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Vanishing viscosity limit for Riemann solutions to zero-pressure gas dynamics with flux perturbation

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Abstract

In this paper, by the viscosity vanishing approach, we consider the Riemann problem for zero-pressure gas dynamics with flux perturbation. The Riemann solutions involve parameterized delta shock wave and constant density state. For the parameterized delta-shock solution, its generalized Rankine–Hugoniot and entropy condition are clarified. While for the constant density solution, the formation of it is rigorously analyzed. Moreover, all of their existence, uniqueness, and stability to reasonable viscous perturbations are shown.

Keywords: Zero-pressure gas dynamics; Constant density solution; Flux perturbation; Delta shock wave; Limiting viscosity approach

1 Introduction

The well-known zero-pressure gas dynamics reads

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + (\rho u^2)_x = 0, \end{cases} \tag{1.1}$$

where $\rho \geq 0$ is the density and u is the velocity. These equations are also called the transport equations, or Euler equations for pressureless fluids, which have been systemically studied by a large number of scholars since 1994. They can be regarded as the direct result by taking the pressure $p = 0$ in the isentropic Euler equations in gas dynamics [1]. They also can be obtained from Boltzmann equations [2] and the flux-splitting scheme of the full compressible Euler equations [3, 4]. System (1.1) is used to model the motion of free particles which stick under collision [5] and the formation of large-scale structures in the universe [6, 7].

For system (1.1), Bouchut [2] presented the existence of measure solutions of the Riemann problem. Weinan et al. [7] discussed the existence of global weak solution and the behavior of such global solution with random initial data. The 1-D and 2-D Riemann problems were constructively solved by Sheng and Zhang [8], and a new kind of discontinuity, called delta shock wave, was found in the Riemann solutions. A delta shock wave is a generalization of an ordinary shock wave, on which at least one of the state variables may develop an extreme concentration in the form of a weighted Dirac delta function with the discontinuity as its support. It is more compressive than an ordinary shock wave and is often used to describe the process of formation of the galaxies in the universe and the process of concentration of particles. In the past over two decades, the investigation of delta shock waves has been increasingly active. Specifically, the study on the stability of delta-shock solution is much more important and interesting.

In the discussing of stability of a delta shock wave, the vanishing viscosity method is one of the most vital ways. In fact, it is a very popular approach to constructing discontinuous solutions of the Cauchy problem for the conservation law

$$u_t + \big(f(u)\big)_x = 0. \tag{1.2}$$

This method consists in viewing (1.2) as the limit of the equation

$$u_t + \big(f(u)\big)_x = \varepsilon u_{xx} \tag{1.3}$$

for $\varepsilon \to 0^+$. The difficulty for this regularization is that (1.3) does not possess space–time expanding invariance ($(x,t) \to (\alpha x, \alpha t), \alpha > 0$). To overcome this difficulty, Dafermos [9], Kalasnikov [10], and Tupciev [11] independently suggested the viscous regularization given by

$$u_t + \big(f(u)\big)_x = \varepsilon t u_{xx}, \tag{1.4}$$

which possesses the desired space–time expanding invariance and admits solutions that depend only on the self-similar independent variable ξ ($\xi = x/t$). For this kind of vanishing viscosity approach, a small amount of diffusion or viscosity makes the mathematical model more realistic in most applications. In addition, the shocks constructed by this method are physical ones, since they satisfy the entropy inequalities.

Specially, in order to obtain the stability of the delta shock wave of (1.1), by using the vanishing viscosity method, Sheng and Zhang [8] considered the following regularized system:

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + (\rho u^2)_x = \varepsilon t u_{xx}, \end{cases} \tag{1.5}$$

where $\varepsilon > 0$ is a small parameter. All of the existence, uniqueness, and stability of solutions were investigated to viscous perturbations. See also Yang [12] for the generalized zero-pressure system and [13–23] for the viscosity vanishing approach on various systems of conservation laws.

In addition, there are a lot of different approaches to studying the formation of a delta shock wave, such as the perturbation of the Coulomb-like friction term [24, 25], the weak asymptotic method [26–28], the shadow wave method [29], and so on. Here, we are pleased to introduce the flux-approximation method proposed by Yang and Liu [30]. The main idea of it is to introduce some small perturbed parameters in the flux function of the system, and then discuss the limits of solutions to the perturbed system by letting perturbed parameters drop to zero. They analyzed the limits of solutions to the perturbed isentropic system

$$\begin{cases} \rho_t + (\rho u - 2\epsilon_1 u)_x = 0, \\ (\rho u)_t + (\rho u^2 - \epsilon_1 u^2 + \epsilon_2 p)_x = 0 \end{cases} \tag{1.6}$$

with $p = \frac{\rho^\gamma}{\gamma}$ ($\gamma > 1$), where $\rho \geq 2\epsilon_1$, ϵ_1, $\epsilon_2 > 0$ are parameters modeling the strength of flux and pressure, respectively. They proved that the limits of Riemann solutions of (1.6) involving two shock waves and two rarefaction waves tend to a delta-shock solution and a vacuum state to the zero-pressure gas dynamics (1.1), respectively. This implies that both the delta-shock and vacuum solutions of (1.1) are stable under some small perturbations of flux. See also the papers [31–33] for more discussions on the flux-approximation method.

System (1.6) is an archetype of hyperbolic systems of conservation laws of the form

$$u_t + \bigl(f(u, \epsilon_1, \epsilon_2)\bigr)_x = 0 \tag{1.7}$$

with $u = (\rho, \rho u)^T$ and $f(u, \epsilon_1, \epsilon_2) = (\rho u - 2\epsilon_1 u, \rho u^2 - \epsilon_1 u^2 + \epsilon_2 p)^T$, where T represents transpose. As $\epsilon_1 = 0$, system (1.6) is nothing but the Euler equations of isentropic gas dynamics with pressure perturbation. By using the vanishing pressure limit method, Chen and Liu [34] identified the stability of the delta shock wave of (1.1) under the pressure perturbation, which was equivalent to the formation of delta shock waves and vacuum states in solutions of system (1.6) as $\epsilon_2 \to 0$. Further, in [35] they also studied vanishing pressure limit of solutions to the nonisentropic fluids. See also Li [36] for the isothermal Euler equations with zero temperature. Now, the vanishing pressure limit method has been widely used and the results were extended to the relativistic Euler equations by Yin et al. [37–39], to the perturbed Aw–Rascle model by Shen and Sun [40], to the modified Chaplygin gas equations by Yang and Wang [41, 42], etc. It is clear that the flux-approximation method is indeed a natural generalization of the vanishing pressure limit method.

While as $\epsilon_2 = 0$, (1.6) becomes the following perturbed zero-pressure gas dynamics:

$$\begin{cases} \rho_t + (\rho u - 2\epsilon_1 u)_x = 0, \\ (\rho u)_t + (\rho u^2 - \epsilon_1 u^2)_x = 0, \end{cases} \tag{1.8}$$

which is a pure flux approximation. The Riemann problem of (1.8) was solved in [30]. The Riemann solutions including a parameterized delta shock wave depending on ϵ_1 and a constant density state ($\rho = 2\epsilon_1$) were obtained. Compared with the transport equations (1.1), the vacuum state here is removed, while the weight of a delta shock wave decreases. These imply that the flux perturbation works in the pressureless fluids.

Motivated by systems (1.5) and (1.8), we are intensely curious if the flux perturbation will have impact on the stability of delta-shock and vacuum state solutions of the zero-pressure gas dynamics under viscosity approach. Therefore, we consider the flux-perturbation viscosity regularized problem

$$\begin{cases} \rho_t + (\rho u - 2\epsilon_1 u)_x = 0, \\ (\rho u)_t + (\rho u^2 - \epsilon_1 u^2)_x = \varepsilon t u_{xx} \end{cases} \tag{1.9}$$

with the initial data

$$(\rho, u)(0, x) = (\rho_\pm, u_\pm) \quad (\pm x > 0). \tag{1.10}$$

Physically, a reasonable perturbation can be used to govern some dynamical behaviors of fluids, so it is worth studying the vanishing viscosity limit for Riemann solutions to the zero-pressure gas dynamics with flux perturbation. As stated in [35], small external forces imposed on the fluids lead to deformation of a fluid particle. The small forces can be regarded as a flux perturbation in terms of mechanics. What is more, although the flux-perturbation parameter ϵ_1 can be considered very small and reflects the strength of the flux, it does not vanish in general. We propose to include this parameter in hope of investigating the effect of a flux approximation to the stability of the delta-shock and vacuum state solutions to the zero-pressure gas dynamics under viscosity approach. Obviously, in contrast to the previous works in [8, 12, 18], we here develop a viscosity approach which contains flux approximation in the considered systems. So this paper to some extent extends the results and proofs in [8].

Using Schauder's fixed point theorem, we first consider the existence of self-similar solution for (1.9) and (1.10) in $[-A, A]$, where A is a sufficiently large real number. Then the obtained solution is extended to the whole interval $(-\infty, +\infty)$. Here, we use a new idea and skill to reach our goal. Furthermore, we investigate the limit of solution of (1.9) and (1.10) when $\varepsilon \to 0^+$. Concretely, when $u_- < u_+$, we rigorously analyze how constant density solution is formed. While if $u_- > u_+$, the limit solution of (1.9) and (1.10) generates the parameterized delta-shock solution of (1.8) and (1.10). At this moment, the limit functions $\rho(x,t)$ is the sum of a step function and a Dirac delta function, $u(x,t)$ is a step function. These facts show that delta-shock and constant density solutions are stable to the reasonable viscous perturbations under flux approximation. It also confirms the mathematical reasonability of the flux perturbation from another perspective. Furthermore, in the process of proof, one can easily observe that the Riemann solutions to (1.9) and (1.10) converge to those of the zero-pressure gas dynamics (1.1) with the same initial data when $\epsilon_1, \varepsilon \to 0$.

