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Pullback attractors for non-autonomous reaction–diffusion equation with infinite delays in $C_{\gamma,L^r(\Omega)}$ or $C_{\gamma,W^{1,r}(\Omega)}$

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Abstract

In this paper, the well-posedness for the non-autonomous reaction–diffusion equation with infinite delays on a bounded domain is established. The existence of pullback attractors for the process in $C_{\gamma,L^r(\Omega)}$ and $C_{\gamma,W^{1,r}(\Omega)}$ is proved, respectively. The noncompact Kuratowski measure is applied to check the asymptotic compactness.

Keywords: Pullback attractor; Reaction–diffusion equation; Infinite delays; Nonautonomous equation

1 Introduction

Let $\Omega \subseteq \mathbb{R}^n$ be a smooth bounded domain. Consider the long-time behavior of the following non-autonomous nonlinear reaction–diffusion equation:

$$\begin{cases} \frac{\partial u}{\partial t} - \Delta u + \lambda u = f(t, u_t) + g(t,x), & \text{in } [\tau, +\infty] \times \Omega, \\ u|_{\partial\Omega} = 0, & t > \tau, \\ u(t,x) = \phi(t-\tau, x), & t \in (-\infty, \tau], x \in \Omega, \end{cases} \tag{1}$$

where $\lambda \geq 0$, and we have the nonlinear term

$$f\bigl(t, u_t(t,x)\bigr) = F\bigl(t, u\bigl(t-\rho(t), x\bigr)\bigr) + \int_{-\infty}^{0} G\bigl(t, z, u(t+z, x)\bigr)\, dz.$$

Suppose there exist two positive constants k_1, k_2, and three positive scalar functions $m_0(\cdot)$, $e^{-r\gamma\rho(t)} m_1(t)$, $m_2(\cdot)e^{-\gamma z}$ which are all in $L^1((-\infty,0],\mathbb{R}^+)$ such that the functions $F \in C(\mathbb{R} \times \mathbb{R};\mathbb{R})$, $\rho \in C(\mathbb{R};[0,+\infty))$, and $G \in C(\mathbb{R} \times (-\infty,0] \times \mathbb{R};\mathbb{R})$ satisfy

$$\bigl|F(t,\upsilon)\bigr|^r \leq |k_1|^r + k_2^r e^{-r\gamma\rho(t)}|\upsilon|^r, \quad \forall t, \upsilon \in \mathbb{R}, \tag{2}$$

$$\bigl|G(t,z,\upsilon)\bigr| \leq m_0(z) + m_1(z)|\nu|, \quad \forall t, \upsilon \in \mathbb{R}, z \in (-\infty, 0], \tag{3}$$

$$\bigl|F(t,\upsilon) - F(t,\nu)\bigr| \leq C_1 e^{-\gamma\rho(t)}|\upsilon - \nu|, \quad \forall t, \upsilon, \nu \in \mathbb{R}, z \in (-\infty, 0], \tag{4}$$

$$\bigl|G(t,z,\upsilon) - G(t,z,\nu)\bigr| \leq C_2 m_2(z)|\upsilon - \nu|, \quad \forall t, \upsilon, \nu \in \mathbb{R}, z \in (-\infty, 0], \tag{5}$$

and the non-autonomous term $g \in L^r_{\mathrm{loc}}(\mathbb{R}; L^r(\Omega))$ $(r > 1)$ satisfies

$$\sup_{\tau \le t} e^{-\delta\tau} \int_{-\infty}^{\tau} \|g(s)\|_X^r e^{\delta s}\, ds < \infty, \quad \forall t \in \mathbb{R}, \tag{6}$$

for each $\delta \in \{\alpha, \alpha - L, r(\delta - \eta)\}$, where α, L, δ, η will be given in Lemma 4.1, the local r-power integral is the Bochner integral. We will denote $m_0 = \int_{-\infty}^0 m_0(s)\, ds$, $m_1 = \int_{-\infty}^0 e^{-\gamma s} m_1(s)\, ds$, and $m_2 = \int_{-\infty}^0 e^{-\gamma s} m_2(s)\, ds$.

Let $C_{\gamma,X}$ denote the Banach space $C((-\infty, 0]; X)$ endowed with the norm

$$\|\phi\|_{C_{\gamma,X}} = \sup_{z \in (-\infty,0]} e^{\gamma z} \|\phi(z)\|_X, \quad \gamma > 0,$$

where X is $L^r(\Omega)$ or $W^{1,r}(\Omega)$.

Given $\tau \in \mathbb{R}$, $T > \tau$ and a function $u : (-\infty, T] \to X$. For each $t \in [\tau, T]$, $u_t : (-\infty, 0] \to X$ denotes the function defined by $u_t(z) = u(t + z)$ for $z \in (-\infty, 0]$. We are interested in the initial condition $\phi \in C_{\gamma,X}$.

Retarded differential equations have been used to research many physical systems with non-instant transmission phenomena such as internet data transmission, other memory processes, and specially biological motivations (e.g. species growth or incubating time on disease models [1, 2]). For autonomous systems with delays, the existence of solutions or global attractors has been studied widely in [3–5] and their qualitative theory has also been well-established. For autonomous systems with variable bounded or unbounded delays, the classical theory extended in [6–13] has been applied to deal with the existence of solution and special attractors. In fact, autonomous systems with variable delays are non-autonomous in essence. Except that time-periodic equations can be dealt with classic theory relatively straightforward manner, the qualitative properties or asymptotic behavior of many general non-autonomous systems are analyzed by new ideas and methods. In recent years, non-autonomous diffusion equations have attracted much attention in mathematical literature. Duong [14] considered a class of flux-limited diffusions with external force and established the comparison and maximum principles. Jung et al. [15] considered the nonlinear singularly perturbed reaction–diffusion problems in the polygonal domain and proposed a boundary layer analysis which fits a domain with corners.

For the reaction–diffusion systems with finite delays, there are also a sires of work [11, 16, 17]. More recently, Wang et al. [10] proved the existence of pullback attractors in the weighted space $C_{\gamma,H^1(\Omega)}$ for the multi-value process generated by (1) based on the concept of the Kuratowski measure of the noncompactness of a bounded set, where the growth of nonlinear term $F(x,v)$ and $G(x,s,v)$ are both linear, and the non-autonomous term $g(t,x) \in L^2_{\mathrm{loc}}(\mathbb{R}; L^2(\Omega))$ satisfies

$$\sup_{\tau \le t} e^{-\eta\tau} \int_{-\infty}^{\tau} \|g(s)\|_{L^2(\Omega)}^2 e^{\eta s}\, ds < +\infty, \quad \forall \eta \in \mathbb{R}, \eta > 0. \tag{7}$$

In the present paper, we will prove the existence of solution and the pullback attractors of (1) in the bounded domain of $C_{\gamma,L^r(\Omega)}$ or $C_{\gamma,W^{1,r}(\Omega)}$ under the conditions (2)–(6) for $r \ge 2$.

The main work of this paper contains three issues. Since the space $L^r(\Omega)$ $(r > 2)$ loses the inner product and orthogonality, canonical projector and approximation methods [10] are both ineffective to prove the existence of solutions and pullback attractors of (1). In order to overcome this difficulty, we adopt the idea of [17] and decompose (1) into two equations to separate the non-autonomous term to establish well-posedness (see Theorem 3.7 and Theorem 3.10). In addition we investigate the existence of pullback absorbing set by using the approximation technique of [9, 10] to overcome difficulties stemming from infinite delays and infinite dimensions. Consequently, for verifying the asymptotic compactness of (1) in $C_{\gamma,L^r(\Omega)}$ ($r > 2$), we employ the weak continuous semigroup theory and finite dimensional approximation method in [16, 18] to construct compact embedding results (see Theorem 5.6). Moreover, by improving smooth effect of the semigroup e^{At}, we prove the dissipativity and the existence of pullback attractors for (1) in $C_{\gamma,W^{1,r}(\Omega)}$ (see Lemma 6.1).

The paper is organized as follows. Section 2 gives some preliminaries concerning the definitions of processes and the pullback attractors of non-autonomous dynamical systems. We also give the definition of ω-limit compact and a suitable non-autonomous frameworks for the discussion of attractors in the future. In Sect. 3, we consider the well-posedness of (1) in $C_{\gamma,L^r(\Omega)}$ and $C_{W^{1,r}(\Omega)}$, respectively. In Sects. 4 and 6, we prove the existence of bounded absorbing sets in both spaces above. In Sects. 5 and 7, the existence of pullback attractors in $C_{\gamma,L^r(\Omega)}$ and $C_{\gamma,W^{1,r}(\Omega)}$ is proved.

2 Preliminaries

Let X be a complete metric space with metric $d_X(\cdot,\cdot)$. Denote by $H_X^*(\cdot,\cdot)$ the Hausdorff semi-distance between two nonempty subsets of a complete metric space X, which is defined by

$$H_X^*(A,B) = \sup_{a\in A}\inf_{b\in B} d_X(a,b).$$

Definition 2.1 A mapping $U(t,\tau): X \to X$, $t > \tau$ in $\mathbb{R}$, is called a process if

(1) $U(\tau,\tau)x = x$, $\forall \tau \in \mathbb{R}$, $x \in X$;

(2) $U(t,s)U(s,\tau)x = U(t,\tau)x$, $\forall \tau \le s \le t \in \mathbb{R}$, $x \in X$.

Definition 2.2 The Kuratowski measure $k(A)$ of noncompactness of the set A is defined by

$$k(A) = \inf\{\delta > 0 \mid A \text{ admits a finite cover by sets whose diameter} \le \delta\}.$$

Definition 2.3 Let $\{U(t,\tau)\}$ be a process on X. We say that $\{U(t,\tau)\}$ is

(1) pullback dissipative, if there exists a family of bounded sets $\mathcal{D} = \{D(t)\}_{t\in\mathbb{R}}$ in X so that, for any bounded set $B \subset X$ and each $t \in \mathbb{R}$, there exists a $S_0 = S_0(B,t) \in \mathbb{R}^+$ such that

$$U(t,t-s)B \subset D(t), \quad \forall s \ge S_0;$$

(2) $\mathcal{D}$-pullback ω-limit compact with respect to each $t \in \mathbb{R}$, if, for any $\varepsilon > 0$, there exists a $S_1 = S_1(\mathcal{D},t,\varepsilon) \in \mathbb{R}^+$ such that

$$k\left(\bigcup_{s\ge S_1} U(t,t-s)D(t-s)\right) \le \varepsilon.$$

Proposition 2.4 *If the process $\{U(t,\tau)\}$ is $\mathcal{D}$-pullback ω-limit compact in X, then $\{U(t,\tau)\}$ is pullback ω-limit compact for any bounded subset B of X.*

It follows from Theorem 3 of [10].

Definition 2.5 A family of nonempty compact subsets $A = \{A(t)\}_{t\in\mathbb{R}}$ of X is called to be a pullback attractor for the process $\{U(t,\tau)\}$ if

(1) $\mathcal{A} = \{A(t)\}_{t\in\mathbb{R}}$ is invariant, i.e.,

$$U(t,\tau)A(t) = A(t), \quad \forall t \geq \tau, \tau \in \mathbb{R};$$

(2) $\mathcal{A}$ is pullback attracting, i.e., for every bounded set B of X and any fixed $t \in \mathbb{R}$,

$$\lim_{s\to+\infty} H_X^*\bigl(U(t,t-s)B, A(t)\bigr) = 0.$$

Definition 2.6 Let $\{U(t,\tau)\}$ be a process on X. We say that $U(t,\tau)\zeta$ is norm-to-weak continuous in ζ for any fixed $t \geq \tau$, $\tau \in \mathbb{R}$, if there exists a sequence $\zeta_n \to \zeta$ in X and $t_n \to t$ such that $U(t_n,\tau)\zeta_n \rightharpoonup U(t,\tau)\zeta$ (weak convergence).

The general existence of pullback attractors has been given as follows [10].

Proposition 2.7 *Let X be a Banach space, and let $\{U(t,\tau)\}$ be a process on X. Let $U(t,\tau)\zeta$ is norm-to-weak continuous in x for fixed $t \geq \tau$, $\tau \in \mathbb{R}$. If, for any fixed $t \in \mathbb{R}$, $\forall T \in \mathbb{R}^+$, $\bigcup_{t\geq T} D(t)$ is bounded, the process $\{U(t,\tau)\}$ is pullback dissipative and $\mathcal{D}$-pullback ω-limit compact with respect to each $t \in \mathbb{R}$, then $\{U(t,\tau)\}$ possesses a pullback attractor in $\mathcal{A} = \{A(t)\}_{t\in\mathbb{R}}$ in X given by*

$$A(t) = \bigcap_{T\in\mathbb{R}^+} \overline{\bigcup_{s\geq T} U(t,t-s)D(t-s)} \subset D(t).$$

3 Existence of solutions

By a solution $u \in C((-\infty,T];X^1)$ of (1), we mean that, for any $T > 0$, $z \in (-\infty,0]$, $\tau < t \leq T$,

$$\begin{aligned} u(t) &= e^{\Delta(t-\tau)}u(\tau) + \int_\tau^t e^{\Delta(t-s)}\bigl[-\lambda u + f(x,u_s) + g(x,s)\bigr]\,ds, \\ &= e^{\Delta(t-\tau)}u(\tau) + \int_\tau^t e^{\Delta(t-s)}\bigl[-\lambda u + f\bigl(x,u(s+z)\bigr) + g(x,s)\bigr]\,ds, \end{aligned} \tag{8}$$

where $u(t) = \phi(t-\tau,x)$, $u(\tau) = \phi(0,x)$, $t \in (-\infty,\tau]$.

Let $A = \Delta$. X^α is the fractional power space associated to the operator Δ. The linear operator $A = \Delta$ with Dirichlet boundary conditions in a bounded and smooth domain Ω can be seen as an unbounded operator in $L^r(\Omega)$, $1 < r < \infty$, with domain $D(A) = W^{2,r}(\Omega) \cap W_0^{1,r}(\Omega)$. In this situation, $-A = -\Delta$ is a sectorial operator and generates an analytic semigroup e^{At} in $L^r(\Omega)$. Denote by $\{E_r^\alpha\}_{\alpha\in\mathbb{R}}$ the fractional power spaces associated to A with the norm $\|u\|_{E_r^\alpha} = \|(-A)^\alpha u\|_{L^r(\Omega)}$, $u \in E_r^\alpha$. Notice that $E_r^0 = L^r(\Omega)$ and $E_r^1 = W^{2,r}(\Omega) \cap W_0^{1,r}(\Omega)$. It follows from [19] that the semigroup e^{At} has the following smooth effect:

$$\bigl\|e^{At}x\bigr\|_{E_r^\beta} \leq t^{-(\beta-\alpha)}\|x\|_{E_r^\alpha}, \quad x \in E_r^\beta, t > 0, 0 \leq \alpha \leq \beta. \tag{9}$$

Since the embedding $E_r^1 \hookrightarrow E_r^0$ is compact, we know from Remark 6.1 of [20] that the resolvent of $-A$ is compact, and the embedding $E_r^\alpha \hookrightarrow E_r^\beta$ is continuous and compact for $\forall\alpha > \beta$.

