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Dynamical analysis of a competition and cooperation system with multiple delays

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Abstract

This paper is concerned with a competition and cooperation system with multiple constant delays relating to economic enterprise. The stability of the unique positive equilibrium is investigated and the existence of Hopf bifurcations is demonstrated by analysing the associated characteristic equation. Furthermore, the explicit formulae determining the stability and the direction of periodic solutions bifurcating from Hopf bifurcations are obtained by applying centre manifold theory and the normal form method. Finally, special attention is paid to some numerical simulations in order to support the theoretical predictions.

MSC: 34K10; 34K20; 37L10

Keywords: Competition and cooperation system; Time delays; Stability; Local Hopf bifurcation

1 Introduction

Delay differential equations (DDEs) which involve delays and change in time [1, 2] exhibit considerably more complex dynamical behaviour than ordinary differential equations (ODEs), since the delays could cause a stable equilibrium to become unstable and fluctuate. Many complicated and large-scale systems in nature and society can be modelled as DDEs due to their flexibility and generality for representing virtually any natural and man-made structure. Research of the dynamical behaviour of DDEs has received much attention in interdisciplinary subjects, including natural sciences [3–6], engineering [7, 8], life sciences [9] and others [10–16]. In recent decades, scientists have focused on the stability and bifurcation phenomena of the continuous-time autonomous predator–prey system with multiple delays (see, for example, [17–22]). In fact, there is a strong relationship between how species co-evolve in nature and how different enterprises co-exist in societal economics, leading to significant research into the delayed competition and cooperation model for business enterprises [23–25], which are governed by the following system of ODEs:

$$\begin{cases} \dot{x_1}(t) = r_1 x_1(t)(1 - \frac{x_1(t)}{K_1} - \frac{\alpha(x_2(t)-c_2)^2}{K_2}), \\ \dot{x_2}(t) = r_2 x_2(t)(1 - \frac{x_2(t)}{K_2} + \frac{\beta(x_1(t)-c_1)^2}{K_1}), \end{cases} \tag{1.1}$$

where $x_1(t)$, $x_2(t)$ denote the output of enterprise x_1 and enterprise x_2 at time t, respectively, $(x_1(t), x_2(t)) \in \mathbb{R}^1 \times \mathbb{R}^1$; r_i $(i = 1, 2)$ represents the intrinsic growth rate for the output of the two enterprises; K_i $(i = 1, 2)$ is a measure of the load capacity of the two enterprises in an unrestricted natural market; α, β are the coefficients of competition of enterprise x_1 and x_2, respectively; c_i $(i = 1, 2)$ denotes the initial production of them. All the parameters assume strictly positive values.

Let $a_1 = \frac{r_1}{K_1}$, $a_2 = \frac{r_2}{K_2}$, $b_1 = \frac{\alpha r_1}{K_2}$, $b_2 = \frac{\beta r_2}{K_1}$, $d_i = r_i - a_i c_i$, $\forall i = 1, 2$, $u(t) = x_1(t) - c_1$, $v(t) = x_2(t) - c_2$. System (1.1) becomes

$$\begin{cases} \frac{du(t)}{dt} = (u(t) + c_1)(d_1 - a_1 u(t) - b_1 v^2(t)), \\ \frac{dv(t)}{dt} = (v(t) + c_2)(d_2 - a_2 v(t) + b_2 u^2(t)), \\ u(0) > 0, \qquad v(0) > 0, \end{cases} \tag{1.2}$$

where $(u(t), v(t)) \in \mathbb{R}^1 \times \mathbb{R}^1$.

Taking into account the influence of the prior history of the enterprises, authors have introduced a time delay, τ, to the feedback in model (1.2) [26], which is a more realistic approach for understanding competition and cooperation dynamics. Delays can induce oscillations and periodic solutions through bifurcations as the delay is increased. Therefore, it is interesting to investigate the following delayed model:

$$\begin{cases} \dot{y_1}(t) = (y_1(t) + c_1)(d_1 - a_1 y_1(t - \tau_1) - b_1 y_2^2(t - \tau_2)), \\ \dot{y_2}(t) = (y_2(t) + c_2)(d_2 - a_2 y_2(t - \tau_1) + b_2 y_1^2(t - \tau_3)), \end{cases} \tag{1.3}$$

where $y_i(t)$ $(i = 1, 2)$ denotes the output of two enterprises at time t, a_i, b_i $(i = 1, 2)$ denote the intraspecific competition rate and interspecific effect rate between them, where a_i, b_i, c_i, d_i $(i = 1, 2)$ are positive constants. τ_1 denotes the interior delays of themselves, τ_i $(i = 2, 3)$ denotes the exterior delays between each other, and τ_i $(i = 1, 2, 3)$ is nonnegative constant delays.

We define $\mathbb{R}_+ \equiv \{x \in \mathbb{R} : x \geq 0\}$, $\operatorname{int}\mathbb{R}_+ \equiv \{x \in \mathbb{R} : x > 0\}$, $\hat{\tau} = \max\{\tau_1, \tau_2, \tau_3\}$. Denote by $C([-\hat{\tau}, 0], \mathbb{R}_+)$ the infinite dimensional Banach space of continuous functions from the interval $[-\hat{\tau}, 0]$ into $\mathbb{R}_+$, equipped with the uniform norm. We assume that the initial data for model (1.3) is taken from

$$X = C\big([-\hat{\tau}, 0], \mathbb{R}_+\big) \times C\big([-\hat{\tau}, 0], \mathbb{R}_+\big). \tag{1.4}$$

The variables $y_1(t)$ and $y_2(t)$ in model (1.3) belong to X for $t \in [-\hat{\tau}, 0]$.

By [1] (Theorem 2.1 and 2.3, Chap. 2, p. 41), solutions of system (1.3) with the initial value in C exist and are unique for all $t > 0$.

Liao [23] assumed τ_i $(i = 1, 2, 3) = \tau$ and Li [24] considered $\tau_1 = 0$, regarding τ and $\tau_2 + \tau_3$ as the bifurcation parameters, respectively. They investigated the existence of the unique positive equilibrium and proved that the Hopf bifurcation can occur as the bifurcation parameter crosses some critical value, and studied the direction of Hopf bifurcation and stability of the periodic solutions. In [27], Liao considered $\tau_2 = \tau_3 \neq \tau_1$, analysed the stability of the positive equilibrium and the existence of local Hopf bifurcation and provided some numerical simulations. However, they did not give the underlying description of the bifurcated periodic solution.

