u},\overline{v})=0\quad\text{and}\quad\lim_{n\to+\infty}E_0(u_{\gamma_n},v_{\gamma_n})=\lim_{n\to+\infty}A_{\gamma_n}>0.$$

Next, we claim A_γ is strictly decreasing for all $\gamma>0$.

Let $\gamma_2>\gamma_1>0$, then, by (9), we have

$$\begin{aligned}t^{2^*-2}_{\gamma_2,u_{\gamma_1},v_{\gamma_1}}&=\frac{\|(u_{\gamma_1},v_{\gamma_1})\|^2_{\mathcal{D}_s(\Omega)}+\int_\Omega(\lambda_1u^2_{\gamma_1}+\lambda_2v^2_{\gamma_1})\,dx}{\int_\Omega(\mu_1|u_{\gamma_1}|^{2^*}+\mu_2|v_{\gamma_1}|^{2^*}+\gamma_2|u_{\gamma_1}|^\alpha|v_{\gamma_2}|^\beta)\,dx}\\&<\frac{\|(u_{\gamma_1},v_{\gamma_1})\|^2_{\mathcal{D}_s(\Omega)}+\int_\Omega(\lambda_1u^2_{\gamma_1}+\lambda_2v^2_{\gamma_1})\,dx}{\int_\Omega(\mu_1|u_{\gamma_1}|^{2^*}+\mu_2|v_{\gamma_1}|^{2^*}+\gamma_1|u_{\gamma_1}|^\alpha|v_{\gamma_2}|^\beta)\,dx}=1.\end{aligned}$$

Consequently,

$$\begin{aligned} A_{\gamma_2} &\le \max_{t>0} E_{\gamma_2}(tu_{\gamma_1}, tv_{\gamma_1}) \\ &= \frac{s}{N} t^{2^*-2}_{\gamma_2,u_{\gamma_1},v_{\gamma_1}} \left(\left\| (u_{\gamma_1}, v_{\gamma_1}) \right\|^2_{\mathcal{D}_s(\Omega)} + \int_\Omega \left(\lambda_1 u^2_{\gamma_1} + \lambda_2 v^2_{\gamma_1} \right) dx \right) \\ &= t^{2^*-2}_{\gamma_2,u_{\gamma_1},v_{\gamma_1}} A_{\gamma_1} < A_{\gamma_1}. \end{aligned}$$

Hence, A_γ is strictly decreasing for $\gamma > 0$. By Lemma 2.4 and the strictly decreasing for A_γ, we have

$$0 < \lim_{n\to+\infty} A_{\gamma_n} \le A_0 \le \min\{m_{\lambda_1,\mu_1}, m_{\lambda_2,\mu_2}\} < \min\left\{ \mu_1^{-\frac{N-2s}{2s}} \frac{s}{N} S_s^{\frac{N}{2s}}, \mu_2^{-\frac{N-2s}{2s}} \frac{s}{N} S_s^{\frac{N}{2s}} \right\}. \tag{25}$$

By the same arguments as prove the first part of Theorem 1.1, we have

$$(u_{\gamma_n}, v_{\gamma_n}) \to (\overline{u}, \overline{v}) \quad \text{strongly in } D_s(\Omega) \times D_s(\Omega).$$

Combining this with (25), one of the following conclusions holds:

(1) $(\overline{u}, 0)$ is a positive ground state solution of

$$\begin{cases} (-\Delta)^s u + \lambda_1 u = \mu_1 |u|^{2^*-2} u & \text{in } \Omega; \\ u = 0 & \text{on } \mathbb{R}^N \setminus \Omega. \end{cases}$$

(2) $(0, \overline{v})$ is a positive ground state solution of

$$\begin{cases} (-\Delta)^s v + \lambda_2 v = \mu_2 |v|^{2^*-2} v & \text{in } \Omega, \\ v = 0 & \text{on } \mathbb{R}^N \setminus \Omega. \end{cases}$$

Since

$$m_{\lambda_1,\mu_1} = \mu_1^{-\frac{N-2s}{2s}} m_{\lambda_1}, \qquad m_{\lambda_2,\mu_2} = \mu_2^{-\frac{N-2s}{2s}} m_{\lambda_2}$$

and

$$\left(\frac{\mu_1}{\mu_2}\right)^{-\frac{N-2s}{2s}} < \frac{m_{\lambda_2}}{m_{\lambda_1}} \quad \text{implies that} \quad m_{\lambda_1,\mu_1} < m_{\lambda_2,\mu_2},$$

by the definition of A_γ, we know that (1) holds.

Similarly, if

$$\left(\frac{\mu_1}{\mu_2}\right)^{-\frac{N-2s}{2s}} > \frac{m_{\lambda_2}}{m_{\lambda_1}} \quad \text{implies that} \quad m_{\lambda_1,\mu_1} > m_{\lambda_2,\mu_2},$$

then (2) occurs. This completes the proof of Theorem 1.1. □

4 Proof of Theorem 1.2

Define

$$X = S_1 \times S_2,$$

where

$$S_i := \{u \in D_s(\Omega) : J'_{\lambda_i,\mu_i}(u) = 0, J_{\lambda_i,\mu_i}(u) = m_{\lambda_i,\mu_i}\}, \tag{26}$$

for $i = 1, 2$. Then we have following lemma.

Lemma 4.1 *X is compact in $\mathcal{D}_s(\Omega)$ and there exist constants $C_2 > C_1 > 0$ such that*

$$C_1 \leq \|u\|^2_{D_s(\Omega)} + \int_\Omega \lambda_1 u^2 \, dx, \qquad \|v\|^2_{D_s(\Omega)} + \int_\Omega \lambda_2 v^2 \, dx \leq C_2, \quad \forall (u, v) \in X.$$

Proof By Remark 2.1, we know S_i is nonempty and $(u_{\lambda_1,\mu_1}, v_{\lambda_2,\mu_2}) \in X$. Next, we claim S_i is compact in $\mathcal{D}_s(\Omega)$. Suppose there exists a sequence $\{u_n\} \subset S_1$, then $\{u_n\}$ is a bounded $(PS)_{m_{\lambda_1,\mu_1}}$ sequence of J_{λ_1,μ_1} and

$$\|u_n\|^2_{D_s(\Omega)} + \int_\Omega \lambda_1 u_n^2 \, dx = \int_\Omega \mu_1 u_n^{2^*} \, dx + o_n(1).$$

Thus, there exists a subsequence u_∞ such that $u_n \rightharpoonup u_\infty$ in $D_s(\Omega)$ and $J'_{\lambda_1,\mu_1}(u_\infty) = 0$.