3.1 Local existence of solutions for (1) in $C_{\gamma,L^r(\Omega)}$ ($1 < r < \infty$)

In order to apply Theorem 1 [18] to prove the existence of a solution for (1), we decompose system (1) into a linear system and a non-autonomous nonlinear system as follows, respectively:

$$\begin{cases} \frac{\partial v}{\partial t} - \Delta v = g(t,x) & \text{in } [\tau, +\infty] \times \Omega, \\ v|_{\partial\Omega} = 0, & t > \tau, \\ v(t,x) = 0, & \tau \in \mathbb{R}, t \in (-\infty, \tau], x \in \Omega, \end{cases} \tag{10}$$

and

$$\begin{cases} \frac{\partial w}{\partial t} - \Delta w = \tilde{f}(x, w_t) + f_1(w) & \text{in } [\tau, +\infty] \times \Omega, \\ w|_{\partial\Omega} = 0, & t > \tau, \\ w(t,x) = \phi(t-\tau, x), & \tau \in \mathbb{R}, t \in (-\infty, \tau], x \in \Omega, \end{cases} \tag{11}$$

where $\tilde{f}(x, w_t) = f(x, w_t + v_t)$, $f_1(w) = -\lambda(w + v)$, $u_t = v_t + w_t$.

Lemma 3.1 ([21]) *For any $\tau \le t_1 < t_2$, $\frac{1}{p} + \frac{1}{q} = 1$,*

$$\left\| \int_{t_1}^{t_2} e^{A(t_2 - s)} g(x,s)\, ds \right\|_{L^r(\Omega)} \le \left\| g(x,t) \right\|_{L^p_{\text{loc}}(\mathbb{R}; L^r(\Omega))} (t_2 - t_1)^{\frac{1}{q}}.$$

Furthermore, Eq. (10) has a unique solution $v(t)$ in the sense of (8) such that

$$v(t) \in C\big([\tau, T_0 + \tau]; L^r(\Omega)\big)$$

satisfies

$$v(t) = \int_\tau^t e^{A(t-s)} g(x,s)\, ds, \tag{12}$$

where T_0 is chosen in Lemma 3.6 later.

Proof

$$\begin{aligned} &\left\| \int_{t_1}^{t_2} e^{A(t_2 - s)} g(x,s)\, ds \right\|_{L^r(\Omega)} \\ &\quad \le \int_{t_1}^{t_2} \left\| g(x,t) \right\|_{L^p_{\text{loc}}(\mathbb{R}; L^r(\Omega))} ds \\ &\quad \le \left(\int_{t_1}^{t_2} ds \right)^{\frac{1}{q}} \left(\int_{t_1}^{t_2} \left\| g(x,t) \right\|_{L^p_{\text{loc}}(\mathbb{R}; L^r(\Omega))} ds \right)^{\frac{1}{p}} \\ &\quad \le \left\| g(x,t) \right\|_{L^p_{\text{loc}}(t_1, t_2; L^r(\Omega))} (t_2 - t_1)^{\frac{1}{q}}. \end{aligned}$$

Note that we can choose $0 < t_2 - t_1 \le 1$. □

Lemma 3.2 *Assuming* (2)–(5) *hold, we have*

$$\left\|\tilde{f}(t,w_t)+f_1(w)\right\|_{X^1} \le C_3(\lambda+1)\left(1+\|w_t\|_{C_{\gamma,L^r(\Omega)}}\right), \tag{13}$$

$$\left\|\tilde{f}(t,w_t)-\tilde{f}(t,v_t)+f_1(w)-f_1(v)\right\|_{X^1} \le C_4(\lambda+1)\|w_t-v_t\|_{C_{\gamma,L^r(\Omega)}}, \tag{14}$$

where $w,v \in C((-\infty,T_0+\tau];L^r(\Omega))$, $t\in(\tau,T_0+\tau]$.

Proof Denote $X_r^\alpha := E_r^{\alpha-1}$, $\alpha\in\mathbb{R}$. Especially, $X_r^1 := L^r(\Omega)$. For any $u,\psi \in C((-\infty,T_0+\tau];L^r(\Omega))$ and any $t\in(\tau,T_0+\tau]$ we get

$$\begin{aligned}\left\|F(t,u_t)\right\|_{X^1} &\le C_5\left(\left\|k_1+k_2e^{-\gamma\rho(t)}u_t\right\|_{X^1}\right)\\ &\le C_5\left(k_1|\Omega|+k_2\|u_t\|_{C_{\gamma,L^r(\Omega)}}\right)\\ &\le C_5\left(1+\|u_t\|_{C_{\gamma,L^r(\Omega)}}\right)\end{aligned} \tag{15}$$

and

$$\begin{aligned}&\left\|\int_{-\infty}^0 G\big(t,z,u(t+z)\big)\,dz\right\|_{L^r(\Omega)}\\ &\quad\le \left\|\int_{-\infty}^0 \left(\left|m_0(z)\right|+m_1(z)\left|u(t+z)\right|\right)dz\right\|_{L^r(\Omega)}\\ &\quad\le m_0|\Omega|+m_1\|u_t\|_{C_{\gamma,L^r(\Omega)}}\\ &\quad\le C_6\left(1+\|u_t\|_{C_{\gamma,L^r(\Omega)}}\right).\end{aligned} \tag{16}$$

Combining with (15) and (16), for any $u,\psi\in C((\tau,T_0+\tau];X^1)$, we have

$$\left\|f(t,u_t)\right\|_{X^1} \le C_3\left(1+\|u_t\|_{C_{\gamma,L^r(\Omega)}}\right). \tag{17}$$

By (4) and (5), we find

$$\begin{aligned}&\left\|f(t,u_t)-f(t,\psi_t)\right\|_{X^1}\\ &\quad\le C_1e^{-\gamma\rho(t)}\left\|u\big(t-\rho(t)\big)-v\big(t-\rho(t)\big)\right\|_{L^r(\Omega)}+C_2\left\|\int_{-\infty}^0 m_1(z)|u_t-\psi_t|\,dz\right\|_{L^r(\Omega)}\\ &\quad\le C_3\|u_t-\psi_t\|_{C_{\gamma,L^r(\Omega)}},\end{aligned} \tag{18}$$

where C_3 and C_4 depend on (k_1,k_2,m_0,m_1,m_2). From (17) and (18), we obtain

$$\left\|\tilde{f}(t,w_t)\right\|_{X^1} \le C_3'\left(\|w_t\|_{C_{\gamma,L^r(\Omega)}}+1\right), \tag{19}$$

and

$$\begin{aligned}\left\|\tilde{f}(t,w_t)-\tilde{f}(t,v_t)\right\|_{L^r(\Omega)} &= \left\|f(t,w_t+v_t)-f(t,w_t+v_t)\right\|_{L^r(\Omega)}\\ &\le C_4'\|w_t-v_t\|_{C_{\gamma,L^r(\Omega)}}.\end{aligned} \tag{20}$$

Hence, (13) and (14) are obvious. □

Lemma 3.3 *If $u \in C((-\infty, T_0+\tau], L^r(\Omega))$, then, for all $t \in (\tau, T_0+\tau]$, $z \in (-\infty, 0]$, we have*

$$\left\| \int_\tau^t e^{A(t-s)} \big(f_1(w) + \tilde{f}(t, w_s)\big)\, ds \right\|_{L^r(\Omega)} \le C(\lambda+1)(t-\tau)\big(\omega(t)+1\big), \tag{21}$$

where

$$\omega(t) = \Big(\|\phi\|_{C_{\gamma,L^r(\Omega)}} + \sup_{\theta\in(\tau,t]} \big\| w(\theta) + v(\theta) \big\|_{L^r(\Omega)} \Big).$$

Proof By (9), it is not difficult to see that

$$\begin{aligned} &\left\| \int_\tau^t e^{A(t-s)} \tilde{f}(t, w_s)\, ds \right\|_{L^r(\Omega)} \\ &\quad \le C(\lambda+1) \int_\tau^t \big(1 + \|w_s + v_s\|_{C_{\gamma,L^r(\Omega)}}\big)\, ds \\ &\quad \le C(\lambda+1) \int_\tau^t \Big(\|\phi\|_{C_{\gamma,L^r(\Omega)}} + \sup_{\theta\in(\tau,s]} \big\| w(\theta) + v(\theta) \big\|_{L^r(\Omega)} \Big)\, ds + C(\lambda+1)(t-\tau) \\ &\quad \le C(\lambda+1)(t-\tau)\omega(t) + C(\lambda+1)(t-\tau). \end{aligned} \tag{22}$$

□

Lemma 3.4 *For any $t \in (\tau, T_0+\tau]$, $z \in (-\infty, 0]$ and any $w, v \in C((-\infty, T_0+\tau], L^r(\Omega))$ be such that $(t-\tau)\|w_t\|_{C_{\gamma,L^r(\Omega)}} \le \mu$, $(t-\tau)\|v_t\|_{C_{\gamma,L^r(\Omega)}} \le \mu$, for some $\mu > 0$. Then we have*

$$\begin{aligned} &\left\| \int_\tau^t e^{A(t-s)} \big[\big(\tilde{f}(s, w_s) - \tilde{f}(s, v_s)\big) + \big(f_1\big(w(s)\big) - f_1\big(v(s)\big)\big) \big]\, ds \right\|_{L^r(\Omega)} \\ &\quad \le C(1+\lambda)(t-\tau) \sup_{\theta\in(\tau,t]} \big\| w(\theta) - v(\theta) \big\|_{L^r(\Omega)}. \end{aligned} \tag{23}$$

Proof

$$\begin{aligned} &\left\| \int_\tau^t e^{A(t-s)} \big[\big(\tilde{f}(s, w_s) - \tilde{f}(s, v_s)\big) + \big(f_1\big(w(s)\big) - f_1\big(v(s)\big)\big) \big]\, ds \right\|_{L^r(\Omega)} \\ &\quad \le C(1+\lambda) \int_\tau^t \|w_s - v_s\|_{C_{\gamma,L^r(\Omega)}}\, ds \\ &\quad \le C(1+\lambda)(t-\tau) \sup_{\theta\in(\tau,t]} \big\| w(\theta) - v(\theta) \big\|_{L^r(\Omega)}. \end{aligned} \tag{24}$$

□

Lemma 3.5 ([22]) *Assume $u : (-\infty, T_0) \to X$ is continuous and $u_\tau = \phi$. If there exists a nondecreasing function $m(t) \ge 0$ such that*

$$\big\|u(t)\big\|_X \le \big\|\phi(\tau)\big\|_X + m(t), \quad \textit{for all } -\infty < t \le T_0,$$

then

$$\sup_{z\in(-\infty,0]} e^{\gamma z} \big\|u(t+z)\big\|_X \le \sup_{z\in(-\infty,0]} e^{\gamma z} \big\|\phi(t+z)\big\|_X + m(t), \quad -\infty < t \le T_0. \tag{25}$$

Lemma 3.6 *Assume (2)–(6) hold. Let $1 < r < \infty$, $z \in (-\infty,0]$. For any $\chi_\tau \in C((-\infty,0]; L^r(\Omega))$, there exist $R(\chi_\tau) > 0$ and $T_0 = T_0(\chi_\tau)$ with the property that, for any $\phi \in B_{C_{\gamma,L^r(\Omega)}}(\chi_\tau,R)$, there exists a continuous function $w(\cdot;\phi(0))$ with $w_\tau = \phi$:*

$$w \in C\big([\tau, T_0+\tau]; L^r(\Omega)\big) \tag{26}$$

such that, for any $t \in [\tau, T_0+\tau]$, w is the unique solution of Eq. (11) in the sense of (8). This solution is a classical solution and for any $t \in (\tau, T_0+\tau]$, satisfies

$$w_t \in C\big((-\infty,0]; L^r(\Omega)\big) \tag{27}$$

and

$$\lim_{t\to\tau^+}(t-\tau)\sup_{z\in(-\infty,0]} e^{\gamma z}\big\|w(t+z,\phi)\big\|_{L^r(\Omega)} = 0, \tag{28}$$

and, moreover, if $\phi_1, \phi_2 \in B_{C_{\gamma,L^r(\Omega)}}(\chi_\tau,R)$ then

$$\sup_{z\in(-\infty,0]} e^{\gamma z}\big\|w(t+z,\phi_1) - v(t+z,\phi_2)\big\|_{L^r(\Omega)} \le M_1 e^{M_2(t-\tau)}\|\phi_1-\phi_2\|_{C_{\gamma,L^r(\Omega)}}. \tag{29}$$

Furthermore, the time of existence is uniform on any bounded set (respectively, compact set) S of $C_{\gamma,L^r(\Omega)}$.

Proof Fix $\mu > 0$ and for any $\tau \in \mathbb{R}$, $\forall t \in (-\infty,\tau]$, let $\|\phi\|_{C_{\gamma,L^r(\Omega)}} \le \mu$. We will use the contraction mapping principle to establish the existence of a solution for (11).

Let

$$K(T_0) = \Big\{w \in C\big((-\infty, T_0+\tau]; L^r(\Omega)\big), t \in (\tau, T_0+\tau] : \sup_{t\in(\tau,T_0+\tau]}\big\|w(t)\big\|_{L^r(\Omega)} \le \mu+1\Big\},$$

with the norm

$$\|w\|_{K(T_0)} = \sup_{t\in(\tau,T_0+\tau]}\big\|w(t)\big\|_{L^r(\Omega)},$$

where T_0 is determined later. So that $(K, \|\cdot\|)$ is a nonempty complete metric space. For each $t \in (\tau, T_0+\tau]$, we introduce the mapping

$$\begin{aligned} &\Phi : K(T_0) \to C\big((-\infty, T_0+\tau]; X^1\big), \\ &\Phi(w)(t) = \begin{cases} e^{\Delta(t-\tau)}w(\tau) + \int_\tau^t e^{\Delta(t-s)}[f_1(w) + \tilde{f}(s,w_s)]\,ds, & t > \tau, \\ w(t,x) = \phi(t-\tau,x), & t \in (-\infty,\tau]. \end{cases} \end{aligned} \tag{30}$$

Let us first prove that Φ is a well-defined map and $\Phi(K(T_0)) \subset K(T_0)$. We start by showing that

$$\text{if } w \in K(T_0), \text{then } \Phi(w) \in C\big((-\infty, T_0+\tau]; L^r(\Omega)\big). \tag{31}$$

Fixing $t_2 \in (\tau, T_0 + \tau]$, and letting $T_0 + \tau \geq t_1 > t_2$, then we have

$$\begin{aligned}
&\big\|(\Phi w)(t_1) - (\Phi w)(t_2)\big\|_{L^r(\Omega)} \\
&\quad \leq \big\|\big(e^{-A(t_1)} - e^{-A(t_2)}\big)w(\tau)\big\|_{L^r(\Omega)} + \bigg\|\int_{t_2}^{t_1} e^{A(t_1-s)}\tilde{f}(s, w_s)\,ds\bigg\|_{L^r(\Omega)} \\
&\qquad + \bigg\|\int_{t_2}^{t_1} e^{A(t_1-s)}f_1\big(w(s)\big)\,ds\bigg\|_{L^r(\Omega)} + \bigg\|\big[I - e^{-A(t_1-t_2)}\big]\int_{\tau}^{t_2} e^{A(t_2-s)}\tilde{f}(s, w_s)\,ds\bigg\|_{L^r(\Omega)}.
\end{aligned}$$

In the above, the first and fourth term trivially go to zero as $t_1 \to t_2$. Let us consider the second term. For this term we have

$$\begin{aligned}
&\bigg\|\int_{t_2}^{t_1} e^{A(t_1-s)}\tilde{f}(s, w_s)\,ds\bigg\|_{L^r(\Omega)} \\
&\quad \leq C\int_{t_2}^{t_1}\big(1 + \|w_s + v_s\|_{C_{\gamma,L^r(\Omega)}}\big)\,ds \\
&\quad \leq C\Big(\|\phi\|_{C_{\gamma,L^r(\Omega)}} + \sup_{s\in(\tau,t_1]}\big\|w(s) + v(s)\big\|_{L^r(\Omega)}\Big)(t_1 - t_2) + C(t_1 - t_2) \\
&\quad \leq C\omega(t)(t_1 - t_2) + C(t_1 - t_2),
\end{aligned}$$

which goes to zero as $t_1 \to t_2^+$. Similarly, the third term also goes to zero as $t_1 \to t_2^+$. The case $t_1 < t_2$ is similar.

