Wang *Boundary Value Problems* (2018) 2018:113
https://doi.org/10.1186/s13661-018-1033-8

Boundary Value Problems
a SpringerOpen Journal

RESEARCH Open Access

The existence of affine-periodic solutions for nonlinear impulsive differential equations

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Abstract

In this paper, we study the existence of affine-periodic solutions of nonlinear impulsive differential equations. The affine-periodic solutions have the form $x(t + T) = Qx(t)$ with some nonsingular matrix Q. We give a theorem on the existence of the affine-periodic solutions, respectively, depending on wether $\det(I - Q)$ (I = identity matrix) is equal to 0 or not.

Keywords: Nonlinear impulsive differential equations; Affine-periodic solutions; Boundary value problem; Topological degree

1 Introduction

The periodicity is a very important property in the study of the impulsive differential equations [1, 2]. However, not all natural phenomena can be described alone by periodicity. Some differential equations often exhibit certain symmetries rather than periodicity. For example, consider the system

$$\dot{x} = f(t,x), \tag{1}$$

where $f : R^1 \times R^n \to R^n$ is continuous, and for some $Q \in GL_n(R)$ (general linear group), satisfies the following affine symmetry:

$$f(t + T, x) = Qf\left(t, Q^{-1}x\right).$$

We call it a (Q, T)-affine-periodic system. For this (Q, T)-affine-periodic system, we are concerned with the existence of (Q, T)-affine-periodic solutions $x(t)$ with

$$x(t + T) = Qx(t), \quad \forall t.$$

It should be pointed out that when $Q = I$ (identity matrix) or $Q = -I$, the solutions are just the pure periodic solutions or antiperiodic ones; when $Q \in SO_n$ (special orthogonal group), the solutions correspond to the solutions with Q-rotating symmetry, particularly to some special quasi-periodic solutions. So the interest to particular kinds of periodic solutions that we are going to study is not purely theoretical. The antiperiodicity property or some quasi-periodicity property, which is obviously a particular case of affine-periodic solutions, has drawn wide attention from physicists and astronomers [3, 4].

Recently, these conceptions and existence results of the solutions have been introduced and proved by Li and his coauthors; see [5] for Levinson's problem, [6] for Lyapunov function type theorems, [7] for averaging methods of affine-periodic solutions, and [8] for some dissipative dynamical systems. The aim of this paper is to touch such a topic for affine-periodic solutions of nonlinear impulsive differential equations.

The paper is organized as follows. We first change the affine-periodic solutions problem to the boundary value problem in Sect. 2. In Sect. 3, when $\det(I - Q) \neq 0$, we give an unique affine-periodic solution by using the Banach contraction mapping principle. Furthermore, via the topological degree theory, we prove the existence of affine-periodic solutions for nonlinear impulsive system when $\det(I - Q) = 0$ in Sect. 4. We give two examples by numerical simulation in Sect. 5.

2 Nonlinear impulsive differential system

In this paper, we investigate the following system:

$$\begin{aligned} &\dot{x} = f(t,x), \quad t \neq t_k, t \in R, \\ &\Delta x = I_k(x), \quad t = t_k, k \in Z. \end{aligned} \tag{2}$$

The system satisfies the following hypotheses **H**:

(1) $f(\cdot) \in C(R \times R^n, R^n)$ and $f(t+T,x) = Qf(t,Q^{-1}x)$ for some $G \in SO_n(R)$.

(2) $I_k(\cdot) \in C(R^n, R^n)$, $t_k < t_{k+1}$ $(k \in Z)$.

(3) There exists $q \in N$ such that $I_{k+q}(x) = QI_k(Q^{-1}x)$ and $t_{k+q} = t_k + T$ $(k \in Z)$.

In system (2), the continuous part corresponds to a nonlinear (Q,T)-affine-periodic system. The discrete component models the affine-periodic impulsive change of $x(t)$.

Lemma 2.1 *The existence of Q-affine-periodic solutions of equation* (2) *is equivalent to the existence of the boundary value problem* (2) *with* $x(T) = Qx(0)$.

Proof Let $x(t)$ be a solution of equation (2) defined on $t \in [0,T]$. Then

$$u(t) = \begin{cases} x(t), & t \in (0,T], \\ Q^j x(t - jT), & t \in (jT, jT + T], \end{cases} \tag{3}$$

is a Q-affine-periodic solution of (2). Indeed, if $t \in (jT, jT + T]$ and $t \neq t_k$, then $t - jT \in (0,T]$, and

$$\begin{aligned} \frac{du(t)}{dt} &= Q^j \frac{dx(t-jT)}{dt} \\ &= Q^j f\bigl(t - jT, x(t-jT)\bigr) \\ &= Q^j \cdot Q^{-j} f\bigl(t, Q^j x(t-jT)\bigr) \\ &= f\bigl(t, u(t)\bigr), \end{aligned} \tag{4}$$

and if $t_k \in (jT, jT+T]$, then $t_{k-jq} = t_k - jT \in (0,T]$ and

$$\begin{aligned}\Delta u(t_k) &= Q^j \Delta x(t_k - jT) \\ &= Q^j I_{k-jq}\big(x(t_k - jT)\big) \\ &= Q^j \cdot Q^{-j} I_k\big(Q^j x(t_k - jT)\big) \\ &= I_k\big(u(t_k)\big).\end{aligned} \tag{5}$$