Since $m_{\lambda_1,\mu_1} \leq \mu_1^{-\frac{N-2s}{2s}} \frac{s}{N} S_s^{\frac{N}{2s}}$ and J_{λ_1,μ_1} satisfies the $(PS)_{m_{\lambda_1,\mu_1}}$ condition, by the same arguments as proving step one in Theorem 1.1, we can obtain $u_n \to u_\infty$ strongly in $\mathcal{D}_s(\Omega)$ and $u_\infty \in S_1$. This proves that S_1 is compact in $\mathcal{D}_s(\Omega)$. Similarly, S_2 is compact in $\mathcal{D}_s(\Omega)$.

Since $X = S_1 \times S_2$ and $m_{\lambda_1,\mu_1} > 0$, $m_{\lambda_2,\mu_2} > 0$, it is easy to see that X is compact and Lemma 4.1 holds. □

By Lemma 2.3 and Remark 2.1, we have

$$\begin{aligned} J_{\lambda_1,\mu_1}(u_{\lambda_1,\mu_1}) &= \max_{t>0} J_{\lambda_1,\mu_1}(t u_{\lambda_1,\mu_1}) = m_{\lambda_1,\mu_1}, \\ J_{\lambda_2,\mu_2}(v_{\lambda_2,\mu_2}) &= \max_{s>0} J_{\lambda_2,\mu_2}(s v_{\lambda_2,\mu_2}) = m_{\lambda_2,\mu_2}. \end{aligned} \tag{27}$$

Thus, there exist $0 < t_0 < 1 < t_1$, $0 < s_0 < 1 < s_1$ such that

$$J_{\lambda_1,\mu_1}(t u_{\lambda_1,\mu_1}) \leq \frac{m_{\lambda_1,\mu_1}}{4} \quad \text{for } t \in (0, t_0] \cup [t_1, +\infty), \tag{28}$$

$$J_{\lambda_2,\mu_2}(s v_{\lambda_2,\mu_2}) \leq \frac{m_{\lambda_2,\mu_2}}{4} \quad \text{for } s \in (0, s_0] \cup [s_1, +\infty). \tag{29}$$

Define

$$\widetilde{\sigma}_1(t) := t u_{\lambda_1,\mu_1} \quad \text{for } 0 \leq t \leq t_1, \qquad \widetilde{\sigma}_2(s) := s v_{\lambda_2,\mu_2} \quad \text{for } 0 \leq s \leq s_1,$$

and

$$\widetilde{\sigma}(t, s) := (\widetilde{\sigma}_1(t), \widetilde{\sigma}_2(s)).$$

Then, there exists a constant $C_0 > 0$ such that

$$\max_{(t,s)\in[0,t_1]\times[0,s_1]} \left\| \widetilde{\sigma}(t,s) \right\|_{\mathcal{D}_s(\Omega)} \le C_0. \tag{30}$$

For convenience we denote $Q = [0,t_1] \times [0,s_1]$. For $\gamma \ge 0$ and C_2 as appearing in Lemma 4.1, we define

$$\widehat{c_\gamma} := \inf_{\sigma\in\widehat{\Gamma}} \max_{(t,s)\in Q} E_\gamma\bigl(\sigma(t,s)\bigr), \qquad d_\gamma = \max_{(t,s)\in Q} E_\gamma\bigl(\widetilde{\sigma}(t,s)\bigr),$$

where

$$\begin{aligned} \widehat{\Gamma} := \Bigl\{ &\sigma \in \mathcal{C}\bigl(Q, \mathcal{D}_s(\Omega)\bigr) : \max_{(t,s)\in Q} \left\| \sigma(t,s) \right\|_{\mathcal{D}_s(\Omega)} \le C_0 + 2C_2 \\ &\sigma(t,s) = \widetilde{\sigma}(t,s), \text{for } (t,s) \in Q \setminus \bigl\{(t_0,t_1) \times (s_0,s_1)\bigr\} \Bigr\}. \end{aligned} \tag{31}$$

Since $\widetilde{\sigma}(t,s) \in \widehat{\Gamma}$, $\widehat{\Gamma}$ is nonempty.

Lemma 4.2 $\lim_{\gamma\to 0} \widehat{c_\gamma} = \lim_{\gamma\to 0} d_\gamma = \widehat{c_0} = m_{\lambda_1,\mu_1} + m_{\lambda_2,\mu_2}$.

Proof On the one hand, since $\gamma > 0$, we have $E_\gamma(\widetilde{\sigma}(t,s)) \le E_0(\widetilde{\sigma}(t,s))$. Consequently

$$\begin{aligned} d_\gamma \le d_0 &= \max_{(t,s)\in Q} E_0\bigl(\widetilde{\sigma}(t,s)\bigr) = \max_{t\in[0,t_1]} J_{\lambda_1,\mu_1}\bigl(\widetilde{\sigma}_1(t)\bigr) + \max_{s\in[0,s_1]} J_{\lambda_2,\mu_2}\bigl(\widetilde{\sigma}_2(s)\bigr) \\ &= J_{\lambda_1,\mu_1}\bigl(\widetilde{\sigma}_1(1)\bigr) + J_{\lambda_2,\mu_2}\bigl(\widetilde{\sigma}_2(1)\bigr) = J_{\lambda_1,\mu_1}(u_{\lambda_1,\mu_1}) + J_{\lambda_2,\mu_2}(v_{\lambda_2,\mu_2}) = m_{\lambda_1,\mu_1} + m_{\lambda_2,\mu_2}. \end{aligned}$$

Since $\widetilde{\sigma} \in \widehat{\Gamma}$, we obtain $\widehat{c_\gamma} \le d_\gamma$, thus

$$\limsup_{\gamma\to 0} \widehat{c_\gamma} \le \liminf_{\gamma\to 0} d_\gamma \le \limsup_{\gamma\to 0} d_\gamma \le d_0, \quad \widehat{c_0} \le d_0. \tag{32}$$