In the more realistic competition and cooperation model, the interior delays exist (i.e. $\tau_1 \neq 0$) and the exterior delays are not necessarily equal (i.e. $\tau_2 \neq \tau_3$). Based on these observations, we studied the system of (1.3) that could better describe the real system behaviour. Compared with the models from the literature [23, 27], the dynamical behaviour of system (1.3) is more complicated than the above models.

In this paper, we have taken the delay $\tau_1 := \tau_2 + \tau_3$ as the bifurcation parameter and show that when τ_1 passes through the critical values, the positive equilibrium loses its stability and a Hopf bifurcation occurs. Furthermore, we give details of the bifurcation values that describe the direction of the Hopf bifurcation and the stability of the bifurcated periodic solution using centre manifold theory and the normal form method introduced by Hassard et al. [28]. Finally, some numerical simulations and conclusions are given to illustrate the theoretical predictions.

2 The existence and the property of the local Hopf bifurcation

In this section, we give the following results about the existence and stability of the positive equilibrium of system (1.3).

Proposition 1 *For system* (1.3), *assume that* a_2, b_1, d_1, d_2 *are positive constants such that*

(H$_1$) $a_2^2 d_1 > b_1 d_2^2$

holds, then the system has a unique positive equilibrium $E^* = (y_1^*, y_2^*)$. *Furthermore, when system* (1.3) *has no delay, i.e.* τ_i $(i = 1, 2, 3) = 0$, *then* E^* *is globally asymptotically stable.*

Proof For system (1.3), assumption (H$_1$) is the parameter condition which ensures the existence of the positive equilibrium E^*. The proof for the existence of E^* is similar to that in [23], we omit it here.

We now prove the global asymptotic stability. When τ_i $(i = 1, 2, 3) = 0$, system (1.3) is reduced to the ODE system (1.1). Defining Dulac function as $B(x_1, x_2) = \frac{1}{x_1 x_2}$, and

$$\begin{aligned} D &= \frac{\partial\{\frac{1}{x_1x_2}[r_1x_1(t)(1-\frac{x_1(t)}{K_1}-\frac{\alpha(x_2(t)-c_2)^2}{K_2})]\}}{\partial x_1} + \frac{\partial\{\frac{1}{x_1x_2}[r_2x_2(t)(1-\frac{x_2(t)}{K_2}+\frac{\beta(x_1(t)-c_1)^2}{K_1})]\}}{\partial x_2} \\ &= -\frac{r_1}{K_1x_2} - \frac{r_2}{x_1K_2}, \end{aligned}$$

we easily get $D < 0$ in the $\operatorname{int}\mathbb{R}_+^2 = \{(x_1, x_2) : x_1 > 0, x_2 > 0\}$ space. By [29] (Theorem 4.1.2, Chap. 4, p. 72), it follows from Dulac's principle that the system has no closed path curve. So E^* is globally asymptotically stable when system (1.3) has no delay and also when the non-negative delays are sufficiently small. □

If a pair of complex roots with negative real parts and non-zero imaginary parts cross the imaginary axis as τ increases, this potentially results in Hopf bifurcation and the positive equilibrium E^* loses stability. Now we discuss the existence of a local Hopf bifurcation occurring at E^*. Let $u_1(t) = y_1(t) - y_1^*$, $u_2(t) = y_2(t) - y_2^*$, then system (1.3) becomes

$$\begin{cases} \dot{u}_1(t) = (u_1(t) + y_1^* + c_1)[-a_1u_1(t-\tau_1) - b_1{u_2}^2(t-\tau_2) - 2b_1y_2^*u_2(t-\tau_2)], \\ \dot{u}_2(t) = (u_2(t) + y_2^* + c_1)[-a_2u_2(t-\tau_1) + b_2{u_1}^2(t-\tau_3) + 2b_2y_2^*u_1(t-\tau_3)], \end{cases} \tag{2.1}$$

the linearization of system (2.1) at E^* is

$$\begin{cases} \dot{u}_1(t) = -a_1(y_1^* + c_1)u_1(t-\tau_1) - 2b_1y_2^*(y_1^* + c_1)u_2(t-\tau_2), \\ \dot{u}_2(t) = 2b_2y_2^*(y_2^* + c_1)u_1(t-\tau_3) - a_2(y_2^* + c_1)u_2(t-\tau_1), \end{cases} \tag{2.2}$$

and the associated characteristic equation of (2.2) is

$$\begin{vmatrix} \lambda + a_1(y_1^* + c_1)e^{-\lambda\tau_1} & 2b_1y_2^*(y_1^* + c_1)e^{-\lambda\tau_2} \\ -2b_2y_2^*(y_2^* + c_1)e^{-\lambda\tau_3} & \lambda + a_2(y_2^* + c_1)e^{-\lambda\tau_1} \end{vmatrix} = 0.$$

For the above characteristic equation, it is hard to do the complete analysis for the distribution of the roots, so we assume that

(H_2) $\tau_2 + \tau_3 = \tau_1$

holds, and $\tau_1 \triangleq \tau$.

Hence, the characteristic equation is equivalent to

$$\lambda^2 + e^{-\lambda\tau}(p\lambda + q) + re^{-2\lambda\tau} = 0, \tag{2.3}$$

where $p = a_1(y_1^* + c_1) + a_2(y_2^* + c_1)$, $q = 4b_1b_2(y_2^*)^2(y_1^* + c_1)(y_2^* + c_1)$, $r = a_1a_2(y_1^* + c_1)(y_2^* + c_1)$.

Since the characteristic equation (2.3) has the same form as equation (2.4) in [30], so by Theorem 2.5 in [30], we can get the following result, which presents the conditions for a Hopf bifurcation to occur in system (1.3).

Proposition 2 *Suppose that* (H_1) *and* (H_2) *hold. Then*

$$\tau_k^j = 1/\omega_k\bigl[\arccos q/\bigl(\omega_k^2 - r\bigr) + 2j\pi\bigr], \quad k = 1,2,3,4, j = 0,1,2,\ldots$$

are Hopf bifurcation values at E^*, *where* $i\omega_k$ ($k = 1,2,3,4$) *are the roots of* (2.3). *And* E^* *is locally asymptotically stable for* $\tau \in [0, \tau_1^0]$ *and unstable where* $\tau > \tau_1^0$.