Let us now show that $\|\Phi(w)(t)\|_{L^r(\Omega)} \leq \mu + 1$, for all $t \in (\tau, T_0 + \tau]$. For $\chi_\tau \in C((-\infty, 0]; L^r(\Omega))$ fixed, choose $r \ll 1$ and $T_0 \leq \frac{1-r}{C(\lambda+1)(1+\omega(t))}$ such that, for any $t \in (\tau, T_0 + \tau]$, by (9), we have $\|e^{A(t-\tau)}\chi_\tau\|_{L^r(\Omega)} \leq \mu$, and $\|e^{A(t-\tau)}r\|_{L^r(\Omega)} \leq r$.

Based on the above fact, we have

$$\begin{aligned}
&\big\|\Phi(w)(t)\big\|_{L^r(\Omega)} \\
&\quad \leq \big\|e^{-A(t-\tau)}w(\tau)\big\|_{L^r(\Omega)} + C(\lambda + 1)(t - \tau) + C(1 + \lambda)(t - \tau)\int_{\tau}^{t}\|w_s\|_{C_{L^r(\Omega)}}\,ds \\
&\quad \leq \big\|e^{-A(t-\tau)}r\big\|_{C_{\gamma,L^r(\Omega)}} + \big\|e^{-A(t-\tau)}\chi_\tau\big\|_{C_{\gamma,L^r(\Omega)}} + C(\lambda + 1)(t - \tau)\big(1 + \omega(t)\big) \\
&\quad \leq r + \|\chi_\tau\|_{C_{\gamma,L^r(\Omega)}} + C(\lambda + 1)(t - \tau)\big(1 + \omega(t)\big) \\
&\quad \leq \mu + r + C(\lambda + 1)(t - \tau)\big(1 + \omega(t)\big).
\end{aligned}$$

On the other hand, it follows from Lemma 3.3 that Φ is a strict contraction in $K(T_0)$ and that

$$\big\|\Phi(w) - \Phi(v)\big\|_{K(T_0)} \leq C(\lambda + 1)(t - \tau)\omega(t)\|w - v\|_{K(T_0)}, \quad t \in [\tau, T_0 + \tau].$$

The simple computations above suggest that we can choose T_0 small enough so that the map Φ is contraction from $K(T_0)$ into itself. By the Banach contraction principle we see that Φ has a unique fixed point in $K(T_0)$. We will denote this fixed point by $w(t,\phi)$ for $t \in (\tau, T_0 + \tau]$, $\phi \in C((-\infty, 0], L^r(\Omega))$, and it is defined for $\|\phi - \chi_\tau\|_{C_{\gamma,L^r(\Omega)}} \leq \rho$. Note that from (31) $w(t,\phi) \in C((-\infty, T_0 + \tau]; L^r(\Omega))$.

Let us prove that $(t - \tau)\|w_t\|_{C_{\gamma,L^r(\Omega)}} \to 0$ as $t \to \tau^+$.

From Lemma 3.3,

$$\begin{aligned}
&(t-\tau)\big\|w(t)\big\|_{L^r(\Omega)}\\
&\quad\le (t-\tau)\big\|e^{A(t-\tau)}\phi(0)\big\|_{L^r(\Omega)}+(t-\tau)\int_\tau^t\big\|e^{A(t-s)}\big(f_1(w)+\tilde{f}(s,w_s)\big)\big\|_{L^r(\Omega)}\,ds\\
&\quad\le (t-\tau)\big\|\phi(0)\big\|_{L^r(\Omega)}+C(1+\lambda)(t-\tau)\int_\tau^t\big(1+\|w_s\|_{C_{\gamma,L^r(\Omega)}}\big)\,ds\\
&\qquad+C(1+\lambda)(t-\tau)\|v_s\|_{C_{\gamma,L^r(\Omega)}}.
\end{aligned}$$

By Lemma 3.5, we obtain

$$\begin{aligned}
&(t-\tau)\|w_t\|_{C_{\gamma,L^r(\Omega)}}\\
&\quad\le (t-\tau)\|\phi\|_{C_{\gamma,L^r(\Omega)}}+C(1+\lambda)(t-\tau)\int_\tau^t\|w_s\|_{C_{\gamma,L^r(\Omega)}}+C(1+\lambda)(t-\tau).
\end{aligned}$$

Thus by the Gronwall inequality, we have

$$\begin{aligned}
&(t-\tau)\|w_t\|_{C_{\gamma,L^r(\Omega)}}\\
&\quad\le (t-\tau)\|\phi\|_{C_{\gamma,L^r(\Omega)}}+C(1+\lambda)(t-\tau)\\
&\qquad+\big(\|\phi\|_{C_{\gamma,L^r(\Omega)}}+C(1+\lambda)\big)(t-\tau)C(1+\lambda)\int_\tau^t\exp\big(C(1+\lambda)\big)(t-s)\,ds\\
&\quad\le \big(\|\phi\|_{C_{\gamma,L^r(\Omega)}}+C(1+\lambda)\big)(t-\tau)\\
&\qquad+C(1+\lambda)\big(\|\phi\|_{C_{\gamma,L^r(\Omega)}}+C(1+\lambda)\big)(t-\tau)^2\exp\big(C(1+\lambda)(t-\tau)\big)\overset{t\to\tau^+}{\to}0.
\end{aligned}$$

Moreover, if $\forall\phi_1,\phi_2\in B_{C_{\gamma,L^r(\Omega)}}(\chi_\tau,r)$, taking into account the estimates of Lemma 3.3 and our choice of T_0, we have

$$\begin{aligned}
&\big\|w\big(t,\phi_1(0)\big)-v\big(t,\phi_2(0)\big)\big\|_{L^r(\Omega)}\\
&\quad\le \big\|e^{A(t-\tau)}\big(\phi_1(0)-\phi_2(0)\big)\big\|_{L^r(\Omega)}\\
&\qquad+\left\|\int_\tau^t e^{A(t-s)}\big[\tilde{f}(s,w_s)-\tilde{f}(s,v_s)+f_1(w)-f_1(v)\big]\,ds\right\|_{L^r(\Omega)}\\
&\quad\le \big\|(\phi_1-\phi_2)\big\|_{C_{\gamma,L^r(\Omega)}}+C(1+\lambda)\int_\tau^t\|w_s-v_s\|_{C_{L^r(\Omega)}}\,ds\\
&\quad\le \big\|(\phi_1-\phi_2)\big\|_{C_{\gamma,L^r(\Omega)}}+C(1+\lambda)(t-\tau)\big\|(\phi_1-\phi_2)\big\|_{C_{\gamma,L^r(\Omega)}}\\
&\qquad+C(1+\lambda)\int_\tau^t\sup_{\theta\in(\tau,s]}\big\|w(\theta)-v(\theta)\big\|_{L^r(\Omega)}\,ds.
\end{aligned}$$

By Lemma 3.5, we have

$$\begin{aligned}
&\sup_{\theta\in(\tau,t]}\big\|w\big(t,\phi_1(0)\big)-v\big(t,\phi_2(0)\big)\big\|_{L^r(\Omega)}\\
&\quad\le \big(1+C(1+\lambda)(t-\tau)\big)e^{C(1+\lambda)(t-\tau)}\big\|(\phi_1-\phi_2)\big\|_{C_{\gamma,L^r(\Omega)}}.
\end{aligned}$$

Furthermore,

$$\begin{aligned}
&\big\|w_t(\cdot,\phi_1)-v_t(\cdot,\phi_2)\big\|_{C_{\gamma,L^r(\Omega)}}\\
&\quad\le\big(1+C(1+\lambda)(t-\tau)\big)\big\|(\phi_1-\phi_2)\big\|_{C_{\gamma,L^r(\Omega)}}e^{C(1+\lambda)(t-\tau)}\\
&\quad\le M_1(t-\tau)\big\|(\phi_1-\phi_2)\big\|_{C_{\gamma,L^r(\Omega)}}e^{M_1(t-\tau)},
\end{aligned}$$

where $M_1 = 1 + C(1 + \lambda)$.

This concludes the existence of the theorem. Notice that, from the existence part, we see that, for any $\phi \in B_{C_{\gamma,L^r(\Omega)}}(\chi_\tau, R)$, there exists a unique solution in the sense of (8), defined in $[\tau, T_0 + \tau]$. The uniqueness of solutions for Eq. (11) is proved. □

Theorem 3.7 *Assume* (2)–(6) *hold. Let* $1 < r < \infty$, $g \in L^r_{\text{loc}}(\mathbb{R}; L^r(\Omega))$ $(r > 1)$, $z \in (-\infty, 0]$. *If* $\nu_\tau \in C((-\infty, 0]; L^r(\Omega))$, *there exist* $0 < R(\nu_\tau) \le R(\chi_\tau)$ *and* $T_0(\nu_\tau) \le T_0(\chi_\tau)$ *with the property that, for any* $\phi \in B_{C_{\gamma,L^r(\Omega)}}(\nu_\tau, R)$, *there exists a continuous function* $u(\cdot;\phi(0))$ *with* $u_\tau = \phi$:

$$u \in C\big([\tau, T_0 + \tau]; L^r(\Omega)\big), \tag{32}$$

which is the unique solution of (1) *in the sense of* (8). *This solution is a classical solution and* $\forall t \in (\tau, T_0 + \tau]$ *it satisfies*

$$u_t \in C\big((-\infty, 0]; L^r(\Omega)\big) \tag{33}$$

and

$$\lim_{t\to\tau^+}(t-\tau)\sup_{z\in(-\infty,0]}e^{\gamma z}\big\|u(t+z,\phi)\big\|_{L^r(\Omega)} = 0; \tag{34}$$

if $\forall \phi_1, \phi_2 \in B_{\gamma,L^r(\Omega)}(\upsilon_\tau, r)$, *then*

$$\sup_{z\in(-\infty,0]}e^{\gamma z}\big\|u_1(t+z,\phi_1)-u_2(t+z,\phi_2)\big\|_{L^r(\Omega)} \le M_1(t-\tau)e^{M_1(t-\tau)}\|\phi_1-\phi_2\|_{C_{\gamma,L^r(\Omega)}}. \tag{35}$$

Furthermore, the time of existence is uniform on any bounded set (*respectively, compact set*) *S of* $C_{\gamma,L^r(\Omega)}$.

Proof By Lemma 3.1 and Lemma 3.6, Eq. (1) has a unique solution $u \in C((-\infty, T_0]; L^r(\Omega))$ satisfying (33)–(35). □

3.2 Local existence of solutions of (1) in $C_{\gamma,W^{1,r}(\Omega)}$ $(1 < r < N)$

Lemma 3.8 ([21]) *For any* $t_1 < t_2$, $0 < \frac{1}{q} - \frac{1}{2}$, *where* $\frac{1}{r} + \frac{1}{q} = 1$, *we have*

$$\begin{aligned}
&\left\|\int_{t_1}^{t_2}e^{A(t_2-s)}g(x,s)\,ds\right\|_{W^{1,r}(\Omega)}\\
&\quad\le\left(\frac{1}{1-\frac{q}{2}}\right)^{\frac{1}{q}}\big\|g(x,t)\big\|_{L^r_b(t_1,t_2;L^r(\Omega))}(t_2-t_1)^{\frac{1}{q}-\frac{1}{2}}.
\end{aligned}$$

Furthermore, Eq. (10) *has a unique solution $v(t)$ in the sense of* (8) *such that*

$$v(t) \in C\bigl([\tau, T_0]; W^{1,r}(\Omega)\bigr) \cap C\bigl([\tau, T_0 + \tau]; W^{2,r}(\Omega)\bigr)$$

satisfies

$$v(t) = \int_\tau^t e^{A(t-s)} g(x,s)\,ds. \tag{36}$$

Proof We have

$$\begin{aligned}
&\left\| \int_{t_1}^{t_2} e^{A(t_2-s)} g(x,s)\,ds \right\|_{W^{1,r}(\Omega)} \\
&\quad \le \left\| \int_{t_1}^{t_2} (t_2 - s)^{-\frac{1}{2}} g(x,s)\,ds \right\|_{L^r(\Omega)} \\
&\quad \le \left(\int_{t_1}^{t_2} (t_2 - s)^{-\frac{q}{2}}\,ds \right)^{\frac{1}{q}} \left(\int_{t_1}^{t_2} \bigl\| g(x,s) \bigr\|^r_{L^r_b(\mathbb{R};L^r(\Omega))}\,ds \right)^{\frac{1}{r}} \\
&\quad \le \left(\frac{1}{1 - \frac{q}{2}} \right)^{\frac{1}{q}} \bigl\| g(x,t) \bigr\|_{L^r_{\mathrm{loc}}(t_1,t_2;L^r(\Omega))} (t_2 - t_1)^{\frac{1}{q}-\frac{1}{2}}.
\end{aligned}$$

□

Lemma 3.9 *Assume* (2)–(6) *hold. Let $1 < r < N$, $z \in (-\infty, 0]$. If $\chi_\tau \in C((-\infty, 0]; W^{1,r}(\Omega))$, there exist $R(\chi_\tau) > 0$ and $T_0(\chi_\tau) > 0$ with the property that $\forall t \in (-\infty, \tau)$ for any $\phi \in B_{C_{\gamma,W^{1,r}(\Omega)}}(\chi_\tau, R)$, there exists a continuous function $w(\cdot; \phi(0))$ with $w_\tau = \phi$:*

$$w \in C\bigl([\tau, T_0 + \tau]; W^{1,r}(\Omega)\bigr), \tag{37}$$

which is the unique solution of (11) *in the sense of* (8). *This solution is a classical solution and $\forall t \in (\tau, T_0 + \tau]$, $z \in (-\infty, 0]$, satisfies*

$$w_t \in C\bigl((-\infty, 0]; W^{1,r}(\Omega)\bigr) \tag{38}$$

and

$$\lim_{t \to \tau^+} (t - \tau) \sup_{z \in (-\infty, 0]} e^{\gamma z} \bigl\| w(t + z, \phi) \bigr\|_{W^{1,r}(\Omega)} = 0, \tag{39}$$

and if $\phi_1, \phi_2 \in B_{C_{\gamma,W^{1,r}(\Omega)}}(\chi_\tau, R)$, then

$$\sup_{z \in (-\infty, 0]} e^{\gamma z} \bigl\| w(t + z, \phi_1) - v(t + z, \phi_2) \bigr\|_{W^{1,r}(\Omega)} \le M_1 T_0 e^{M_1(t-\tau)} \|\phi_1 - \phi_2\|_{C_{\gamma,W^{1,r}(\Omega)}}. \tag{40}$$

Furthermore, the time of existence is uniform on any bounded set (*respectively, compact set*) *S of $C_{\gamma,W^{1,r}(\Omega)}$.*

Proof For $\forall t \in (\tau, T_0 + \tau]$, $z \in (-\infty, 0]$ and any $w, v \in C((-\infty, T_0 + \tau]; W^{1,r}(\Omega))$, using (2),(3), we obtain (13) and (14). The remaining part of the proof is similar to Lemma 3.6.