On the other hand, for any $\sigma(t,s) = (\sigma_1(t,s), \sigma_2(t,s)) \in \widehat{\Gamma}$, we define $\Upsilon(\sigma) : [t_0,t_1] \times [s_0,s_1] \to \mathbb{R}^2$ by

$$\Upsilon(\sigma) := \bigl(J_5\bigl(\sigma_1(t,s)\bigr) - J_6\bigl(\sigma_2(t,s)\bigr), J_5\bigl(\sigma_1(t,s)\bigr) + J_6\bigl(\sigma_2(t,s)\bigr) - 2\bigr),$$

where $J_5, J_6 : D_s(\Omega) \to \mathbb{R}$ are defined by

$$J_5(u) = \begin{cases} \frac{\int_\Omega \mu_1 |u|^{2^*}\,dx}{\|u\|^2_{D_s(\Omega)} + \int_\Omega \lambda_1 |u|^2\,dx}, & \text{if } u \ne 0, \\ 0, & \text{if } u = 0, \end{cases} \tag{33}$$

and

$$J_6(u) = \begin{cases} \frac{\int_\Omega \mu_2 |u|^{2^*}\,dx}{\|u\|^2_{D_s(\Omega)} + \int_\Omega \lambda_2 |u|^2\,dx}, & \text{if } u \ne 0, \\ 0, & \text{if } u = 0. \end{cases} \tag{34}$$

By the Sobolev embedding theorem $D_s(\Omega) \hookrightarrow L^{2^*}(\Omega)$, for any $u \in D_s(\Omega)$, we have

$$\int_\Omega \mu_i |u|^{2^*}\,dx \le C\left(\|u\|^2_{D_s(\Omega)} + \int_\Omega \lambda_i |u|^2\,dx\right)^{\frac{2^*}{2}}, \quad i = 1,2.$$

Consequently, we can deduce J_5, J_6 are continuous and

$$\begin{aligned}\Upsilon(\widetilde{\sigma})(t,s) = \Bigg(&\frac{t^{2^*-2}\int_\Omega \mu_1|u_{\lambda_1,\mu_1}|^{2^*}\,dx}{\|u_{\lambda_1,\mu_1}\|^2_{D_s(\Omega)} + \int_\Omega \lambda_1|u_{\lambda_1,\mu_1}|^2\,dx} - \frac{s^{2^*-2}\int_\Omega \mu_2|v_{\lambda_2,\mu_2}|^{2^*}\,dx}{\|v_{\lambda_2,\mu_2}\|^2_{D_s(\Omega)} + \int_\Omega \lambda_2|v_{\lambda_2,\mu_2}|^2\,dx},\\ &\frac{t^{2^*-2}\int_\Omega \mu_1|u_{\lambda_1,\mu_1}|^{2^*}\,dx}{\|u_{\lambda_1,\mu_1}\|^2_{D_s(\Omega)} + \int_\Omega \lambda_1|u_{\lambda_1,\mu_1}|^2\,dx} - \frac{s^{2^*-2}\int_\Omega \mu_2|v_{\lambda_2,\mu_2}|^{2^*}\,dx}{\|v_{\lambda_2,\mu_2}\|^2_{D_s(\Omega)} + \int_\Omega \lambda_2|v_{\lambda_2,\mu_2}|^2\,dx} - 2\Bigg).\end{aligned}$$

Since

$$\int_\Omega \mu_1|u_{\lambda_1,\mu_1}|^{2^*}\,dx = \|u_{\lambda_1,\mu_1}\|^2_{D_s(\Omega)} + \int_\Omega \lambda_1|u_{\lambda_1,\mu_1}|^2\,dx$$

and

$$\int_\Omega \mu_2|v_{\lambda_2,\mu_2}|^{2^*}\,dx = \|v_{\lambda_2,\mu_2}\|^2_{D_s(\Omega)} + \int_\Omega \lambda_2|v_{\lambda_2,\mu_2}|^2\,dx.$$

Thus, $\Upsilon(\widetilde{\sigma})(1,1) = (0,0)$. By direct calculation, we have

$$\deg\big(\Upsilon(\widetilde{\sigma}), [t_0,t_1]\times[s_0,s_1], (0,0)\big) = 1.$$

By (31), we know that, for any $(t,s) \in \partial([t_0,t_1]\times[s_0,s_1])$, $\Upsilon(\widetilde{\sigma})(t,s) = \Upsilon(\sigma)(t,s) \neq (0,0)$. Therefore

$$\deg\big(\Upsilon(\sigma), [t_0,t_1]\times[s_0,s_1], (0,0)\big) = \deg\big(\Upsilon(\widetilde{\sigma}), [t_0,t_1]\times[s_0,s_1], (0,0)\big) = 1.$$

Then there exist $(t_2,s_2) \in [t_0,t_1]\times[s_0,s_1]$ such that $\Upsilon(\sigma)(t_2,s_2) = (0,0)$, thus

$$J_5\big(\sigma_1(t_2,s_2)\big) = J_6\big(\sigma_2(t_2,s_2)\big) = 1.$$

This implies

$$\sigma_i(t_2,s_2) \in \mathbb{M}_i \quad \text{and} \quad \sigma_i(t_2,s_2) \neq 0 \quad \text{for } i = 1,2.$$

By (7) and $\sigma_i(t_2,s_2) \in \mathbb{M}_i$, we have

$$\begin{aligned}\max_{(t,s)\in Q} E_0\big(\sigma(t,s)\big) \ge E_0\big(\sigma(t_2,s_2)\big)\\ &= J_{\lambda_1,\mu_1}\big(\sigma_1(t_2,s_2)\big) + J_{\lambda_2,\mu_2}\big(\gamma_2(t_2,s_2)\big)\\ &\ge m_{\lambda_1,\mu_1} + m_{\lambda_2,\mu_2} = d_0.\end{aligned}$$

Therefore $\widehat{c_0} \ge d_0$, combining this with (32), we obtain $\widehat{c_0} = d_0$.