Remark 1 The characteristic equation (2.3) has some pairs of purely imaginary roots denoted by $\lambda = \pm i\omega_k$ with $\tau = \tau_k^j$ under the condition of (H_1), (H_2). Define $\tau^0 = \tau_{k_0}^0 = \min_{1\le k\le 4}\{\tau_k^0\}$, $\omega_0 = \omega_{k_0}$, where $k_0 \in \{1,2,3,4\}$. Then τ^0 is the first value of τ such that (2.3) has purely imaginary roots. For convenience, we denote τ_k^j by τ^j ($j = 0,1,2,\ldots$) for fixed $k \in \{1,2,3,4\}$.

Remark 2 Let $\lambda(\tau) = \alpha(\tau) \pm i\omega(\tau)$ be the roots of (2.3) near $\tau = \tau^j$ satisfying $\alpha(\tau^j) = 0$, $\omega(\tau^j) = \omega_0$ ($j = 0,1,2,\ldots$). By the theory of DDEs, for $\forall\tau_k^j$, $\exists\varepsilon > 0$ s.t. $\lambda(\tau)$ in $|\tau - \tau_k^j| < \varepsilon$ about τ is continuous and differentiable. The transversality condition $\frac{d\operatorname{Re}\lambda(\tau)}{d\tau}|_{\tau=\tau_j} > 0$ is satisfied (more details are provided in [30]).

In the previous part, it was shown that system (2.1) undergoes a Hopf bifurcation under certain conditions. Here we will derive explicit formulae determining the direction of the Hopf bifurcation and the stability of the periodic solutions bifurcating from E^* at τ^j ($j = 0,1,2,\ldots$), by employing centre manifold theory and the normal form method. For convenience, denote τ^j by $\widetilde{\tau}$ and $\tau = \widetilde{\tau} + \mu$, $\mu \in \mathbb{R}$, then $\mu = 0$ is the Hopf bifurcation value for system (1.3), where $\widetilde{\tau} = \widetilde{\tau}_2 + \widetilde{\tau}_3$, $\tau = \widetilde{\tau}_2 + \widetilde{\tau}_3 + \mu$. Without loss of generality, assume $\widetilde{\tau}_2 < \widetilde{\tau}_3$.

The discussion will be divided into five steps as follows.

Step 1. Transform system (2.1) into the abstract ODE.

System (2.1) can locally be represented as the following DDE in $C = C([-\widetilde{\tau}, 0], R^2)$:

$$\dot{u}(t) = L_\mu(u_t) + F(\mu, u_t), \tag{2.4}$$

where $u(t) = (u_1(t), u_2(t))^T$, $u_t(\theta) = u(t+\theta)$, $L_\mu : C \to R$ is a bounded linear operator and $F : R \times C \to R$ is continuous and differentiable with

$$L_\mu \phi = (\widetilde{\tau} + \mu) \begin{pmatrix} -a_1(y_1^* + c_1)\phi_1(-\tau_1) + 2b_1 y_2^*(y_1^* + c_1)\phi_2(-\tau_2) \\ 2b_2 y_2^*(y_2^* + c_1)\phi_1(-\tau_3) - a_2(y_2^* + c_1)\phi_2(-\tau_1) \end{pmatrix},$$

and

$$\begin{aligned} F(\mu, \phi) &= (\widetilde{\tau} + \mu) \\ &\quad \times \begin{pmatrix} -a_1\phi_1(0)\phi_1(-\tau_1) - b_1\phi_1(0)\phi_2^2(-\tau_2) - 2b_1 y_2^*\phi_1(0)\phi_2(-\tau_2) - b_1(y_1^* + c_1)\phi_2^2(-\tau_2) \\ -a_2\phi_2(0)\phi_2(-\tau_1) + b_2\phi_2(0)\phi_2^1(-\tau_3) + 2b_2 y_2^*\phi_2(0)\phi_1(-\tau_3) + b_2(y_2^* + c_1)\phi_1^2(-\tau_3) \end{pmatrix}, \end{aligned}$$

where $\phi = (\phi_1(\theta), \phi_2(\theta)) \in C$.

By the Riesz representation theorem, there exists a 2×2 matrix whose elements are a bounded variation function $\eta(\theta, \mu)$ in $\theta \in [-\widetilde{\tau}, 0]$ such that

$$L_\mu \phi = \int_{-\widetilde{\tau}}^0 d\eta(\theta, \mu)\phi(\theta), \quad \phi \in C,$$

where $\eta(\theta, \mu)$ can be chosen as

$$\eta(\theta, \mu) = \begin{cases} (\widetilde{\tau} + \mu)(M + N + P), & \theta \in [-\widetilde{\tau}_2, 0), \\ (\widetilde{\tau} + \mu)(M + N), & \theta \in (-\widetilde{\tau}_3, -\widetilde{\tau}_2), \\ (\widetilde{\tau} + \mu)M, & \theta \in (-\widetilde{\tau}, -\widetilde{\tau}_3], \\ 0, & \theta = -\widetilde{\tau}, \end{cases}$$

with

$$M = \begin{pmatrix} -a_1(y_1^* + c_1) & 0 \\ 0 & -a_2(y_2^* + c_1) \end{pmatrix}, \qquad N = \begin{pmatrix} 0 & -2b_1 y_2^*(y_1^* + c_1) \\ 0 & 0 \end{pmatrix},$$

$$P = \begin{pmatrix} 0 & 0 \\ 2b_2 y_2^*(y_2^* + c_1) & 0 \end{pmatrix}.$$

For $\phi \in C$, let

$$A(\mu)\phi(\theta) = \begin{cases} d\phi(\theta)/d\theta, & \theta \in [-\widetilde{\tau}, 0), \\ \int_{-\widetilde{\tau}}^0 d\eta(\mu, \theta)\phi(\theta), & \theta = 0, \end{cases}$$

$$R(\mu)\phi(\theta) = \begin{cases} 0, & \theta \in [-\widetilde{\tau}, 0), \\ F(\mu, \phi), & \theta = 0, \end{cases}$$

then system (2.4) is equivalent to the following abstract operator equation:

$$\dot{u}(t) = A(\mu)u_t + R(\mu)u_t. \tag{2.5}$$

Step 2. Calculate the eigenfunctions of $A = A(0)$ and the adjoint operator A^* corresponding to $i\omega_0\widetilde{\tau}$ and $-i\omega_0\widetilde{\tau}$.