Let $x(t)$ be any solution of (2) with $x(T) = Qx(0)$. Then $x(t)$ has the form

$$x(t) = x(0) + \int_0^t f\big(s, x(s)\big)\,ds + \sum_{0 \le t_k < t} I_k\big(x(t_k)\big).$$

Denote $x(0)$ by x_0. Then we have

$$(I-Q)x_0 = -\left[\int_0^T f\big(s,x(s)\big)\,ds + \sum_{0\le t_k<T} I_k\big(x(t_k)\big)\right]. \tag{6}$$

□

3 Noncritial case

$\det(I-Q) \neq 0$.

In this case, $(I-Q)^{-1}$ exists. Then

$$x_0 = -(I-Q)^{-1}\left[\int_0^T f\big(s,x(s)\big)\,ds + \sum_{0\le t_k<T} I_k\big(x(t_k)\big)\right]. \tag{7}$$

So, the existence of Q-affine-periodic solutions of equation (2) is equivalent to the existence of solutions of the following impulsive integral equation:

$$\begin{aligned} x(t) = {}& -(I-Q)^{-1}\left[\int_0^T f\big(s,x(s)\big)\,ds + \sum_{0\le t_k<T} I_k\big(x(t_k)\big)\right] \\ & + \int_0^t f\big(s,x(s)\big)\,ds + \sum_{0\le t_k<t} I_k\big(x(t_k)\big). \end{aligned} \tag{8}$$

Let

$$\mathrm{X} = \big\{x : [0,T] \to R^n : x(t) \text{ is continuous on } [0,T]\big\},$$

and define the norm $\|x\| = \sup_{t\in[0,T]} |x(t)|$. It is easy to see that X is a Banach space with norm $\|x\|$. We also define the norm of the matrix $\|X(t)\| = \|(x_1(t), x_2(t), \ldots, x_n(t))\| = \max_{i=1,2,\ldots,n} \|x_i\|$. Then we have the following theorem.

Theorem 3.1 *Let a function $p \in L([0,T], R^+)$ and nonnegative constants α_k $(k = 1,2,\ldots,q)$ be such that*

$$\big|f(t,y) - f(t,x)\big| \le p(t)|y-x|, \quad \forall t \in [0,T], x, y \in R^n,$$

$$\bigl|I_k(y)-I_k(x)\bigr|\le a_k|y-x|,\quad \alpha_k\in R(k=1,2,\dots,q),x,y\in R^n,$$

and

$$\left(\int_0^T p(s)\,ds+\sum_{k=1}^{q}a_k\right)<\frac{1}{\|(I-Q)^{-1}\|+1}.$$

Then system (2) *has an unique Q-affine-periodic solution.*

Proof Define

$$\begin{aligned}A\bigl(x(t)\bigr)={}&-(I-Q)^{-1}\left[\int_0^T f\bigl(s,x(s)\bigr)\,ds+\sum_{0\le t_k<T}I_k\bigl(x(t_k)\bigr)\right]\\&+\int_0^t f\bigl(s,x(s)\bigr)\,ds+\sum_{0\le t_k<t}I_k\bigl(x(t_k)\bigr).\end{aligned}$$

Then

$$\begin{aligned}&\bigl|A\bigl(y(t)\bigr)-A\bigl(x(t)\bigr)\bigr|\\&\quad=\Bigg|-(I-Q)^{-1}\left[\int_0^T\bigl(f\bigl(s,y(s)\bigr)-f\bigl(s,x(s)\bigr)\bigr)\,ds+\sum_{0\le t_k<T}\bigl(I_k\bigl(y(t_k)-I_k\bigl(x(t_k)\bigr)\bigr)\right]\\&\qquad+\int_0^t\bigl(f\bigl(s,y(s)\bigr)-f\bigl(s,x(s)\bigr)\bigr)\,ds+\sum_{0\le t_k<t}\bigl(I_k\bigl(y(t_k)\bigr)-I_k\bigl(x(t_k)\bigr)\bigr)\Bigg|\\&\quad\le\bigl\|(I-Q)^{-1}\bigr\|\left(\int_0^T p(s)\,ds+\sum_{k=1}^{q}a_k\right)|y-x|\\&\qquad+\left(\int_0^t p(s)\,ds+\sum_{k=1}^{q'}a_k\right)|y-x|\\&\quad\le\bigl(\bigl\|(I-Q)^{-1}\bigr\|+1\bigr)\left(\int_0^T p(s)\,ds+\sum_{k=1}^{q}a_k\right)|y-x|.\end{aligned}\tag{9}$$

So, if $(\int_0^T p(s)\,ds+\sum_{k=1}^{q}a_k)<\frac{1}{\|(I-Q)^{-1}\|+1}$, then by the Banach contraction mapping principle system (2) has a unique Q-affine-periodic solution. □

4 Critial case

$$\det(I-Q)=0.$$

To investigate the existence of solutions of system (2), the following auxiliary equation is often considered:

$$\begin{aligned}&\dot{x}=\lambda f(t,x),\quad t\ne t_k,t\in R,\\&\Delta x=\lambda I_k(x),\quad t=t_k,k\in Z.\end{aligned}\tag{10}$$

Then we give the following existence theorem for (Q,T)-affine-periodic solutions by using the topological degree theory [6, 7, 9–11].