This paper is organized as follows. In Sect. 2, for readers' convenience, we present some preliminary knowledge of system (1.8) and (1.10). Section 3 shows the existence of solution to the viscous system (1.9) and (1.10). Then, as the viscosity vanishes, we discuss the limit of solutions of the viscous system in Sect. 4. Finally, a brief conclusion is presented in Sect. 5.

2 Riemann solutions to system (1.8) and (1.10)

We briefly recall the Riemann solutions for system (1.8) and (1.10) in this section. See [30] for more details.

The characteristic roots and corresponding right characteristic vectors of the system are $\lambda = u$ and $r = (1,0)^T$, with $\nabla\lambda \cdot \vec{r} = 0$. So the system is full linear degenerate and elementary waves only involve contact discontinuities.

Under the self-similar transformation $\xi = x/t$, besides the constant state solution, the system provides the singular solution $\rho = 2\epsilon_1$, $u = \xi$, while the elementary wave has only contact discontinuity $J : \xi = u_- = u_+$.

For case $u_- < u_+$, one can construct solution of the Riemann problem as follows:

$$(\rho, u)(\xi) = \begin{cases} (\rho_-, u_-), & -\infty < \xi < u_-, \\ (2\epsilon_1, \xi), & u_- \leq \xi \leq u_+, \\ (\rho_+, u_+), & u_+ < \xi < +\infty, \end{cases} \tag{2.1}$$

which includes two contact discontinuities and a constant-density state besides two constant states.

For the case $u_- > u_+$, the solution contains a delta shock wave. In order to define the delta-shock solution of (1.8) and (1.10) in the sense of distributions, a two-dimensional weighted delta function $w(s)\delta_S$ supported on a smooth curve S parameterized as $t = t(s)$, $x = x(s)$ ($a \leq s \leq b$) is introduced as

$$\bigl\langle w\bigl(t(s)\bigr)\delta_S, \varphi(t,x)\bigr\rangle = \int_a^b w\bigl(t(s)\bigr)\varphi\bigl(t(s), x(s)\bigr)\sqrt{x'(s)^2 + t'(s)^2}\,\mathrm{d}s \tag{2.2}$$

for all test functions $\varphi \in C_0^\infty([0,+\infty) \times (-\infty,+\infty))$.

By this definition, the delta shock wave type solution in the sense of distributions is defined as

$$\rho = \rho_0(x,t) + w\delta_s, \qquad u = u_0(x,t),$$

where $S = \{(\sigma t, t) : 0 \leq t < \infty\}$,

$$\rho_0(x,t) = \rho_- + [\rho]H(x - \sigma t), \qquad u_0(x,t) = u_- + [u]H(x - \sigma t),$$
$$w(t) = \frac{t}{\sqrt{1+\sigma^2}}\bigl(\sigma[\rho] - [\rho u - 2\epsilon_1 u]\bigr),$$

in which $[G] = G_+ - G_-$ expresses the jump of the quality G across the curve S, σ is the tangential derivative of the curve S, and $H(x)$ is the Heaviside function.

The solution (ρ, u) constructed above satisfies that

$$\langle \rho, \varphi_t\rangle + \langle \rho u - 2\epsilon_1 u, \varphi_x\rangle = 0, \tag{2.3}$$
$$\langle \rho u, \varphi_t\rangle + \bigl\langle \rho u^2 - \epsilon_1 u^2, \varphi_x\bigr\rangle = 0 \tag{2.4}$$

for all test functions $\varphi \in C_0^\infty([0,+\infty) \times (-\infty,+\infty))$, where

$$\langle \rho, \varphi\rangle = \int_0^{+\infty}\int_{-\infty}^{+\infty} \rho_0\varphi\,\mathrm{d}x\,\mathrm{d}t + \langle w\delta_S, \varphi\rangle,$$
$$\langle \rho u, \varphi\rangle = \int_0^{+\infty}\int_{-\infty}^{+\infty} \rho_0 u_0\varphi\,\mathrm{d}x\,\mathrm{d}t + \langle \sigma w\delta_S, \varphi\rangle,$$

u, u^2 and ρu^2 have similar integral identities as above.

With this definition, the delta-shock solution of (1.8) and (1.10) has the form

$$(\rho, u)(t,x) = \begin{cases} (\rho_-, u_-), & x < x(t), \\ (w(t)\delta(x - x(t)), u_\delta), & x = x(t), \\ (\rho_+, u_+), & x > x(t), \end{cases} \tag{2.5}$$

where $x(t)$, $w(t)$, and u_δ satisfy the following generalized Rankine–Hugoniot relation:

$$\begin{cases} \frac{\mathrm{d}x}{\mathrm{d}t} = u_\delta = \sigma, \\ \frac{\mathrm{d}\sqrt{1+\sigma^2}w(t)}{\mathrm{d}t} = \sigma[\rho] - [\rho u - 2\epsilon_1 u], \\ \frac{\mathrm{d}\sqrt{1+\sigma^2}w(t)\sigma}{\mathrm{d}t} = \sigma[\rho u] - [\rho u^2 - \epsilon_1 u^2]. \end{cases} \tag{2.6}$$

Besides, the discontinuity should satisfy the entropy condition

$$u_+ < \sigma < u_-.$$

Then we solve the generalized Rankine–Hugoniot relation (2.6) with initial conditions $t = 0$: $x(0) = 0$, $w(0) = 0$. By a routine calculation, when $[\rho] \neq 0$, one can easily obtain that

$$x(t) = \frac{[\rho u] + \sqrt{[\rho u]^2 - [\rho]([2\epsilon_1 u]\sigma + [\rho u^2 - \epsilon_1 u^2])}}{[\rho]} t,$$

$$u_\delta = \sigma = \frac{[(\rho - \epsilon_1)u] + \sqrt{(\rho_- - \epsilon_1)(\rho_+ - \epsilon_1)}(u_- - u_+)}{[\rho]},$$

$$w(t) = \frac{[\epsilon_1 u] + \sqrt{(\rho_- - \epsilon_1)(\rho_+ - \epsilon_1)}(u_- - u_+)}{\sqrt{1+\sigma^2}} t,$$

while when $[\rho] = 0$, $x(t) = \frac{u_- + u_+}{2} t$, $u_\delta = \sigma = \frac{u_- + u_+}{2}$, $w(t) = \frac{[2\epsilon_1 u - \rho u]}{\sqrt{1+\sigma^2}} t$.

3 Existence of solutions to the viscous system (1.9) and (1.10)

In this section, we show that the viscosity regularized problem (1.9) with initial data (1.10) has a smooth self-similar solution. Equivalently, we consider the boundary value problem

$$\begin{cases} -\xi\rho_\xi + (\rho u - 2\epsilon_1 u)_\xi = 0, \\ -\xi(\rho u)_\xi + (\rho u^2 - \epsilon_1 u^2)_\xi = \varepsilon u_{\xi\xi} \end{cases} \tag{3.1}$$

and

$$(\rho, u)(\pm\infty) = (\rho_\pm, u_\pm). \tag{3.2}$$

For system (3.1) and (3.2), we have the following theorem of existence.

Theorem 3.1 *There exists a weak solution*

$$(\rho, u) \in L^1(-\infty, +\infty) \times C^2(-\infty, +\infty)$$

for the boundary value problem (3.1), (3.2).

In order to prove this theorem, we first consider the existence of solutions of system (3.1), (3.2) in the interval $[-A, A]$, where A is a sufficiently large real number, with the boundary condition

$$(\rho, u)(\pm A) = (\rho_\pm, u_\pm). \tag{3.3}$$

The main idea is to use Schauder's fixed point theorem, so we take

$$B = C^2[-A,A], \qquad K = \{U \mid U \in B, U(\pm A) = u_\pm, U \text{ is monotone}\}.$$

Obviously, K is a bounded convex closed set in B, a Banach space.

Lemma 3.2 *For any $U \in K$, the problem*

$$\begin{cases} -\xi\rho_\xi + (\rho U - 2\epsilon_1 U)_\xi = 0, \\ \rho(\pm A) = \rho_\pm, \end{cases} \tag{3.4}$$

possesses a weak solution $\rho \in L^1[-A,A]$.

(i) *When $u_- > u_+$,*

$$\rho(\xi) = \begin{cases} \rho_1(\xi), & -A \le \xi < \xi_\sigma, \\ \rho_2(\xi), & \xi_\sigma < \xi \le A, \end{cases} \tag{3.5}$$

where ξ_σ is a unique solution of equation

$$U(\xi_\sigma) = \xi_\sigma, \tag{3.6}$$

$\rho_1(\xi)$ is increasing in $(-A,\xi_\sigma)$, while $\rho_2(\xi)$ is decreasing in (ξ_σ,A).

(ii) *When $u_- < u_+$,*

$$\rho(\xi) = \begin{cases} \rho_1(\xi), & -A \le \xi < \xi_{\sigma_1}, \\ 2\epsilon_1, & \xi_{\sigma_1} \le \xi \le \xi_{\sigma_2}, \\ \rho_2(\xi), & \xi_{\sigma_2} < \xi \le A, \end{cases} \tag{3.7}$$

where $\xi_{\sigma_1} \le \xi_{\sigma_2}$ satisfying

$$\xi_{\sigma_1} = \min\{\xi \mid U(\xi) = \xi\}, \qquad \xi_{\sigma_2} = \max\{\xi \mid U(\xi) = \xi\} \tag{3.8}$$

and

$$\lim_{\xi\to\xi_{\sigma_1}^-} \rho_1(\xi) = \lim_{\xi\to\xi_{\sigma_2}^+} \rho_2(\xi) = 2\epsilon_1, \tag{3.9}$$

$\rho_1(\xi)$ is decreasing in $(-A,\xi_{\sigma_1})$, while $\rho_2(\xi)$ is increasing in (ξ_{σ_2},A).