For $\psi \in C([0,\widetilde{\tau}],(C^2)^*)$, where $(C^2)^*$ is the two-dimensional complex space of row vectors, we define the adjoint operator A^* of A

$$A^*\psi(s) = \begin{cases} -d\psi(s)/ds, & s \in (0,\widetilde{\tau}], \\ \int_{-\widetilde{\tau}}^{0} d\eta^T(\mu,t)\psi(-t), & s = 0, \end{cases}$$

and the bilinear form is given by

$$\langle \psi(s),\phi(\theta)\rangle = \overline{\psi}(0)\phi(0) - \int_{-\widetilde{\tau}}^{0}\int_{\xi=0}^{\theta} \overline{\psi}^T(\xi-\theta)\,d\eta(\theta)\phi(\xi)\,d\xi,$$

where $\eta(\theta) = \eta(\theta,0)$. Then $A = A(0)$ and $A^*(0)$ are adjoint operators.

By [27], $\pm i\omega_0\widetilde{\tau}$ are eigenvalues of $A(0)$, so they are also eigenvalues of $A^*(0)$. Suppose that $q(\theta) = (1,\alpha)^T e^{i\omega_0\theta}$ is the eigenfunction of $A(0)$ corresponding to the eigenvalue $i\omega_0\widetilde{\tau}$ and $q^*(s) = G(\beta,1)e^{i\omega_0 s}$ is the eigenfunction of A^* corresponding to the eigenvalue $-i\omega_0\widetilde{\tau}$, where

$$\begin{aligned}
\alpha &= -\bigl[-i\omega_0 + a_2\bigl(y_2^* + c_1\bigr)i\omega_0 e^{-i\omega_0\widetilde{\tau}}\bigr]/2b_1y_2^*\bigl(y_1^* + c_1\bigr)e^{-i\omega_0\widetilde{\tau}}, \\
\beta &= -\bigl[i\omega_0 + a_1\bigl(y_1^* + c_1\bigr)i\omega_0 e^{-i\omega_0\widetilde{\tau}}\bigr]/2b_1y_2^*\bigl(y_1^* + c_1\bigr)e^{-i\omega_0\widetilde{\tau}}, \\
G &= \bigl\{\overline{\beta} + \alpha - 2\alpha\overline{\beta}b_1y_2^*\bigl(y_1^* + c_1\bigr)\widetilde{\tau}_2 e^{i\omega_0\widetilde{\tau}_2} + 2\alpha\overline{\beta}b_2y_2^*\bigl(y_2^* + c_1\bigr)\widetilde{\tau}_3 e^{i\omega_0\widetilde{\tau}_3} \\
&\quad + \bigl[-\overline{\beta}a_1\bigl(y_1^* + c_1\bigr) - \alpha_2\bigl(y_2^* + c_1\bigr)\bigr]\widetilde{\tau}e^{i\omega_0\widetilde{\tau}}\bigr\}^{-1},
\end{aligned}$$

which assures that $\langle q^*(s),q(\theta)\rangle = 1$, $\langle q^*(s),\overline{q}(\theta)\rangle = 0$.

Step 3. Obtain the reduced system on the centre manifold.

In this part, we will use the same notations as in [28] and compute the coordinates to describe the centre manifold $\mathbf{C}_0$ at $\mu = 0$ (a local centre manifold is in general not unique, and the dimension of local centre manifold is 2). Let $u_t \in C$ be the solution of system (2.5) when $\mu = 0$, and define

$$z(t) = \langle q^*,u_t\rangle, \qquad W(t,\theta) = u_t(\theta) - z(t)q(\theta) - \overline{z}(t)\overline{q}(\theta), \tag{2.6}$$

where z and $\overline{z}$ are local coordinates for the centre manifold $\mathbf{C}_0$ in the direction of q^* and $\overline{q}^*$. On the centre manifold $\mathbf{C}_0$, we have $W(t,\theta) = W(z(t),\overline{z}(t),\theta)$, where

$$W(z,\overline{z},\theta) = W_{20}(\theta)z^2/2 + W_{11}(\theta)z\overline{z} + W_{02}(\theta)\overline{z}^2/2 + \cdots. \tag{2.7}$$

The existence of a centre manifold enables us to reduce (2.5) to an ODE on $\mathbf{C}_0$. Note that W is real if u_t is real, we consider only real solutions. For solution $u_t \in \mathbf{C}_0$ of system (2.5) at $\mu = 0$,

$$\begin{aligned} \dot{z}(t) &= \langle q^*, \dot{u}_t \rangle = \langle q^*, A(u_t) + R(u_t) \rangle = \langle A^*(q^*), u_t \rangle + \langle q^*, R(u_t) \rangle \\ &= i\omega_0 \widetilde{\tau} z(t) + \overline{q}^*(0) \cdot f\bigl(0, W(z, \overline{z}, \theta) + 2\,\mathrm{Re}\{z(t) q(\theta)\}\bigr) \\ &= i\omega_0 \widetilde{\tau} z(t) + \overline{q}^*(0) \cdot f(0, u_t), \end{aligned} \tag{2.8}$$

with

$$\overline{q}^*(0) \cdot f(0, u_t) \triangleq g(z, \overline{z}).$$

Rewriting (2.8), we obtain that the reduced system on $\mathbf{C}_0$ is described by

$$\dot{z}(t) = i\omega_0 \widetilde{\tau} z(t) + g(z, \overline{z}), \tag{2.9}$$

where

$$g(z, \overline{z}) = g_{20}(\theta) z^2/2 + g_{11}(\theta) z\overline{z} + g_{02}(\theta) \overline{z}^2/2 + g_{21}(\theta) z^2 \overline{z}/2 + \cdots. \tag{2.10}$$

We will mainly discuss equation (2.9) in the following part.

Step 4. Obtain the values of g_{20}, g_{11}, g_{02}, g_{21} in (2.10).

In this part, we calculate the coefficients $W_{20}(\theta)$, $W_{11}(\theta)$, $W_{02}(\theta)$, … and substitute them in (2.8) to get the reduced system (2.9) on $\mathbf{C}_0$.