Theorem 4.1 *Let $D \subset R^n$ be a bounded open set. Assume that the following hypotheses hold for system* (10):

(H1) *For each $\lambda \in (0,1]$, every Q-affine-periodic solution $x(t)$ of system* (10) *satisfies*

$$x(t) \notin \partial D \quad \text{for all } t;$$

(H2) *the Brouwer degree,*

$$\deg\bigl(g, D \cap \operatorname{Ker}(I-Q), 0\bigr) \neq 0 \quad \text{if } \operatorname{Ker}(I-Q) \neq 0,$$

where

$$g(a) = \frac{1}{T}\left[\int_0^T Pf(s,a)\,ds + \sum_{0 \le t_k < T} PI_k\bigl(x(t_k)\bigr)\right],$$

with an orthogonal projection $P : R^n \to \operatorname{Ker}(I-Q)$.

Then system (2) *has at least one Q-affine-periodic solution $x_*(t) \in D$ for all t.*

Proof Consider the auxiliary equation (10) with the boundary value condition $x(T) = Qx(t)$, where $\lambda \in (0,1]$. Let $x(t)$ be any solution of (10) with $x(T) = Qx(0)$. Then

$$\begin{aligned}(I-Q)x_0 \\ = -\lambda\left[\int_0^T f\bigl(s,x(s)\bigr)\,ds + \sum_{0 \le t_k < T} I_k\bigl(x(t_k)\bigr)\right].\end{aligned} \tag{11}$$

In this case, $(I-Q)^{-1}$ does not exist. By coordinate transformation, without loss of generality, we can just let

$$Q = \begin{pmatrix} I & 0 \\ 0 & Q_1 \end{pmatrix}, \tag{12}$$

where $(I-Q_1)^{-1}$ exists. Here $Q = Q_1 \oplus I$.

Let $P : R^n \to \operatorname{Ker}(I-Q)$ be the orthogonal projection. Then

$$\begin{aligned}(I-Q)x_0 &= (I-Q)\bigl(x_{\ker}^0 + x_\perp^0\bigr) \\ &= -\lambda\left[\int_0^T f\bigl(s,x(s)\bigr)\,ds + \sum_{0 \le t_k < T} I_k\bigl(x(t_k)\bigr)\right] \\ &= -\lambda\left[\int_0^T Pf\bigl(s,x(s)\bigr)\,ds + \sum_{0 \le t_k < T} PI_k\bigl(x(t_k)\bigr)\right] \\ &\quad - \lambda\left[\int_0^T (I-P)f\bigl(s,x(s)\bigr)\,ds + \sum_{0 \le t_k < T} (I-P)I_k\bigl(x(t_k)\bigr)\right],\end{aligned} \tag{13}$$

where $x_{\ker}^0 \in \operatorname{Ker}(I-Q)$, $x_\perp^0 \in \operatorname{Im}(I-Q)$ and $x_0 = x_{\ker}^0 + x_\perp^0$.

Let $L_p = (I-Q)|_{\mathrm{Im}(I-Q)}$. It is easy to see that L_p^{-1} exists. Thus equation (13) is equivalent to

$$(I-Q)x_{\ker}^0 = -\lambda\left[\int_0^T Pf\bigl(s,x(s)\bigr)\,ds + \sum_{0\le t_k<T} PI_k\bigl(x(t_k)\bigr)\right] = 0,$$

$$(I-Q)x_{\perp}^0 = -\lambda\left[\int_0^T (I-P)f\bigl(s,x(s)\bigr)\,ds + \sum_{0\le t_k<T} (I-P)I_k\bigl(x(t_k)\bigr)\right].$$

Thus we have

$$x_{\perp}^0 = \lambda L_p^{-1}\left[\int_0^T (I-P)f\bigl(s,x(s)\bigr)\,ds + \sum_{0\le t_k<T} (I-P)I_k\bigl(x(t_k)\bigr)\right].$$

For $x \in \mathrm{X}$ such that $x(t) \in \overline{D}$ for all $t \in [0,T]$, we define the operator $\mathtt{T}(x_{\ker}^0, x, \lambda)$ by

$$\mathtt{T}\bigl(x_{\ker}^0, x, \lambda\bigr) = \begin{pmatrix} x_{\ker}^0 + \frac{1}{T}[\int_0^T Pf(s,x(s))\,ds + \sum_{0\le t_k<T} PI_k(x(t_k))] \\ x_{\ker}^0 - \lambda L_p^{-1}[\int_0^T (I-P)f(s,x(s))\,ds + \sum_{0\le t_k<T}(I-P)I_k(x(t_k))] \\ + \lambda[\int_0^t f(s,x(s))\,ds + \sum_{0\le t_k<t} I_k(x(t_k))] \end{pmatrix}, \tag{14}$$

where $\lambda \in [0,1]$. We claim that each fixed point x of $\mathtt{T}$ in X is a solution of (10) with $x(T) = Qx(0)$.