Formulae of $\rho_1(\xi)$ and $\rho_2(\xi)$ in (3.5) and (3.7) can be given as

$$\rho_1(\xi) = 2\epsilon_1 + (\rho_- - 2\epsilon_1)\exp\left(\int_{-A}^{\xi} \frac{-U'(s)}{U(s)-s}\right) ds \tag{3.10}$$

and

$$\rho_2(\xi) = 2\epsilon_1 + (\rho_+ - 2\epsilon_1)\exp\left(\int_{\xi}^{A} \frac{U'(s)}{U(s)-s}\right) ds, \tag{3.11}$$

where $' = d/d\xi$.

Proof (i) When $u_- > u_+$, the equation in (3.4) can be rewritten as

$$\big(U(\xi)-\xi\big)\rho_\xi + (\rho - 2\epsilon_1)U_\xi = 0. \tag{3.12}$$

Obviously, the singularity point of it is given by the solution of equation (3.6). $U(\xi)$ is decreasing, the uniqueness of singularity point can be easily obtained, we denote it by ξ_σ. The solution (3.5) with formulae (3.10) and (3.11) can be obtained by integrating (3.12) from $-A$ to ξ or ξ to A, respectively. The monotonicity of $\rho_1(\xi)$ and $\rho_2(\xi)$ is apparent from formulae of solution. Besides this, one easily obtains

$$\lim_{\xi\to\xi_\sigma^-}\rho_1(\xi) = +\infty, \qquad \lim_{\xi\to\xi_\sigma^+}\rho_2(\xi) = +\infty. \tag{3.13}$$

Next, we show that $\rho(\xi)$ is a weak solution of (3.4) and $\rho \in L^1[-A,A]$. Integrating (3.12) on $[-A,\xi]$ for $-A<\xi<\xi_\sigma$, we have

$$\big(U(\xi)-\xi\big)\rho_1(\xi) + \int_{-A}^{\xi}\rho_1(s)\,ds = (A+u_-)\rho_- + 2\epsilon_1 U(\xi) - 2\epsilon_1 u_-. \tag{3.14}$$

Set

$$p(\xi) = \int_{-A}^{\xi}\rho_1(s)\,ds, \quad A_1 = (A+u_-)\rho_-, a(\xi) = U(\xi)-\xi. \tag{3.15}$$

Equation (3.14) can be written as

$$\begin{cases} a(\xi)p'(\xi) + p(\xi) = A_1 + 2\epsilon_1 U(\xi) - 2\epsilon_1 u_-, \\ p(-A) = 0. \end{cases} \tag{3.16}$$

Solving (3.16), we obtain

$$\begin{aligned} p(\xi) = {}&(A_1 - 2\epsilon_1 u_-)\left\{1-\exp\left(\int_{-A}^{\xi}\frac{-ds}{a(s)}\right)\right\} \\ &+ \exp\left(\int_{-A}^{\xi}\frac{-ds}{a(s)}\right)\int_{-A}^{\xi}\frac{2\epsilon_1 U(r)}{a(r)}\exp\left(\int_{-A}^{r}\frac{ds}{a(s)}\right)dr. \end{aligned}$$

For the term $\int_{-A}^{\xi}\frac{2\epsilon_1 U(r)}{a(r)}\exp(\int_{-A}^{r}\frac{ds}{a(s)})\,dr$ in the expression above, using the second mean value theorem for integrals, we have

$$\begin{aligned} p(\xi) = {}&(A_1 - 2\epsilon_1 u_-)\left\{1-\exp\left(\int_{-A}^{\xi}\frac{-ds}{a(s)}\right)\right\} \\ &+ \exp\left(\int_{-A}^{\xi}\frac{-ds}{a(s)}\right)\left\{U(-A)\int_{-A}^{\zeta}\frac{2\epsilon_1}{a(r)}\exp\left(\int_{-A}^{r}\frac{ds}{a(s)}\right)dr \right. \\ &\left. + U(\xi)\int_{\zeta}^{\xi}\frac{2\epsilon_1}{a(r)}\exp\left(\int_{-A}^{r}\frac{ds}{a(s)}\right)dr\right\} \end{aligned}$$

$$
\begin{aligned}
&= (A_1 - 2\epsilon_1 u_-)\left\{1 - \exp\left(\int_{-A}^{\xi} \frac{-ds}{a(s)}\right)\right\} \\
&\quad + 2\epsilon_1 u_-\left(\exp\left(\int_{\zeta}^{\xi} \frac{-ds}{a(s)}\right) - \exp\left(\int_{-A}^{\xi} \frac{-ds}{a(s)}\right)\right) \\
&\quad + 2\epsilon_1 U(\xi)\left(1 - \exp\left(\int_{\zeta}^{\xi} \frac{-ds}{a(s)}\right)\right),
\end{aligned}
$$

where $\zeta \in (-A, \xi)$.

Because $a(\xi) > 0$ and $a(\xi) = O(|\xi - \xi_\sigma|)$ as $\xi \to \xi_\sigma^-$, one has

$$
\lim_{\xi \to \xi_\sigma^-} \int_{-A}^{\xi} \rho_1(s)\, ds = A_1 - 2\epsilon_1 u_- + 2\epsilon_1 U(\xi_\sigma). \tag{3.17}
$$

Hence

$$
\lim_{\xi \to \xi_\sigma^-} \big(U(\xi)\big) - \xi)\rho_1(\xi) = 0. \tag{3.18}
$$

For $\xi_\sigma < \xi < A$, in the same way as above, the following results

$$
\lim_{\xi \to \xi_\sigma^+} \int_{\xi}^{A} \rho_2(s)\, ds = A_2 - 2\epsilon_1 u_+ + 2\epsilon_1 U(\xi_\sigma) \tag{3.19}
$$

and

$$
\lim_{\xi \to \xi_\sigma^+} \big(U(\xi)\big) - \xi)\rho_2(\xi) = 0 \tag{3.20}
$$

hold, where $A_2 = (u_+ - A)\rho_+$. Equations (3.17) and (3.19) mean that $\rho \in L^1[-A, A]$.

For arbitrary $\psi \in C_0^\infty[-A, A]$, we verify that

$$
\int_{-A}^{A} (\xi\rho - \rho U + 2\epsilon_1 U)\psi'\, d\xi + \int_{-A}^{A} \rho\psi\, d\xi = 0. \tag{3.21}
$$

In fact, for any ξ_1, ξ_2 satisfying $-A < \xi_1 < \xi_\sigma < \xi_2 < A$, we have

$$
\begin{aligned}
I &= \int_{-A}^{A} (\xi\rho - \rho U + 2\epsilon_1 U)\psi'\, d\xi + \int_{-A}^{A} \rho\psi\, d\xi \\
&= \left(\int_{-A}^{\xi_1} + \int_{\xi_1}^{\xi_2} + \int_{\xi_2}^{A}\right)\big((\xi\rho - \rho U + 2\epsilon_1 U)\psi' + \rho\psi\big)\, d\xi \\
&= I_1 + I_2 + I_3.
\end{aligned}
$$

By simple calculation, one can obtain

$$
\begin{aligned}
I_1 &= \big(\big(\xi_1 - U(\xi_1)\big)\rho(\xi_1) + 2\epsilon_1 U(\xi_1)\big)\phi(\xi_1), \\
I_3 &= -\big(\big(\xi_2 - U(\xi_2)\big)\rho(\xi_2) + 2\epsilon_1 U(\xi_2)\big)\phi(\xi_2).
\end{aligned}
$$

Considering the monotonicity of $U(\xi)$, from (3.18) and (3.20), we get

$$\begin{aligned} |I_1+I_3| &\le \big|\big(\xi_1-U(\xi_1)\big)\rho(\xi_1)\phi(\xi_1)\big| + \big|\big(\xi_2-U(\xi_2)\big)\rho(\xi_2)\phi(\xi_2)\big| \\ &\quad + 2\epsilon_1\big|\big(U(\xi_1)\phi(\xi_1)-U(\xi_2)\phi(\xi_2)\big)\big| \\ &\to 0, \quad \text{as } \xi_1\to\xi_\sigma^-, \xi_2\to\xi_\sigma^+. \end{aligned}$$

Since $\rho\in L^1[-A,A]$, we can prove that

$$|I_2| \le \int_{\xi_1}^{\xi_2} \big|-(U-\xi)\psi'+\psi\big||\rho| + 2\epsilon_1\int_{\xi_1}^{\xi_2}|U|\,d\xi \to 0, \quad \text{as } \xi_1\to\xi_\sigma^-, \xi_2\to\xi_\sigma^+.$$

Noting I is independent of ξ_1 and ξ_2, so $I=0$, that is, (3.21) holds. Therefore, $\rho(\xi)$ defined in (3.5) is a weak solution of (3.4).

(ii) When $u_-<u_+$, we can obtain $\xi_{\sigma_1}\le\xi_{\sigma_2}$ because $U(\xi)$ is increasing. The solution $\rho(\xi)$ of (3.4) to be (3.7) with (3.10) and (3.11) can be easily obtained by using the same method as case (i).