By the definition of $\widehat{c}_\gamma$ and d_γ, we have

$$\widehat{c}_\gamma \leq d_\gamma \leq d_0.$$

Next, we prove $\liminf_{\gamma\to 0}\widehat{c}_\gamma \geq d_0$. Assume by contradiction that $\liminf_{\gamma\to 0}\widehat{c}_\gamma < d_0$. Then there exist $\epsilon > 0$, $\gamma_n \to 0$ and $\sigma_n = (\sigma_{n,1}, \sigma_{n,2}) \in \widehat{\Gamma}$ such that

$$\max_{(t,s)\in Q} E_{\gamma_n}\bigl(\sigma_n(t,s)\bigr) \leq d_0 - 2\epsilon.$$

By the definition of $\widehat{\Gamma}$ in (31), there exists n_0 large enough such that

$$\max_{(t,s)\in Q} \frac{1}{2^*}\gamma_n \left| \int_\Omega \bigl|\sigma_{n,1}(t,s)\bigr|^\alpha \bigl|\sigma_{n,2}(t,s)\bigr|^\beta \, dx \right| \leq C\gamma_n \leq \epsilon, \quad \forall n \geq n_0.$$

Thus, $\max_{(t,s)\in Q} E_0(\sigma_n(t,s)) \leq \max_{(t,s)\in Q} E_{\gamma_n}(\sigma_n(t,s)) + \epsilon \leq d_0 - \epsilon$, $\forall n \geq n_0$. Since $\widehat{c}_0 \leq d_0$, this is a contradiction. Therefore $\liminf_{\gamma\to 0}\widehat{c}_\gamma \geq d_0$. Combining this again with (32), we complete the proof. □

Define

$$X^\delta := \bigl\{(u,v) \in \mathcal{D}_s(\Omega) : \operatorname{dist}\bigl((u,v), X\bigr) \leq \delta\bigr\}, \qquad E_\gamma^c := \bigl\{(u,v) \in \mathcal{D}_s(\Omega) : E_\gamma(u,v) \leq c\bigr\}.$$

Lemma 4.3 *Let $d > 0$ be a fixed number and let $\{(u_n, v_n)\} \subset X^d$ be a sequence. Then up to a subsequence, $(u_n, v_n) \rightharpoonup (u_0, v_0) \in X^{2d}$.*

Proof By Lemma 4.1 and the definition of X^d, there exists a sequence $\{(\overline{u}_n, \overline{v}_n)\} \subset X$ such that

$$\operatorname{dist}\bigl((u_n, v_n), X\bigr) = \operatorname{dist}\bigl((u_n, v_n), (\overline{u}_n, \overline{v}_n)\bigr) \leq d.$$

By Lemma 4.1, we also know that there exist $(\overline{u}, \overline{v}) \in X$ such that $(\overline{u}_n, \overline{v}_n) \to (\overline{u}, \overline{v})$ strongly in $\mathcal{D}_s(\Omega)$. Consequently, when n is sufficiently large, we have

$$\operatorname{dist}\bigl((\overline{u}_n, \overline{v}_n), (\overline{u}, \overline{v})\bigr) \leq d.$$

Thus, $\{(u_n, v_n)\}$ is bounded and up to a subsequence, $(u_n, v_n) \rightharpoonup (u_0, v_0)$ in $\mathcal{D}_s(\Omega)$. Since $B_{2d}(\overline{u}, \overline{v})$ is weakly closed in $\mathcal{D}_s(\Omega)$, we get $(u_0, v_0) \in B_{2d}(\overline{u}, \overline{v}) \subset X^{2d}$. □

Lemma 4.4 *Let $d_1 := \frac{1}{2}(\frac{Nm_{\lambda_1,\mu_1}}{s})^{\frac{1}{2}}$ and $d \in (0, d_1)$. Suppose that there exist sequences $\{\gamma_j\}$, with $\gamma_j > 0$ and $\gamma_j \to 0$, and $\{(u_j, v_j)\} \subset X^d$ satisfying*

$$\lim_{j\to+\infty} E_{\gamma_j}(u_j, v_j) \leq \widehat{c}_0, \qquad \lim_{j\to+\infty} E'_{\gamma_j}(u_j, v_j) = 0.$$

Then (u_j, v_j) converges strongly to an element $(u, v) \in X$.

Proof By the choice of d_1 and Lemma 4.3 $(u_j,v_j) \rightharpoonup (u,v) \in X^{2d}$, we can deduce that $u \not\equiv 0$ and $v \not\equiv 0$. Since $\{(u_j,v_j)\}$ is bounded and $\lim_{j\to+\infty} E'_{\gamma_j}(u_j,v_j) = 0$, for all $(\varphi,\psi) \in \mathcal{D}_s(\Omega)$,

$$\begin{aligned}\big\langle E'_0(u,v),(\varphi,\psi)\big\rangle &= \langle u,\varphi\rangle_{D_s(\Omega)} + \langle v,\psi\rangle_{D_s(\Omega)} + \int_\Omega (\lambda_1 u\varphi + \lambda_2 v\psi)\,dx \\ &\quad - \int_\Omega \big(\mu_1|u|^{2^*-2}u\varphi + \mu_2|v|^{2^*-2}v\psi\big)\,dx \\ &= \lim_{j\to+\infty}\bigg[\big\langle E'_{\gamma_j}(u_j,v_j),(\varphi,\psi)\big\rangle + \frac{\alpha\gamma_j}{2^*}\int_\Omega |u_j|^{\alpha-2}u_j\varphi|v_j|^\beta\,dx \\ &\quad + \frac{\beta\gamma_j}{2^*}\int_\Omega |u_j|^\alpha|v_j|^{\beta-2}v_j\psi\,dx\bigg] \\ &= 0,\end{aligned}$$

where

$$\langle u,\varphi\rangle_{D_s(\Omega)} = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|y-x|^{N+2s}}\,dx\,dy.$$

Hence, $E'_0(u,v) = 0$. Since $(u_j,v_j) \in X^d$ for all j, we have

$$\begin{aligned}&\big\langle E'_0(u_j,v_j),(\varphi,\psi)\big\rangle \\ &\quad= \big\langle E'_{\gamma_j}(u_j,v_j),(\varphi,\psi)\big\rangle + \frac{\alpha\gamma_j}{2^*}\int_\Omega |u_j|^{\alpha-2}u_j\varphi|v_j|^\beta\,dx + \frac{\beta\gamma_j}{2^*}\int_\Omega |u_j|^\alpha|v_j|^{\beta-2}v_j\psi\,dx \\ &\quad= O(1)\big\|(\varphi,\psi)\big\|_{\mathcal{D}_s(\Omega)}.\end{aligned}$$

