Tian et al. *Boundary Value Problems* (2018) 2018:115
https://doi.org/10.1186/s13661-018-1034-7

Boundary Value Problems
a SpringerOpen Journal

RESEARCH

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Weighted Lorentz estimates for nonlinear elliptic obstacle problems with partially regular nonlinearities

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Abstract

A global estimate in the framework of weighted Lorentz spaces is reached for the gradient of weak solution to a class of nonlinear elliptic obstacle problems with partially regular nonlinearities in a bounded Reifenberg flat domain. We mainly assume that the nonlinearities are merely measurable in one spatial variable and have small BMO seminorms in the remaining variables and that the underlying domain is flat in the sense of Reifenberg. As an application, we also present a global Lorentz estimate for the gradients of weak solutions to Dirichlet problems of relevant nonlinear elliptic equations under controlled growth in Reifenberg domains based on the bootstrap argument.

MSC: 35B65; 35D30; 35J87

Keywords: Nonlinear elliptic obstacle problems; Measurable nonlinearities; Weighted Lorentz spaces; Reifenberg flat domain; Controlled growth

1 Introduction

The main goal of this paper is finding the minimal regular nonlinearities to study a global estimate in weighted Lorentz spaces for the gradient of weak solution to a nonlinear elliptic obstacle problem over a bounded nonsmooth domain. Let Ω be a bounded nonsmooth domain of $\mathbb{R}^{d\geq 2}$ to be specified later. For a given obstacle $\Psi \in W^{1,2}(\Omega)$ with $\Psi \leq 0$ a.e. on $\partial\Omega$, we denote the *set of admissible functions* by

$$\mathcal{A} = \{\phi \in W_0^{1,2}(\Omega) : \phi \geq \Psi \text{ a.e. in } \Omega\}.$$

For $u \in \mathcal{A}$, we focus on considering the following variational inequalities:

$$\int_\Omega \mathbf{a}(Du,x)\cdot D(\phi-u)\,dx \geq \int_\Omega \mathbf{f}\cdot D(\phi-u)\,dx \quad \text{for all } \phi \in \mathcal{A}, \tag{1.1}$$

where the inhomogeneous term $\mathbf{f}$ is a given vector-valued function in $L^2(\Omega;\mathbb{R}^d)$, and the nonlinearities $\mathbf{a}(\xi,x) : \mathbb{R}^d \times \Omega \to \mathbb{R}^d$ are, as usual, so-called Carathéodory functions satisfying the following conditions:

$$\begin{cases} \mathbf{a}(\xi,x) \text{ measurable in } x \in \Omega \text{ for all } \xi \in \mathbb{R}^d, \\ \mathbf{a}(\xi,x) \text{ differentiable in } \xi \in \mathbb{R}^d \text{ for almost all } x \in \Omega. \end{cases}$$

We call such a function $u \in \mathcal{A}$ weak solution to the variational inequalities (1.1). To ensure the solvability in $L^2(\Omega)$ to (1.1), it is quite necessary to impose additional assumptions on the given datum.

(H1) (*ellipticity and growth*) There exist two constants $0 < \lambda \le \Lambda < \infty$ such that

$$\begin{cases} \langle D_\xi \mathbf{a}(\xi,x)\eta \cdot \eta\rangle \ge \lambda|\eta|^2, \\ |\mathbf{a}(\xi,x)| + |\xi||D_\xi \mathbf{a}(\xi,x)| \le \Lambda|\xi| \end{cases} \tag{1.2}$$

for a.e. $x \in \Omega$ and $\xi, \eta \in \mathbb{R}^d$.

It is clear that relations (1.2) immediately yield the following *monotonicity conditions*:

$$\mathbf{a}(0,x) = 0 \quad \text{and} \quad \bigl\langle \mathbf{a}(\xi,x) - \mathbf{a}(\eta,x), \xi - \eta \bigr\rangle \ge \lambda|\xi - \eta|^2. \tag{1.3}$$

With the nonlinearities satisfying (1.2), by way of classical estimate we make sure that there exists a unique weak solution $u \in \mathcal{A}$ to the variational inequality (1.1) with the usual L^2 estimate

$$\|Du\|_{L^2(\Omega)} \le C\bigl(\|\mathbf{f}\|_{L^2(\Omega)} + \|D\Psi\|_{L^2(\Omega)}\bigr), \tag{1.4}$$

where the constant C is independent of $u, \mathbf{f}, \Psi$, and Ω; see Lemma 2.1 in [6]. In this paper, we are interested in the Calderón–Zygmund-type theory in the scale of weighted Lorentz spaces regrading the variational inequality (1.1) by imposing some minimal regular assumptions on the given datum. More precisely, we are interested in finding small partially BMO requirements on the nonlinearities and Reifenberg flat geometric structure of the domain to ensure the Calderón–Zygmund estimate for the gradient of weak solution in the weighted Lorentz spaces $L^{(p,q)}_\omega(\Omega)$, which essentially shows that