It follows from (2.6) that

$$\begin{aligned} u_t(\theta) = u(t+\theta) &= W(t, \theta) + 2\,\mathrm{Re}\{z(t), q(\theta)\} \\ &= W_{20}(\theta) z^2/2 + W_{11}(\theta) z\overline{z} + W_{02}(\theta) \overline{z}^2/2 + (1, \alpha)^T e^{i\omega_0 \widetilde{\tau}\theta} z \\ &\quad + (1, \overline{\alpha})^T e^{-i\omega_0 \widetilde{\tau}\theta} \overline{z} + \cdots. \end{aligned}$$

And we have

$$\begin{aligned} &u_1(t) = z + \overline{z} + W^{(1)}(t, 0), \qquad u_2(t) = z\alpha + \overline{z\alpha} + W^{(2)}(t, 0), \\ &u_1(t - \widetilde{\tau}) = z e^{-i\omega_0 \widetilde{\tau}} + \overline{z} e^{i\omega_0 \widetilde{\tau}} + W^{(1)}(-\widetilde{\tau}), \\ &u_2(t - \widetilde{\tau}) = z\alpha e^{-i\omega_0 \widetilde{\tau}} + \overline{z\alpha} e^{i\omega_0 \widetilde{\tau}} + W^{(2)}(-\widetilde{\tau}), \\ &u_2(t - \widetilde{\tau}_2) = z\alpha e^{-i\omega_0 \widetilde{\tau}_2} + \overline{z\alpha} e^{i\omega_0 \widetilde{\tau}_2} + W^{(2)}(-\widetilde{\tau}_2), \\ &u_1(t - \widetilde{\tau}_3) = z e^{-i\omega_0 \widetilde{\tau}_3} + \overline{z} e^{i\omega_0 \widetilde{\tau}_3} + W^{(1)}(-\widetilde{\tau}_3), \\ &u_2(t - \widetilde{\tau}_3) = z\alpha e^{-i\omega_0 \widetilde{\tau}_3} + \overline{z\alpha} e^{i\omega_0 \widetilde{\tau}_3} + W^{(2)}(-\widetilde{\tau}_3). \end{aligned} \tag{2.11}$$

It follows that together with $F(\mu, \phi)$ we get

$$f(0, u_t) = \widetilde{\tau} \begin{pmatrix} -a_1 u_1(t) u_1(t - \widetilde{\tau}) - b_1 u_1(t) u_2^2(t - \widetilde{\tau}_2) - 2b_1 y_2^* u_1(t) u_2(t - \widetilde{\tau}_2) - b_1(y_1^* + c_1) u_2^2(t - \widetilde{\tau}_2) \\ -a_2 u_2(t) u_2(t - \widetilde{\tau}) + b_2 u_2(t) u_2^1(t - \widetilde{\tau}_3) + 2b_2 y_2^* u_2(t) u_1(t - \widetilde{\tau}_3) + b_2(y_2^* + c_1) u_1^2(t - \widetilde{\tau}_3) \end{pmatrix}. \tag{2.12}$$

Substituting (2.11) into (2.12), then this substitution into (2.8), and comparing the coefficients with (2.10), we obtain

$$\begin{aligned}
g_{20} &= 2\widetilde{\tau}\overline{G}\{\overline{\beta}[-a_1e^{-i\omega_0\widetilde{\tau}} - 2b_1y_2^*\alpha e^{-i\omega_0\widetilde{\tau}_2} - b_1(y_1^*+c_1)\alpha^2e^{-2i\omega_0\widetilde{\tau}_2}] \\
&\quad - a_2\alpha^2e^{-i\omega_0\widetilde{\tau}} + 2b_2y_2^*\alpha e^{-i\omega_0\widetilde{\tau}_3} + b_2(y_2^*+c_1)e^{-2i\omega_0\widetilde{\tau}}\}, \\
g_{11} &= \widetilde{\tau}\overline{G}\{\overline{\beta}[-a_1(e^{i\omega_0\widetilde{\tau}}+e^{-i\omega_0\widetilde{\tau}}) - 2b_1y_2^*(\overline{\alpha}e^{i\omega_0\widetilde{\tau}}+\alpha e^{-i\omega_0\widetilde{\tau}}) - 2b_1(y_1^*+c_1)\alpha\overline{\alpha}] \\
&\quad - a_2\alpha\overline{\alpha}(e^{i\omega_0\widetilde{\tau}}+e^{-i\omega_0\widetilde{\tau}}) + 2b_2y_2^*(\alpha e^{i\omega_0\widetilde{\tau}_3}+\overline{\alpha}e^{-i\omega_0\widetilde{\tau}_3}) + 2b_2(y_2^*+c_1)\}, \\
g_{02} &= 2\widetilde{\tau}\overline{G}\{\overline{\beta}[-a_1e^{i\omega_0\widetilde{\tau}} - 2b_1y_2^*\overline{\alpha}e^{i\omega_0\widetilde{\tau}} - b_1(y_1^*+c_1)\overline{\alpha}^2e^{2i\omega_0\widetilde{\tau}}] - a_2\overline{\alpha}^2e^{i\omega_0\widetilde{\tau}} \\
&\quad + 2b_2y_2^*\overline{\alpha}e^{i\omega_0\widetilde{\tau}} + b_2(y_2^*+c_1)\}, \\
g_{21} &= \widetilde{\tau}\overline{G}\{\overline{\beta}[-a_1(W_{11}^{(1)}(-\widetilde{\tau}) + W_{20}^{(1)}(-\widetilde{\tau})/2 + e^{-i\omega_0\widetilde{\tau}}W_{11}^{(1)}(0) + e^{i\omega_0\widetilde{\tau}}W_{20}^{(1)}(0)/2) \\
&\quad - b_1(2\alpha\overline{\alpha}+\overline{\alpha}^2e^{2i\omega_0\widetilde{\tau}_2}) - 2b_1y_2^*(W_{11}^{(2)}(-\widetilde{\tau}) + W_{20}^{(2)}(-\widetilde{\tau})/2 + \alpha e^{-i\omega_0\widetilde{\tau}}W_{11}^{(1)}(0) \\
&\quad + \overline{\alpha}e^{i\omega_0\widetilde{\tau}}W_{20}^{(1)}(0)/2) - b_1(y_1^*+c_1)(\alpha e^{-i\omega_0\widetilde{\tau}}W_{11}^{(2)}(-\widetilde{\tau}) + \overline{\alpha}e^{i\omega_0\widetilde{\tau}}W_{20}^{(2)}(-\widetilde{\tau})/2)] \\
&\quad - a_2(\alpha W_{11}^{(2)}(-\widetilde{\tau}) + \overline{\alpha}W_{20}^{(2)}(-\widetilde{\tau})/2 + \alpha e^{-i\omega_0\widetilde{\tau}}W_{11}^{(2)}(0) + \overline{\alpha}e^{i\omega_0\widetilde{\tau}}W_{20}^{(2)}(0)/2) \\
&\quad + b_2(2\alpha + e^{-2i\omega_0\widetilde{\tau}}) + 2b_2y_2^*[\alpha W_{11}^{(1)}(-\widetilde{\tau}) + W_{20}^{(1)}(-\widetilde{\tau})/2 + e^{-i\omega_0\widetilde{\tau}}W_{11}^{(1)}(0) \\
&\quad + e^{i\omega_0\widetilde{\tau}}W_{20}^{(1)}(0)] + b_2(y_2^*+c_1)[2e^{-i\omega_0\widetilde{\tau}}W_{11}^{(1)}(-\widetilde{\tau}_3) + e^{i\omega_0\widetilde{\tau}}W_{20}^{(1)}(-\widetilde{\tau}_3)]\}.
\end{aligned} \tag{2.13}$$