We have

$$\begin{aligned}\widehat{c}_0 &\geq \lim_{j\to+\infty} E_{\gamma_j}(u_j,v_j) \\ &= \lim_{j\to+\infty} E_0(u_j,v_j) - \lim_{j\to+\infty}\frac{\gamma_j}{2^*}\int_\Omega |u_j|^\alpha|v_j|^\beta\,dx \\ &= \lim_{j\to+\infty} E_0(u_j,v_j) := m.\end{aligned}\tag{35}$$

So $\{(u_j,v_j)\}$ is a $(PS)_m$ sequence of E_0 with $m := \lim_{j\to+\infty} E_0(u_j,v_j)$. Thus, we have

$$\begin{aligned}E_0(u,v) &= \frac{1}{2}\big\|(u,v)\big\|^2_{\mathcal{D}_s(\Omega)} + \frac{1}{2}\int_\Omega \big(\lambda_1 u^2 + \lambda_2 v^2\big)\,dx - \frac{1}{2^*}\int_\Omega \big(\mu_1|u|^{2^*} + \mu_2|v|^{2^*}\big)\,dx \\ &= \frac{s}{N}\bigg[\big\|(u,v)\big\|^2_{\mathcal{D}_s(\Omega)} + \int_\Omega \big(\lambda_1 u^2 + \lambda_2 v^2\big)\,dx\bigg] \\ &\leq \frac{s}{N}\liminf_{j\to+\infty}\bigg[\big\|(u_j,v_j)\big\|^2_{\mathcal{D}_s(\Omega)} + \int_\Omega \big(\lambda_1 u_j^2 + \lambda_2 v_j^2\big)\,dx\bigg] \\ &= \liminf_{j\to+\infty}\bigg[E_0(u_j,v_j) - \frac{1}{2^*}\big\langle E'_0(u_j,v_j),(u_j,v_j)\big\rangle\bigg] = m.\end{aligned}$$

Then, by Lemma 4.2, we have $m \geq E_0(u,v) \geq \widehat{c}_0$. Combining this with (35), we get $m = E_0(u,v) = \widehat{c}_0$. This implies $(u_j,v_j) \to (u,v)$ strongly in $\mathcal{D}_s(\Omega)$ and $(u,v) \in X$. □

Lemma 4.5 *Let d_1 be as in Lemma 4.4. For a small $\delta \in (0, d_1)$, there exist constants $0 < \sigma < 1$ and $\gamma_1 > 0$ such that $\|E'_\gamma(u,v)\| \geq \sigma$ for any $(u,v) \in E_\gamma^{d_\gamma} \cap (X^\delta \setminus X^{\frac{\delta}{2}})$ and $\gamma \in (0, \gamma_1)$.*

Proof Assume by contradiction. Suppose there exist a number $\delta_0 \in (0, d_1)$, a positive sequence $\{\gamma_j\}$ with $\lim_{j\to+\infty} \gamma_j = 0$ and a sequence $\{(u_j, v_j)\} \in E_{\gamma_j}^{d_{\gamma_j}} \cap (X^{\delta_0} \setminus X^{\frac{\delta_0}{2}})$ such that $\lim_{j\to+\infty} E'_{\gamma_j}(u_j, v_j) = 0$. Then, by Lemma 4.2, we have

$$\lim_{j\to+\infty} E_{\gamma_j}(u_j, v_j) \leq \widehat{c}_0, \quad \{(u_j, v_j)\} \subset X^{\delta_0}, \delta_0 < d_1,$$

and

$$\lim_{j\to+\infty} E'_{\gamma_j}(u_j, v_j) = 0.$$

Then, by Lemma 4.4, we know there exist $(u,v) \in X$ such that $(u_j, v_j) \to (u,v)$ strongly in $\mathcal{D}_s(\Omega)$. Hence, $\operatorname{dist}((u_j, v_j), X) \to 0$ as $j \to +\infty$. This contradicts $(u_j, v_j) \notin X^{\frac{\delta_0}{2}}$. □

In the next part of this paper, we let $0 < \sigma < 1$, $\gamma_1 > 0$ and $\delta \in (0, \frac{d_1}{2})$ such that the conclusions in Lemma 4.5 hold.

Lemma 4.6 *There exist $\gamma_2 \in (0, \gamma_1)$ and $\varsigma > 0$ such that, for any $\gamma \in (0, \gamma_2)$,*

$$E_\gamma\big(\widetilde{\sigma}(t,s)\big) \geq \widehat{c}_\gamma - \varsigma \quad \textit{implies that} \quad \widetilde{\sigma}(t,s) \in X^{\frac{\delta}{2}}. \tag{36}$$

Proof Suppose by contradiction that there exist $\gamma_n \to 0$, $\varsigma_n \to 0$ and $(t_n, s_n) \in Q$ such that

$$E_{\gamma_n}\big(\widetilde{\sigma}(t_n,s_n)\big) \geq \widehat{c}_{\gamma_n} - \varsigma_n \quad \textit{and} \quad \widetilde{\sigma}(t_n,s_n) \notin X^{\frac{\delta}{2}}. \tag{37}$$

We assume $(t_n, s_n) \to (\bar{t}, \bar{s}) \in Q$. Since

$$E_0\big(\widetilde{\sigma}(t_n,s_n)\big) \geq E_{\gamma_n}\big(\widetilde{\sigma}(t_n,s_n)\big) \geq \widehat{c}_{\gamma_n} - \varsigma_n, \tag{38}$$

we take the limit on both sides of (38), we have

$$E_0\big(\widetilde{\sigma}(\bar{t},\bar{s})\big) \geq \lim_{n\to+\infty} \widehat{c}_{\gamma_n}.$$

By Lemma 4.2, we have

$$E_0\big(\widetilde{\sigma}(\bar{t},\bar{s})\big) \geq \lim_{n\to+\infty} \widehat{c}_{\gamma_n} = m_{\lambda_1,\mu_1} + m_{\lambda_2,\mu_2}.$$

Combining this with (27) and (32), we can deduce that $(\bar{t}, \bar{s}) = (1,1)$. Hence,

$$\lim_{n\to+\infty} \big\|\widetilde{\sigma}(t_n,s_n) - \widetilde{\sigma}(1,1)\big\| = 0.$$

However, $\widetilde{\sigma}(1,1) = (u_{\lambda_1,\mu_1}, v_{\lambda_2,\mu_2}) \in X$, which contradicts (37). □