$$\|Du\|_{L^{(p,q)}_\omega(\Omega)} \le C\bigl(\|\mathbf{f}\|_{L^{(p,q)}_\omega(\Omega)} + \|D\Psi\|_{L^{(p,q)}_\omega(\Omega)}\bigr) \tag{1.5}$$

for $\omega \in A_{p/2}$, $2 < p < \infty$ and $0 < q \le \infty$, where the constant C is independent of $u, \mathbf{f}$, and Ψ.

A key ingredient under consideration concerning the nonlinearities $\mathbf{a}(\xi,x)$, apart from C^1 in ξ is that we also require them to be *small BMO of codimension one* with respect to the spatial variable x, which means that, in a neighborhood of each point in Ω, there is a local coordinate system such that $\mathbf{a}(\xi,x)$ is only measurable in one direction and has small bounded mean oscillation in the remaining $(n-1)$ orthogonal directions. In fact, this was first introduced by Kim and Krylov [21], and later employed by Dong and Kim [11–13] and Byun and Wang [8] in the study of weighted L^p theory for divergence and nondivergence of linear elliptic and parabolic equations/systems. It has actually proved to be a sort of minimal regular requirement imposed on the leading coefficients of the elliptic operator to ensure a satisfactory Calderón–Zygmund theory for all $p > 1$. Here, we would also like to point out that Byun and Palagachev [8] derived a global weighted $W^{1,p}$-estimate for $2 < p < \infty$ to Dirichlet problems of linear elliptic equations in Reifenberg flat domains, provided that the coefficients are *vanishing of codimension one* (also called small partially BMO) based on a different geometric approach instead of the pointwise estimates of sharp functions from Dong, Kim and Krylov's papers. Furthermore, we also remarked that Byun et al. have employed their argument to derive L^p estimates to Dirichlet problems of quasilinear principal coefficients $a_{ij}(x,u)$ (see [9]) and nonlinearities $\mathbf{a}(x,Du)$ (see [7]) with *small partially BMO "coefficients"* in x-variables. Very recently, Erhardt [14] obtained a local Calderón–Zygmund estimate for localizable solutions of parabolic obstacle problems with nonstandard growth, and Liang and Zheng [22]showed the $W^{1,\gamma(\cdot)}$-regularity for nonlinear nonuniformly elliptic equations with *small BMO coefficients*.

As a refined version of Lebesgue spaces, Lorentz spaces are a two-parameter scale of spaces [3, 24]. The regularity in Lorentz spaces concerning partial differential equations was originated from Talenti's work [28] based on symmetrization. Since then, there is a lot of papers to study the Lorentz regularity of various problems of PDEs; see some recent references in [2, 4, 5, 23], and we also refer the reader to Xiao [32], who characterized a nonnegative Radon measure μ on $\mathbb{R}^d$ to produce a continuous map I_α from the Lorentz space $L^{(p,1)}$ to the Lebesgue space L^p_μ. We would like to mention that Baroni [4, 5] showed the Lorentz estimates for evolutionary p-Laplacian systems and obstacle parabolic p-Laplacian, respectively, by using the *large-M-inequality principle* introduced by Acerbi and Mingione [1]. Meanwhile, Mengesha and Phuc [23] and Zhang and Zhou [33] attained gradient estimates in weighted Lorentz spaces for quasilinear elliptic p-Laplacian and $p(x)$-Laplacian equations based on a rather different geometrical approach used in [8], respectively. Tian and Zheng [29, 30] very recently derived a globally weighted Lorentz estimate and a variable Lorentz estimate to linear elliptic problems over Reifenberg flat domains under the assumptions of partially BMO coefficients, respectively. In addition, Zhang and Zheng [34] studied weighted Lorentz estimates of the Hessian of strong solution for nondivergent linear elliptic equations with partially BMO coefficients. We notice that in these papers concerning nonlinear problems mentioned, an important regular assumption on the "nonlinearity coefficients" is VMO or small BMO in all x beyond the settings of linear PDEs.