□

Theorem 3.10 *Assume* (2)–(6) *hold. Let* $1 < r < \infty$, $r > 1$, $z \in (-\infty, 0]$. *If* $\nu_\tau \in C((-\infty,0]; W^{1,r}(\Omega))$, *there exist* $0 < R(\nu_\tau) \leq R(\chi_\tau)$ *and* $T_0(\nu_\tau) \leq T_0(\chi_\tau)$ *with the property that for any* $\phi \in B_{C_{\gamma,W^{1,r}(\Omega)}}(\nu_\tau, R)$, *there exists a continuous function* $u(\cdot;\phi(0))$ *with* $u_\tau = \phi$:

$$u \in C\big([\tau, T_0 + \tau]; W^{1,r}(\Omega)\big), \tag{41}$$

which is the unique solution of (11) *in the sense of* (8). *This solution is a classical solution and* $\forall t \in [\tau, T_0 + \tau]$ *it satisfies*

$$u_t \in C\big((-\infty,0]; W^{1,r}(\Omega)\big), \qquad \lim_{t\to\tau^+}(t-\tau) \sup_{z\in(-\infty,0]} e^{\gamma z}\big\|u(t+z,\phi)\big\|_{W^{1,r}(\Omega)} = 0, \tag{42}$$

and if $\phi_1, \phi_2 \in B_{C_{\gamma,W^{1,r}(\Omega)}}(\nu_\tau, R)$, *then*

$$\begin{aligned} &\sup_{z\in(-\infty,0]} e^{\gamma z}\big\|u(t+z,\phi_1) - u(t+z,\phi_2)\big\|_{W^{1,r}(\Omega)} \\ &\quad \leq M_1(t-\tau)e^{M_1(t-\tau)}\|\phi_1 - \phi_2\|_{C_{\gamma,W^{1,r}(\Omega)}}. \end{aligned} \tag{43}$$

Furthermore, the time of existence is uniform on any bounded set (*respectively, compact set*) *S of* $C_{\gamma,W^{1,r}(\Omega)}$.

Proof It follows from Lemmas 3.8 and 3.9. The proof is similar to Theorem 3.7. Here we omit the details. □

4 Uniform estimates in $C_{\gamma,L^r(\Omega)}$

Lemma 4.1 *Assume that* (2), (3), *and* (6) *hold*, $g \in L^r_{\mathrm{loc}}(\mathbb{R}; L^r(\Omega))$, *and there exists a positive constant* α *such that*

$$\big(\lambda - (\varepsilon_2 + m_1 + \varepsilon_4)(r-1) - \alpha\big) > 0 \tag{44}$$

and

$$L := \left(m_1 + \frac{2^r k_2^r}{\lambda^{(r-1)}}\right) < \alpha \leq r\gamma. \tag{45}$$

Then, for any initial data $\phi \in C_{\gamma,L^r(\Omega)}$, *any solution* u_t *of Eq.* (1) *satisfies*

$$\begin{aligned} \|u_t\|^r_{C_{\gamma,L^r(\Omega)}} \leq{}& re^{\alpha\tau}e^{-\alpha t}\|\phi\|^r_{C_{\gamma,L^r(\Omega)}} + \frac{\alpha}{\alpha - L}C_\Omega + \varepsilon_4^{-(r-1)}e^{-\alpha t}\int_{-\infty}^t e^{\alpha s}\big\|g(s)\big\|^r_{L^r(\Omega)}\,ds \\ &+ re^{(\alpha-L)\tau}e^{(L-\alpha)t}\|\phi\|^r_{C_{\gamma,L^r(\Omega)}} \\ &+ \varepsilon_4^{-(r-1)}e^{(L-\alpha)t}\int_{-\infty}^t \big(e^{(\alpha-L)s}\big\|g(s)\big\|^r_{L^r(\Omega)}\big)\,ds, \end{aligned} \tag{46}$$

where ε_2, ε_4 *will be determined later on.*

Proof Multiplying (1) by $|u(t)|^{r-2}u(t)$ and integrating by parts, we get

$$\frac{1}{r}\frac{d}{dt}\big\|u(t)\big\|^r_{L^r(\Omega)} + \frac{4(r-1)}{r^2}\int_\Omega\big|\nabla\big(\big|u(t)\big|^{\frac{r}{2}}\big)\big|^2\,dx + \int_\Omega \lambda\big|u(t)\big|^r\,dx$$

$$= \int_\Omega F\big(t, u\big(x, t-\rho(t)\big)\big)\big|u(t)\big|^{r-2}u(t)\,dx + \int_\Omega\int_{-\infty}^0 G\big(t,s,u(t+s)\big)\big|u(t)\big|^{r-2}u(t)\,ds\,dx$$
$$+ \int_\Omega g(t,x)\big|u(t)\big|^{r-2}u(t)\,dx. \tag{47}$$

We fix two positive parameters ε_1 and ε_4 that will be chosen later. Then, by assumptions (2), (6) and Young's inequality, we have

$$\begin{aligned}
&\int_\Omega F\big(t, u\big(x, t-\rho(t)\big)\big)|u|^{r-2}u\,dx \\
&\quad\le \int_\Omega \big|F\big(t, u\big(x, t-\rho(t)\big)\big)\big|\big|u(t)\big|^{(r-1)}\,dx \\
&\quad\le \frac{2^r\varepsilon_1^{-(r-1)}}{r}|k_1|^r|\Omega|^r + \frac{2^r\varepsilon_1^{-(r-1)}}{r}k_2^r\|u_t\|^r_{C_{\gamma,L^r(\Omega)}} + \varepsilon_1\left(\frac{r-1}{r}\right)\big\|u(t)\big\|^r_{L^r(\Omega)}
\end{aligned} \tag{48}$$

and

$$\begin{aligned}
\int_\Omega g(t,x)\big|u(t)\big|^{r-2}u(t)\,dx &\le \int_\Omega \big|g(t,x)\big|\big|u(t)\big|^{(r-1)}\,dx \\
&\le \frac{\varepsilon_4^{-(r-1)}}{r}\big\|g(t)\big\|^r_{L^r(\Omega)} + \varepsilon_4\left(\frac{r-1}{r}\right)\big\|u(t)\big\|^r_{L^r(\Omega)}.
\end{aligned} \tag{49}$$

Therefore

$$\begin{aligned}
&\frac{d}{dt}\big\|u(t)\big\|^r_{L^r(\Omega)} + \frac{4(r-1)}{r}\int_\Omega\big|\nabla\big(\big|u(t)\big|^{\frac{r}{2}}\big)\big|^2\,dx + \big(r\lambda - (\varepsilon_1+\varepsilon_4)(r-1)\big)\big\|u(t)\big\|^r_{L^r(\Omega)}\,dx \\
&\quad\le \varepsilon_1^{-(r-1)}\big(k_1|\Omega|^r + k_2^r\|u_t\|^r_{C_{\gamma,L^r(\Omega)}}\big) + r\int_\Omega\int_{-\infty}^0 G\big(t,s,u(t+s)\big)\big|u(t)\big|^{r-2}u(t)\,ds\,dx \\
&\qquad + \varepsilon_4^{-(r-1)}\big\|g(t)\big\|^r_{L^r(\Omega)}.
\end{aligned} \tag{50}$$

Let $\alpha > 0$, it will also be determined later. Then

$$\begin{aligned}
&\frac{d}{dt}\big(e^{\alpha t}\big\|u(t)\big\|^r_{L^r(\Omega)}\big) \\
&\quad= \alpha e^{\alpha t}\big\|u(t)\big\|^r_{L^r(\Omega)} + e^{\alpha t}\frac{d}{dt}\big\|u(t)\big\|^r_{L^r(\Omega)} \\
&\quad\le -\frac{4(r-1)}{r}e^{\alpha t}\int_\Omega\big|\nabla\big(\big|u(t)\big|^{\frac{r}{2}}\big)\big|^2\,dx - \big(r\lambda - (\varepsilon_1+\varepsilon_4)(r-1) - \alpha\big)e^{\alpha t}\big\|u(t)\big\|^r_{L^r(\Omega)} \\
&\qquad + \varepsilon_1^{-(r-1)}e^{\alpha t}|k_1|^r|\Omega|^r + \varepsilon_1^{-(r-1)}e^{\alpha t}k_2^r\|u_t\|^r_{C_{\gamma,L^r(\Omega)}} + \varepsilon_4^{-(r-1)}e^{\alpha t}\big\|g(t)\big\|^r_{L^r(\Omega)} \\
&\qquad + re^{\alpha t}\int_\Omega\int_{-\infty}^0 G\big(t,s,u(t+s)\big)\big|u(t)\big|^{r-2}u(t)\,ds\,dx.
\end{aligned} \tag{51}$$

Integrating from τ to t, we have

$$\begin{aligned}
&e^{\alpha t}\big\|u(t)\big\|^r_{L^r(\Omega)} \\
&\quad\le e^{\alpha\tau}\big\|u(\tau)\big\|^r_{L^r(\Omega)} - \int_\tau^t\big(r\lambda - (\varepsilon_1+\varepsilon_4)(r-1) - \alpha\big)e^{\alpha s}\big\|u(s)\big\|^r_{L^r(\Omega)}\,dx
\end{aligned}$$

$$
\begin{aligned}
&+ \varepsilon_1^{-(r-1)}|k_1|^r|\Omega|^r \frac{e^{\alpha t}}{\alpha} + \varepsilon_1^{-(r-1)}k_2^r \int_\tau^t e^{\alpha s}\|u_t\|^r_{C_{\gamma,L^r(\Omega)}}\,ds \\
&+ r\int_\tau^t e^{\alpha s}\int_\Omega\int_{-\infty}^0 G\bigl(s,z,u(s+z)\bigr)\bigl|u(s)\bigr|^{r-2}u(s)\,dz\,dx\,ds \\
&+ \varepsilon_4^{-(r-1)}\int_\tau^t e^{\alpha s}\bigl\|g(s)\bigr\|^r_{L^r(\Omega)}\,ds.
\end{aligned}
\tag{52}
$$

By assumption (3), (6) and Young's inequality, we obtain

$$
\begin{aligned}
& r\left|\int_\tau^t e^{\alpha s}\int_\Omega\int_{-\infty}^0 G\bigl(s,z,u(s+z)\bigr)\bigl|u(s)\bigr|^{r-2}u(s)\,dz\,dx\,ds\right| \\
&\quad \le r\int_\tau^t e^{\alpha s}\int_\Omega\int_{-\infty}^0 \bigl|G\bigl(s,z,u(s+z)\bigr)\bigr|\bigl|u(s)\bigr|^{r-1}\,dz\,dx\,ds \\
&\quad \le \varepsilon_2^{-(r-1)}m_0^r|\Omega|^r\int_\tau^t e^{\alpha s}\,ds + \varepsilon_2(r-1)\int_\tau^t e^{\alpha s}\bigl\|u(s)\bigr\|^r_{L^r(\Omega)}\,ds \\
&\qquad + \varepsilon_3^{-(r-1)}m_1\int_\tau^t e^{\alpha s}\|u_s\|^r_{C_{\gamma,L^r(\Omega)}}\,ds + \varepsilon_3(r-1)m_1\int_\tau^t e^{\alpha s}\bigl\|u(s)\bigr\|^r_{L^r(\Omega)}\,ds \\
&\quad \le \varepsilon_2^{-(r-1)}m_0^r\|\Omega\|^r_{L^r(\Omega)}\frac{e^{\alpha t}}{\alpha} + \varepsilon_2(r-1)\int_\tau^t e^{\alpha s}\bigl\|u(s)\bigr\|^r_{L^r(\Omega)}\,ds \\
&\qquad + \varepsilon_3^{-(r-1)}m_1\int_\tau^t e^{\alpha s}\|u_s\|^r_{C_{\gamma,L^r(\Omega)}}\,ds + \varepsilon_3(r-1)m_1\int_\tau^t e^{\alpha s}\bigl\|u(s)\bigr\|^r_{L^r(\Omega)}\,ds,
\end{aligned}
\tag{53}
$$

where ε_2 and ε_3 are other positive constants to be determined later.

Combining (52)–(53) we conclude that

$$
\begin{aligned}
& e^{\alpha t}\bigl\|u(t)\bigr\|^r_{L^r(\Omega)} \\
&\quad \le e^{\alpha\tau}\bigl\|u(\tau)\bigr\|^r_{L^r(\Omega)} + \left(\frac{k_1|\Omega|^r}{\varepsilon_1^{(r-1)}\alpha} + \frac{m_0^r|\Omega|^r}{\varepsilon_2^{(r-1)}\alpha}\right)e^{\alpha t} \\
&\qquad - \bigl(r\lambda - (\varepsilon_1+\varepsilon_2+\varepsilon_3 m_1+\varepsilon_4)(r-1)-\alpha\bigr)\int_\tau^t e^{\alpha s}\bigl\|u(s)\bigr\|^r_{L^r(\Omega)}\,ds \\
&\qquad + \left(\frac{m_1}{\varepsilon_3^{(r-1)}} + \frac{k_2^r}{\varepsilon_1^{(r-1)}}\right)\int_\tau^t e^{\alpha s}\|u_s\|^r_{C_{\gamma,L^r(\Omega)}}\,ds + \frac{1}{\varepsilon_4^{(r-1)}}\int_\tau^t e^{\alpha s}\bigl\|g(s)\bigr\|^r_{L^r(\Omega)}\,ds.
\end{aligned}
\tag{54}
$$

Choosing $\varepsilon_1 = \lambda$, $\varepsilon_3 = 1$, we now can choose positive constants ε_2 and ε_4 small enough such that $(\lambda - (\varepsilon_2 + \bar{m}_1 + \varepsilon_4)(r-1) - \alpha) > 0$. Then

$$
\begin{aligned}
& e^{\alpha t}\bigl\|u(t)\bigr\|^r_{L^r(\Omega)} \\
&\quad \le e^{\alpha\tau}\bigl\|u(\tau)\bigr\|^r_{L^r(\Omega)} + \left(\frac{k_1|\Omega|^r}{\lambda^{(r-1)}\alpha} + \frac{m_0^r|\Omega|^r}{\varepsilon_2^{(r-1)}\alpha}\right)e^{\alpha t} \\
&\qquad + \left(m_1 + \frac{k_2^r}{\lambda^{(r-1)}}\right)\int_\tau^t e^{\alpha s}\|u_s\|^r_{C_{L^r(\Omega)}}\,ds + \varepsilon_4^{-(r-1)}\int_\tau^t e^{\alpha s}\bigl\|g(s)\bigr\|^r_{L^r(\Omega)}\,ds.
\end{aligned}
\tag{55}
$$

Now set $t+\theta$ instead of t, where $\theta \in (-\infty, 0]$. By the assumption (45), we have $\alpha \le r\gamma$. Multiplying (55) by $e^{-\alpha(t+\theta)}$ and $e^{r\gamma\theta}e^{-r\gamma\theta}$, it follows that

$$\begin{aligned}\sup_{\theta\in(\tau-t,0]} e^{r\gamma\theta}\big\|u(t+\theta)\big\|^r_{L^r(\Omega)} &\le e^{-\alpha t}e^{\alpha\tau}\|\phi\|^r_{C_{\gamma,L^r(\Omega)}} + C_\Omega + \frac{e^{-\alpha t}}{\varepsilon_4^{(r-1)}}\int_\tau^t e^{\alpha s}\big\|g(s)\big\|^r_{L^r(\Omega)}\,ds \\ &\quad + \left(m_1 + \frac{k_2^r}{\lambda^{(r-1)}}\right)e^{-\alpha t}\int_\tau^t e^{\alpha s}\|u_s\|^r_{C_{\gamma,L^r(\Omega)}}\,ds, \end{aligned} \tag{56}$$

where

$$C_\Omega = \left(\frac{k_1|\Omega|^r}{\lambda^{(r-1)}\alpha} + \frac{m_0^r|\Omega|^r}{\varepsilon_2^{(r-1)}\alpha}\right). \tag{57}$$

Note that

$$\begin{aligned} e^{r\gamma\theta}\big\|u(t+\theta)\big\|^r_{L^r(\Omega)} &= e^{r\gamma\theta}\big\|\phi(t+\theta-\tau)\big\|^r_{L^r(\Omega)} = e^{-r\gamma(t-\tau)}e^{r\gamma(t+\theta-\tau)}\big\|\phi(t+\theta-\tau)\big\|^r_{L^r(\Omega)} \\ &\le e^{-r\gamma(t-\tau)}\|\phi\|^r_{C_{\gamma,L^r(\Omega)}} \le e^{-\alpha(t-\tau)}\|\phi\|^r_{C_{\gamma,L^r(\Omega)}}, \quad \forall\theta\in(-\infty,\tau-t]. \end{aligned}$$