Next, we set

$$\varsigma_0 := \min\left\{\frac{\varsigma}{2}, \frac{m_{\lambda_1,\mu_1}}{4}, \frac{\delta\sigma^2}{8}\right\}, \tag{39}$$

where δ, σ are given in Lemma 4.5, ς is from Lemma 4.6. By Lemma 4.2, we know that there exists $\gamma_0 \in (0, \gamma_2]$ such that

$$|\widehat{c}_\gamma - d_\gamma| < \varsigma_0, \qquad \left|\widehat{c}_\gamma - (m_{\lambda_1,\mu_1} + m_{\lambda_2,\mu_2})\right| < \varsigma_0, \quad \forall \gamma \in (0, \gamma_0). \tag{40}$$

Lemma 4.7 *For fixed $\gamma \in (0, \gamma_0)$, there exist $\{(u_n, v_n)\}_{n=1}^{\infty} \subset X^\delta \cap E_\gamma^{d_\gamma}$ such that*

$$E'_\gamma(u_n, v_n) \to 0 \quad \textit{in } \mathcal{D}_s(\Omega) \textit{ as } n \to +\infty.$$

Proof Assume by contradiction, for fixed $\gamma \in (0, \gamma_0)$, that there exists $0 < l(\gamma) < 1$ such that

$$\left\|E'_\gamma(u, v)\right\| \geq l(\gamma) \quad \text{on } X^\delta \cap E_\gamma^{d_\gamma}.$$

Then there exists a pseudo-gradient vector field T_γ in $\mathcal{D}_s(\Omega)$ which is defined on a neighborhood Z_γ of $X^\delta \cap E_\gamma^{d_\gamma}$ such that, for any $(u, v) \in Z_\gamma$,

$$\begin{aligned} &\left\|T_\gamma(u, v)\right\| \leq 2\min\left\{1, \left\|E'_\gamma(u, v)\right\|\right\}, \\ &\left\langle E'_\gamma(u, v), T_\gamma(u, v)\right\rangle \geq \min\left\{1, \left\|E'_\gamma(u, v)\right\|\right\}\left\|E'_\gamma(u, v)\right\|. \end{aligned}$$

Let η_γ be a Lipschitz continuous function on $\mathcal{D}_s(\Omega)$ such that

$$0 \leq \eta_\gamma \leq 1, \quad \eta_\gamma = 1 \quad \text{on } X^\delta \cap E_\gamma^{d_\gamma} \quad \text{and} \quad \eta_\gamma = 0 \quad \text{on } \mathcal{D}_s(\Omega) \setminus Z_\gamma.$$

Let ξ_γ be a Lipschitz continuous function on $\mathbb{R}$ such that

$$0 \leq \xi_\gamma \leq 1, \quad \xi_\gamma(l) \equiv 1 \quad \text{if } |l - \widehat{c}_\gamma| \leq \frac{\varsigma}{2} \quad \text{and} \quad \xi_\gamma(l) \equiv 0 \quad \text{if } |l - \widehat{c}_\gamma| \geq \varsigma.$$

Let

$$e_\gamma(u, v) := \begin{cases} -\eta_\gamma(u, v)\xi_\gamma(E_\gamma(u, v))T_\gamma(u, v), & \text{if } (u, v) \in Z_\gamma, \\ 0, & \text{if } (u, v) \in H \setminus Z_\gamma. \end{cases} \tag{41}$$

Then there exists a global solution $\psi_\gamma : \mathcal{D}_s(\Omega) \times [0, +\infty) \to \mathcal{D}_s(\Omega)$ for the initial value problem

$$\begin{cases} \frac{d}{d\theta}\psi_\gamma(u, v, \theta) = e_\gamma(\psi_\gamma(u, v, \theta)), \\ \psi_\gamma(u, v, 0) = (u, v). \end{cases} \tag{42}$$

Then we can deduce that ψ_γ has the following properties:

(1) $\psi_\gamma(u, v, \theta) = (u, v)$ if $\theta = 0$ or $(u, v) \in \mathcal{D}_s(\Omega) \setminus Z_\gamma$ or $|E_\gamma(u, v) - \widehat{c}_\gamma| \geq \varsigma$.

(2) $\|\frac{d}{d\theta}\psi_\gamma(u, v, \theta)\| \leq 2$.

(3) $\frac{d}{d\theta}E_\gamma(\psi_\gamma(u, v, \theta)) = \langle E'_\gamma(\psi_\gamma(u, v, \theta)), e_\gamma(\psi_\gamma(u, v, \theta))\rangle \leq 0$.

In order to prove Lemma 4.7, we use the above properties, Lemma 4.5 and Lemma 4.6, then divide two step to prove it.

Step one. We show that, for any $(t,s)\in Q$, there exists $\theta_{t,s}\in[0,+\infty)$ such that $\psi_\gamma(\widetilde{\sigma}(t,s),\theta_{t,s})\in E_\gamma^{\widehat{c}_\gamma-\varsigma_0}$, where ς_0 is seen in (39).

Suppose by contradiction that there exists $(t,s)\in Q$ such that

$$E_\gamma\big(\psi_\gamma\big(\widetilde{\sigma}(t,s),\theta\big)>\widehat{c}_\gamma-\varsigma_0\quad\text{for any }\theta\geq 0.$$

Since $\varsigma_0<\varsigma$, by Lemma 4.6, we have $\widetilde{\sigma}(t,s)\in X^{\frac{\delta}{2}}$. By (40), we get

$$E_\gamma\big(\widetilde{\sigma}(t,s)\big)\leq d_\gamma<\widehat{c}_\gamma+\varsigma_0.$$

By the property (3), we have

$$\widehat{c}_\gamma-\varsigma_0<E_\gamma\big(\psi_\gamma\big(\widetilde{\sigma}(t,s),\theta\big)\leq d_\gamma<\widehat{c}_\gamma+\varsigma_0,\quad\forall\theta\geq 0.$$

This implies $\xi_\gamma(E_\gamma(\psi_\gamma(\widetilde{\sigma}(t,s),\theta)))\equiv 1$. If $\psi_\gamma(\widetilde{\sigma}(t,s),\theta)\in X^\delta$ for all $\theta\geq 0$, then

$$\eta_\gamma\big(\psi_\gamma\big(\widetilde{\sigma}(t,s),\theta\big)\big)\equiv 1\quad\text{and}\quad\big\|E'_\gamma\big(\psi_\gamma\big(\widetilde{\sigma}(t,s),\theta\big)\big)\big\|\geq l(\gamma)\quad\text{for all }\theta>0.$$

Consequently,

$$E_\gamma\left(\psi_\gamma\left(\widetilde{\sigma}(t,s),\frac{\varsigma}{l^2(\gamma)}\right)\right)\leq\widehat{c}_\gamma+\frac{\varsigma}{2}-\int_0^{\frac{\varsigma}{l^2(\gamma)}}l^2(\gamma)\,dt=\widehat{c}_\gamma-\frac{\varsigma}{2},$$

which is a contradiction. Thus, there exists $\theta_{t,s}>0$ such that $\psi_\gamma(\widetilde{\sigma}(t,s),\theta_{t,s})\notin X^\delta$.