Motivated by recent progress [7, 9] in particular involved in *partially regular coefficients* to nonlinear problems, in the present paper, we essentially want to study the Lorentz estimates (1.5) to the variational inequalities (1.1) and relevant nonlinear elliptic equations with controlled growth under the minimal assumption with partially regular nonlinearities $\mathbf{a}(\xi,x)$. More precisely, we assume that there is no regular requirement on the nonlinearities $\mathbf{a}(\xi,x)$ with respect to the variable x_1, which implies that the nonlinearities $\mathbf{a}(\xi,x)$ might have jumps along the x_1 variable, whereas the nonlinearities $\mathbf{a}(\xi,x)$ are controlled in terms of small BMO, such as small multipliers of the Heaviside step function, along the remaining variables. Of course, our consideration is a natural outgrowth of Byun and Kim's paper [7] concerning the Calderón–Zygmund estimate for nonlinear elliptic problems with measurable nonlinearities. Here we would like to mention that this is a kind of minimal regular requirement on the "coefficients" even for the settings of linear equations in accordance with the famous counterexample by Ural'tseva [31], who constructed an example of an equation in $\mathbb{R}^d$ ($d \geq 3$) with the coefficients depending only on the first two coordinates, so that we get that there is no unique solvability in Sobolev spaces $W^{1,p}$ for any $p > 1$. Its particular interest under consideration is due to a subtle link with application to medium composite materials [20]. Also, these are closely related to some important problems arising in modeling of deformations in composite materials, in the mechanics of membranes and films of simple nonhomogeneous materials that form linear laminated medium [26].

Before stating main results, let us recall some basic concepts and facts. In the context, let us denote a type point by $x = (x_1, \dots, x_d) = (x_1, x') \in \mathbb{R}^d$ with $x' = (x_2, \dots, x_d)$. Set

$$B_r(x) = \{y \in \mathbb{R}^d : |x - y| < r\}, \qquad B'_r(x') = \{y' \in \mathbb{R}^{d-1} : |x' - y'| < r\},$$

and a typical cylinder

$$Q_r(x) = (x_1 - r, x_1 + r) \times B'_r(x').$$

For convenience, we sometimes write $B_r = B_r(0)$ and $B'_r = B'_r(0')$. We denote the average of f on Q_r with $r > 0$ by

$$\fint_{Q_r} f(x)\,dx = \frac{1}{|Q_r|} \int_{Q_r} f(x)\,dx,$$

where $|Q_r|$ is the d-dimensional Lebesgue measure of Q_r, and we also denote the $(d-1)$-dimensional average with respect to x' by

$$\bar{f}_{B'_r}(x_1) = \fint_{B'_r} f(x_1, x')\,dx' = \frac{1}{|B'_r|} \int_{B'_r} f(x_1, x')\,dx'$$

with $|B'_r|$ as the $(d-1)$-dimensional Lebesgue measure of B'_r. We are now in a position to impose an additional partially regular assumption on the nonlinearities $\mathbf{a}(\xi, x)$ just like in [7]. For this, we recall the function $\beta(\mathbf{a}, Q_r)(x)$ on Q_r with $r > 0$ defined by

$$\beta(\mathbf{a}, Q_r)(x) = \sup_{\xi \in \mathbb{R}^d \setminus \{0\}} \frac{|\mathbf{a}(\xi, x) - \bar{\mathbf{a}}_{B'_r}(\xi, x_1)|}{|\xi|}.$$

Assumption 1.1 We say that $(\mathbf{a}(\xi, x), \Omega)$ is (δ, R_0)-vanishing of codimension one if for every point $x_0 \in \Omega$, there exists a constant $R_0 > 0$ such that, for any $0 < r \le R_0$ with

$$\operatorname{dist}(x_0, \partial\Omega) = \min_{z \in \partial\Omega} \operatorname{dist}(x_0, z) > \sqrt{2} r,$$

there exists a coordinate system depending only on x_0 and r, whose variables are still denoted by x, such that, in the new coordinate system with x_0 as the origin,

$$\fint_{Q_r} \left|\beta(\mathbf{a}, Q_r)(x)\right|^2 dx \le \delta^2;$$

whereas, for $x_0 \in \overline{\Omega}$ with

$$\operatorname{dist}(x_0, \partial\Omega) = \min_{z \in \partial\Omega} \operatorname{dist}(x_0, z) = \operatorname{dist}(x_0, z_0) \le \sqrt{2} r,$$

where $z_0 \in \partial\Omega$, there exists a coordinate system depending on x_0 and $0 < r < R_0$ such that, in the new coordinate system with z_0 as the origin,

$$Q_{3r} \cap \{x_1 \geq 3\delta r\} \subset Q_{3r} \cap \Omega \subset Q_{3r} \cap \{x_1 \geq -3\delta r\} \tag{1.6}$$

and

$$\fint_{Q_{3r}} \left|\beta(\mathbf{a}, Q_{3r})(x)\right|^2 dx \leq \delta^2, \tag{1.7}$$

where $a(x,\xi)$ is zero extended from $Q_{3r} \cap \Omega$ to Q_{3r}, and the parameter $\delta > 0$ will be specified later.