Since there are $W_{20}(\theta)$ and $W_{11}(\theta)$ in g_{21}, we still need to compute them.

From (2.5) and (2.6), we have

$$\dot{W} = \dot{u}_t - \dot{z}q - \dot{\overline{z}}\overline{q} = \begin{cases} AW - 2\,\mathrm{Re}\{gq(\theta)\}, & \theta \in [-1,0), \\ AW - 2\,\mathrm{Re}\{gq(0)\} + f_0, & \theta = 0, \end{cases} \tag{2.14}$$

where

$$f_0 = f_{z^2}z^2/2 + f_{z\overline{z}}z\overline{z} + f_{\overline{z}^2}\overline{z}^2/2 + f_{z^2\overline{z}}z^2\overline{z}/2\cdots.$$

On the other hand, near the origin, on the centre manifold $\mathbf{C}_0$, according to (2.7), we obtain

$$\begin{aligned}
\dot{W} &= W_z\dot{z} + W_{\overline{z}}\dot{\overline{z}} = [W_{20}(\theta)z + W_{11}(\theta)\overline{z}]\dot{z} + [W_{11}(\theta)z + W_{02}(\theta)\overline{z}]\dot{\overline{z}} \\
&= [W_{20}(\theta)z + W_{11}(\theta)\overline{z}](i\omega_0z + g(z,\overline{z})) \\
&\quad + [W_{11}(\theta)z + W_{02}(\theta)\overline{z}](\overline{g}(z,\overline{z}) - i\omega_0\overline{z}) + \cdots.
\end{aligned} \tag{2.15}$$

Substituting (2.7) into the right-hand side of (2.14), equating terms of $\frac{z^2}{2}$ and $z\overline{z}$ of (2.14) with (2.15), we obtain

$$(2i\omega_0I - A)W_{20}(\theta) = \begin{cases} -g_{20}q(\theta) - \overline{g}_{02}\overline{q}(\theta), & \theta \in [-\widetilde{\tau},0), \\ -g_{20}q(0) - \overline{g}_{02}\overline{q}(0) + f_{z^2}, & \theta = 0, \end{cases} \tag{2.16}$$

$$-AW_{11}(\theta) = \begin{cases} -g_{11}q(\theta) - \overline{g}_{11}\overline{q}(\theta), & \theta \in [-\widetilde{\tau},0), \\ -g_{11}q(0) - \overline{g}_{11}\overline{q}(0) + f_{z\overline{z}}, & \theta = 0. \end{cases} \tag{2.17}$$

According to the definition of A and from (2.16), (2.17) for $\theta \in [-\widetilde{\tau}, 0)$, we get

$$\dot{W}_{20}(\theta) = 2i\omega_0 W_{20}(\theta) + g_{20}q(\theta) + \overline{g}_{02}\overline{q}(\theta),$$
$$\dot{W}_{11}(\theta) = g_{11}q(\theta) + \overline{g}_{11}\overline{q}(\theta).$$

Solving for $W_{20}(\theta)$ and $W_{11}(\theta)$, we obtain

$$W_{20}(\theta) = ig_{20}/\omega_0 \cdot q(0)e^{i\omega_0\theta} + i\overline{g}_{02}/3\omega_0 \cdot \overline{q}(0)e^{-i\omega_0\theta} + E_1 e^{2i\omega_0\theta}, \tag{2.18}$$
$$W_{20}(\theta) = -ig_{11}/\omega_0 \cdot q(0)e^{i\omega_0\theta} + i\overline{g}_{11}/\omega_0 \cdot \overline{q}(0)e^{-i\omega_0\theta} + E_2, \tag{2.19}$$

where $E_1 = (E_1^{(1)}, E_1^{(2)})^T \in R^2$ and $E_2 = (E_2^{(1)}, E_2^{(2)})^T \in R^2$ are constant vectors.

In what follows we shall seek appropriate E_1 and E_2 in (2.18) and (2.19), respectively. According to the definition of A and (2.16), (2.17) for $\theta = 0$, we have

$$\int_{-\widetilde{\tau}}^{0} d\eta(\theta)W_{20}(\theta) = 2i\omega_0 W_{20}(0) + g_{20}q(0) + \overline{g}_{02}\overline{q}(0) - f_{z^2}, \tag{2.20}$$
$$\int_{-\widetilde{\tau}}^{0} d\eta(\theta)W_{11}(\theta) = g_{11}q(0) + \overline{g}_{11}\overline{q}(0) - f_{z\bar{z}}, \tag{2.21}$$