When $\xi_{\sigma_1}\le\xi\le\xi_{\sigma_2}$, we rewrite the first equation in (3.4) as $(\xi-u)\rho_\xi+\rho(1-u_\xi)+2\epsilon_1 u_\xi-2\epsilon_1=\rho-2\epsilon_1$, that is, $((\xi-u)\rho)'+2\epsilon_1(u-\xi)'=\rho-2\epsilon_1$. Hence

$$\int_{\xi_{\sigma_1}}^{\xi_{\sigma_2}}\big(\rho(\xi)-2\epsilon_1\big)\,d\xi = (\xi-u)\rho(\xi)\Big|_{\xi_{\sigma_1}}^{\xi_{\sigma_2}} + 2\epsilon_1(u-\xi)\Big|_{\xi_{\sigma_1}}^{\xi_{\sigma_2}} = 0, \tag{3.22}$$

which implies that $\rho(\xi)=2\epsilon_1$. In addition, with the help of (3.10) and noting that

$$\begin{aligned} \int_{-A}^{\xi}\frac{U'(s)}{U(s)-s}\,ds &= U'(\zeta_1)\int_{-A}^{\xi}\frac{ds}{U(s)-s} \\ &\ge U'(\zeta_1)\int_{-A}^{\xi}\frac{ds}{U(\xi)-s} \\ &= -U'(\zeta_1)\ln\frac{U(\xi)-\xi}{U(\xi)+A} \\ &\to +\infty, \quad \text{as } \xi\to\xi_{\sigma_1}^-, \end{aligned}$$

where $-A<\xi<\xi_{\sigma_1}$, $-A\le\zeta_1\le\xi$, we get the first half of (3.9). Similarly, the second half can also be obtained. Moreover, the monotonicity of $\rho_1(\xi)$ and $\rho_2(\xi)$ is obvious. The proof of Lemma 3.2 is completed. □

Define an operator $T:K\to B$ as follows: for any $U\in K$, $u=TU$ is the unique solution of boundary value problem

$$\begin{cases} \varepsilon u''=\rho(U,\xi)(U-\xi)u', \\ u(\pm A)=u_\pm, \end{cases} \tag{3.23}$$

where $\rho(U,\xi)$ is defined in (3.5) or (3.7). Integrating the first equation of (3.23) twice on $[-A,\xi)$, we get the solution as follows:

$$u(\xi)=\frac{(u_+-u_-)\int_{-A}^{\xi}\exp(\int_{-A}^{\tau}(\rho(U-s)/\varepsilon)\,ds)\,d\tau}{\int_{-A}^{A}\exp(\int_{-A}^{\tau}(\rho(U-s)/\varepsilon)\,ds)\,d\tau}+u_-. \tag{3.24}$$

Lemma 3.3 *$T : K \to K$ is a continuous operator in B.*

Proof Take $U_n \to U$ (in B) ($n \to \infty$), U_n, $U \in K$. Then

$$TU_n = u_n, \qquad TU = u$$

satisfy (3.23), and we have

$$\begin{cases} \varepsilon(u_n - u)'' = \rho_n(U_n - \xi)(u_n - u)' + (\rho_n(U_n - \xi) - \rho(U - \xi))u', \\ (u_n - u)(\pm A) = 0. \end{cases} \tag{3.25}$$

Then it follows that

$$\begin{aligned} (u_n - u)'(\xi) = {}& -\frac{\int_{-A}^{A} \int_{-A}^{t} q_n(\tau) \exp(\int_{\tau}^{t} p_n(s)\,ds)\,d\tau\,dt}{\int_{-A}^{A} \exp(\int_{-A}^{\tau} p_n(s)\,ds)\,d\tau} \\ & \times \exp\left(\int_{-A}^{\xi} p_n(s)\,ds\right) + \int_{-A}^{\xi} q_n(\tau) \exp\left(\int_{\tau}^{\xi} p_n(s)\,ds\right) d\tau, \end{aligned} \tag{3.26}$$

$$\begin{aligned} (u_n - u)(\xi) = {}& -\frac{\int_{-A}^{A} \int_{-A}^{t} q_n(\tau) \exp(\int_{\tau}^{t} p_n(s)\,ds)\,d\tau\,dt}{\int_{-A}^{A} \exp(\int_{-A}^{\tau} p_n(s)\,ds)\,d\tau} \\ & \times \int_{-A}^{\xi} \exp\left(\int_{-A}^{\tau} p_n(s)\,ds\right) d\tau \\ & + \int_{-A}^{\xi} \int_{-A}^{t} q_n(\tau) \exp\left(\int_{\tau}^{t} p_n(s)\,ds\right) d\tau\,dt, \end{aligned} \tag{3.27}$$

where $\varepsilon p_n = \rho_n(U_n - \xi)$, $\varepsilon q_n = (\rho_n(U_n - \xi) - \rho(U - \xi))u'$. From the first equation of (3.4), we have

$$\bigl(\rho(U - \xi) - 2\epsilon_1 U\bigr)' = -\rho < 0, \qquad \bigl(\rho_n(U_n - \xi) - 2\epsilon_1 U_n\bigr)' = -\rho_n < 0 \quad (n = 1, 2, \ldots).$$

Then $\rho(U - \xi) - 2\epsilon_1 U$ and $\rho_n(U_n - \xi) - 2\epsilon_1 U_n$ ($n = 1, 2, \ldots$) are monotone decreasing and continuous functions. We can rewrite εq_n as follows:

$$\varepsilon q_n = \bigl(\rho_n(U_n - \xi) - 2\epsilon_1 U_n\bigr) - \bigl(\rho(U - \xi) - 2\epsilon_1 U\bigr) + 2\epsilon_1(U_n - U)u'.$$

Because the sequence of monotone functions (continuous or discontinuous) which converges to a continuous function must converge uniformly, we have that $q_n(\xi)$ converges to zero uniformly. From (3.25), (3.26), and (3.27), we can get that

$$u_n \to u \quad (\text{in } B), \text{ as } n \to \infty.$$

Therefore $T : K \to B$ is continuous in B.

Furthermore, from (3.23) we have

$$u'(\xi) = \frac{(u_+ - u_-) \exp(\int_{-A}^{\xi} (\rho(U - s)/\varepsilon)\,ds}{\int_{-A}^{A} \exp(\int_{-A}^{\tau} (\rho(U - s)/\varepsilon)\,ds)\,d\tau}. \tag{3.28}$$

It is obvious that $u = TU$ is monotone. So we get $TK \subset K$. The proof of this lemma is completed. □

Lemma 3.4 *TK is precompact in B.*

Proof According to the continuity of T and the Ascoli–Arzela theorem [43], we still need to show the boundedness of TK in B.

When $u_- > u_+$, for any $U \in K$, we have

$$u'(\xi) = u'(-A)\exp\left(\int_{-A}^{\xi} \frac{\rho(U-s)}{\varepsilon}\,ds\right).$$

By Lemma 3.2, when $s < \xi_\sigma$, it yields that

$$0 < \rho(U-s) = \rho_-(u_- + A) - \int_{-A}^{s} \rho(\xi)\,d\xi + 2\epsilon_1\bigl(U(\xi) - u_-\bigr) < \rho_-(u_- + A).$$

While when $s > \xi_\sigma$, we get

$$0 > \rho(U-s) = \rho_+(u_+ - A) + \int_{s}^{A} \rho(\xi)\,d\xi + 2\epsilon_1\bigl(U(\xi) - u_+\bigr) > \rho_+(u_+ - A).$$

Thus we need only to consider the uniform boundedness of $u'(-A)$. From (3.23) we obtain

$$u''(\xi) < 0, \quad \xi \in [-A, \xi_\sigma).$$

Then it follows that

$$u'(\xi) < u'(-A) < 0, \quad \xi \in [-A, \xi_\sigma),$$

and

$$u_- - u_+ > u(-A) - u(\xi_\sigma) = u'(\zeta_2)(-A - \xi_\sigma) > u'(\zeta_2)(-A - u_+), \quad \zeta_2 \in [-A, \xi_\sigma).$$

So

$$0 > u'(-A) > u'(\zeta_2) > -\frac{u_- - u_+}{A + u_+}.$$

These above imply that $u'(\xi)$ is uniformly bounded for K.

When $u_- < u_+$, from (3.23) and (3.24), we have

$$0 < u'(\xi) < u'(\xi_{\sigma_1}) \le 1, \quad \xi \in [-A, \xi_{\sigma_1})$$

and

$$0 < u'(\xi) < u'(\xi_{\sigma_2}) \le 1, \quad \xi \in (\xi_{\sigma_2}, A].$$

Therefore, $u(\xi)$, $u'(\xi)$, and $u''(\xi)$ are all uniformly bounded for K, that is, TK is a bounded set in B. We complete the proof of this lemma. □

From the above lemmas, by virtue of Schauder's fixed point theorem, we get the following result.

Theorem 3.5 *There exists a weak solution*

$$(\rho, u) \in L^1[-A, A] \times C^2[-A, A]$$

for the boundary value problem (3.1) *and* (3.2).

Now we extend the solution of (3.1) and (3.2) in $[-A, A]$ to the whole interval $(-\infty, +\infty)$. The following lemma is necessary.

Lemma 3.6 *The solution* $(\rho, u)(\xi)$ *of system* (3.1) *and* (3.2) *satisfies*

(i) $u(\xi)$, $u'(\xi)$ *have uniform bounds independent of* A;

(ii) $|u''(\xi)| \le C(\varepsilon)$, $\xi \in [-A, A]$, *where* $C(\varepsilon)$ *is a constant only dependent on* ε;

(iii) $\rho_A(u, \xi)$ *converges as* $A \to +\infty$, *where* $\rho_A(u, \xi)$ *can be expressed as*

$$\rho_A(u,\xi) = \begin{cases} \rho_1(\xi) = 2\epsilon_1 + (\rho_- - 2\epsilon_1)\exp(\int_{-A}^{\xi} \frac{-u'(s)}{u(s)-s})\,ds, \\ \quad \xi \in (-A, \xi_\sigma) \text{ or } \xi \in (-A, \xi_{\sigma_1}), \\ \rho_2(\xi) = 2\epsilon_1 + (\rho_+ - 2\epsilon_1)\exp(\int_{\xi}^{A} \frac{u'(s)}{u(s)-s})\,ds, \\ \quad \xi \in (\xi_\sigma, A) \text{ or } \xi \in (\xi_{\sigma_2}, A). \end{cases}$$

Proof We only consider the case $u_- > u_+$, one can prove the case $u_- < u_+$ in a similar way. For this case, we have

$$u_+ < \xi_\sigma < u_-.$$

(i) Take $-A < \xi_1 < u_+$. From the first equation in (3.23), we can obtain

$$u'(\xi) = u'(\xi_1)\exp\left(\int_{\xi_1}^{\xi} \frac{\rho(u-s)}{\varepsilon}\,ds\right).$$