In fact, if x is a fixed point of $\mathtt{T}$, we have

$$\begin{pmatrix} x_{\ker}^0 \\ x(t) \end{pmatrix} = \begin{pmatrix} x_{\ker}^0 + \frac{1}{T}[\int_0^T Pf(s,x(s))\,ds + \sum_{0\le t_k<T} PI_k(x(t_k))] \\ x_{\ker}^0 - \lambda L_p^{-1}[\int_0^T (I-P)f(s,x(s))\,ds + \sum_{0\le t_k<T}(I-P)I_k(x(t_k))] \\ + \lambda[\int_0^t f(s,x(s))\,ds + \sum_{0\le t_k<t} I_k(x(t_k))] \end{pmatrix}.$$

Thus

$$\frac{1}{T}\left[\int_0^T Pf\bigl(s,x(s)\bigr)\,ds + \sum_{0\le t_k<T} PI_k\bigl(x(t_k)\bigr)\right] = 0, \tag{15}$$

$$\begin{aligned} x(t) = x_{\ker}^0 &- \lambda L_p^{-1}\left[\int_0^T (I-P)f\bigl(s,x(s)\bigr)\,ds + \sum_{0\le t_k<T} (I-P)I_k\bigl(x(t_k)\bigr)\right] \\ &+ \lambda\left[\int_0^t f\bigl(s,x(s)\bigr)\,ds + \sum_{0\le t_k<t} I_k\bigl(x(t_k)\bigr)\right]. \end{aligned} \tag{16}$$

By equation (16) we know that

$$x_0 = x_{\ker}^0 - \lambda L_p^{-1}\left[\int_0^T (I-P)f\bigl(s,x(s)\bigr)\,ds + \sum_{0\le t_k<T} (I-P)I_k\bigl(x(t_k)\bigr)\right].$$

According to $(I-Q)x_{\ker}^0 = 0$, we have

$$\begin{aligned} Qx_0 &= Qx_{\ker}^0 - \lambda QL_p^{-1}\left[\int_0^T (I-P)f\bigl(s,x(s)\bigr)\,ds + \sum_{0\le t_k<T}(I-P)I_k\bigl(x(t_k)\bigr)\right] \\ &= x_{\ker}^0 - \lambda QL_p^{-1}\left[\int_0^T (I-P)f\bigl(s,x(s)\bigr)\,ds + \sum_{0\le t_k<T}(I-P)I_k\bigl(x(t_k)\bigr)\right]. \end{aligned}$$

Since equation (15) holds, we have

$$\begin{aligned} &(I-Q)L_p^{-1}\left[\int_0^T (I-P)f\bigl(s,x(s)\bigr)\,ds + \sum_{0\le t_k<T}(I-P)I_k\bigl(x(t_k)\bigr)\right] \\ &\quad= \left[\int_0^T (I-P)f\bigl(s,x(s)\bigr)\,ds + \sum_{0\le t_k<T}(I-P)I_k\bigl(x(t_k)\bigr)\right] \\ &\quad= \left[\int_0^T (I-P)f\bigl(s,x(s)\bigr)\,ds + \sum_{0\le t_k<T}(I-P)I_k\bigl(x(t_k)\bigr)\right] \\ &\qquad+ \left[\int_0^T Pf\bigl(s,x(s)\bigr)\,ds + \sum_{0\le t_k<T}PI_k\bigl(x(t_k)\bigr)\right] \\ &\quad= \int_0^T f\bigl(s,x(s)\bigr)\,ds + \sum_{0\le t_k<T}I_k\bigl(x(t_k)\bigr). \end{aligned}$$

Thus

$$\begin{aligned} &\lambda QL_p^{-1}\left[\int_0^T (I-P)f\bigl(s,x(s)\bigr)\,ds + \sum_{0\le t_k<T}(I-P)I_k\bigl(x(t_k)\bigr)\right] \\ &\quad= \lambda L_p^{-1}\left[\int_0^T (I-P)f\bigl(s,x(s)\bigr)\,ds + \sum_{0\le t_k<T}(I-P)I_k\bigl(x(t_k)\bigr)\right] \\ &\qquad- \lambda\left[\int_0^T f\bigl(s,x(s)\bigr)\,ds + \sum_{0\le t_k<T}I_k\bigl(x(t_k)\bigr)\right]. \end{aligned}$$

Then

$$\begin{aligned} Qx_0 &= x_{\ker}^0 - \lambda QL_p^{-1}\left[\int_0^T (I-P)f\bigl(s,x(s)\bigr)\,ds + \sum_{0\le t_k<T}(I-P)I_k\bigl(x(t_k)\bigr)\right] \\ &= x_{\ker}^0 - \lambda L_p^{-1}\left[\int_0^T (I-P)f\bigl(s,x(s)\bigr)\,ds + \sum_{0\le t_k<T}(I-P)I_k\bigl(x(t_k)\bigr)\right] \\ &\quad+ \lambda\left[\int_0^T f\bigl(s,x(s)\bigr)\,ds + \sum_{0\le t_k<T}I_k\bigl(x(t_k)\bigr)\right] = x(T). \end{aligned} \tag{17}$$