where $\eta(\theta) = \eta(0, \theta)$ and

$$f_{z^2} = \begin{pmatrix} -a_1 e^{-i\omega_0\widetilde{\tau}} - 2b_1 y_2^*\alpha e^{-i\omega_0\widetilde{\tau}_2} - b_1(y_1^* + c_1)\alpha^2 e^{-2i\omega_0\widetilde{\tau}_2} \\ -a_2\alpha^2 e^{-i\omega_0\widetilde{\tau}} + 2b_2 y_2^*\alpha e^{-i\omega_0\widetilde{\tau}_3} + b_2(y_2^* + c_1)e^{-2i\omega_0\widetilde{\tau}} \end{pmatrix},$$
$$f_{z\bar{z}} = \begin{pmatrix} -a_1(e^{i\omega_0\widetilde{\tau}} + e^{-i\omega_0\widetilde{\tau}}) - 2b_1 y_2^*(\overline{\alpha}e^{i\omega_0\widetilde{\tau}} + \alpha e^{-i\omega_0\widetilde{\tau}}) - 2b_1(y_1^* + c_1)\alpha\overline{\alpha} \\ -a_2\alpha\overline{\alpha}(e^{i\omega_0\widetilde{\tau}} + e^{-i\omega_0\widetilde{\tau}}) + 2b_2 y_2^*(\alpha e^{i\omega_0\widetilde{\tau}_3} + \overline{\alpha}e^{-i\omega_0\widetilde{\tau}_3}) + 2b_2(y_2^* + c_1) \end{pmatrix}.$$

Substituting (2.18) into (2.20), we obtain

$$\left(2i\omega_0 I - \int_{-\widetilde{\tau}}^{0} e^{2i\omega_0\theta}\, d\eta(\theta)\right)E_1 = f_{z^2},$$

that is

$$\begin{pmatrix} 2i\omega_0 + a_1(y_1^* + c_1)e^{-i\omega_0\widetilde{\tau}} & 2b_1 y_2^*(y_1^* + c_1)e^{-i\omega_0\widetilde{\tau}_2} \\ -2b_2 y_2^*(y_2^* + c_1)e^{-i\omega_0\widetilde{\tau}_3} & 2i\omega_0 + a_2(y_2^* + c_1)e^{-i\omega_0\widetilde{\tau}} \end{pmatrix} E_1 = f_{z^2}. \tag{2.22}$$

Similarly, substituting (2.19) into (2.21), we get

$$\int_{-\widetilde{\tau}}^{0} d\eta(\theta)E_2 = f_{z\bar{z}},$$

that is

$$\begin{pmatrix} a_1(y_1^* + c_1)e^{-i\omega_0\widetilde{\tau}} & 2b_1 y_2^*(y_1^* + c_1)e^{-i\omega_0\widetilde{\tau}_2} \\ -2b_2 y_2^*(y_2^* + c_1)e^{-i\omega_0\widetilde{\tau}_3} & a_2(y_2^* + c_1)e^{-i\omega_0\widetilde{\tau}} \end{pmatrix} E_2 = f_{z\bar{z}}. \tag{2.23}$$

We have obtained the values of E_1 and E_2 as (2.22) and (2.23) and, ultimately, the reduced system (2.9).

Step 5. Obtain the key values μ_2, β_2, T_2 to determine the property of the Hopf bifurcation.

As with the calculation of the ODE Hopf bifurcation parameter and as in [28], according to the analysis above and the expressions of g_{20}, g_{11}, g_{02} and g_{21}, we can compute the following values:

$$\begin{aligned} &c_1(0) = i/2\omega_0\widetilde{\tau}\bigl(g_{11}g_{20} - 2|g_{11}|^2 - |g_{02}|^2/3\bigr) + g_{21}/2, \\ &\mu_2 = -\operatorname{Re}\bigl\{c_1(0)\bigr\}/\alpha'(\widetilde{\tau}), \\ &\beta_2 = 2\operatorname{Re}\bigl\{c_1(0)\bigr\}, \\ &T_2 = -\bigl[\operatorname{Im}\bigl\{c_1(0)\bigr\} + \mu_2\omega'(\widetilde{\tau})\bigr]/\omega_0, \end{aligned} \tag{2.24}$$

where $\lambda(\tau) = \alpha(\tau) \pm i\omega(\tau)$ is the characteristic root of (2.3), which is a continuous differentiable family. $\alpha'(\widetilde{\tau})$ and $\omega'(\widetilde{\tau})$ can be obtained by taking the derivative of the two sides of (2.3) and taking values at $\widetilde{\tau}$.

These formulae give a description of the Hopf bifurcation periodic solution of system (1.3) at $\tau = \tau^j$ ($j = 0, 1, 2, \dots$) on the centre manifold. Thus, we can obtain the following results according to the discussion about properties of Hopf bifurcating periodic solutions of dynamical system in [30].

Proposition 3 *Assume that* (H_1) *and* (H_2) *hold. Then*

(i) *μ_2 determines the direction of the Hopf bifurcation. If $\mu_2 > 0$ ($\mu_2 < 0$), then the Hopf bifurcation is supercritical (subcritical);*

(ii) *β_2 determines the stability of the bifurcating periodic solutions. If $\beta_2 < 0$ ($\beta_2 > 0$), then bifurcating periodic solution is stable (unstable);*

(iii) *T_2 determines the period of the bifurcating periodic solutions. If $T_2 > 0$ ($T_2 < 0$), then periods of the periodic solutions increase (decrease).*

3 Numerical simulations and conclusions

In this section, we shall give some numerical simulations to support the theoretical analysis discussed in the previous section. We also present our conclusions and limitations of the analysis.

Firstly, we study the following specific model:

$$\begin{cases} \dot{y_1}(t) = (y_1(t) + 1)(0.6 - 0.2y_1(t - \tau) - 0.4y_2^2(t - \tau/3)), \\ \dot{y_2}(t) = (y_2(t) + 1)(0.4 - 0.6y_2(t - \tau) + 0.2y_1^2(t - 2\tau/3)), \end{cases} \tag{3.1}$$

with initial values $(y_1(t), y_2(t)) = (0.2, 0.2)$, which satisfies ($H_1$), ($H_2$). By computing, $E^* = (1, 1)$ and by [30], $h(z) = z^4 - 2.56z^3 + 0.3584z^2 - 2.1627z - 0.3244 = 0$, which has only one positive root $z = 2.7341$, get $\omega = 1.6535$, $\tau^0 = 0.5847$. By Proposition 2, we find that E^* is asymptotically stable when $0 \le \tau < \tau^0 = 0.5847$, as Figs. 1(a)–(d) illustrate, and E^* is unstable when $\tau > \tau^0 = 0.5847$, as shown in Figs. 2(a)–(d) which are generated by dde23 [31], a Matlab tool that integrates DDEs.