Because

$$u''(\xi) < 0, \quad \xi \in (-A, \xi_\sigma),$$

it follows that

$$0 > u'(\xi_1) > u'(\xi), \quad \xi \in (\xi_1, \xi_\sigma).$$

Since

$$u_- - u_+ > u(\xi_1) - u(\xi_\sigma) = u'(\zeta_3)(\xi_1 - \xi_\sigma) > u'(\zeta_3)(\xi_1 - u_+), \quad \zeta_3 \in (\xi_1, \xi_\sigma),$$

we get

$$u'(\zeta_3) > \frac{u_- - u_+}{\xi_1 - u_+}.$$

Then

$$0 > u'(\xi_1) > \frac{u_- - u_+}{\xi_1 - u_+}.$$

In addition,

$$\begin{aligned}
\rho(\xi_1) &= 2\epsilon_1 + (\rho_- - 2\epsilon_1)\exp\left(\int_{-A}^{\xi_1} \frac{-u'}{u-s}\,ds\right) \\
&= 2\epsilon_1 + (\rho_- - 2\epsilon_1)\exp\left(\int_{-A}^{\xi_1} \frac{-(u-s)' - 1}{u-s}\,ds\right) \\
&= 2\epsilon_1 + (\rho_- - 2\epsilon_1)\frac{u_- + A}{u(\xi_1) - \xi_1}\exp\left(\int_{-A}^{\xi_1} \frac{-ds}{u-s}\right) \\
&\le 2\epsilon_1 + (\rho_- - 2\epsilon_1)\frac{u_- + A}{u(\xi_1) - \xi_1}\exp\left(\int_{-A}^{\xi_1} \frac{-ds}{u_- - s}\right) \\
&= 2\epsilon_1 + (\rho_- - 2\epsilon_1)\frac{u_- - \xi_1}{u(\xi_1) - \xi_1}.
\end{aligned}$$

If $\xi < \xi_1$, then

$$\exp\left(\int_{\xi_1}^{\xi} \frac{\rho(u-s)}{\varepsilon}\,ds\right) < 1.$$

While if $\xi_1 < \xi < \xi_\sigma$, we have

$$\begin{aligned}
\rho(u-\xi) &= \rho(\xi_1)\bigl(u(\xi_1) - \xi_1\bigr) + 2\epsilon_1\bigl(u(\xi) - u(\xi_1)\bigr) - \int_{\xi_1}^{\xi} \rho\,ds \\
&\le \rho(\xi_1)\bigl(u(\xi_1) - \xi_1\bigr) \le \rho_-(u_- - \xi_1).
\end{aligned}$$

So we obtain that

$$\exp\left(\int_{\xi_1}^{\xi} \frac{\rho(u-s)}{\varepsilon}\,ds\right) \le \exp\left(\frac{\rho_-(u_- - \xi_1)^2}{\varepsilon}\right).$$

When $\xi > \xi_\sigma$,

$$\begin{aligned}
\int_{\xi_1}^{\xi} \frac{\rho(u-s)}{\varepsilon}\,ds &= \int_{\xi_1}^{\xi_\sigma} \frac{\rho(u-s)}{\varepsilon}\,ds + \int_{\xi_\sigma}^{\xi} \frac{\rho(u-s)}{\varepsilon}\,ds \\
&\le \int_{\xi_1}^{\xi_\sigma} \frac{\rho(u-s)}{\varepsilon}\,ds.
\end{aligned}$$

Therefore, $u(\xi)$ and $u'(\xi)$ have uniform bounds independent of A.

(ii) We can get this result from the first equation of (3.23) and (i).

(iii) Similar to the estimate on $\rho_1(\xi)$ in (i), we can obtain that

$$\rho_1(\xi) \le 2\epsilon_1 + (\rho_- - 2\epsilon_1)\frac{u_- - \xi}{u(\xi) - \xi}, \quad \xi \in (-A, \xi_\sigma)$$

and

$$\rho_2(\xi) \le 2\epsilon_1 + (\rho_+ - 2\epsilon_1)\frac{u_+ - \xi}{u(\xi) - \xi}, \quad \xi \in (\xi_\sigma, A),$$

which show that $\rho_A(u,\xi)$ converges as $A \to +\infty$. We finish the proof of Lemma 3.6. □

From the above discussion, for any $L > 0$, $\{u_A(\xi)\}$ is a compact set in $C^1[-L,L]$ if $A > L$. Hence there exists a subsequence $\{u_{A_i}(\xi)\}$ such that

$$\lim_{A_i \to +\infty} u_{A_i}(\xi) = u(\xi), \qquad \lim_{A_i \to +\infty} u'_{A_i}(\xi) = u'(\xi), \quad \xi \in (-L, +L).$$

By Helly's selection theorem, we get a subsequence, also denoted by $\{u_{A_i}(\xi)\}$, such that

$$\lim_{A_i \to +\infty} u_{A_i}(\xi) = u(\xi), \qquad \lim_{A_i \to +\infty} u'_{A_i}(\xi) = u'(\xi), \quad \xi \in (-\infty, +\infty).$$

Theorem 3.7 *Let $\varepsilon \le \varepsilon_0$. Then $u(\xi)$ satisfies*

$$\begin{cases} \varepsilon u'' = \rho(u,\xi)(u-\xi)u', \\ u(\pm\infty) = u_\pm, \end{cases} \tag{3.29}$$

and

$$\rho(\xi) = \begin{cases} \rho_1(\xi), & -\infty < \xi < \xi_{\sigma_1}, \\ 2\epsilon_1, & \xi_{\sigma_1} \le \xi \le \xi_{\sigma_2}, \\ \rho_2(\xi), & \xi_{\sigma_2} < \xi < +\infty, \end{cases} \tag{3.30}$$

when $u_- < u_+$, while

$$\rho(\xi) = \begin{cases} \rho_1(\xi), & -\infty < \xi < \xi_\sigma, \\ \rho_2(\xi), & \xi_\sigma < \xi < +\infty, \end{cases} \tag{3.31}$$

when $u_- > u_+$, where

$$\rho_1(\xi) = 2\epsilon_1 + (\rho_- - 2\epsilon_1)\exp\left(\int_{-\infty}^{\xi} \frac{-u'(s)}{u(s)-s}\right) ds, \tag{3.32}$$

$$\rho_2(\xi) = 2\epsilon_1 + (\rho_+ - 2\epsilon_1)\exp\left(\int_{\xi}^{+\infty} \frac{u'(s)}{u(s)-s}\right) ds, \tag{3.33}$$

and ξ_σ, ξ_{σ_1}, and ξ_{σ_2} satisfy

$$\xi_{\sigma_1} = \min\{\xi_\sigma \,|\, u(\xi_\sigma) = \xi_\sigma\}, \qquad \xi_{\sigma_2} = \max\{\xi_\sigma \,|\, u(\xi_\sigma) = \xi_\sigma\}.$$

Proof Denote $(\rho_A(\xi), u_A(\xi))$ by the solution of (3.1), (3.3). When $u_- > u_+$, integrating (3.23) from ξ_2 to ξ, ξ_2 is a fixed point. Then we have

$$\begin{aligned}\varepsilon\big(u_A'(\xi) - u_A'(\xi_2)\big) = {} & \rho_A(\xi)\big(u_A(\xi) - \xi\big)u_A(\xi) \\ & - \rho_A(\xi_2)\big(u_A(\xi_2) - \xi_2\big)u_A(\xi_2) - \epsilon_1\big(u_A^2(\xi) - u_A^2(\xi_2)\big) + \int_{\xi_2}^{\xi} \rho_A u_A \, ds\end{aligned}$$

whenever ξ_σ is between ξ_2 and ξ or not. Letting $A \to +\infty$, by the Lebesgue convergence theorem it follows that

$$\begin{aligned}\varepsilon\big(u'(\xi) - u'(\xi_2)\big) = {} & \rho(\xi)\big(u(\xi) - \xi\big)u(\xi) - \rho(\xi_2)\big(u(\xi_2) - \xi_2\big)u(\xi_2) \\ & - \epsilon_1\big(u^2(\xi) - u^2(\xi_2)\big) + \int_{\xi_2}^{\xi} \rho u \, ds.\end{aligned} \tag{3.34}$$

When $u_- < u_+$, we can get the same formula as above. The right-hand side of (3.34) is continuous, so we get

$$u' \in C^1(-\infty, +\infty).$$

Differentiating (3.34) with respect to ξ yields

$$\varepsilon u'' = \rho(-\xi + u)u',$$

and from (3.24) we have

$$u(-\infty) = u_-, \qquad u(+\infty) = u_+.$$

The formulae of $\rho(\xi)$ in (3.30)–(3.33) can be obtained from the lemmas above. The proof is completed. □

4 The limit solutions of (1.9), (1.10) as viscosity vanishes

In this section, we investigate the behavior of solution of (3.1), (3.2) as $\varepsilon \to 0$.

Case 1. $u_- > u_+$.

Lemma 4.1 *Let ξ_σ^ε be the unique point satisfying*

$$u^\varepsilon\big(\xi_\sigma^\varepsilon\big) = \xi_\sigma^\varepsilon, \qquad \xi_\sigma = \lim_{\varepsilon\to 0^+} \xi_\sigma^\varepsilon$$

(pass to a subsequence if necessary). Then, for any $\eta > 0$,

$$\begin{aligned} & \lim_{\varepsilon\to 0^+} u_\xi^\varepsilon(\xi) = 0 \quad \text{for } |\xi - \xi_\sigma| \geq \eta, \\ & \lim_{\varepsilon\to 0^+} u^\varepsilon(\xi) = u_+ \quad \text{for } \xi \geq \xi_\sigma + \eta, \\ & \lim_{\varepsilon\to 0^+} u^\varepsilon(\xi) = u_- \quad \text{for } \xi \leq \xi_\sigma - \eta \end{aligned} \tag{4.1}$$

uniformly in the intervals above.