By equations (16) and (17), equation (11) holds. Thus,

$$x_\perp^0 = -\lambda L_p^{-1}\left[\int_0^T (I-P)f\bigl(s,x(s)\bigr)\,ds + \sum_{0\le t_k<T}(I-P)I_k\bigl(x(t_k)\bigr)\right].$$

Then,

$$\begin{aligned} x(t) &= x_{\mathrm{ker}}^0 - \lambda L_p^{-1}\left[\int_0^T (I-P)f\bigl(s,x(s)\bigr)\,ds + \sum_{0\le t_k<T}(I-P)I_k\bigl(x(t_k)\bigr)\right] \\ &\quad + \lambda\left[\int_0^t f\bigl(s,x(s)\bigr)\,ds + \sum_{0\le t_k<t} I\bigl(x(t_k)\bigr)\right] \\ &= x_{\mathrm{ker}}^0 + x_{\perp}^0 + \lambda\left[\int_0^t f\bigl(s,x(s)\bigr)\,ds + \sum_{0\le t_k<t} I_k\bigl(x(t_k)\bigr)\right] \\ &= x_0 + \lambda\left[\int_0^t f\bigl(s,x(s)\bigr)\,ds + \sum_{0\le t_k<t} I_k\bigl(x(t_k)\bigr)\right]. \end{aligned}$$

This means that the fixed point x is a solution of (10) with $x(T) = Qx(0)$.

Now, we need to prove the existence of the fixed point of T. Take a constant M such that $M > \sup_{t\in[0,T],x\in\overline{D}} |f(t,x)|$, and let

$$\mathrm{X}_\lambda = \left\{x \in \mathrm{X} : \left|\frac{x(t)-x(r)}{t-r}\right| \le \lambda M \text{ for all } t, r \in (t_k, t_{k+1}], t \ne r\right\}.$$

Then, it is easy to make a retraction $\alpha_\lambda : \mathrm{X} \to \mathrm{X}_\lambda$.

Define an operator $\widehat{\mathrm{T}}(x_{\mathrm{ker}}^0, x, \lambda)$ by

$$\begin{aligned} &\widehat{\mathrm{T}}\bigl(x_{\mathrm{ker}}^0, x, \lambda\bigr) \\ &\quad = \begin{pmatrix} x_{\mathrm{ker}}^0 + \frac{1}{T}[\int_0^T Pf(s,\alpha_\lambda\circ x(s))\,ds + \sum_{0\le t_k<T} PI_k(\alpha_\lambda\circ x(t_k))] \\ \alpha_\lambda\circ x_{\mathrm{ker}}^0 - \lambda L_p^{-1}[\int_0^T (I-P)f(s,\alpha_\lambda\circ x(s))\,ds + \sum_{0\le t_k<T}(I-P)I_k(\alpha_\lambda\circ x(t_k))] \\ + \lambda[\int_0^t f(s,\alpha_\lambda\circ x(s))\,ds + \sum_{0\le t_k<t} I_k(\alpha_\lambda\circ x(t_k))] \end{pmatrix}. \end{aligned} \tag{18}$$

Since $P : R^n \to \mathrm{Ker}(I-Q)$, it is easy to see that

$$\frac{1}{T}\left[\int_0^T Pf\bigl(s,x(s)\bigr)\,ds + \sum_{0\le t_k<T} PI_k\bigl(x(t_k)\bigr)\right] \in \mathrm{Ker}(I-Q).$$

Also,

$$\frac{1}{T}\left[\int_0^T Pf\bigl(s,\alpha_\lambda\circ x(s)\bigr)\,ds + \sum_{0\le t_k<T} PI_k\bigl(\alpha_\lambda\circ x(t_k)\bigr)\right] \in \mathrm{Ker}(I-Q).$$

Let us consider the homotopy

$$H\bigl(x_{\mathrm{ker}}^0, x, \lambda\bigr) = \widehat{\mathrm{T}}\bigl(x_{\mathrm{ker}}^0, x, \lambda\bigr), \tag{19}$$

$$\bigl(x_{\mathrm{ker}}^0, x, \lambda\bigr) \in \bigl(D \cap \mathrm{Ker}(I-Q) \times \widetilde{D} \times [0,1]\bigr), \tag{20}$$

where $\widetilde{D} = \{x \in X : x(t) \in D \text{ for all } t \in [0,T]\}$.

We claim that

$$0 \notin (id - H)\big(\partial\big(\big(D \cap \operatorname{Ker}(I - Q) \times \widetilde{D}\big) \times [0,1]\big). \tag{21}$$

Suppose, on the contrary, that there exists $(\widehat{x}^0_{\ker}, \widehat{x}, \widehat{\lambda}) \in \partial((D \cap \operatorname{Ker}(I - Q) \times \widetilde{D}) \times [0,1]$ such that $(id - H)(\widehat{x}^0_{\ker}, \widehat{x}, \widehat{\lambda}) = 0$. Since $\widehat{x}^0_{\ker} \in \partial D$ is contradictory to (H_1) and since $\partial(D \cap \operatorname{Ker}(I - Q)) \subset \partial D$, we have that $\widehat{x}^0_{\ker} \notin \partial(D \cap \operatorname{Ker}(I - Q))$. In other words, $\widehat{x} \in \partial D$. Then (21) can be proved as follows.