Since $\widetilde{\sigma}(t,s)\in X^{\frac{\delta}{2}}$, there exist $0<\theta^1_{t,s}<\theta^2_{t,s}\leq\theta_{t,s}$ such that

$$\psi_\gamma\big(\widetilde{\sigma}(t,s),\theta^1_{t,s}\big)\in\partial X^{\frac{\delta}{2}},\qquad\psi_\gamma\big(\widetilde{\sigma}(t,s),\theta^2_{t,s}\big)\in\partial X^\delta$$

and

$$\psi_\gamma\big(\widetilde{\sigma}(t,s),\theta\big)\in X^\delta\setminus X^{\frac{\delta}{2}}\quad\text{for all }\theta\in\big(\theta^1_{t,s},\theta^2_{t,s}\big).$$

Then, by Lemma 4.5, we have $\|E'_\gamma(\psi_\gamma(\widetilde{\sigma}(t,s),\theta))\|\geq\sigma$ for all $\theta\in(\theta^1_{t,s},\theta^2_{t,s})$. Then, by the property (2), we have

$$\frac{\delta}{2}\leq\big\|\psi_\gamma\big(\widetilde{\sigma}(t,s),\theta^2_{t,s}\big)-\psi_\gamma\big(\widetilde{\sigma}(t,s),\theta^1_{t,s}\big)\big\|\leq 2\big|\theta^2_{t,s}-\theta^1_{t,s}\big|,$$

thus, $|\theta^2_{t,s}-\theta^1_{t,s}|\geq\frac{\delta}{4}$. Consequently,

$$\begin{aligned}E_\gamma\big(\psi_\gamma\big(\widetilde{\sigma}(t,s),\theta^2_{t,s}\big)\big)&\leq E_\gamma\big(\psi_\gamma\big(\widetilde{\sigma}(t,s),\theta^1_{t,s}\big)\big)+\int_{\theta^1_{t,s}}^{\theta^2_{t,s}}\frac{d}{d\theta}E_\gamma\big(\psi_\gamma(u,v,\theta)\big)\,d\theta\\&\leq\widehat{c}_\gamma+\varsigma_0-\sigma^2\big(\theta^2_{t,s}-\theta^1_{t,s}\big)\leq\widehat{c}_\gamma+\varsigma_0-\frac{\delta\sigma^2}{4}\\&\leq\widehat{c}_\gamma-\varsigma_0,\end{aligned}$$

which is a contradiction.

By step one we can define $T(t,s) := \inf\{\theta \geq 0 : E_\gamma(\psi_\gamma(\widetilde{\sigma}(t,s),\theta)) \leq \widehat{c}_\gamma - \varsigma_0\}$ and let $\sigma(t,s) := \psi_\gamma(\widetilde{\sigma}(t,s), T(t,s))$. Then $E_\gamma(\sigma(t,s)) \leq \widehat{c}_\gamma - \varsigma_0$ for all $(t,s) \in Q$.

Step two. We claim $\sigma(t,s) \in \widehat{\Gamma}$.

By (27)–(28) and (39), (40), for any $(t,s) \in Q \setminus (t_0,t_1) \times (s_0,s_1)$, we have

$$E_\gamma\big(\widetilde{\sigma}(t,s)\big) \leq E_0\big(\widetilde{\sigma}(t,s)\big) = J_{\lambda_1,\mu_1}\big(\widetilde{\sigma}_1(t)\big) + J_{\lambda_2,\mu_2}\big(\widetilde{\sigma}_2(s)\big)$$
$$\leq \frac{m_{\lambda_1,\mu_1}}{4} + m_{\lambda_2,\mu_2} \leq m_{\lambda_1,\mu_1} + m_{\lambda_2,\mu_2} - 3\varsigma_0 < \widehat{c}_\gamma - \varsigma_0,$$

which implies that $T(t,s) = 0$ and so $\sigma(t,s) = \widetilde{\sigma}(t,s)$.

By the definition of $\widehat{\Gamma}$ in (31), we need to prove that $\|\sigma(t,s)\|_{\mathcal{D}_s(\Omega)} \leq 2C_2 + C_0$ for all $(t,s) \in Q$ and $T(t,s)$ is continuous with respect to (t,s).

For any $(t,s) \in Q$, if $E_\gamma(\widetilde{\sigma}(t,s)) \leq \widehat{c}_\gamma - \varsigma_0$, we have $T(t,s) = 0$ and so $\sigma(t,s) = \widetilde{\sigma}(t,s)$. By (30), we have $\|\sigma(t,s)\|_{\mathcal{D}_s(\Omega)} \leq C_0 < 2C_2 + C_0$.