Here we point out that the boundary geometric structure (1.6) implies that Ω is a (δ, R)-*Reifenberg flat domain*. It is also obvious that this is an *A-type domain* with the relations

$$\sup_{0<r\leq R} \sup_{y\in\Omega} \frac{|B_r(y)|}{|B_r(y) \cap \Omega|} \leq \left(\frac{2}{1-\delta}\right)^n \leq \left(\frac{16}{7}\right)^n \tag{1.8}$$

for $0 < \delta < \frac{1}{8}$ by a scaling transformation [10].

Considering that our estimates are concerned with the weighted Lorentz spaces, it is necessary to recall some basic definitions involved in weight functions and Lorentz spaces.

Definition 1.2 For $1 < s < \infty$, a nonnegative function $\omega(x) \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ is called a weight in Muckenhoupt class A_s, denoted by $\omega \in A_s$, if

$$[\omega]_s = \sup_B \left(\fint_B \omega(x)\,dx\right)\left(\fint_B \omega^{\frac{-1}{s-1}}(x)\,dx\right)^{s-1} < \infty, \tag{1.9}$$

where the supremum is taken over all balls $B \subset \mathbb{R}^d$, and the constant $[\omega]_s$ is referred to be the A_s constant of ω.

For a given measurable set $E \subset \mathbb{R}^d$ and weight ω, we set

$$\omega(E) = \int_E \omega(x)\,dx.$$

Definition 1.3 Let E be an open subset in $\mathbb{R}^d$, and let ω be a weight function. The weighted Lorentz space $L^{(p,q)}_\omega(E)$ with $p \in [1, +\infty)$ and $q \in (0, +\infty)$ is the set of measurable functions $g : E \to \mathbb{R}^d$ such that

$$\|g\|_{L^{(p,q)}_\omega(E)} := \left(p\int_0^\infty \left(\gamma^p \omega\left(\left\{x \in E : \left|g(x)\right| > \gamma\right\}\right)\right)^{\frac{q}{p}} \frac{d\gamma}{\gamma}\right)^{\frac{1}{q}} < +\infty.$$

For $q = \infty$, the space $L^{(p,\infty)}_\omega(E)$ is the classical Marcinkiewicz space with quasinorm

$$\|g\|_{L^{(p,\infty)}_\omega(E)} := \sup_{\gamma>0}\left(\gamma^p \omega\left(\left\{x \in E : \left|g(x)\right| > \gamma\right\}\right)\right)^{\frac{1}{p}} < +\infty.$$

It is rather clear that if $p = q$, then the Lorentz space $L^{(p,p)}_\omega(E)$ is nothing but the usual weighted Lebesgue space $L^p_\omega(E)$, which is equivalently defined by

$$\|g\|_{L^p_\omega(E)} = \left(\int_E |g(x)|^p \omega(x)\,dx\right)^{\frac{1}{p}} < +\infty;$$

more specifically, if $\omega(x) = 1$, then $\omega(E) = \int_E dx = |E|$, which implies

$$L^{(p,q)}_\omega(E) = L^{(p,q)}(E) \quad \text{and} \quad L^p_\omega(E) = L^p(E). \tag{1.10}$$

We are now ready to summarize our main result.

Theorem 1.4 *Let $\omega \in A_{p/2}$ be a weight function with $2 < p < \infty$. For $0 < q \le \infty$ and $R_0 > 0$, there exists a positive constant $\delta = \delta(d,p,q,\lambda,\Lambda,[\omega]_{p/2})$ such that $(\mathbf{a}(\xi,x),\Omega)$ satisfies (δ,R_0)-vanishing of codimension one (Assumption 1.1). If $|\mathbf{f}|$ and $|D\Psi| \in L^{(p,q)}_\omega(\Omega)$, then, for a weak solution $u \in \mathcal{A}$ of variational inequalities (1.1) satisfying* (H1), *we have $|Du| \in L^{(p,q)}_\omega(\Omega)$ with the estimate*

$$\|Du\|_{L^{(p,q)}_\omega(\Omega)} \le C\big(\|\mathbf{f}\|_{L^{(p,q)}_\omega(\Omega)} + \|D\Psi\|_{L^{(p,q)}_\omega(\Omega)}\big), \tag{1.11}$$

where the constant C is independent of u, $\mathbf{f}$, and Ψ.