(i) When $\widehat{\lambda} = 0$, by the definition of the set X_λ we have

$$\mathrm{X}_0 = \left\{x \in X : \left|\frac{x(t) - x(r)}{t - r}\right| \leq 0 \text{ for all } t, r \in (t_k, t_{k+1}], t \neq r\right\}.$$

Hence $\alpha_0 \circ x(t) \equiv \alpha_0 \circ x(t_{k+1})$ for all $t \in (t_k, t_{k+1}]$. Since $(id - H)(\widehat{x}^0_{\ker}, \widehat{x}, 0) = 0$, we have

$$\begin{pmatrix} \widehat{x}^0_{\ker} \\ \widehat{x}(t) \end{pmatrix} = \begin{pmatrix} \widehat{x}^0_{\ker} + \frac{1}{T}[\int_0^T Pf(s, \alpha_\lambda \circ x(s))\, ds + \sum_{0 \leq t_k < T} PI_k(\alpha_\lambda \circ x(t_k))] \\ \alpha_0 \circ \widehat{x}^0_{\ker} \end{pmatrix}. \tag{22}$$

This means that $\widehat{x}(t) \equiv \widehat{x}(0)$ for all $t \in [0, T]$. Taking $\widehat{x}(t) = p$, we have $\alpha_0 \circ \widehat{x}^0_{\ker} = \widehat{x}(t) = p$. Consequently,

$$\frac{1}{T}\left[\int_0^T Pf\big(s, \alpha_\lambda \circ x(s)\big)\, ds + \sum_{0 \leq t_k < T} PI_k\big(\alpha_\lambda \circ x(t_k)\big)\right] = 0,$$

and this is equivalent to $g(p) = 0$ by the definition of $g(a)$. Notice that $\widehat{x} \in \partial\widetilde{D}$ and $\widetilde{D} = \{x \in D$ for all $t \in [0, T]\}$. Then there exists $t_0 \in [0, T]$ such that $\widehat{x(t)}_0 \in \partial D$. Since $\widehat{x}(t) \equiv p$ for all $t \in [0, T]$, we obtain that $p \in \partial D$. Thus, we have $p \in \partial D$ and $g(p) = 0$. It is contradictory to (H_2) because the Brouwer degree $\deg(g, D, 0) \neq 0$.

(ii) When $\widehat{\lambda} \in (0, 1]$, as $0 = (id - H)(\widehat{x}^0_{\ker}, \widehat{x}, \widehat{\lambda})$, we have

$$\begin{aligned} &\begin{pmatrix} \widehat{x}^0_{\ker} \\ \widehat{x}(t) \end{pmatrix} \\ &= \begin{pmatrix} \widehat{x}^0_{\ker} + \frac{1}{T}[\int_0^T Pf(s, \alpha_{\widehat{\lambda}} \circ x(s))\, ds + \sum_{0 \leq t_k < T} PI_k(\alpha_{\widehat{\lambda}} \circ x(t_k))] \\ \alpha_{\widehat{\lambda}} \circ x^0_{\ker} - \widehat{\lambda} L_p^{-1}[\int_0^T (I - P)f(s, \alpha_{\widehat{\lambda}} \circ x(s))\, ds + \sum_{0 \leq t_k < T}(I - P)I_k(\alpha_{\widehat{\lambda}} \circ x(t_k))] \\ + \widehat{\lambda}[\int_0^t f(s, \alpha_{\widehat{\lambda}} \circ x(s))\, ds + \sum_{0 \leq t_k < t} I_k(\alpha_{\widehat{\lambda}} \circ x(t_k))] \end{pmatrix}. \end{aligned}$$

Thus

$$\frac{1}{T}\left[\int_0^T Pf\big(s, \alpha_{\widehat{\lambda}} \circ x(s)\big)\, ds + \sum_{0 \leq t_k < T} PI_k\big(\alpha_{\widehat{\lambda}} \circ x(t_k)\big)\right] = 0$$

and

$$\begin{aligned} \widehat{x}(t) = {}& \alpha_{\widehat{\lambda}} \circ x^0_{\ker} - \widehat{\lambda} L_p^{-1}\left[\int_0^T (I - P)f\big(s, \alpha_{\widehat{\lambda}} \circ x(s)\big)\, ds + \sum_{0 \leq t_k < T} (I - P)I_k\big(\alpha_{\widehat{\lambda}} \circ x(t_k)\big)\right] \\ &+ \widehat{\lambda}\left[\int_0^t f\big(s, \alpha_{\widehat{\lambda}} \circ x(s)\big)\, ds + \sum_{0 \leq t_k < t} I_k\big(\alpha_{\widehat{\lambda}} \circ x(t_k)\big)\right]. \end{aligned} \tag{23}$$