If $E_\gamma(\widetilde{\sigma}(t,s)) > \widehat{c}_\gamma - \varsigma_0$, then, by Lemma 4.6, we have $\widetilde{\sigma}(t,s) \in X^{\frac{\delta}{2}}$ and

$$\widehat{c}_\gamma - \varsigma_0 < E_\gamma\big(\psi_\gamma\big(\widetilde{\sigma}(t,s),\theta\big) \leq d_\gamma < \widehat{c}_\gamma + \varsigma_0, \quad \forall\theta \in \big[0, T(t,s)\big).$$

This implies $\xi_\gamma(E_\gamma(\psi_\gamma(\widetilde{\sigma}(t,s),\theta))) \equiv 1$ for $\theta \in [0, T(t,s))$. If $\psi_\gamma(\widetilde{\sigma}(t,s), T(t,s)) \notin X^\delta$, then there exist $0 < \theta^1_{t,s} < \theta^2_{t,s} < T(t,s)$ as above. Then we can prove that

$$E_\gamma(\psi_\gamma\big(\widetilde{\sigma}(t,s),\theta^2_{t,s}\big) \leq \widehat{c}_\gamma - \varsigma_0,$$

which contradicts the definition of $T(t,s)$. Therefore,

$$\sigma(t,s) = \psi_\gamma\big(\widetilde{\sigma}(t,s), T(t,s)\big) \in X^\delta.$$

Then there exist $(u,v) \in X$ such that $\|\sigma(t,s) - (u,v)\|_{\mathcal{D}_s(\Omega)} \leq \delta \leq \frac{C_0}{2}$. By Lemma 4.1, we have

$$\big\|\sigma(t,s)\big\|_{\mathcal{D}_s(\Omega)} \leq \big\|(u,v)\big\|_{\mathcal{D}_s(\Omega)} + \frac{C_0}{2} \leq 2C_2 + C_0.$$

In order to prove the continuity of $T(t,s)$, we fix any $(\widetilde{t},\widetilde{s}) \in Q$. First, we assume that $E_\gamma(\sigma(\widetilde{t},\widetilde{s})) < \widehat{c}_\gamma - \varsigma_0$. Then, by the definition of $T(t,s)$, we have $T(\widetilde{t},\widetilde{s}) = 0$, that is,

$$E_\gamma\big(\widetilde{\sigma}(\widetilde{t},\widetilde{s})\big) < \widehat{c}_\gamma - \varsigma_0.$$

By the continuity of $\widetilde{\sigma}$, there exists $\tau > 0$ such that, for any $(t,s) \in (\widetilde{t} - \tau, \widetilde{t} + \tau) \times (\widetilde{s} - \tau, \widetilde{s} + \tau) \cap Q$, we have $E_\gamma(\widetilde{\sigma}(t,s)) < \widehat{c}_\gamma - \varsigma_0$, that is, $T(t,s) = 0$ and T is continuous at $(\widetilde{t},\widetilde{s})$. Now, we assume that $E_\gamma(\sigma(\widetilde{t},\widetilde{s})) = \widehat{c}_\gamma - \varsigma_0$. Then from the previous proof we have

$$\sigma(\widetilde{t},\widetilde{s}) = \psi_\gamma\big(\widetilde{\sigma}(\widetilde{t},\widetilde{s}), T(\widetilde{t},\widetilde{s})\big) \in X^\delta,$$

and so

$$\big\|E'_\gamma(\psi_\gamma\big(\widetilde{\sigma}(\widetilde{t},\widetilde{s}), T(\widetilde{t},\widetilde{s})\big)\big\| \geq l(\gamma) > 0.$$

Then, for any $\omega > 0$, we have

$$E_\gamma\big(\psi_\gamma\big(\widetilde{\sigma}(\widetilde{t},\widetilde{s}), T(\widetilde{t},\widetilde{s}) + \omega\big) < \widehat{c}_\gamma - \varsigma_0.$$

By the continuity of ψ_γ, there exists $\tau > 0$ such that, for any $(t,s) \in (\widetilde{t} - \tau, \widetilde{t} + \tau) \times (\widetilde{s} - \tau, \widetilde{s} + \tau) \cap Q$, we have $E_\gamma(\psi_\gamma(\widetilde{\sigma}(t,s)), T(\widetilde{t},\widetilde{s}) + \omega)) < \widehat{c}_\gamma - \varsigma_0$, so $T(t,s) \leq T(\widetilde{t},\widetilde{s}) + \omega$. It follows that

$$0 < \limsup_{(t,s)\to(\widetilde{t},\widetilde{s})} T(t,s) \leq T(\widetilde{t},\widetilde{s}).$$

If $T(\widetilde{t},\widetilde{s}) = 0$, we have

$$\lim_{(t,s)\to(\widetilde{t},\widetilde{s})} T(t,s) = T(\widetilde{t},\widetilde{s}).$$

If $T(\widetilde{t},\widetilde{s}) > 0$, then, for any $0 < \omega < T(\widetilde{t},\widetilde{s})$, by the same arguments, we have

$$E_\gamma\big(\psi_\gamma\big(\widetilde{\sigma}(\widetilde{t},\widetilde{s}), T(\widetilde{t},\widetilde{s}) - \omega\big) > \widehat{c}_\gamma - \varsigma_0.$$

By the continuity of ψ_γ again, we have

$$\lim_{(t,s)\to(\widetilde{t},\widetilde{s})} T(t,s) = T(\widetilde{t},\widetilde{s}).$$

So T is continuous at $(\widetilde{t},\widetilde{s})$. This completes the proof of step two.

Now, we have proved that $\sigma(t,s) \in \widehat{\Gamma}$ and $\max_{(t,s)\in Q} E_\gamma(\sigma(t,s)) \leq \widehat{c}_\gamma - \varsigma_0$, which contradicts the definition of $\widehat{c}_\gamma$. This completes the proof. □

Proof of Theorem 1.2 Let us fix $d_1 := \frac{1}{2}(\frac{Nm_{\lambda_1,\mu_1}}{s})^{\frac{1}{2}}$. By Lemma 4.7, there exists some $\gamma_0 > 0$ such that, for any fixed $\gamma \in (0,\gamma_0)$, a Palais–Smale sequence $\{(u_n^\gamma, v_n^\gamma)\}$ with $(u_n^\gamma, v_n^\gamma) \in X^\delta$ exists. Since X is compact, we can deduce that $\{(u_n^\gamma, v_n^\gamma)\}$ is bounded in $\mathcal{D}_s(\Omega)$. By Lemma 4.3, there exist $(u_\gamma, v_\gamma) \in X^d$ such that $(u_n^\gamma, v_n^\gamma) \rightharpoonup (u_\gamma, v_\gamma)$ weakly in $\mathcal{D}_s(\Omega)$. Therefore, $E'_\gamma(u_\gamma, v_\gamma) = 0$. By the choice of d, we have $u_\gamma \not\equiv 0$ and $v_\gamma \not\equiv 0$. Hence, (u_γ, v_γ) is the desired solution to (3). □

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Authors' contributions
The authors declare that the study was realized in collaboration with equal responsibility. All authors read and approved the final manuscript.

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