As a consequence of Theorem 1.4, by taking a special weight we also get the following Lorentz–Morrey estimate for the gradient of weak solution to variational inequalities (1.1). Let us recall the so-called *Lorentz–Morrey spaces* $L^{p,q;\theta}(E)$ for $1 < p < \infty$, $0 < q \le \infty$, and $0 < \theta \le d$. We say that $g(x) \in L^{(p,q)}(E)$ belongs to $\mathcal{L}^{p,q;\theta}(E)$ if for $\delta = \operatorname{diam}(E)$, we have

$$\|g\|_{\mathcal{L}^{p,q;\theta}(E)} := \sup_{z\in E, 0<r<\delta} r^{\frac{\theta-d}{p}} \|g\|_{L^{(p,q)}(B_r(z)\cap E)} < +\infty.$$

Clearly, $\mathcal{L}^{p,q;\theta}(E) \subset \mathcal{L}^{(p,q)}(E)$ for all $\theta \in (0,d]$. For $p = q$, the space $\mathcal{L}^{p,q;\theta}(E)$ is the usual Morrey space $\mathcal{L}^{p;\theta}(E)$; see [15, 16].

Corollary 1.5 *For $2 < p < \infty, 0 < q \le \infty$, $0 < \theta \le d$, and $R_0 > 0$, there exists a positive constant $\delta = \delta(d,p,q,\theta,\lambda,\Lambda)$ such that $(\mathbf{a}(\xi,x),\Omega)$ satisfies (δ,R_0)-vanishing of codimension one (Assumption 1.1). If $|\mathbf{f}|$ and $|D\Psi| \in \mathcal{L}^{p,q;\theta}(\Omega)$, then, for weak solution $u \in \mathcal{A}$ of variational inequalities (1.1) satisfying* (H1), *we have $|Du| \in \mathcal{L}^{p,q;\theta}(\Omega)$ with the estimate*

$$\|Du\|_{\mathcal{L}^{p,q;\theta}(\Omega)} \le C\big(\|\mathbf{f}\|_{\mathcal{L}^{p,q;\theta}(\Omega)} + \|D\Psi\|_{\mathcal{L}^{p,q;\theta}(\Omega)}\big), \tag{1.12}$$

where the constant C is independent of u, $\mathbf{f}$, and Ψ.

Finally, as an application of main Theorem 1.4, we further present a global Lorentz estimate to the following Dirichlet problem for nonlinear elliptic equations with controlled growth under very weak assumptions on given datum. Let us consider the Dirichlet problem

$$\begin{cases} -\operatorname{div}(\mathbf{a}(Du,x)) = B(Du,x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.13}$$

where the nonlinearities $\mathbf{a}(\xi,x)$: $\mathbb{R}^d \times \Omega \to \mathbb{R}^d$ satisfy the hypothesis (1.2), and $B(\xi,x)$: $\mathbb{R}^d \times \Omega \to \mathbb{R}$ satisfies the following controlled growth condition:

(H2) *(controlled growth)* There exist a constant $\mu > 0$ and a nonnegative function

$$\psi \in L^{(p,q)}(\Omega) \tag{1.14}$$

with $p \geq \frac{2d}{d+2}$ and $0 < q \leq \infty$ such that

$$\left|B(\xi,x)\right| \leq \mu\left(\psi(x) + |\xi|^{\frac{d+2}{d}}\right) \tag{1.15}$$

for a.e. $x \in \Omega$ and all $\xi \in \mathbb{R}^d$, where inequality (1.15) is usually said to be the controlled growth.

This problem is inspired by the following achievements on this topic. It is well known that nonlinear PDEs with controlled growth were always very important research subjects coming from variational problems [17, 18]. Regarding the setting with discontinuous coefficients, Zheng and Feng [35] showed an optimal Hölder regularity of weak solutions to quasilinear elliptic systems under controlled growth with VMO coefficients. Later, Dong and Kim [12] obtained an L^p estimate for quasilinear elliptic equations under controlled growth with coefficients satisfying VMO in spatial variables. Also, Byun and Palagachev derived a refined Morrey regularity for the gradient of weak solution to a quasilinear elliptic equation with lower-order term of Riccati type under the assumption of partially BMO nonlinearity in x (small BMO in the remainders except an independent variable, say x_1), and very recently they also dealt with the Sobolev–Morrey estimate for general quasilinear equations of p-Laplacian type with BMO nonlinearities in all x under controlled growth. Here, our aim under consideration is to attain a global Lorentz gradient estimate to problem (1.13) over Reifenberg flat domain under a very weak regular Assumption 1.1, based on an elaborate bootstrap argument, which implies that

$$\psi \in L^{(p,q)}(\Omega) \quad \Longrightarrow \quad Du \in L^{(p^*,q)}(\Omega)$$

with

$$p^* = \begin{cases} \frac{dp}{d-p} \geq 2 & \text{if } p < d, \\ \text{any number} > p & \text{if } p \geq d. \end{cases} \tag{1.16}$$

More precisely, we have the following:

Theorem 1.6 *Let $u \in W_0^{1,2}(\Omega)$ be a weak solution to Dirichlet problem* (1.13) *with nonlinear structural conditions* (H1) *and* (H2) *and $R_0 > 0$. There exists a positive constant $\delta = \delta(d,p,q,\lambda,\Lambda,\mu)$ such that if $(\mathbf{a}(\xi,x),\Omega)$ satisfies (δ,R_0)-vanishing of codimension one as in Assumption* 1.1, *then we have that the gradient Du belongs to an appropriate Lorentz space*:

$$Du \in L^{(p^*,q)}(\Omega)$$

for any $p \geq \frac{2d}{d+2}$ and $0 < q \leq \infty$.