Note that

$$
\begin{aligned}
&\left|\frac{x(t)-x(r)}{t-r}\right| \\
&\quad = \frac{1}{|t-r|}\left|\widehat{\lambda}\int_0^t f\bigl(s,\alpha_{\widehat{\lambda}}\circ\widehat{x}(s)\bigr)\,ds - \widehat{\lambda}\int_0^r f\bigl(s,\alpha_{\widehat{\lambda}}\circ\widehat{x}(s)\bigr)\,ds\right| \\
&\quad = \frac{1}{|t-r|}\left|\widehat{\lambda}\int_r^t f\bigl(s,\alpha_{\widehat{\lambda}}\circ\widehat{x}(s)\bigr)\,ds\right| \\
&\quad \le \lambda M.
\end{aligned}
$$

By the definition of X_λ we obtain $\widehat{x}\in\mathrm{X}_{\widehat{\lambda}}$, which means that $\alpha_{\widehat{\lambda}}\circ\widehat{x}=\widehat{x}$. Now we can rewrite equation (23) as

$$
\begin{aligned}
\widehat{x}(t) = {}& x^0_{\ker} - \widehat{\lambda}L_p^{-1}\biggl[\int_0^T (I-P)f\bigl(s,x(s)\bigr)\,ds + \sum_{0\le t_k<T}(I-P)I_k\bigl(x(t_k)\bigr)\biggr] \\
&+ \widehat{\lambda}\biggl[\int_0^t f\bigl(s,x(s)\bigr)\,ds + \sum_{0\le t_k<t} I_k\bigl(x(t_k)\bigr)\biggr].
\end{aligned}
$$

By a similar discussion of equation (16) we can prove that $\widehat{x}(t)$ is a solution of equation (10). By hypothesis (H_1) we know that $\widehat{x}(t)\notin\partial\widetilde{D}$ for any $t\in[0,T]$. This is a contradiction to $\widehat{x}\in\partial\widetilde{D}$.

By (i) and (ii) we obtain that

$$
0\notin(id-H)\bigl(\partial\bigl(\bigl(D\cap\mathrm{Ker}(I-Q)\bigr)\times\widetilde{D}\bigr)\times[0,1]\bigr).
$$

Therefore, by the homotopy invariance and the theory of Brouwer degree we have

$$
\begin{aligned}
&\deg\bigl(id-H\bigl(x^0_{\ker},\cdot,1\bigr),\bigl(D\cap\mathrm{Ker}(I-Q)\bigr)\times\widetilde{D},0\bigr) \\
&\quad = \deg\bigl(id-H\bigl(x^0_{\ker},\cdot,0\bigr),\bigl(D\cap\mathrm{Ker}(I-Q)\bigr)\times\widetilde{D},0\bigr) \\
&\quad = \deg\bigl(g,D\cap\mathrm{Ker}(I-Q),0\bigr)\ne 0.
\end{aligned}
$$

This means that there exists $\widehat{x}_*\in\widetilde{D}$ such that

$$
\begin{pmatrix}\widehat{x}^0_{*\ker}\\ \widehat{x}_*(t)\end{pmatrix} = \widehat{T}\bigl(\widehat{x}^0_{*\ker},\widehat{x}_*(t),1\bigr). \tag{24}
$$

Similarly to the proof in (ii), we get $\widehat{x}_*\in\mathrm{X}_\lambda$. Then

$$
\widehat{T}\bigl(\widehat{x}^0_{*\ker},\widehat{x}_*(t),1\bigr) = T\bigl(\widehat{x}^0_{*\ker},\widehat{x}_*(t),1\bigr). \tag{25}
$$

By equations (24) and (25) we obtain that $\widehat{x}_*$ is a fixed point of T in X. Thus, $\widehat{x}_*$ is a solution of system (2) with boundary value condition $x(T)=Qx(0)$. □

5 Numerical simulation

Example 1 Consider the system

$$\begin{aligned} &\dot{x} = -|x|^2 x + (\sin \pi t, \cos \pi t)^T, \quad t \neq N, \\ &\Delta x = \left(\frac{1}{e} - 1\right)x, \quad t = N. \end{aligned} \tag{26}$$

Set

$$Q = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

In this example, $Q = -I$. System (26) has an antiperiodic solution (see Fig. 1).

Example 2 Consider the system

$$\begin{aligned} &\dot{x} = -|x|^2 x + (\sin t, \cos t, 1)^T, \quad t \neq N, \\ &\Delta x = \left(\frac{1}{e} - 1\right)x, \quad t = N. \end{aligned} \tag{27}$$