Let $L := m_1 + \frac{2^r k_2^r}{\lambda^{(r-1)}} < \alpha$. Then it yields

$$\begin{aligned} e^{\alpha t}\|u_t\|^r_{C_{\gamma,L^r(\Omega)}} &\le re^{\alpha\tau}\|\phi\|^r_{C_{\gamma,L^r(\Omega)}} + C_\Omega e^{\alpha t} + \varepsilon_4^{-(r-1)}\int_\tau^t e^{\alpha s}\big\|g(s)\big\|^r_{L^r(\Omega)}\,ds \\ &\quad + \left(m_1 + \frac{2^r k_2^r}{\lambda^{(r-1)}}\right)\int_\tau^t e^{\alpha s}\|u_s\|^r_{C_{\gamma,L^r(\Omega)}}\,ds \\ &\le re^{\alpha\tau}\|\phi\|^r_{C_{\gamma,L^r(\Omega)}} + C_\Omega e^{\alpha t} + \varepsilon_4^{-(r-1)}\int_\tau^t e^{\alpha s}\big\|g(s)\big\|^r_{L^r(\Omega)}\,ds \\ &\quad + L\int_\tau^t e^{\alpha s}\|u_s\|^r_{C_{\gamma,L^r(\Omega)}}\,ds. \end{aligned}$$

By Fubini's theorem and Grownwall's lemma, we find that

$$\begin{aligned} e^{\alpha t}\|u_t\|^r_{C_{\gamma,L^r(\Omega)}} &\le re^{\alpha\tau}\|\phi\|^r_{C_{\gamma,L^r(\Omega)}} + \varepsilon_4^{-(r-1)}\int_\tau^t e^{\alpha s}\big\|g(s)\big\|^r_{L^r(\Omega)}\,ds \\ &\quad + re^{(\alpha-L)\tau}e^{Lt}\|\phi\|^r_{C_{\gamma,L^r(\Omega)}} + \frac{\alpha}{\alpha-L}C_\Omega e^{\alpha t} \\ &\quad + \varepsilon_4^{-(r-1)}e^{Lt}\int_\tau^t \left(e^{(\alpha-L)s}\big\|g(s)\big\|^r_{L^r(\Omega)}\right)ds. \end{aligned} \tag{58}$$

Hence, (6) and condition (45) imply that

$$\begin{aligned} \|u_t\|^r_{C_{\gamma,L^r(\Omega)}} &\le Cre^{-\alpha t}\|\phi\|^r_{C_{\gamma,L^r(\Omega)}} + \frac{\alpha}{\alpha-L}C_\Omega + \varepsilon_4^{-(r-1)}e^{-\alpha t}\int_{-\infty}^t e^{\alpha s}\big\|g(s)\big\|^r_{L^r(\Omega)}\,ds \\ &\quad + re^{(\alpha-L)\tau}e^{(L-\alpha)t}\|\phi\|^r_{C_{\gamma,L^r(\Omega)}} \\ &\quad + \varepsilon_4^{-(r-1)}e^{(L-\alpha)t}\int_{-\infty}^t \left(e^{(\alpha-L)s}\big\|g(s)\big\|^r_{L^r(\Omega)}\right)ds \\ &\triangleq R_{1,C_{\gamma,L^r(\Omega)}}(t,\phi,g,\alpha,L). \end{aligned} \tag{59}$$

For each $t \in \mathbb{R}$, let

$$B_{R_{1,C_{\gamma,L^r(\Omega)}}}(t) = \{u \in C_{\gamma,L^r(\Omega)} \mid \|u\|^r_{C_{\gamma,L^r(\Omega)}} \le R_{1,C_{\gamma,L^r(\Omega)}}(t,\phi,g,\alpha,L)\}, \tag{60}$$

which implies that the family of bounded sets $B = \{B_{R_{1,C_{\gamma,L^r(\Omega)}}}(t)\}_{t\in\mathbb{R}}$ is pullback absorbing for the process $\{U(t,\tau)\}$ on $C_{\gamma,L^r(\Omega)}$. □

5 Existence of the pullback attractors in $C_{\gamma,L^r(\Omega)}$ ($r > 2$)

In this section, we will discuss the case where the external forcing term g belongs only to $L^r_{\mathrm{loc}}(\mathbb{R}, L^r(\Omega))$. Inspired by the idea for proving the existence of global attractors in $L^r(\Omega)$, we modify Theorem 5.11 [18] to prove the existence of the pullback attractors in $C_{\gamma,L^r(\Omega)}$.

Lemma 5.1 *Hypotheses* (2), (3), (6) *hold, and* $g \in C(\mathbb{R}; L^2(\Omega))$. *Then there exists a pullback attractor* $\{\mathcal{A}_{C_{\gamma,L^2(\Omega)}}(t)\}_{t\in\mathbb{R}}$ *for the processes* $\{U(t,\tau)\}$ *on* $C_{\gamma,L^2(\Omega)}$ *generated by the solution of Eq.* (1).

Proof By Theorem 13 [10], the processes $\{U(t,\tau)\}$ on $C_{\gamma,H^1(\Omega)}$ associated with Eq. (1) has a pullback attractor $\mathcal{A}_{C_{\gamma,H^1(\Omega)}}$. From the Sobolev embedding theorem $H^1(\Omega) \hookrightarrow\hookrightarrow L^2(\Omega)$ and $C_{\gamma,H^1(\Omega)} \subseteq C_{\gamma,L^2(\Omega)}$, $\mathcal{A}_{C_{\gamma,H^1(\Omega)}}$ is a pullback attractor for the processes $\{U(t,\tau)\}$ on $C_{\gamma,L^2(\Omega)}$. □

Lemma 5.2 *Let* $\{U(t,\tau)\}$ *associated with Eq.* (1) *be an evolution process on* $C_{\gamma,L^r(\Omega)}$ *with a pullback absorbing set* $\mathcal{D} = \{D(t)\}_{t\in\mathbb{R}}$ *on* $C_{\gamma,L^r(\Omega)}$. *Then, for each* $t \in \mathbb{R}$, *for any* $\varepsilon > 0$, *and any pullback absorbing set* $\mathcal{D} \subset C_{\gamma,L^r(\Omega)}$, *there exist* $T = T(\mathcal{D},t,\varepsilon) > 0$, $M = M(\varepsilon) > 0$ *such that*

$$m\big(\Omega_t^{\cdot}\big(\big|U(t,t+z)u^0(t+z)\big| \ge M\big)\big) \le \varepsilon, \quad \text{for any } -z \le T, \text{and } u_t^0(\cdot) \in \mathcal{D},$$

where $m(e)$ *denotes the Lebesgue measure of* $e \subset \Omega$ *and* $\Omega_t^{\cdot}(|u_t(z)| \ge M) \triangleq \bigcup_{z\in(-\infty,0]}\{x \in \Omega \mid |u(t+z,x)| \ge M\}$.

Proof From the assumption that $\{U(t,\tau)\}$ has a pullback absorbing set in $C_{\gamma,L^r(\Omega)}$, we know that there exists a positive constant M_0, such that, for each $t \in \mathbb{R}$ and for any pullback absorbing set $\mathcal{D}$ of $C_{\gamma,L^r(\Omega)}$, we can find a positive constant T which depends on $\mathcal{D}$, such that

$$\big\|U(t,t+z)u^0(t+z)\big\|^r_{C_{\gamma,L^r(\Omega)}} \le M_0, \quad \text{for any } -z \ge T, \text{and } u_t^0(\cdot) \in \mathcal{D}.$$

So, we have

$$\begin{aligned} 2M_0 &\ge 2 \sup_{z\in(-\infty,0]} e^{\gamma z} \int_\Omega \big|U(t,t+z)u^0(t+z)\big|^r\,dx \\ &\ge \sup_{z\in(-\infty,-T_1]} e^{\gamma z} \int_{\Omega_t^{\cdot}(\{|u(t+z)|\ge M_1\})} \big|U(t,t+z)u^0(t+z)\big|^r\,dx \\ &\quad + \sup_{z\in(-T_1,0]} e^{\gamma z} \int_{\Omega_t^{\cdot}(\{|u(t+z)|\ge M_1\})} \big|U(t,t+z)u^0(t+z)\big|^r\,dx \end{aligned}$$

$$\begin{aligned}
&\geq e^{-\gamma T_1}\left(\int_{\Omega_t^{\cdot}(\{|U(t,t+z)u^0(t+z)|\geq M_1\})} M_1^r\,dx + \int_{\Omega_t^{\cdot}(\{|U(t,t+z)u^0(t+z)|\geq M_1\})} M_1^r\,dx\right)\\
&\geq 2e^{-\gamma T_1}M_1^r m\bigl(\Omega\bigl(\bigl\{\bigl|U(t,t+z)u^0(t+z)\bigr|\geq M_1\bigr\}\bigr)\bigr).
\end{aligned}$$

This inequality implies that $m(\Omega_t^{\cdot}(\{|U(t,t+z)u^0(t+z)|\geq M_1\}))\leq\varepsilon$, if we choose M_1 large enough such that $M_1\geq(\frac{M_0}{e^{-\gamma T_1}\varepsilon})^{\frac{1}{r}}$. □

Lemma 5.3 *For each $t\in\mathbb{R}$, any $\varepsilon>0$, the pullback absorbing set $\mathcal{D}$ of process $\{U(t,\tau)\}$ associated with Eq. (1) on $C_{\gamma,L^r(\Omega)}$ $(r>0)$ has a finite ε-net in $C_{\gamma,L^r(\Omega)}$, if there exists a positive constant $M=M(\varepsilon)$ which depends on ε, such that*

(i) *$\mathcal{D}$ has a finite $(3M)^{(2-r)/2}(\frac{\varepsilon}{2})^{\frac{r}{2}}$-net in $C_{\gamma,L^2(\Omega)}$,*

(ii)
$$\begin{aligned}
&\left(\sup_{z\in(-\infty,0]} e^{\gamma z}\int_{\Omega_t^z(\{|u(t+z)|\geq M\})}\bigl|u(t+z)\bigr|^r\,dx\right)^{\frac{1}{r}}\\
&\quad<2^{-(2r+2)/r}\varepsilon,\quad \textit{for any } u_t(\cdot)\in\mathcal{D}.
\end{aligned}\tag{61}$$

Proof For each $t\in\mathbb{R}$, any fixed $\varepsilon>0$, it follows from the assumptions that $\mathcal{D}$ has a finite $\frac{(3M)^{(2-r)}}{2\varepsilon^{r/2}}$-net in $C_{\gamma,L^2(\Omega)}$, that is, there exist $u_t^1,\dots,u_t^k\in\mathcal{D}$, such that, for each $u_t(\cdot)\in\mathcal{D}$, we can find some u_t^i $(1\leq i\leq k)$ satisfying

$$\begin{aligned}
\bigl\|u(t+z)-u^i(t+z)\bigr\|_{L^2(\Omega)}^2 &\leq \sup_{z\in(-\infty,0]} e^{\gamma z}\bigl\|u(t+z)-u^i(t+z)\bigr\|_{L^2(\Omega)}^2\\
&=\sup_{z\in(-\infty,0]} e^{\gamma z}\bigl\|u_t-u_t^i\bigr\|_{L^2(\Omega)}^2<(3M)^{(2-r)}\left(\frac{\varepsilon}{2}\right)^r.
\end{aligned}\tag{62}$$

Then, obviously, we have

$$\begin{aligned}
&\bigl\|u_t-u_t^i\bigr\|_{C_{\gamma,L^r(\Omega)}}^r\\
&\quad\leq \sup_{z\in(-\infty,0]} e^{\gamma z}\int_{\Omega_t^z(|u(t+z)-u^i(t+z)|\geq 3M)}\bigl|u(t+z)-u^i(t+z)\bigr|^r\,dx\\
&\qquad+\sup_{z\in(-\infty,0]} e^{\gamma z}\int_{\Omega_t^z(|u(t+z)-u^i(t+z)|\leq 3M)}\bigl|u(t+z)-u^i(t+z)\bigr|^r\,dx
\end{aligned}\tag{63}$$

and

$$\begin{aligned}
&\sup_{z\in(-\infty,0]} e^{\gamma z}\int_{\Omega_t^z(|u(t+z)-u^i(t+z)|\leq 3M)}\bigl|u(t+z)-u^i(t+z)\bigr|^r\,dx\\
&\quad\leq(3M)^{r-2}\sup_{z\in(-\infty,0]} e^{\gamma z}\int_{\Omega_t^z(|u_t-u_{it}|\leq 3M)}\bigl|u_t-u_t^i\bigr|^2\,dx,\\
&\quad\leq(3M)^{r-2}(3M)^{2-r}\left(\frac{\varepsilon}{2}\right)^r=\left(\frac{\varepsilon}{2}\right)^r.
\end{aligned}\tag{64}$$

On the other hand, set

$$\Omega_1^z=\Omega_t^z\left(\bigl|u(t+z)\bigr|\geq\frac{3M}{2}\right)\cap\Omega_t^z\left(\bigl|u^i(t+z)\bigr|\leq\frac{3M}{2}\right),$$

$$\Omega_2^z = \Omega_t^z\left(\left|u(t+z)\right| \le \frac{3M}{2}\right) \cap \Omega_t^z\left(\left|u^i(t+z)\right| \ge \frac{3M}{2}\right),$$
$$\Omega_3^z = \Omega_t^z\left(\left|u(t+z)\right| \ge \frac{3M}{2}\right) \cap \Omega_t^z\left(\left|u^i(t+z)\right| \ge \frac{3M}{2}\right),$$

then we have

$$\Omega_t^z\left(\left|u(t+z)\right| \ge 3M\right) \subset \Omega_1^z \cup \Omega_2^z \cup \Omega_3^z.$$

From the simple facts that $|u(t+z) - u^i(t+z)| \le 2|u(t+z)|$ in Ω_1^z and $|u(t+z) - u^i(t+z)| \le 2|u^i(t+z)|$ in Ω_2^z, combining with (61), we have

$$\begin{aligned}
&\sup_{z\in(-\infty,0]} e^{\gamma z} \int_{\Omega_t^z(|u(t+z)-u^i(t+z)|\ge 3M)} \left|u(t+z) - u^i(t+z)\right|^r dx \\
&\quad\le \sup_{z\in(-\infty,0]} e^{\gamma z} \int_{\Omega_1^z} \left|u(t+z) - u^i(t+z)\right|^r dx + \sup_{z\in(-\infty,0]} e^{\gamma z} \int_{\Omega_2^z} \left|u(t+z) - u^i(t+z)\right|^r dx \\
&\qquad + \sup_{z\in(-\infty,0]} e^{\gamma z} \int_{\Omega_3^z} \left|u(t+z) - u^i(t+z)\right|^r dx \\
&\quad\le 2^r \sup_{z\in(-\infty,0]} e^{\gamma z} \left(\int_{\Omega_t^z(|u(t+z)|\ge M)} \left|u(t+z)\right|^r dx + \int_{\Omega_t^z(|u^i(t+z)|\ge M)} \left|u^i(t+z)\right|^r dx \right. \\
&\qquad \left. + \int_{\Omega_t^z(|u(t+z)|\ge M)} \left|u(t+z)\right|^r dx + \int_{\Omega_t^z(|u^i(t+z)|\ge M)} \left|u^i(t+z)\right|^r dx \right) \\
&\quad\le 2^{r+2} \cdot 2^{(2r+2)} \varepsilon^r = \left(\frac{\varepsilon}{2}\right)^r.
\end{aligned} \tag{65}$$

Substituting (64) and (65) into (63), we can deduce that

$$\sup_{z\in(-\infty,0]} e^{\gamma z} \left\|u(t+z) - u^i(t+z)\right\|_{L^r(\Omega)} \le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

which means that $\mathcal{D}$ has a finite ε-net in $C_{\gamma,L^r(\Omega)}$. □

Lemma 5.4 *Let $\mathcal{D}$ be a pullback absorbing set in $C_{\gamma,L^r(\Omega)}$ $(r \ge 1)$. If $\mathcal{D}$ has a finite ε-net in $C_{\gamma,L^r(\Omega)}$ $(r \ge 1)$ then there exists a positive $M = M(B,\varepsilon)$, such that, for any $u_t(\cdot) \in \mathcal{D}$, $z \in (-\infty, 0]$, we can find*

$$\sup_{z\in(-\infty,0]} e^{\gamma z} \int_{\Omega_t^z(|u(t+z)|\ge M)} \left|u(t+z)\right|^r dx \le 2^{r+1}\varepsilon^r.$$

Proof Since $\mathcal{D}$ has a finite ε-net in $C_{\gamma,L^r(\Omega)}$ $(r \ge 1)$, for each $t \in \mathbb{R}$, we know that there exist $u_t^1, \dots, u_t^k \in \mathcal{D}$, such that, for any $u_t(\cdot) \in \mathcal{D}$, we can find some u_t^i $(1 \le i \le k)$ satisfying

$$\sup_{z\in(-\infty,0]} e^{\gamma z} \int_{\Omega_t^z(|u(t+z)|\ge M)} \left|u(t+z) - u^i(t+z)\right|^r dx \le \varepsilon^r. \tag{66}$$

Simultaneously, for the fixed $\varepsilon > 0$, there exists a $\delta > 0$, such that, for each u_t^i, $1 \le i \le k$, we have

$$\sup_{z\in(-\infty,0]} e^{\gamma z} \int_e \left|u^i(t+z)\right|^r dx \le \varepsilon^r, \tag{67}$$

provided that $m(e) < \delta$ $(e \subset \Omega)$.