Here, we would like to point out that there is no usual restriction with $p > \frac{d}{2}$ to Dirichlet problem (1.13) since we do not employ the boundedness of weak solution of (1.13). In fact, this makes the weak solution of (1.13) possibly unbounded since it is invalid for the De Giorgi–Moser–Nash iterating argument in $p \leq \frac{d}{2}$. Finally, we complete the proof by enhancing the index of gradient integrability of weak solution to the linearized problem in accordance with a successive application of the bootstrapping argument. Also, we would remark that in the particular case $p = q$, Theorem 1.6 is just a classical Calderón–Zygmund property of (1.13) also in the framework of Lebesgue scales as in [12].

The remainder of this paper is organized as follows. We denote by $C(d, \mu, \Lambda, \ldots)$ and $N_i(d, \mu, \Lambda, \partial\Omega, \ldots)$ for $i = 1, 2, \ldots$ universal constants depending only on prescribed quantities and possibly varying from line to line. We recall some usual auxiliary results in the next section. In Sect. 3, we show local interior and boundary estimates of the gradient to weak solution of the reference problem to variational inequality (1.1). We prove main Theorem 1.4 and Corollary 1.5 in Sect. 4. Finally, we give a proof of Theorem 1.6 regarding the Dirichlet problem (1.13) with controlled growth in Sect. 5.

2 Preliminaries

This section mainly presents some usual preliminary facts. We begin with recalling the following invariance properties of variational inequality (1.1) under scaling, translation, and normalization. It is obvious that we get them similarly to Lemma 3.1 in [7] since we only add the restriction on the obstacle function.

Lemma 2.1 *For all $M, \tau > 0$, we define the normalization by*

$$\mathbf{a}_M^\tau(\xi, x) := \frac{\mathbf{a}(M\xi, \tau x + x_0)}{M}, \qquad u_M^\tau(x) = \frac{u(\tau x + x_0)}{\tau M},$$
$$\mathbf{f}_M^\tau(x) = \frac{\mathbf{f}(\tau x + x_0)}{M}, \qquad \Psi_M^\tau(x) = \frac{\Psi(\tau x + x_0)}{\tau M}$$

and

$$\Omega^\tau = \{(x - x_0)/\tau : x \in \Omega\}, \qquad \mathcal{A}_M^\tau = \{\phi \in W_0^{1,2}(\Omega^\tau) : \phi \geq \Psi_M^\tau \text{ a.e. in } x \in \Omega^\tau\}.$$

Then

(i) *If $u \in \mathcal{A}$ is a weak solution of*

$$\int_\Omega \mathbf{a}(Du, x) \cdot D(\phi - u)\, dx \geq \int_\Omega \mathbf{f} \cdot D(\phi - u)\, dx,$$

then $u_M^\tau \in \mathcal{A}_M^\tau$ is also a weak solution of

$$\int_{\Omega^\tau} \mathbf{a}_M^\tau(Du_M^\tau, x) \cdot D(\phi - u_M^\tau)\, dx \geq \int_{\Omega^\tau} \mathbf{f}_M^\tau \cdot D(\phi - u_M^\tau)\, dx.$$

(ii) *If $(\mathbf{a}(\xi, x), \Omega)$ is (δ, R_0)-vanishing of codimension one, then $(\mathbf{a}_M^\tau(\xi, x), \Omega^\tau)$ is $(\delta, R_0/\tau)$-vanishing of codimension one. Moreover, $\mathbf{a}_M^\tau(\xi, x)$ satisfies (1.2) for the same constants λ and Λ.*

The following doubling-type property of the A_s weights is a useful way to the transformation between the Lebesgue measure and weight measure A_s; see [8, 27].

Lemma 2.2 *Let $\omega \in A_s$ with $1 < s < \infty$. Then there exist positive constants C and $\sigma \in (0,1)$, depending only on d, s, and $[\omega]_s$, such that*

$$\frac{1}{C}\left(\frac{|E|}{|B|}\right)^s \leq \frac{\omega(E)}{\omega(B)} \leq C\left(\frac{|E|}{|B|}\right)^\sigma$$

for any ball B and measurable subset $E \subset B$.