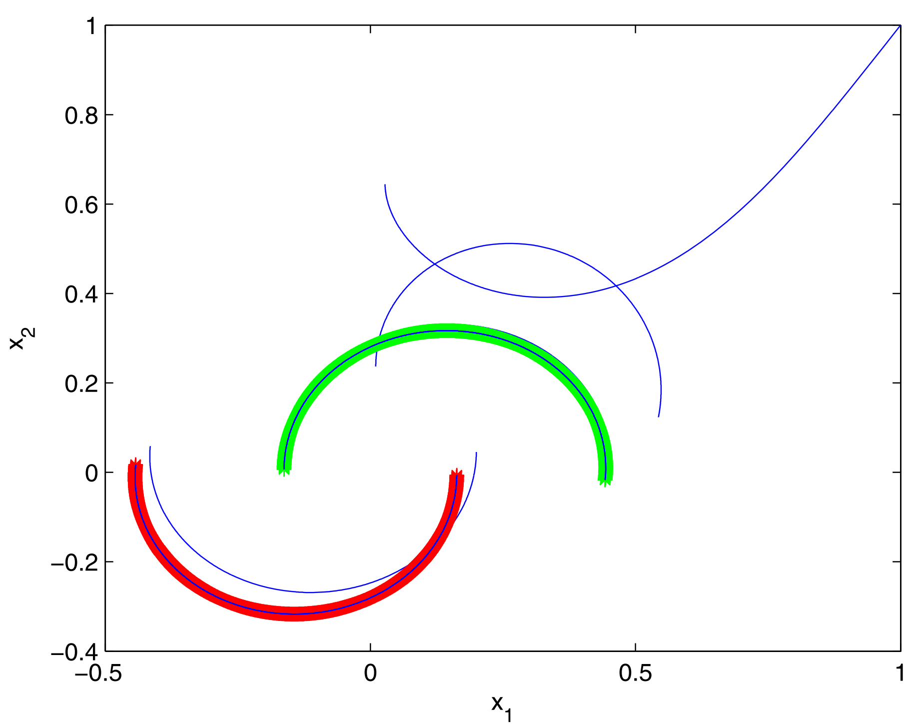

Figure 1 The antiperiodic solution of system (26). The green line is the trajectory of $x(t)$ for $t \in (0,1]$, the red line corresponds to the trajectory of $Qx(t)$ for $t \in (0,1]$

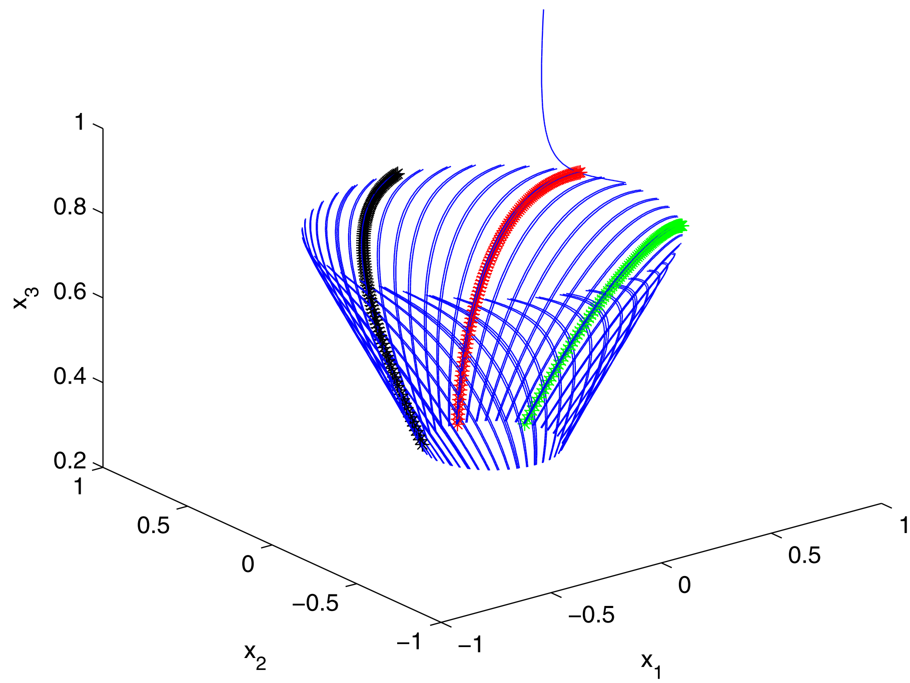

Figure 2 The (Q,T)-affine-periodic solution of system (27). The black line is the trajectory of $x(t)$ for $t \in (0,1]$, the red line corresponds to the trajectory of $Qx(t)$ for $t \in (0,1]$, and the green line corresponds to the trajectory of $Q^2x(t)$ for $t \in (0,1]$. It is easy to see that $x(t)$ is a quasi-periodic solution of system (27)

Set

$$Q = \begin{pmatrix} \cos(2\pi - 1) & -\sin(2\pi - 1) & 0 \\ \sin(2\pi - 1) & \cos(2\pi - 1) & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Similarly to Example 1, system (26) has a $(Q,1)$-affine-periodic solution., which is a quasi-periodic solution (see Fig. 2).

Acknowledgements
The author expresses sincere thanks to the anonymous referees for their comments.

Funding
The author is supported by the fund of the "Thirteen Five" Scientific and Technological Research Planning Project of the Department of Education of Jilin Province (JJKH20170301KJ).

Abbreviations
I, identity matrix; GL, general linear group; SO, special orthogonal group.

Availability of data and materials
Data sharing not applicable to this paper as no datasets were generated or analyzed during the current study.

Competing interests
The author has no competing interests. There is no conflict of interest exists in the submission of this manuscript.

Author's contributions
The author declares that the work described was an original research, which has not been published previously and is not under consideration for publication elsewhere. Author read and approved the final manuscript.

Publisher's Note

Received: 12 March 2018 Accepted: 6 July 2018 Published online: 18 July 2018

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