On the other hand, since $\mathcal{D}$ is bounded in $C_{\gamma,L^r(\Omega)}$ ($r \geq 1$), for the fixed $\delta > 0$ above, there exists $M > 0$, such that $m(\Omega_t^{\cdot}(|u(t+z)| \geq M)) < \delta$ holds for each $u_t \in B$. So, $m(\Omega_t^z(|u(t+z)| \geq M)) < \delta$ also holds for each $u_t \in B$.

Therefore,

$$\begin{aligned}
&\sup_{z\in(-\infty,0]} e^{\gamma z}\int_{\Omega_t^z(|u(t+z)|\geq M)} \left|u(t+z)\right|^r dx \\
&\quad= \sup_{z\in(-\infty,0]} e^{\gamma z}\int_{\Omega_t^z(|u(t+z)|\geq M)} \left|u(t+z)-u^i(t+z)+u^i(t+z)\right|^r dx \\
&\quad\leq 2^r \sup_{z\in(-\infty,0]} e^{\gamma z}\int_{\Omega_t^z(|u(t+z)|\geq M)} \left|u(t+z)-u^i(t+z)\right|^r dx \\
&\qquad+ 2^r \sup_{z\in(-\infty,0]} e^{\gamma z}\int_{\Omega_t^z(|u(t+z)|\geq M)} \left|u^i(t+z)\right|^r dx \\
&\quad\leq 2^{r+1}\varepsilon^r.
\end{aligned} \tag{68}$$

□

Lemma 5.5 *For each $t \in \mathbb{R}$, for any $\varepsilon > 0$ and any pullback absorbing set $\mathcal{D} \in C_{\gamma,L^2(\Omega)}$, there exist two positive constants $T_3 = T_3(B,\varepsilon) = \max\{T_1, T_2\}$ and $M = M(\varepsilon)$, such that*

$$\sup_{z\in(-\infty,0]} e^{\gamma z}\int_{\Omega_t^z(|u(t+z)|\geq M)} \left|u(t+z)\right|^r dx < C\varepsilon, \quad \textit{for any } -z \geq T_3, u_t^0(\cdot) \in \mathcal{D}, \tag{69}$$

where the constant C is independent of ε and $\mathcal{D}$.

Proof For each $t \in \mathbb{R}$, any fixed $\varepsilon > 0$, there exists $\delta > 0$ such that if $e \subset \Omega$ and $m(e) \leq \delta$, then

$$\int_e \left|\phi(x)\right|^r dx \leq C\varepsilon, \tag{70}$$

where $\phi(x), g(x) \in L^r(\Omega)$. Moreover, from Lemmas 5.1, 5.2 and 5.4, we know that there exist $T = T(\mathcal{D},\varepsilon) > 0$ and $M = M(\varepsilon)$, for each $-z \geq T$, $u_t(\cdot) \in D$, we have

$$m\big(\Omega_t^z\big(\left|u(t+z)\right| \geq M\big)\big) < \min\{\varepsilon,\delta\}, \quad \text{for each } t \in \mathbb{R}, \tag{71}$$

and

$$\sup_{z\in(-\infty,0]} e^{\gamma z}\int_{\Omega_t^z(|u(t+z)|)\geq M} \left|u(t+z)\right|^2 < 8\varepsilon. \tag{72}$$

Thus, we also have

$$\int_{\Omega_t^0(|u(t)|\geq M)} \left|u(t)\right|^2 < 8\varepsilon, \quad \text{for } t \in [T,+\infty]. \tag{73}$$

Multiplying (1) by $(u-M)_+^{r-1}$ and integrating over $\Omega_t^0 = \Omega_t^0(u > M)$, we have

$$\int_{\Omega_t^0(u>M)} \frac{\partial u}{\partial t}(u-M)_+^{r-1}\,dx - \int_{\Omega_t^0(u>M)} \Delta u(u-M)_+^{r-1}\,dx$$

$$
\begin{aligned}
&+\int_{\Omega_t^0(u>M)}\lambda u(u-M)_+^{r-1}\,dx\\
&=\int_{\Omega_t^0(u>M)}f(t,u_t)(u-M)_+^{r-1}\,dx+\int_{\Omega_t^0(u>M)}g(t,x)(u-M)_+^{r-1}\,dx.
\end{aligned}
\tag{74}
$$

After integrating over $\Omega_t^0(u>M)$, (74) becomes

$$
\begin{aligned}
&\frac{1}{r}\frac{d}{dt}\big\|(u-M)_+\big\|_{L^r(\Omega)}^r-\int_{\Omega_t^0(u>M)}\Delta u(u-M)_+^{r-1}\,dx+\lambda\int_{\Omega_t^0(u>M)}u(u-M)_+^{r-1}\,dx\\
&=\int_{\Omega_t^0(u>M)}F\big(t,u\big(x,t-\rho(t)\big)\big)(u-M)_+^{r-1}\,dx+\int_{\Omega_t^0(|u|>M)}g(t,x)(u-M)_+^{r-1}\,dx\\
&+\int_{\Omega_t^0(u>M)}\int_{-\infty}^0\big|G\big(s,z,u(s+z)\big)\big|(u-M)_+^{r-1}\,dz\,dx,
\end{aligned}
\tag{75}
$$

where

$$
(u-M)_+=\begin{cases}u-M, & u\ge M,\\ 0, & u\le M.\end{cases}
$$

Let $\Omega_{1,t}^0=\Omega_t^0(u>M)$, then we have

$$
\begin{aligned}
&\frac{1}{r}\frac{d}{dt}\big\|(u-M)_+\big\|_{L^r(\Omega)}^r-\int_{\Omega_{1,t}^0}\Delta u(u-M)_+^{r-1}\,dx+\lambda\int_{\Omega_{1,t}^0}u(u-M)_+^{r-1}\,dx\\
&=\int_{\Omega_{1,t}^0}F\big(t,u\big(x,t-\rho(t)\big)\big)(u-M)_+^{r-1}\,dx+\int_{\Omega_{1,t}^0}g(t,x)(u-M)_+^{r-1}\,dx\\
&+\int_{\Omega_{1,t}^0}\int_{-\infty}^0 G\big(s,z,u(s+z)\big)(u-M)_+^{r-1}\,dz\,dx.
\end{aligned}
$$

We now estimate every term of (75). First, we obtain

$$
-\int_{\Omega_{1,t}^0}\Delta u(u-M)_+^{r-1}\,dx=(r-1)\int_{\Omega_1^0}\nabla u\big|(u-M)_+\big|^{r-2}\nabla u\,dx\ge 0
\tag{76}
$$

and

$$
\lambda\int_{\Omega_{1,t}^0}u(u-M)_+^{r-1}\,dx\ge\lambda\big\|(u-M)_+\big\|_{L^r(\Omega)}^r.
\tag{77}
$$

By the assumption (2), (3), (6) and Young's inequality, we have

$$
\begin{aligned}
&\int_{\Omega_{1,t}^0}F\big(t,u\big(x,t-\rho(t)\big)\big)(u-M)_+^{r-1}\,dx\\
&\le\frac{\varepsilon_1^{-(r-1)}}{r}\int_{\Omega_{1,t}^0}\big|F\big(x,u\big(x,t-\rho(t)\big)\big)\big|^r\,dx+\frac{(r-1)\varepsilon_1}{r}\int_{\Omega_{1,t}^0}(u-M)_+^r\,dx\\
&\le\frac{\varepsilon_1^{-(r-1)}}{r}\int_{\Omega_{1,t}^0}|k_1|^r\,dx+\frac{k_2^r\varepsilon_1^{-(r-1)}}{r}\int_{\Omega_{1,t}^0}e^{-r\gamma\rho(t)}\big|u\big(x,t-\rho(t)\big)\big|^r\,dx
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{(r-1)\varepsilon_1}{r}\int_{\Omega_{1,t}^0}(u-M)_+^r\,dx\\
&\le\frac{\varepsilon_1^{-(r-1)}}{r}|k_1|^r|_{L^r(\Omega_{1,t}^0)}+\frac{k_2^r\varepsilon_1^{-(r-1)}}{r}\|u_t\|_{C_{\gamma,L^r(\Omega_1^0)}}^r+\frac{(r-1)\varepsilon_1}{r}\big\|(u-M)_+\big\|_{L^r(\Omega_{1,t}^0)}^r,
\end{aligned}
\tag{78}
$$

$$
\begin{aligned}
&\int_{\Omega_{1,t}^0}\int_{-\infty}^0 G\big(x,z,u(s+z)\big)(u-M)_+^{r-1}\,dz\,dx\\
&\le\int_{\Omega_{1,t}^0}\int_{-\infty}^0\big|m_0(z)\big|\big|(u-M)_+\big|^{r-1}\,dz\,dx+\int_{\Omega_{1,t}^0}\int_{-\infty}^0 m_1(z)|u(t+z)(u-M)_+^{r-1}\,dz\,dx\\
&\le\frac{\varepsilon_2^{-(r-1)}}{r}\int_{\Omega_{1,t}^0}|m_0|^r\,dx+\frac{(r-1)\varepsilon_2}{r}\int_{\Omega_1}(u-M)_+^r\,dx\\
&\quad+\frac{\bar m_1\varepsilon_3^{-(r-1)}}{r}\int_{\Omega_{1,t}^0}\big|u(t+z)\big|^r\,dx+\frac{\bar m_1(r-1)\varepsilon_3}{r}\int_{\Omega_{1,t}^0}(u-M)_+^r\,dx\\
&\le\frac{\varepsilon_2^{-(r-1)}}{r}|m_0|_{L^r(\Omega_{1,t}^0)}^r+\frac{(r-1)\varepsilon_2}{r}\big\|(u-M)_+\big\|_{L^r(\Omega_{1,t}^0)}^r\\
&\quad+\frac{\bar m_1(r-1)\varepsilon_3}{r}\big\|u(t+z)\big\|_{L^r(\Omega_{1,t}^0)}^r,
\end{aligned}
\tag{79}
$$

and

$$
\begin{aligned}
\int_{\Omega_{1,t}^0}g(t,x)(u-M)_+^{r-1}\,dx&\le\int_{\Omega_{1,t}^0}\big|g(t,x)\big|(u-M)_+^{r-1}\,dx\\
&\le\frac{\varepsilon_4^{-(r-1)}}{r}\int_{\Omega_{1,t}^0}\big|g(t,x)\big|^r\,dx+\frac{(r-1)\varepsilon_4}{r}\int_{\Omega_{1,t}^0}(u-M)_+^r\,dx\\
&\le\frac{\varepsilon_4^{-(r-1)}}{r}\big\|g(t,x)\big\|_{L^r(\Omega_{1,t}^0)}^r+\frac{(r-1)\varepsilon_4}{r}\big\|(u-M)_+\big\|_{L^r(\Omega_{1,t}^0)}^r.
\end{aligned}
\tag{80}
$$

Combining with (76)–(80), we can conclude that

$$
\begin{aligned}
&\frac{d}{dt}\big\|(u-M)_+\big\|_{L^r(\Omega)}^r+r(r-1)\int_{\Omega_{1,t}^0}\nabla u(u-M)_+^{r-2}\nabla u\,dx\\
&\quad+r\lambda\int_{\Omega_{1,t}^0}u(u-M)_+^{r-1}\,dx\\
&\le\varepsilon_1^{-(r-1)}\int_{\Omega_{1,t}^0}|k_1|^r\,dx+\varepsilon_2^{-(r-1)}\int_{\Omega_{1,t}^0}|m_0|^r\,dx\\
&\quad+(r-1)(\varepsilon_1+\varepsilon_2+m_1\varepsilon_3+\varepsilon_4)\int_{\Omega_{1,t}^0}(u-M)_+^r\,dx\\
&\quad+k_2^r\varepsilon_1^{-(r-1)}e^{-r\gamma\rho(t)}\int_{\Omega_{1,t}^0}\big|u\big(x,t-\rho(t)\big)\big|^r\,dx\\
&\quad+m_1\varepsilon_3^{-(r-1)}\int_{\Omega_{1,t}^0}e^{\gamma z}\big|u(t+z)\big|^r\,dx+\varepsilon_4^{-(r-1)}\int_{\Omega_{1,t}^0}\big|g(t,x)\big|^r\,dx.
\end{aligned}
\tag{81}
$$

We also have

$$
\begin{aligned}
&\frac{d}{dt}\big\|(u-M)_+\big\|_{L^r(\Omega)}^r\\
&\le-r\lambda\big\|(u-M)_+\big\|_r^r+\varepsilon_1^{-(r-1)}|k_1|_{L^r(\Omega_{1,t}^0)}^r+\varepsilon_2^{-(r-1)}|m_0|_{L^r(\Omega_{1,t}^0)}^r
\end{aligned}
$$

$$
\begin{aligned}
&+ (r-1)(\varepsilon_1+\varepsilon_2+m_1\varepsilon_3+\varepsilon_4)\big\|(u-M)_+\big\|^r_{L^r(\Omega^0_{1,t})} + k_2^r\varepsilon_1^{-(r-1)}\|u_t\|^r_{C_{\gamma,L^r(\Omega^0_{1,t})}} \\
&+ m_1\varepsilon_3^{-(r-1)}\|u_t\|^r_{C_{\gamma,L^r(\Omega^0_{1,t})}} + \varepsilon_4^{-(r-1)}\big\|g(t,x)\big\|^r_{L^r(\Omega^0_{1,t})}.
\end{aligned}
\tag{82}
$$