Here and after, denote u^ε, ρ^ε as u, ρ when there is no confusion.

Proof Take $\xi_3 = \xi_\sigma + \frac{\eta}{2}$, and let ε be so small that $\xi_\sigma < \xi_3 - \frac{\eta}{4}$. Integrating the first equation of (3.1) twice on $[\xi_3, \xi]$, we get

$$u(\xi_3) - u(\xi) = -u'(\xi_3) \int_{\xi_3}^{\xi} \exp\left(\int_{\xi_3}^{\tau} \frac{\rho(u(s) - s)}{\varepsilon} \, ds \right) d\tau.$$

When $\xi > \xi_\sigma$,

$$\begin{aligned}
\rho(\xi) &= 2\epsilon_1 + (\rho_+ - 2\epsilon_1) \exp\left(\int_{\xi}^{+\infty} \frac{u'(s)}{u(s) - s} \, ds \right) \\
&= 2\epsilon_1 + \lim_{A \to +\infty} (\rho_+ - 2\epsilon_1) \exp\left(\int_{\xi}^{A} \frac{(u(s) - s)' + 1}{u(s) - s} \, ds \right) \\
&\leq 2\epsilon_1 + \lim_{A \to +\infty} (\rho_+ - 2\epsilon_1) \frac{u_+ - A}{u(\xi) - \xi} \exp\left(\int_{\xi}^{A} \frac{ds}{u_+ - s} \right) \\
&= 2\epsilon_1 + \lim_{A \to +\infty} (\rho_+ - 2\epsilon_1) \frac{u_+ - A}{u(\xi) - \xi} \frac{u_+ - \xi}{u_+ - A} \\
&= 2\epsilon_1 + (\rho_+ - 2\epsilon_1) \frac{u_+ - \xi}{u(\xi) - \xi},
\end{aligned}$$

we have

$$\begin{aligned}
\rho(u - \xi) &\geq 2\epsilon_1 (u - \xi) + (\rho_+ - 2\epsilon_1)(u_+ - \xi) \\
&\geq 2\epsilon_1 (u_+ - \xi) + (\rho_+ - 2\epsilon_1)(u_+ - \xi) \\
&= \rho_+ (u_+ - \xi), \quad \xi \in (\xi_\sigma, +\infty).
\end{aligned} \tag{4.2}$$

Then

$$\begin{aligned}
u(\xi_3) - u(\xi) &\geq -u'(\xi_3) \int_{\xi_3}^{\xi} \exp\left(\int_{\xi_3}^{\tau} \frac{\rho_+ (u_+ - s)}{\varepsilon} \, ds \right) d\tau \\
&= -u'(\xi_3) \int_{\xi_3}^{\xi} \exp\left(\frac{\rho_+}{\varepsilon} \left((u_+ - \xi_3)(\tau - \xi_3) - \frac{1}{2} (\tau - \xi_3)^2 \right) \right) d\tau \\
&= -u'(\xi_3) \int_{0}^{\xi - \xi_3} \exp\left(\frac{\rho_+}{\varepsilon} \left((u_+ - \xi_3)\tau - \frac{1}{2} \tau^2 \right) d\tau \right).
\end{aligned}$$

Letting $\xi \to +\infty$, it follows that

$$\begin{aligned}
u_- - u_+ &\geq -u'(\xi_3) \int_{0}^{+\infty} \exp\left(\frac{v_+}{2\varepsilon} \left(2(u_+ - \xi_3)\tau - \tau^2 \right) d\tau \right) \\
&\geq -u'(\xi_3) \sqrt{\varepsilon} A_3,
\end{aligned}$$

where A_3 is a constant independent of ε. Thus

$$\left| u'(\xi_3) \right| \leq \frac{u_- - u_+}{\sqrt{\varepsilon} A_3}.$$

So, one can check that

$$\left|u'(\xi)\right| \le \frac{u_- - u_+}{\sqrt{\varepsilon}A_3} \exp\left(\int_{\xi_3}^{\xi} \frac{\rho(u-s)}{\varepsilon}\,ds\right).$$

Again note that when $\xi > \xi_3$,

$$\begin{aligned}
\rho(\xi) &= 2\epsilon_1 + (\rho_+ - 2\epsilon_1)\exp\left(\int_{\xi}^{+\infty} \frac{u'(s)}{u(s)-s}\right) ds \\
&= 2\epsilon_1 + \lim_{A\to+\infty}(\rho_+ - 2\epsilon_1)\exp\left(\int_{\xi}^{A} \frac{(u(s)-s)'+1}{u(s)-s}\,ds\right) \\
&\ge 2\epsilon_1 + \lim_{A\to+\infty}(\rho_+ - 2\epsilon_1)\frac{u_+ - A}{u(\xi)-\xi}\exp\left(\int_{\xi}^{A} \frac{ds}{u(\xi_3)-s}\right) \\
&= 2\epsilon_1 + \lim_{A\to+\infty}(\rho_+ - 2\epsilon_1)\frac{u_+ - A}{u(\xi)-\xi}\frac{u(\xi_3)-\xi}{u(\xi_3)-A} \\
&= 2\epsilon_1 + (\rho_+ - 2\epsilon_1)\frac{u(\xi_3)-\xi}{u(\xi)-\xi},
\end{aligned}$$

we have

$$\begin{aligned}
\rho(u-\xi) &\le 2\epsilon_1\big(u(\xi)-\xi\big) + (\rho_+ - 2\epsilon_1)\big(u(\xi_3)-\xi\big) \\
&\le 2\epsilon_1\big(u(\xi_3)-\xi\big) + (\rho_+ - 2\epsilon_1)\big(u(\xi_3)-\xi\big) \\
&= \rho_+\big(u(\xi_3)-\xi\big), \quad \xi \in (\xi_3, +\infty).
\end{aligned} \tag{4.3}$$

Then

$$\left|u'(\xi)\right| \le \frac{u_- - u_+}{\sqrt{\varepsilon}A_3} \exp\left(-\frac{\rho_+}{\varepsilon}\int_{\xi_3}^{\xi} \big(s - u(\xi_3)\big)\,ds\right), \tag{4.4}$$

which implies that

$$\lim_{\varepsilon\to 0^+} u_\xi^\varepsilon(\xi) = 0, \quad \text{uniformly for } \xi \ge \xi_\sigma + \eta.$$

Next, we pick ξ_4 such that $\xi > \xi_4 \ge \xi_\sigma + \eta$. From

$$u(\xi_4) - u(\xi) = -u'(\xi_4)\int_{\xi_4}^{\xi} \exp\left(\int_{\xi_4}^{\tau} \frac{\rho(u-s)}{\varepsilon}\,ds\right) d\tau,$$

we get

$$\begin{aligned}
\left|u(\xi_4) - u(\xi)\right| &\le \left|u'(\xi_4)\right| \int_{\xi_4}^{\xi} \exp\left(\int_{\xi_4}^{\tau} \frac{-A_4}{2\varepsilon}\,ds\right) d\tau \\
&\le \frac{2\varepsilon}{A_4}\left|u'(\xi_4)\right|\left\{1 - \exp\left(\frac{A_4}{2\varepsilon}(\xi_4 - \xi)\right)\right\},
\end{aligned}$$

where $A_4 = 2\rho_+(\xi_4 - u(\xi_4))$. Letting $\xi \to +\infty$, we conclude that

$$\left|u(\xi_4) - u_+\right| \le \frac{2\varepsilon}{A_4}\left|u'(\xi_4)\right|,$$

which implies that

$$\lim_{\varepsilon\to 0^+} u^\varepsilon(\xi) = u_+, \quad \text{uniformly for } \xi \geq \xi_\sigma + \eta.$$

The result for $\xi \leq \xi_\sigma$ can be obtained in the same way. The proof is completed. □

Lemma 4.2 *For any* $\eta > 0$,

$$\lim_{\varepsilon\to 0^+} \rho^\varepsilon(\xi) = \begin{cases} \rho_-, & -\infty < \xi < \xi_\sigma - \eta, \\ \rho_+, & \xi_\sigma + \eta < \xi < +\infty, \end{cases} \tag{4.5}$$

uniformly.

Proof From (4.2) and (4.3), for any $\xi > \xi_5 > \xi_\sigma + \eta$, it follows that

$$2\epsilon_1 + (\rho_+ - 2\epsilon_1)\frac{u(\xi_5) - \xi}{u(\xi) - \xi} \leq \rho(\xi) \leq 2\epsilon_1 + (\rho_+ - 2\epsilon_1)\frac{u_+ - \xi}{u(\xi) - \xi},$$

which yields

$$\lim_{\varepsilon\to 0^+} \rho^\varepsilon(\xi) = \rho_+, \quad \text{uniformly for } \xi > \xi_\sigma + \eta.$$

In a similar way, the rest can be obtained. This completes the proof. □

Next, we study in more detail the limiting behavior of ρ^ε in the neighborhood of $\xi = \xi_\sigma$ as $\varepsilon \to 0^+$. Denote

$$\sigma = \xi_\sigma = \lim_{\varepsilon\to 0^+} \xi_\sigma^\varepsilon = \lim_{\varepsilon\to 0^+} u^\varepsilon\left(\xi_\sigma^\varepsilon\right) = u(\sigma). \tag{4.6}$$