Next, we present a summary of embedding relations involving the Lorentz spaces that will be useful in the proofs; see [23, Proposition 3.9], [4, Sect. 3.2], and [19, Sects. 1.1 and 1.4].

Proposition 2.3 *Let E be a bounded measurable subset of $\mathbb{R}^d$, and let ω be an A_s weight for $1 < s < \infty$. Then*:

(i) *If $0 < q_2 \leq \infty$ and $1 < p_1 < p_2 < \infty$, then $L_\omega^{(p_2,q_2)}(E) \subset L_\omega^{p_1}(E)$ and*

$$\|g\|_{L_\omega^{p_1}(E)} \leq C(p_1,p_2,q_2)\omega(E)^{\frac{1}{p_1}-\frac{1}{p_2}}\|g\|_{L_\omega^{(p_2,q_2)}(E)}. \tag{2.1}$$

(ii) *If $1 < p < \infty$ and $0 < q_1 < q_2 \leq \infty$, then $L_\omega^{(p,q_1)}(E) \subset L_\omega^{(p,q_2)}(E) \subset L_\omega^{(p,\infty)}(E)$ and*

$$\|g\|_{L_\omega^{(p,q_2)}(E)} \leq C(p,q_1,q_2)\|g\|_{L_\omega^{(p,q_1)}(E)}. \tag{2.2}$$

(iii) *If $|g|^\sigma \in L_\omega^{(p,q)}(E)$ for some $0 < \sigma < \infty$, then $g \in L_\omega^{(\sigma p,\sigma q)}(E)$ with the estimate*

$$\big\||g|^\sigma\big\|_{L_\omega^{(p,q)}(E)} = \|g\|^\sigma_{L_\omega^{(\sigma p,\sigma q)}(E)}.$$

(iv) *If $f, g \in L_\omega^{(p,q)}(E)$, then $f + g \in L_\omega^{(p,q)}(E)$ with the estimate*

$$\|f+g\|_{L_\omega^{(p,q)}(E)} \leq C(p,q)\big(\|f\|_{L_\omega^{(p,q)}(E)} + \|g\|_{L_\omega^{(p,q)}(E)}\big).$$

Consequently, for a weight $\omega \in A_s$ with $1 < s < \infty$, if $1 < p < \infty$ and $0 < q \leq \infty$, then, for any bounded domain E, by Proposition 2.3 we have

$$L_\omega^{(p,q)}(E) \subset L_\omega^{(p,\infty)}(E) \subset L_\omega^{p-\varepsilon}(E) \subset L^1(E)$$

for any $\varepsilon > 0$ such that $p - \varepsilon > 1$.

It is an important tool for us to describe an elementary characterization of functions in the scale of weighted Lorentz spaces based on the level set of a distributional function; see [23, Lemma 3.12]. For completeness, here we briefly prove it by the classical measure theory, which is similar to the idea of proof in [23, Lemma 3.12].

Lemma 2.4 *Suppose that ω is an A_s weight for some $1 < s < \infty$ and g is a nonnegative measurable function in a bounded domain $E \subset \mathbb{R}^d$. Let $\theta > 0$ and $T > 1$ be constants. Then,* *for* $1 \le p < \infty$ *and* $0 < q < \infty$, *we have*

$$g(x) \in L_{\omega}^{(p,q)}(E) \quad \Longleftrightarrow \quad S := \sum_{k\ge 1} T^{kq}\omega\bigl(\bigl\{x \in E : \bigl|g(x)\bigr| > \theta T^k\bigr\}\bigr)^{\frac{q}{p}} < +\infty$$

and

$$C^{-1}S \le \bigl\|g(x)\bigr\|_{L_{\omega}^{(p,q)}(E)}^q \le C\bigl(\omega(E)^{\frac{q}{p}} + S\bigr), \tag{2.3}$$

where $C > 0$ *is a constant depending only on* θ, T, p, *and* q. *Analogously, for* $1 < p < \infty$ *and* $q = \infty$, *we have*

$$C^{-1}L \le \bigl\|g(x)\bigr\|_{L_{\omega}^{(p,\infty)}(E)} \le C\bigl(\omega(E)^{\frac{1}{p}} + L\bigr), \tag{2.4}$$

where

$$L := \sup_{k\ge 1} T^k\omega\bigl(\bigl\{x \in E : \bigl|g(x)\bigr| > \theta T^k\bigr\}\bigr)^{\frac{1}{p}}.$$