Let $\alpha > 0$, which will also be determined later. Then

$$
\begin{aligned}
&\frac{d}{dt}e^{\alpha t}\big\|(u-M)_+\big\|^r_{L^r(\Omega)} \\
&\quad = \alpha e^{\alpha t}\big\|(u-M)_+\big\|^r_r + e^{\alpha t}\frac{d}{dt}\big\|(u-M)_+\big\|^r_r \\
&\quad \le -\big(r\lambda-\alpha-(r-1)(\varepsilon_1+\varepsilon_2+m_1\varepsilon_3+\varepsilon_4)\big)e^{\alpha t}\big\|(u-M)_+\big\|^r_{L^r(\Omega)} \\
&\qquad + \big(\varepsilon_1^{-(r-1)}|k_1|^r_{L^r(\Omega^0_{1,t})} + \varepsilon_2^{-(r-1)}|m_0|^r_{L^r(\Omega^0_{1,t})}\big)e^{\alpha t} + \varepsilon_4^{-(r-1)}e^{\alpha t}\big\|g(t,x)\big\|^r_{L^r(\Omega^0_{1,t})} \\
&\qquad + \big(k_2^r\varepsilon_1^{-(r-1)} + m_1\varepsilon_3^{-(r-1)}\big)e^{\alpha t}\|u_t\|^r_{C_{\gamma,L^r(\Omega^0_{1,t})}}.
\end{aligned}
\tag{83}
$$

Let $A = (r\lambda-\alpha-(r-1)(\varepsilon_1+\varepsilon_2+m_1\varepsilon_3+\varepsilon_4))$. By Gronwall's inequality, we have

$$
\begin{aligned}
&e^{\alpha t}\big\|(u-M)_+\big\|^r_{L^r(\Omega^1_{t,0})} \\
&\quad \le e^{-A(t-\tau)}e^{\alpha\tau}\big\|\big(u(\tau)-M\big)_+\big\|^r_{C_{\gamma,L^r(\Omega)}} + \varepsilon_4^{-(r-1)}e^{-At}\int_{-\infty}^t e^{(A+\alpha)s}\big\|g(s,x)\big\|^r_{L^r(\Omega^0_{1,t})}\,ds \\
&\qquad + \big(k_2^r\varepsilon_1^{-(r-1)} + m_1\varepsilon_3^{-(r-1)}\big)e^{-At}\int_\tau^t e^{(A+\alpha)s}\|u_s\|^r_{C_{\gamma,L^r(\Omega^0_{1,t})}}\,ds \\
&\qquad + \big(\varepsilon_1^{-(r-1)}|k_1|^r_{L^r(\Omega^0_{1,t})} + \varepsilon_2^{-(r-1)}|m_0|^r_{L^r(\Omega^0_{1,t})}\big)\frac{e^{\alpha t}}{A+\alpha}.
\end{aligned}
\tag{84}
$$

Thanks to (46), and letting $\alpha_1 > \alpha \ge \alpha^*$, we can deduce that

$$
\begin{aligned}
&\big(k_2^r\varepsilon_1^{-(r-1)} + m_1\varepsilon_3^{-(r-1)}\big)e^{-At}\int_\tau^t e^{(A+\alpha)s}\|u_s\|^r_{C_{\gamma,L^r(\Omega^1_{t,0})}}\,ds \\
&\quad \le \big(k_2^r\varepsilon_1^{-(r-1)} + m_1\varepsilon_3^{-(r-1)}\big)\bigg(\frac{re^{\alpha\tau}}{A}\|\phi\|^r_{C_{\gamma,L^r(\Omega^1_{t,0})}} + \frac{\alpha C_{\Omega^1_{t,0}}e^{\alpha t}}{(A+\alpha)(\alpha-L)} \\
&\qquad + \varepsilon_4^{-(r-1)}\frac{1}{A}\int_{-\infty}^t e^{\alpha s}\big\|g(s)\big\|^r_{L^r(\Omega^1_{t,0})}\,ds + \frac{re^{(\alpha-L)\tau}e^{Lt}}{(A+L)}\|\phi\|^r_{C_{\gamma,L^r(\Omega^1_{t,0})}} \\
&\qquad + \varepsilon_4^{-(r-1)}\frac{e^{Lt}}{(A+L)}\int_{-\infty}^t \big(e^{(\alpha-L)s}\big\|g(s)\big\|^r_{L^r(\Omega^1_{t,0})}\big)\,ds\bigg).
\end{aligned}
\tag{85}
$$

Multiplying (84) by $e^{-\alpha t}$, we have

$$
\begin{aligned}
&\big\|(u-M)_+\big\|^r_{L^r(\Omega^1_{t,0})} \\
&\quad \le e^{-A(t-\tau)}e^{\alpha\tau}e^{-\alpha t}\big\|\big(u(\tau)-M\big)_+\big\|^r_{C_{\gamma,L^r(\Omega^0_{1,t})}} + \frac{\varepsilon_1^{-(r-1)}|k_1|^r_{L^r(\Omega^0_{1,t})}}{A+\alpha} \\
&\qquad + \varepsilon_4^{-(r-1)}e^{-(A+\alpha)t}\int_{-\infty}^t e^{(A+\alpha)s}\big\|g(s,x)\big\|^r_{L^r(\Omega^0_{1,t})}\,ds + \frac{\varepsilon_2^{-(r-1)}|m_0|^r_{L^r(\Omega^0_{1,t})}}{A+\alpha}
\end{aligned}
$$

$$
\begin{aligned}
&+ \left(k_2^r \varepsilon_1^{-(r-1)} + m_1 \varepsilon_3^{-(r-1)}\right)\left(\frac{re^{\alpha\tau}e^{-\alpha t}}{A}\|\phi\|^r_{C_{\gamma,L^r(\Omega^0_{1,t})}} + \frac{\alpha C_{\Omega^0_{1,t}}}{(A+\alpha)(\alpha-L)}\right.\\
&+ \varepsilon_4^{-(r-1)}\frac{1}{A}e^{-\alpha t}\int_{-\infty}^t e^{\alpha s}\|g(s)\|^r_{L^r(\Omega^0_{1,t})}\,ds + \frac{re^{(\alpha-L)\tau}e^{-(\alpha-L)t}}{(A+L)}\|\phi\|^r_{C_{\gamma,L^r(\Omega^0_{1,t})}}\\
&\left.+ \varepsilon_4^{-(r-1)}\frac{e^{-(\alpha-L)t}}{(A+L)}\int_{-\infty}^t \left(e^{(\alpha-L)s}\|g(s)\|^r_{L^r(\Omega^0_{1,t})}\right)ds\right)\\
\le{}& e^{\alpha\tau}e^{-\alpha t}\|(\phi-M)_+\|^r_{C_{\gamma,L^r(\Omega^0_{1,t})}} + Ce^{-(A+\alpha)t}\int_{-\infty}^t e^{(A+\alpha)s}\|g(s,x)\|^r_{L^r(\Omega^0_{1,t})}\,ds\\
&+ Cm\left(\Omega^0_{1,t}\right) + Ce^{\alpha\tau}e^{-\alpha t}\|\phi\|^r_{C_{\gamma,L^r(\Omega^0_{1,t})}} + CC_{\Omega^0_{1,t}}\\
&+ Ce^{-\alpha t}\int_{-\infty}^t e^{\alpha s}\|g(s)\|^r_{L^r(\Omega^0_{1,t})}\,ds + Ce^{(\alpha-L)\tau}e^{-(\alpha-L)t}\|\phi\|^r_{C_{\gamma,L^r(\Omega^0_{1,t})}}\\
&+ Ce^{-(\alpha-L)t}\int_{-\infty}^t \left(e^{(\alpha-L)s}\|g(s)\|^r_{L^r(\Omega^0_{1,t})}\right)ds. \qquad (86)
\end{aligned}
$$

Now replacing t by $t+z$, similar to the arguments in Lemma 4.1, in view of (45), we have

$$
\begin{aligned}
&e^{r\gamma z}\|(u_t-M)_+\|^r_{L^r(\Omega^z_{1,t})}\\
&\le e^{\alpha\tau}e^{-\alpha t}e^{(r\gamma-\alpha)z}\|(\phi-M)_+\|^r_{C_{\gamma,L^r(\Omega^z_{1,t})}} + Ce^{-(A+\alpha)t}\int_{-\infty}^t e^{(A+\alpha)s}\|g(s,x)\|^r_{L^r(\Omega^z_{1,t})}\,ds\\
&\quad+ Cm\left(\Omega^z_{1,t}\right)e^{(r\gamma-\alpha)z} + Ce^{\alpha\tau}e^{-\alpha t}\|\phi\|^r_{C_{\gamma,L^r(\Omega^z_{1,t})}} + CC_{\Omega^z_{1,t}}\\
&\quad+ Ce^{-\alpha t}e^{(r\gamma-\alpha)z}\int_{-\infty}^t e^{\alpha s}\|g(s)\|^r_{L^r(\Omega^z_{1,t})}\,ds + Ce^{(\alpha-L)\tau}e^{-(\alpha-L)t}e^{(r\gamma+L-\alpha)z}\|\phi\|^r_{C_{\gamma,L^r(\Omega^z_{1,t})}}\\
&\quad+ Ce^{-(\alpha-L)t}e^{(r\gamma+L-\alpha)z}\int_{-\infty}^t \left(e^{(\alpha-L)s}\|g(s)\|^r_{L^r(\Omega^z_{1,t})}\right)ds. \qquad (87)
\end{aligned}
$$

Furthermore, by (57) and (70), we have

$$
\begin{aligned}
&\|(u_t-M)_+\|^r_{C_{\gamma,L^r(\Omega^z_{1,t})}}\\
&\le e^{\alpha\tau}e^{-\alpha t}\varepsilon + C\varepsilon e^{-(A+\alpha)t}\int_{-\infty}^t e^{(A+\alpha)s}\,ds + C\varepsilon + Ce^{\alpha\tau}e^{-\alpha t}\varepsilon + C\varepsilon\\
&\quad+ Ce^{-\alpha t}\varepsilon\int_{-\infty}^t e^{\alpha s}\,ds + Ce^{(\alpha-L)\tau}e^{-(\alpha-L)t}\varepsilon + Ce^{-(\alpha-L)t}\varepsilon\int_{-\infty}^t e^{(\alpha-L)s}\,ds\\
&\le e^{\alpha\tau}e^{-\alpha t}\varepsilon + C\varepsilon + C\varepsilon + Ce^{\alpha\tau}e^{-\alpha t}\varepsilon + C\varepsilon + C\varepsilon + Ce^{(\alpha-L)\tau}e^{-(\alpha-L)t}\varepsilon + C\varepsilon\\
&\le C\varepsilon, \qquad (88)
\end{aligned}
$$

where $\alpha > L$. Repeating the same steps above, just taking $(u(t+z)-M)_-$ instead of $(u(t+z)-M)_+$, we deduce that

$$
\left\|\left(u(t+z)-M\right)_-\right\|^r_{C_{\gamma,L^r(\Omega^z_{1,t})}} \le C\varepsilon. \qquad (89)
$$

From (88), (89) and Lemma 5.1, we know the hypotheses of Lemma 5.3 are all satisfied. Therefore the process $\{U(t,\tau)\}$ generated by Eq. (1) is $\mathcal{D}$-pullback ω-limit compact. □

Theorem 5.6 *Suppose in addition to the hypotheses in Lemma 4.1 that $g \in C(\mathbb{R}, L^r(\Omega))$. Then the processes $\{U(t,\tau)\}$ on $C_{\gamma,L^r(\Omega)}$ generated by the solution of Eq. (1) with $u_0 \in C_{\gamma,L^r(\Omega)}$ has the $\mathcal{D}$-pullback attractors $\{\mathcal{A}_{C_{\gamma,L^r(\Omega)}}(t)\}_{t\in\mathbb{R}}$.*

Proof From Theorem 7.1, Lemmas 4.1, 5.1 and 5.5, now for every bounded subset B in $C_{\gamma,L^r(\Omega)}$, the process generated by Eq. (1) has the pullback attractors in $C_{\gamma,L^r(\Omega)}$. □

6 Uniform estimates in $C_{\gamma,W^{1,r}(\Omega)}$

Let semigroup e^{At} has the following higher smooth effect [19]:

$$\|e^{At}x\|_{E_r^\beta} \le Mt^{-(\beta-\alpha)}e^{-\delta t}\|x\|_{E_r^\alpha}, \quad x \in E_r^\beta, t > 0, 0 \le \alpha \le \beta, 0 < \delta < \lambda_1. \tag{90}$$

Lemma 6.1 *Suppose the conditions of Lemma 4.1 hold and*

$$\alpha < r(\delta - \eta) \le r\gamma, \quad r > 2, \tag{91}$$

holds, the family of processes $\{U_g(t,\tau)\}$ is uniformly dissipative in $C_{\gamma,W^{1,r}(\Omega)}$, where $g(x,t) \in L^r_{\mathrm{loc}}(\mathbb{R}; L^r(\Omega))$, $\eta > 0$ will be determined later.

Proof Choosing α_1 with $\alpha < \alpha_1$ and using (46), we obtain

$$\begin{aligned}
&\int_\tau^t e^{-\alpha_1(t-s)}\|u_s\|^r_{C_{L^r(\Omega)}}\,ds \\
&\quad\le \int_\tau^t e^{-\alpha_1(t-s)}\biggl(re^{\alpha\tau}e^{-\alpha s}\|\phi\|^r_{C_{\gamma,L^r(\Omega)}} + \frac{\alpha}{\alpha - L}C_\Omega \\
&\qquad + \varepsilon_4^{-(r-1)}e^{-\alpha s}\int_{-\infty}^s e^{\alpha l}\bigl\|g(l)\bigr\|^r_{L^r(\Omega)}\,dl + re^{(\alpha-L)\tau}e^{(L-\alpha)s}\|\phi\|^r_{C_{\gamma,L^r(\Omega)}} \\
&\qquad + \varepsilon_4^{-(r-1)}e^{(L-\alpha)s}\int_{-\infty}^s e^{(\alpha-L)l}\bigl\|g(l)\bigr\|^r_{L^r(\Omega)}\,dl\biggr)\,ds \\
&\quad\le \frac{C}{\alpha_1-\alpha}e^{\alpha\tau}\|\phi\|^r_{C_{\gamma,L^r(\Omega)}} + C + \frac{C}{\alpha_1-\alpha}e^{-\alpha t}\int_{-\infty}^t e^{\alpha s}\bigl\|g(s)\bigr\|^r_{L^r(\Omega)}\,ds \\
&\qquad + \frac{Ce^{(\alpha-L)\tau}e^{(L-\alpha)t}}{\alpha_1-\alpha+L}\|\phi\|^r_{C_{\gamma,L^r(\Omega)}} + \frac{Ce^{(L-\alpha)t}}{\alpha_1-\alpha+L}\int_{-\infty}^t e^{(\alpha-L)s}\bigl\|g(s)\bigr\|^r_{L^r(\Omega)}\,ds \\
&\quad\triangleq Q(\alpha_1,\alpha,L,\tau,\phi,g_0,t).
\end{aligned} \tag{92}$$

It is obvious that $Q(\alpha_1,\alpha,L,\tau,\phi,g_0,t)$ is bounded, as $\tau \to -\infty$. From the well-posedness of (1), we know that the solution of (1) satisfies

$$u(t) = e^{A(t-\tau)}u(\tau) + \int_\tau^t e^{A(t-s)}\bigl[-\lambda u + f(x,u_s) + g(x,s)\bigr]\,ds. \tag{93}$$