Then

$$u_+ < \sigma < u_-. \tag{4.7}$$

Now we take $\xi_1 < \sigma < \xi_2$, $\psi \in C_0^\infty[\xi_1, \xi_2]$ such that $\psi(\xi) \equiv \psi(\sigma)$ for ξ in a neighborhood Ω of $\xi = \sigma$ (ψ is called a sloping test function). When $0 < \varepsilon < \varepsilon_0$, $\xi_\sigma^\varepsilon \in \Omega$. From (3.1) we have

$$\int_{\xi_1}^{\xi_2} \left(\rho^\varepsilon\left(\xi - u^\varepsilon\right) + 2\epsilon_1 u^\varepsilon\right)\psi'\,d\xi + \int_{\xi_1}^{\xi_2} \rho^\varepsilon \psi\,d\xi = 0, \tag{4.8}$$

$$\int_{\xi_1}^{\xi_2} \left(\rho^\varepsilon\left(\xi - u^\varepsilon\right)u^\varepsilon + \epsilon_1\left(u^\varepsilon\right)^2\right)\psi'\,d\xi + \int_{\xi_1}^{\xi_2} \rho^\varepsilon u^\varepsilon \psi\,d\xi = \varepsilon\int_{\xi_1}^{\xi_2} u^\varepsilon \psi''\,d\xi. \tag{4.9}$$

For any α_1 and α_2 near σ with $\alpha_1 < \sigma < \alpha_2$, from Lemmas 4.1 and 4.2, we immediately obtain that

$$\begin{aligned}
&\lim_{\varepsilon\to0^+}\int_{\xi_1}^{\xi_2}\bigl(\rho^\varepsilon\bigl(\xi-u^\varepsilon\bigr)+2\epsilon_1u^\varepsilon\bigr)\psi'\,d\xi\\
&=\lim_{\varepsilon\to0^+}\int_{\xi_1}^{\alpha_1}\bigl(\rho^\varepsilon\bigl(\xi-u^\varepsilon\bigr)+2\epsilon_1u^\varepsilon\bigr)\psi'\,d\xi+\lim_{\varepsilon\to0^+}\int_{\alpha_2}^{\xi_2}\bigl(\rho^\varepsilon\bigl(\xi-u^\varepsilon\bigr)+2\epsilon_1u^\varepsilon\bigr)\psi'\,d\xi\\
&=\int_{\xi_1}^{\alpha_1}\bigl(\rho_-(\xi-u_-)+2\epsilon_1u_-\bigr)\psi'\,d\xi+\int_{\alpha_2}^{\xi_2}\bigl(\rho_+(\xi-u_+)+2\epsilon_1u_+\bigr)\psi'\,d\xi\\
&=(\rho_+u_+-\rho_+\alpha_2-2\epsilon_1u_+-\rho_-u_-+\rho_-\alpha_1+2\epsilon_1u_-)\psi(\sigma)\\
&\quad-\int_{\xi_1}^{\alpha_1}\rho_-\psi(\xi)\,d\xi-\int_{\alpha_2}^{\xi_2}\rho_+\psi(\xi)\,d\xi.
\end{aligned}$$

Letting $\alpha_1\to\sigma^-$ and $\alpha_2\to\sigma^+$, we conclude that

$$\begin{aligned}
&\lim_{\varepsilon\to0^+}\int_{\xi_1}^{\xi_2}\bigl(\rho^\varepsilon\bigl(\xi-u^\varepsilon\bigr)+2\epsilon_1u^\varepsilon\bigr)\psi'\,d\xi\\
&=\bigl(-\sigma[\rho]+[\rho u-2\epsilon_1u]\bigr)\psi(\sigma)-\int_{\xi_1}^{\xi_2}H(\xi-\sigma)\psi(\xi),
\end{aligned}\tag{4.10}$$

where

$$H(x)=\begin{cases}\rho_-, & x<0,\\ \rho_+, & x>0.\end{cases}$$

Returning to (4.8), we get

$$\lim_{\varepsilon\to0^+}\int_{\xi_1}^{\xi_2}\bigl(\rho^\varepsilon-H(\xi-\sigma)\bigr)\psi(\xi)\,d\xi=\bigl(\sigma[\rho]-[\rho u-2\epsilon_1u]\bigr)\psi(\sigma)\tag{4.11}$$

for all sloping test function $\psi\in C_0^\infty[\xi_1,\xi_2]$. For an arbitrary $\varphi\in C_0^\infty[\xi_1,\xi_2]$, we take a sloping test function ψ such that $\psi(\sigma)=\varphi(\sigma)$ and

$$\max|\psi-\varphi|<\mu,\quad \text{for } \xi\in[\xi_1,\xi_2], \mu>0.$$

Considering that $\rho^\varepsilon\in L^1[\xi_1,\xi_2]$ uniformly, we find that

$$\begin{aligned}
&\lim_{\varepsilon\to0^+}\int_{\xi_1}^{\xi_2}\bigl(\rho^\varepsilon-H(\xi-\sigma)\bigr)\varphi(\xi)\,d\xi\\
&=\lim_{\varepsilon\to0^+}\int_{\xi_1}^{\xi_2}\bigl(\rho^\varepsilon-H(\xi-\sigma)\bigr)\psi(\xi)\,d\xi+O(\mu)\\
&=\bigl(\sigma[\rho]-[\rho u-2\epsilon_1u]\bigr)\psi(\sigma)+O(\mu)\\
&=\bigl(\sigma[\rho]-[\rho u-2\epsilon_1u]\bigr)\varphi(\sigma)+O(\mu).
\end{aligned}$$

Sending $\mu \to 0$, we find that (4.11) holds for all $\psi \in C_0^\infty[\xi_1, \xi_2]$. Thus, the limit function of $\rho^\varepsilon(\xi)$ is the sum of a step function and a Dirac delta function with strength $\sigma[\rho] - [\rho u - 2\epsilon_1 u]$.

Similarly, from (4.9) we can obtain

$$\lim_{\varepsilon \to 0^+} \int_{\xi_1}^{\xi_2} \bigl(\rho^\varepsilon u^\varepsilon - \tilde{H}(\xi - \sigma)\bigr)\psi(\xi)\, d\xi = \bigl(\sigma[\rho u] - \bigl[\rho u^2 - \epsilon_1 u^2\bigr]\bigr)\psi(\sigma) \tag{4.12}$$

for $\psi \in C_0^\infty[\xi_1, \xi_2]$, where

$$\tilde{H}(x) = \begin{cases} \rho_- u_-, & x < 0, \\ \rho_+ u_+, & x > 0. \end{cases}$$

Thus $\rho^\varepsilon u^\varepsilon$ converges in the weak star topology of $C_0^\infty(R^1)$, and the limit function is a step function plus a Dirac delta function with strength $\sigma[\rho u] - [\rho u^2 - \epsilon_1 u^2]$.

If we take the test function as $\psi/(\tilde{u}^\varepsilon + \rho)$ in (4.9), where $\tilde{u}^\varepsilon$ is a modified function satisfying $u^\varepsilon(\sigma)$ in Ω and u^ε outside Ω, and let $\rho \to 0$, then we can get the other formula as follows:

$$\lim_{\varepsilon \to 0} \int_{\xi_1}^{\xi_2} \bigl(\rho^\varepsilon - H(\xi - \sigma)\bigr)\psi(\xi)\, d\xi \cdot u(\sigma) = \bigl(\sigma[\rho u] - \bigl[\rho u^2 - \epsilon_1 u^2\bigr]\bigr), \tag{4.13}$$

where $\psi \in C_0^\infty(\xi_1, \xi_2)$.

Denote $u_\delta = \lim_{\varepsilon \to 0^+} u^\varepsilon(\xi_\sigma^\varepsilon) = u(\sigma)$, compared (4.13) with (4.11), we have

$$u_\delta\bigl(u_\delta[\rho] - [\rho u - 2\epsilon_1 u]\bigr) = u_\delta[\rho u] - \bigl[\rho u^2 - \epsilon_1 u^2\bigr], \tag{4.14}$$

that is,

$$u_\delta^2[\rho] - u_\delta\bigl([\rho u - 2\epsilon_1 u] + [\rho u]\bigr) + [\rho u - 2\epsilon_1 u] = 0.$$

When $[\rho] \neq 0$, we can get

$$u_\delta = \frac{[(\rho - \epsilon_1)u] \pm \sqrt{(\rho_- - \epsilon_1)(\rho_+ - \epsilon_1)}(u_- - u_+)}{[\rho]}.$$

Because $u_+ < u^\varepsilon(\xi_\sigma^\varepsilon) = \xi_\alpha^\varepsilon < u_-$, we take

$$u_\delta = \frac{[(\rho - \epsilon_1)u] + \sqrt{(\rho_- - \epsilon_1)(\rho_+ - \epsilon_1)}(u_- - u_+)}{[\rho]}. \tag{4.15}$$

Let w_0 be the strength of Dirac delta function in ρ, then

$$w_0 = u_\delta[\rho] - [\rho u - 2\epsilon_1 u] = [\epsilon_1 u] + \sqrt{(\rho_- - \epsilon_1)(\rho_+ - \epsilon_1)}(u_- - u_+). \tag{4.16}$$

Then we have the following theorem.

Theorem 4.3 *When $u_- > u_+$, let $(\rho^\varepsilon(\xi), u^\varepsilon(\xi))$ be the solution to (3.1), (3.2). Then*

$$u(\xi) = \lim_{\varepsilon\to 0^+} u^\varepsilon(\xi) = \begin{cases} u_-, & \xi < u_\delta, \\ u_\delta, & \xi = u_\delta, \\ u_+, & \xi > u_\delta, \end{cases} \tag{4.17}$$

$\rho^\varepsilon(\xi)$ converges in the weak star topology of $C_0^\infty(R^1)$, and the limit function is a sum of a step function and a Dirac delta function with strength w_0, where u_δ and w_0 are expressed by (4.15) and (4.16).

This theorem shows the stability of delta shock waves for (1.8) and (1.10) under viscous perturbations.

Case 2. $u_- < u_+$.