According to the above numerical simulations and from an economic viewpoint, we conclude that a critical duration time of the two enterprise outputs exists. When the duration time is less than the critical delay, the cooperation between the two enterprises is

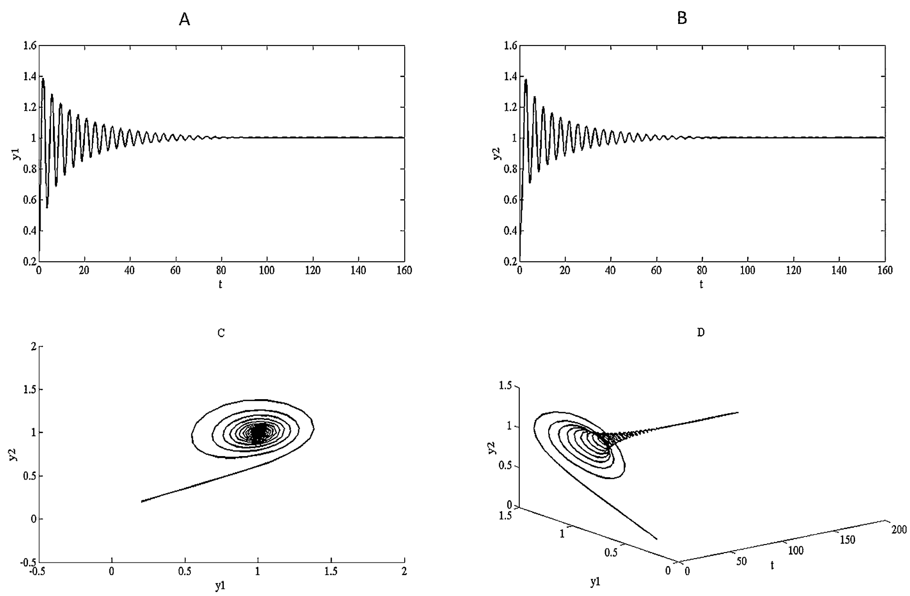

Figure 1 The trajectory graph of system (3.1) with $\tau_1 = 0.56$, $\tau_2 = 0.56/3$, $\tau_3 = 1.12/3$ in (**a**) the t–y_1 plane and in (**b**) the t–y_2 plane. The phase graph of system (3.1) with $\tau_1 = 0.56$, $\tau_2 = 0.56/3$, $\tau_3 = 1.12/3$ in (**c**) the y_1–y_2 plane and in (**d**) the t–y_1–y_2 plane

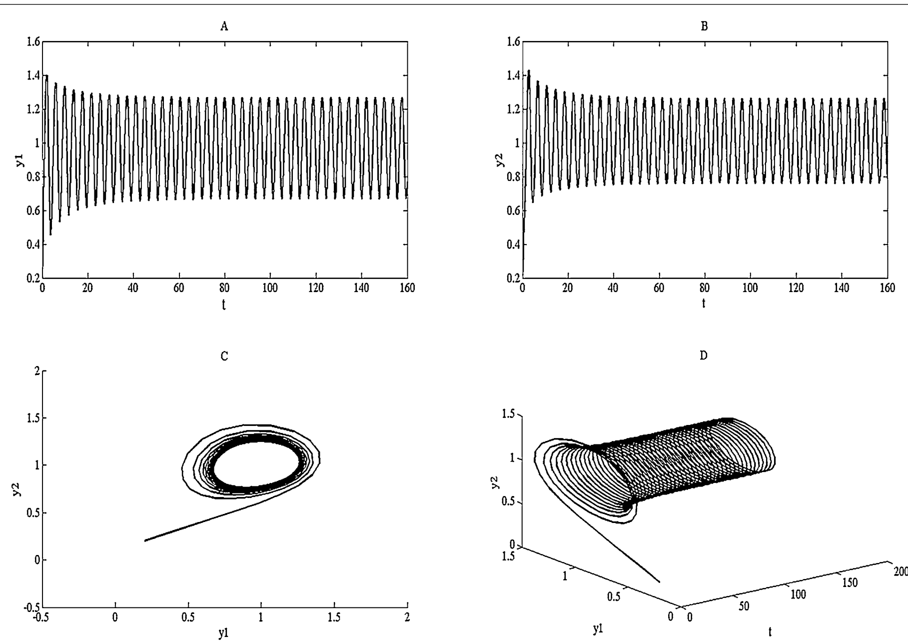

Figure 2 The trajectory graph of system (3.1) with $\tau_1 = 0.6$, $\tau_2 = 0.6/3$, $\tau_3 = 1.2/3$ in (**a**) the t–y_1 plane and in (**b**) the t–y_2 plane. The phase graph of system (3.1) with $\tau_1 = 0.6$, $\tau_2 = 0.6/3$, $\tau_3 = 1.2/3$ in (**c**) the y_1–y_2 plane and in (**d**) the t–y_1–y_2 plane

very effective; though the competition between them exists, they can coexist and have developed over a long time. Alternatively, when the competition between the two enterprises is much stronger than their effective cooperation, the result will ultimately force a merger or a closure by one of the enterprises. Therefore, entrepreneurs must have a shrewd understanding of market forces and economic laws in order to maintain a viable and successful enterprise.

However, here we have only considered the problem of a local Hopf bifurcation and do not give the conditions ensuring the existence of a global Hopf bifurcation for large values of the delay. Furthermore, we do not consider systems with a spatial variable, which is the diffusive model subject to a suitable boundary condition. We intend to make the comparison between the two models, and find what is the influence on the dynamical behaviour with different delays and diffusive terms [32–38], and then illustrate with the theoretical predictions. Lastly, our problem is only restricted to the theoretical analysis of such economical phenomena. It may be timely and necessary to make field investigations and experimental studies for real-world scenarios, and this is left for further study.

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Authors' contributions
The main idea of this paper was proposed by XZ and she prepared the manuscript initially and performed all the steps of the proofs in this research. All authors read and approved the final manuscript.

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Xin Zhang's research interests are delay differential equations, dynamical systems, bifurcation theory and mathematical biology. The research interests of Zizhen Zhang mainly are bifurcation theory and population dynamics.
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