Therefore, using (90) and choosing $\alpha_1 > 0$, $\eta > 0$, $q = \frac{r}{r-1} < 2$, $r > 2$ such that $0 < \alpha < r(\delta - \eta) = \alpha_1 < r\gamma$, for each $t \ge \tau$ we obtain

$$\bigl\|u(t)\bigr\|_{W^{1,r}(\Omega)} = \biggl\|e^{A(t-\tau)}u(\tau) + \int_\tau^t e^{A(t-\tau)}\bigl[-\lambda u + f(x,u_s) + g(x,s)\bigr]\,ds\biggr\|_{W^{1,r}(\Omega)}$$

$$
\begin{aligned}
&\le \left\| e^{A(t-\tau)}u(\tau)\right\|_{W^{1,r}(\Omega)} + \lambda \int_\tau^t \left\| e^{A(t-s)}u\right\|_{W^{1,r}(\Omega)}\,ds \\
&\quad + \int_\tau^t \left\| e^{A(t-s)}f(x,u_s)\right\|_{W^{1,r}(\Omega)}\,ds + \int_\tau^t \left\| e^{A(t+z-s)}g(x,s)\right\|_{W^{1,r}(\Omega)}\,ds \\
&\le M_1 e^{-\delta(t-\tau)}\left\| u(\tau)\right\|_{W^{1,r}(\Omega)} + \lambda M_2 \int_\tau^t (t-s)^{-\frac{1}{2}}e^{-\delta(t-s)}\|u\|_{L^r(\Omega)}\,ds \\
&\quad + M_3 \int_\tau^t (t-s)^{-\frac{1}{2}}e^{-\delta(t-s)}\left\| F\bigl(s,u\bigl(s-\rho(s)\bigr)\bigr)\right\|_{L^r(\Omega)}\,ds \\
&\quad + M_4 \int_\tau^t (t-s)^{-\frac{1}{2}}e^{-\delta(t-s)}\left\| \int_{-\infty}^0 G\bigl(s,z,u(s+z)\bigr)\,dz\right\|_{L^r(\Omega)}\,ds \\
&\quad + M_5 \int_\tau^t (t-s)^{-\frac{1}{2}}e^{-\delta(t-s)}\left\| g(x,s)\right\|_{L^r(\Omega)}\,ds.
\end{aligned} \tag{94}
$$

Then, by (46), (92), Hold's inequality and Young's inequality, we have

$$
\begin{aligned}
&\lambda M_2 \int_\tau^t (t-s)^{-\frac{1}{2}}e^{-\delta(t-s)}\|u\|_{L^r(\Omega)}\,ds \\
&\le \lambda M_2 \left(\int_\tau^t (t-s)^{-\frac{1}{2}q}e^{-q\eta(t-s)}\,ds\right)^{\frac{1}{q}} \times \left(\int_\tau^t e^{-r(\delta-\eta)(t-s)}\|u\|_{L^r(\Omega)}^r\right)^{\frac{1}{r}} \\
&\le \frac{\lambda M_2}{q}\left(\int_\tau^t (t-s)^{-\frac{1}{2}q}e^{-q\eta(t-s)}\,ds\right) + \frac{\lambda M_2}{r}\left(\int_\tau^t e^{-r(\delta-\eta)(t-s)}\|u\|_{L^r(\Omega)}^r\,ds\right) \\
&\le \frac{\lambda M_2\Gamma(1-\frac{q}{2})}{q^{2-\frac{1}{2}q}\eta^{1-\frac{1}{2}q}} + \frac{\lambda M_2}{r}\left(\int_\tau^t e^{-r(\delta-\eta)(t-s)}\|u\|_{L^r(\Omega)}^r\,ds\right) \\
&\le \frac{\lambda M_2\Gamma(1-\frac{q}{2})}{q^{2-\frac{1}{2}q}\eta^{1-\frac{1}{2}q}} + \frac{\lambda M_2}{r}Q\bigl(r(\delta-\eta),\tau,\phi,g_0,t\bigr) \\
&\triangleq \frac{\lambda M_2\Gamma(1-\frac{q}{2})}{q^{2-\frac{1}{2}q}\eta^{1-\frac{1}{2}q}} + R_{2,W^{1,r}(\Omega)}\bigl(r(\delta-\eta),\tau,\phi,g_0,t\bigr).
\end{aligned} \tag{95}
$$

Similarly, combining (2), (3), and (6), we have

$$
\begin{aligned}
&M_3 \int_\tau^t (t-s)^{-\frac{1}{2}}e^{-\delta(t-s)}\left\| F\bigl(x,u\bigl(s-\rho(s)\bigr)\bigr)\right\|_{L^r(\Omega)}\,ds \\
&\le M_3 \left(\int_\tau^t (t-s)^{-\frac{1}{2}q}e^{-q\eta(t-s)}\,ds\right)^{\frac{1}{q}} \times \left(\int_\tau^t e^{-r(\delta-\eta)(t-s)}\|F\|_{L^r(\Omega)}^r\right)^{\frac{1}{r}} \\
&\le \frac{M_3}{q}\left(\int_\tau^t (t-s)^{-\frac{1}{2}q}e^{-q\eta(t-s)}\,ds\right) + \frac{M_3}{r}\left(\int_\tau^t e^{-r(\delta-\eta)(t-s)}\|F\|_{L^r(\Omega)}^r\,ds\right) \\
&\le \frac{M_3\Gamma(1-\frac{q}{2})}{q^{2-\frac{1}{2}q}\eta^{1-\frac{1}{2}q}} + \frac{M_3}{r}\int_\tau^t e^{-r(\delta-\eta)(t-s)}\Bigl(k_1^r\|\Omega\|_{L^r(\Omega)}^r + k_2^r e^{-r\gamma\rho(t)}\left\| u\bigl(s-\rho(s)\bigr)\right\|_{L^r(\Omega)}^r\Bigr)\,ds \\
&\le \frac{M_3\Gamma(1-\frac{q}{2})}{q^{2-\frac{1}{2}q}\eta^{1-\frac{1}{2}q}} + \frac{M_3k_1^r|\Omega|_{L^r(\Omega)}^r}{r^2(\delta-\eta)} + \frac{k_2^rM_3}{r}Q\bigl(r(\delta-\eta),\tau,\phi,g_0,h,t\bigr) \\
&\triangleq \frac{M_3\Gamma(1-\frac{r}{2})}{q^{2-\frac{1}{2}q}\eta^{1-\frac{1}{2}q}} + \frac{M_3k_1^r|\Omega|_{L^r(\Omega)}^r}{r^2(\delta-\eta)} + R_{3,W^{1,r}(\Omega)}\bigl(r(\delta-\eta),\tau,\phi,g_0,t\bigr),
\end{aligned} \tag{96}
$$

$$
\begin{aligned}
& M_4 \int_\tau^t (t-s)^{-\frac{1}{2}} e^{-\delta(t-s)} \left\| \int_{-\infty}^0 G\big(s,z,u(s+z)\big)\,dz \right\|_{L^r(\Omega)} ds \\
& \quad \le M_4 \left(\int_\tau^t (t-s)^{-\frac{1}{2}q} e^{-q\eta(t-s)}\,ds \right)^{\frac{1}{q}} \\
& \qquad \times \left(\int_\tau^t e^{-r(\delta-\eta)(t-s)} \left\| \int_{-\infty}^0 \big(m_0(z) + m_1(z)\big|u(s+z_0)\big|\big)\,dz \right\|_{L^r(\Omega)}^r \right)^{\frac{1}{r}} \\
& \quad \le \frac{M_4\Gamma(1-\frac{q}{2})}{q^{2-\frac{1}{2}q}\eta^{1-\frac{1}{2}q}} + \frac{M_4}{r}\left(m_0^r|\Omega|^r \int_\tau^t e^{-r(\delta-\eta)(t-s)}\,ds + m_1^r \int_\tau^t e^{-r(\delta-\eta)(t-s)} \|u_s\|_{C_{\gamma,L^r(\Omega)}}^r\,ds \right) \\
& \quad \triangleq \frac{M_4\Gamma(1-\frac{q}{2})}{q^{2-\frac{1}{2}q}\eta^{1-\frac{1}{2}q}} + \frac{2^{r-1}M_4 m_0^r|\Omega|^r}{r^2(\delta-\eta)} + R_{4,W^{1,r}(\Omega)}\big(r(\delta-\eta),\tau,\phi,g_0,t\big),
\end{aligned}
\tag{97}
$$

and

$$
\begin{aligned}
& \int_\tau^t \big\| e^{A(t-s)} g(x,s) \big\|_{W^{1,r}(\Omega)}\,ds \\
& \quad \le M_5 \int_\tau^t (t-s)^{-\frac{1}{2}} e^{-\delta(t-s)} \|g\|_{L^r(\Omega)}\,ds \\
& \quad \le M_5 \int_\tau^t (t-s)^{-\frac{1}{2}} e^{-(\delta-\eta)(t-s)} e^{-\delta(t-s)} \|g\|_{L^r(\Omega)}\,ds \\
& \quad \le M_5 \left(\int_\tau^t (t-s)^{-\frac{1}{2}} e^{-q\delta(t-s)}\,ds \right)^{\frac{1}{q}} \times \left(\int_\tau^t e^{-r(\delta-\eta)(t-s)} \|g\|_{L^r(\Omega)}^r\,ds \right)^{\frac{1}{r}} \\
& \quad \le \frac{M_5}{q}\left(\int_\tau^t (t-s)^{-\frac{1}{2}} e^{-q\delta(t-s)}\,ds \right) + \frac{M_5}{r}\left(\int_{-\infty}^t e^{-r(\delta-\eta)(t-s)} \|g\|_{L^r(\Omega)}^r\,ds \right) \\
& \quad \triangleq \frac{M_5\Gamma(1-\frac{q}{2})}{q^{2-\frac{1}{2}q}\eta^{1-\frac{1}{2}q}} + R_{5,W^{1,r}(\Omega)}\big(r(\delta-\eta),\tau,q,g,t\big).
\end{aligned}
\tag{98}
$$

Similar to the arguments in Lemma 4.1, for each $t \in \mathbb{R}$, we can conclude that by (91)

$$
\begin{aligned}
& \sup_{z\in[-\infty,0]} e^{-r\gamma z} \big\| u(t+z) \big\|_{W^{1,r}(\Omega)} \\
& \quad \le M_1 e^{-\delta(t-\tau)} \big\| u(\tau) \big\|_{W^{1,r}(\Omega)} + \frac{(\lambda M_2 + M_3 + M_4 + M_5)\Gamma(1-\frac{r}{2})}{r^{2-\frac{1}{2}r}\eta^{1-\frac{1}{2}r}} \\
& \qquad + R_{2,W^{1,r}(\Omega)}\big(r(\delta-\eta),\tau,\phi,g_0,t\big) + \frac{M_3 k_1^r|\Omega|^r}{r^2(\delta-\eta)} \\
& \qquad + R_{3,W^{1,r}}\big(r(\delta-\eta),\phi,\tau,g_0,t\big) + \frac{2^{r-1}M_4 m_0^r|\Omega|^r}{r^2(\delta-\eta)} \\
& \qquad + R_{4,W^{1,r}}\big(r(\delta-\eta),\tau,\phi,g_0,t\big) + R_{5,W^{1,r}(\Omega)}\big(r(\delta-\eta),\tau,q,g,t\big) \\
& \quad \triangleq R_{6,W^{1,r}(\Omega)}\big(r(\delta-\eta),\tau,r,\phi,g_0,t\big), \quad \text{for each } t \in \mathbb{R}.
\end{aligned}
\tag{99}
$$

Hence, we can see that $\sup_{z\in[-\infty,0]} e^{-r\gamma z}\|u(t+z)\|_{W^{1,r}(\Omega)}$ is bounded, for each $t \in \mathbb{R}$, $z \in (-\infty,0]$, as $\tau \to -\infty$, which implies the process $\{U(t,\tau)\}$ has pullback absorbing sets in $C_{\gamma,W^{1,r}(\Omega)}$. □

7 Existence of the pullback attractors in $C_{\gamma,W^{1,r}(\Omega)}$

Theorem 7.1 *Suppose in additional to the hypotheses in Lemma 6.1 and $g(s) \in C(\mathbb{R}, W^{1,r}(\Omega))$, $F \in C^1(\mathbb{R} \times \mathbb{R}; \mathbb{R})$, $G \in C^1(\mathbb{R} \times \mathbb{R} \times \mathbb{R}; \mathbb{R})$, $\frac{\partial F}{\partial x}$, $\frac{\partial G}{\partial x}$ are both bounded. Then the processes $\{U(t,\tau)\}$ on $C_{\gamma,W^{1,r}(\Omega)}$ generated by the solution of Eq. (1) with $\phi \in C_{\gamma,W^{1,r}(\Omega)}$ has the pullback attractors $\mathcal{A}_{C_{\gamma,W^{1,r}(\Omega)}}$.*

Proof We divide the proof into three steps.

Step 1. Taking gradient operator ∇ to act on (1), we can obtain

$$\begin{aligned} \frac{\partial \nabla u}{\partial t} - \Delta \nabla u + \lambda \nabla u = {} & \frac{\partial F}{\partial x} + \frac{\partial F}{\partial u} \nabla u\bigl(t-\rho(t),x\bigr) + \int_{-\infty}^{0} \frac{\partial G}{\partial x}\,dz \\ & + \int_{-\infty}^{0} \frac{\partial G}{\partial u} \nabla u(t+z,x)\,dz + \nabla g(t,x). \end{aligned} \tag{100}$$

Multiplying (100) by $|\nabla u|^{r-2}\nabla u$ and integrating by parts, we get

$$\begin{aligned} & \frac{1}{r}\frac{d}{dt}\bigl\|\nabla u(t)\bigr\|_{L^r(\Omega)}^r + \frac{4(r-1)}{r^2}\int_\Omega \bigl|\nabla\bigl(\bigl|\nabla u(t)\bigr|^{\frac{r}{2}}\bigr)\bigr|^2\,dx + \int_\Omega \lambda\bigl|\nabla u(t)\bigr|^r\,dx \\ & \quad = \int_\Omega \frac{\partial F}{\partial x}\bigl|\nabla u(t)\bigr|^{r-2}\nabla u(t)\,dx + \int_\Omega \frac{\partial F}{\partial u}\nabla u\bigl(t-\rho(t),x\bigr)|\nabla u|^{r-2}\nabla u\,dx \\ & \qquad + \int_\Omega\int_{-\infty}^0 \frac{\partial G}{\partial x}\bigl|\nabla u(t)\bigr|^{r-2}\nabla u(t)\,dz\,dx + \int_\Omega\int_{-\infty}^0 \frac{\partial G}{\partial u}\nabla u(t+z,x)|\nabla u|^{r-2}\nabla u\,dz\,dx \\ & \qquad + \int_\Omega \nabla g(t,x)\bigl|\nabla u(t)\bigr|^{r-2}\nabla u(t)\,dx. \end{aligned} \tag{101}$$

By the same arguments as Lemma 4.1, we also obtain the process $\{U(t,\tau)\}$ generating by (100) has pullback absorbing sets in $C_{\gamma,W^{1,r}(\Omega)}$.

Step 2. According to Theorem 15 [10], Eq. (1) has a pullback attractor $\mathcal{A}_{C_{\gamma,H^1(\Omega)}}$. Hence, by the same arguments as Theorem 5.6, we also obtain the process $\{U(t,\tau)\}$ generating by Eq. (100) on $C_{\gamma,L^2(\Omega)}$ is ω-limit compact.

Step 3. Combining step 1, step 2, and Lemma 6.1, as the proof of Theorem 5.6, we find that the process $\{U(t,\tau)\}$ generated by Eq. (100) on $C_{\gamma,W^{1,r}(\Omega)}$ has pullback absorbing sets and is $\mathcal{D}$ pullback ω-limit compact. Thus, we know from Theorem 5.6 the process $\{U(t,\tau)\}$ generating by Eq. (1) has the pullback attractors $\mathcal{A}_{C_{\gamma,W^{1,r}(\Omega)}}$. □

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Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors read and approved the final manuscript. YR finished the manuscript and JL made the content correction and English language checking.

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