Lemma 4.4 *For any $\eta > 0$,*

$$\lim_{\varepsilon\to 0^+} u_\xi^\varepsilon = 0, \quad \text{for } \xi \le u_- - \eta \text{ or } \xi \ge u_+ + \eta$$

$$\lim_{\varepsilon\to 0^+} (\rho^\varepsilon, u^\varepsilon)(\xi) = \begin{cases} (\rho_-, u_-) & \text{for } \xi < u_- - \eta, \\ (2\epsilon_1, \xi) & \text{for } u_- - \eta \le \xi \le u_+ + \eta, \\ (\rho_+, u_+) & \text{for } \xi > u_+ + \eta, \end{cases}$$

uniformly in the above intervals.

Proof Taking $\xi_3 = u_+ + \eta$ and integrating the first equation of (3.1) twice on $[\xi_3, \xi]$, we get

$$\begin{aligned} u(\xi) - u(\xi_3) &= u'(\xi_3) \int_{\xi_3}^{\xi} \exp\left(\int_{\xi_3}^{\tau} \frac{\rho(u-s)}{\varepsilon}\, ds \right) d\tau \\ &> u'(\xi_3) \int_{\xi_3}^{\xi} \exp\left(\int_{\xi_3}^{\tau} \frac{\rho_+(u_- - s)}{\varepsilon}\, ds \right) d\tau \\ &= u'(\xi_3) \int_{\xi_3}^{\xi} \exp\left(\frac{\rho_+}{\varepsilon} \left((u_- - \xi_3)(\tau - \xi_3) - \frac{1}{2}(\tau - \xi_3)^2 \right) \right) d\tau \\ &= u'(\xi_3) \int_{0}^{\xi - \xi_3} \exp\left(\frac{\rho_+}{2\varepsilon} \left(2(u_- - \xi_3)\tau - \tau^2 \right) \right) d\tau. \end{aligned}$$

Letting $\xi \to +\infty$, it follows that

$$\begin{aligned} u_+ - u_- &\ge u'(\xi_3) \int_0^{+\infty} \exp\left(\frac{\rho_+}{2\varepsilon} \left(2(u_- - \xi_3)\tau - \tau^2 \right) \right) d\tau \\ &\ge u'(\xi_3)\sqrt{\varepsilon} A_5, \end{aligned}$$

where A_5 is a constant independent of ε. Thus

$$\left| u'(\xi_3) \right| \le \frac{u_+ - u_-}{\sqrt{\varepsilon} A_5}.$$

So we get

$$\left|u'(\xi)\right| \leq \frac{u_+ - u_-}{\sqrt{\varepsilon}A_5} \exp\left(\int_{\xi_3}^{\xi} \frac{\rho(u-s)}{\varepsilon}\,ds\right).$$

Noting that when $\xi > \xi_\sigma$,

$$\begin{aligned}
\rho(\xi) &= 2\epsilon_1 + (\rho_+ - 2\epsilon_1)\exp\left(\int_{\xi}^{+\infty} \frac{u'(s)}{u(s)-s}\,ds\right)\\
&= 2\epsilon_1 + \lim_{A\to+\infty}(\rho_+ - 2\epsilon_1)\exp\left(\int_{\xi}^{A} \frac{(u(s)-s)'+1}{u(s)-s}\,ds\right)\\
&\geq 2\epsilon_1 + \lim_{A\to+\infty}(\rho_+ - 2\epsilon_1)\frac{u_+ - A}{u(\xi)-\xi}\exp\left(\int_{\xi}^{A} \frac{ds}{u_+ - s}\right)\\
&= 2\epsilon_1 + \lim_{A\to+\infty}(\rho_+ - 2\epsilon_1)\frac{u_+ - A}{u(\xi)-\xi}\frac{u_+ - \xi}{u_+ - A}\\
&= 2\epsilon_1 + (\rho_+ - 2\epsilon_1)\frac{u_+ - \xi}{u(\xi)-\xi},
\end{aligned}$$

we obtain

$$\rho(u-\xi) \leq \rho_+(u_+ - \xi), \quad \xi \in (\xi_\sigma, +\infty). \tag{4.18}$$

Therefore

$$\begin{aligned}
\left|u'(\xi)\right| &\leq \frac{u_+ - u_-}{\sqrt{\varepsilon}A_5}\exp\left(\int_{\xi_3}^{\xi} \frac{\rho_+(u_+ - s)}{\varepsilon}\,ds\right)\\
&= \frac{u_+ - u_-}{\sqrt{\varepsilon}A_5}\exp\left(-\frac{\rho_+}{2\varepsilon}\left((u_+ - \xi)^2 - (u_+ - \xi_3)^2\right)\right),
\end{aligned}$$

which means that

$$\lim_{\varepsilon\to 0^+} u_\xi^\varepsilon(\xi) = 0, \quad \text{uniformly for } \xi \geq u_+ + \eta.$$

Next, we take ξ_4 such that $\xi > \xi_4 \geq u_+ + \eta$. Noting that

$$u(\xi) - u(\xi_4) = u'(\xi_4)\int_{\xi_4}^{\xi} \exp\left(\int_{\xi_4}^{\tau} \frac{\rho(u-s)}{\varepsilon}\,ds\right)d\tau,$$

we get

$$\begin{aligned}
\left|u(\xi) - u(\xi_4)\right| &\leq \left|u'(\xi_4)\right|\int_{\xi_4}^{\xi} \exp\left(\int_{\xi_4}^{\tau} \frac{\rho_+(u_+ - s)}{\varepsilon}\,ds\right)d\tau\\
&= \left|u'(\xi_4)\right|\int_{\xi_4}^{\xi} \exp\left(\frac{\rho_+}{2\varepsilon}\left(2(u_+ - \xi_4)(s-\xi_4) - (s-\xi_4)^2\right)\right)ds\\
&= \left|u'(\xi_4)\right|\int_{0}^{\xi-\xi_4} \exp\left(\frac{\rho_+}{2\varepsilon}\left(2(u_+ - \xi_4)s - s^2\right)\right)ds.
\end{aligned}$$

Letting $\xi \to +\infty$, we obtain

$$\left|u(\xi_4) - u_+\right| \leq \left|u'(\xi_4)\right|\sqrt{\varepsilon}A_6,$$

where A_6 is a constant independent of ε, which implies that

$$\lim_{\varepsilon\to 0^+} u^\varepsilon(\xi) = u_+, \quad \text{uniformly for } \xi \geq u_+ + \eta.$$

Furthermore, from Lemma 3.2(ii) and (4.18), for $\xi > u_+ + \eta$, we get that

$$\rho_+ \geq \rho(\xi) \geq \frac{\rho_+(u_+ - \xi)}{u(\xi) - \xi} \to \rho_+ \quad \text{for } \varepsilon \to 0^+.$$

Thus

$$\lim_{\varepsilon\to 0^+} \rho^\varepsilon(\xi) = \rho_+, \quad \text{uniformly for } \xi > u_+ + \eta.$$

Analogously, we can obtain the result for $\xi < u_+ + \eta$.

Now we consider the limit solution on $[u_-, u_+]$. Set

$$F(\xi) = u(\xi) - \xi.$$

Then from Lemma 3.2(ii) we have

$$F'(\xi) = \big(u(\xi) - \xi\big)' = u'(\xi) - 1 \leq 0,$$

where $\xi \in [u_-, u_+]$. Hence

$$u(u_+ + \eta) \leq u(\xi) \leq u(u_- - \eta),$$

namely

$$u(u_+ + \eta) - (u_+ + \eta) \leq u(\xi) - \xi \leq u(u_- - \eta) - (u_- - \eta),$$

which yields

$$-\eta \leq \lim_{\varepsilon\to 0^+} \big(u(\xi) - \xi\big) \leq \eta.$$

Since η is arbitrary, we conclude that

$$\lim_{\varepsilon\to 0^+} \big(u(\xi) - \xi\big) = 0.$$

This immediately shows that

$$\lim_{\varepsilon\to 0^+} \rho^\varepsilon(\xi) = 2\epsilon_1, \quad \text{uniformly for } u_- - \eta \leq \xi \leq u_+ + \eta. \qquad \square$$

Thus we have the following theorem.

Theorem 4.5 *Let* $(\rho^\varepsilon, u^\varepsilon)$ *be the solution of* (3.1) *and* (3.2) *and* $u_- < u_+$. *Then*

$$\bigl(\rho(\xi), u(\xi)\bigr) = \lim_{\varepsilon\to 0^+}\bigl(\rho^\varepsilon, u^\varepsilon\bigr)(\xi) = \begin{cases} (\rho_-, u_-), & \xi < u_-, \\ (2\epsilon_1, \xi), & u_- \le \xi \le u_+, \\ (\rho_+, \rho_+), & \xi > u_+. \end{cases} \tag{4.19}$$

The theorem shows that the constant density solution is stable under viscous perturbation.

Remark When the two perturbed parameters ϵ_1 and ε vanish simultaneously, one can observe that the Riemann solutions to (1.9) and (1.10) converge to those of the zero-pressure gas dynamics (1.1) with the same initial data, which shows that the parameterized delta-shock and constant density solutions to the zero-pressure gas dynamics with flux perturbation are stable to the reasonable viscous perturbations and flux perturbations.

5 Conclusion

In order to explore the impact of flux perturbation on the stability of delta-shock and vacuum state solutions to the zero-pressure gas dynamics under viscosity approach, we propose the perturbed zero-pressure gas dynamics model, which contains viscosity and flux approximation simultaneously. This is quite different from the previous works [8, 30] that only involve viscosity or flux perturbation (or pressure perturbation). The vanishing viscosity limits for Riemann solutions to the flux-approximation pressureless system are investigated and the formation of constant density solution and parameterized delta-shock solution is rigorously analyzed. It is proved that the parameterized delta-shock and constant density solutions to the zero-pressure gas dynamics with flux perturbation are stable to the reasonable viscous perturbations. Moreover, our work to some extent confirms the mathematical reasonability of the flux perturbation proposed in [30–33] from another perspective.

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Authors' contributions
ZYY carried out the study, drafted the manuscript, and approved the initial and revised version. YJG carried out the language polishing. All authors read and approved the final manuscript.

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