Proof We begin with the case $0 < q < \infty$. Let $\theta > 0$ and $T > 1$. From the definition of the weighted Lorentz spaces we have

$$\begin{aligned}
\|g\|_{L_{\omega}^{(p,q)}(E)}^q &= p\int_0^{\theta T} \bigl(\alpha^p\omega\bigl(\bigl\{x \in E : \bigl|g(x)\bigr| > \alpha\bigr\}\bigr)\bigr)^{\frac{q}{p}} \frac{d\alpha}{\alpha} \\
&\quad + p\int_{\theta T}^{\infty} \bigl(\alpha^p\omega\bigl(\bigl\{x \in E : \bigl|g(x)\bigr| > \alpha\bigr\}\bigr)\bigr)^{\frac{q}{p}} \frac{d\alpha}{\alpha} \\
&:= I_1 + I_2.
\end{aligned}$$

To estimate I_1, we get

$$I_1 \le \omega(E)^{\frac{q}{p}} p\int_0^{\theta T} \alpha^{q-1}\,d\alpha \le \frac{p}{q}(\theta T)^q\omega(E)^{\frac{q}{p}}.$$

To estimate I_2, we get

$$\begin{aligned}
I_2 &= p\int_{\theta T}^{\infty} \alpha^{q-1}\omega\bigl(\bigl\{x \in E : \bigl|g(x)\bigr| > \alpha\bigr\}\bigr)^{\frac{q}{p}}\,d\alpha \\
&= p\sum_{k\ge 1}\int_{\theta T^k}^{\theta T^{k+1}} \alpha^{q-1}\omega\bigl(\bigl\{x \in E : \bigl|g(x)\bigr| > \alpha\bigr\}\bigr)^{\frac{q}{p}}\,d\alpha \\
&\le \frac{p}{q}\sum_{k\ge 1}\omega\bigl(\bigl\{x \in E : \bigl|g(x)\bigr| > \theta T^k\bigr\}\bigr)^{\frac{q}{p}}\bigl(\theta T^k\bigr)^q\bigl(T^q - 1\bigr) \\
&= \frac{p}{q}\theta^q\bigl(T^q - 1\bigr)\sum_{k\ge 1} T^{kq}\omega\bigl(\bigl\{x \in E : \bigl|g(x)\bigr| > \theta T^k\bigr\}\bigr)^{\frac{q}{p}} \\
&:= \frac{p}{q}\theta^q\bigl(T^q - 1\bigr)S.
\end{aligned}$$

Now, putting the estimates of I_1 and I_2 together, we deduce

$$\|g\|^q_{L^{(p,q)}_\omega(E)} \le C\bigl(\omega(E)^{\frac{q}{p}} + S\bigr), \tag{2.5}$$

where C depends only on q, p, θ, and T. Conversely, we observe that

$$\begin{aligned} &p\int_{\theta T^{k-1}}^{\theta T^k} \alpha^{q-1}\omega\bigl(\bigl\{x\in E:\bigl|g(x)\bigr|>\alpha\bigr\}\bigr)^{\frac{q}{p}}\,d\alpha \\ &\quad\ge \frac{p}{q}\omega\bigl(\bigl\{x\in E:\bigl|g(x)\bigr|>\theta T^k\bigr\}\bigr)^{\frac{q}{p}}\bigl(\theta T^k\bigr)^q\left(1-\frac{1}{T^q}\right) \\ &\quad= \frac{p}{q}\theta^q\left(1-\frac{1}{T^q}\right)T^{kq}\omega\bigl(\bigl\{x\in E:\bigl|g(x)\bigr|>\theta T^k\bigr\}\bigr)^{\frac{q}{p}}. \end{aligned}$$

Therefore, by summing up with $k \ge 1$ we have

$$\begin{aligned} &\frac{p}{q}\theta^q\left(1-\frac{1}{T^q}\right)\sum_{k\ge 1}T^{kq}\omega\bigl(\bigl\{x\in E:\bigl|g(x)\bigr|>\theta T^k\bigr\}\bigr)^{\frac{q}{p}} \\ &\quad\le \sum_{k\ge 1} p\int_{\theta T^{k-1}}^{\theta T^k}\alpha^{q-1}\omega\bigl(\bigl\{x\in E:\bigl|g(x)\bigr|>\alpha\bigr\}\bigr)^{\frac{q}{p}}\,d\alpha \\ &\quad= p\int_\theta^\infty \bigl(\alpha^p\omega\bigl(\bigl\{x\in E:\bigl|g(x)\bigr|>\alpha\bigr\}\bigr)\bigr)^{\frac{q}{p}}\frac{d\alpha}{\alpha} \\ &\quad\le p\int_0^\infty \bigl(\alpha^p\omega\bigl(\bigl\{x\in E:\bigl|g(x)\bigr|>\alpha\bigr\}\bigr)\bigr)^{\frac{q}{p}}\frac{d\alpha}{\alpha} = \|g\|^q_{L^{(p,q)}_\omega(E)}, \end{aligned}$$

which implies

$$\|g\|^q_{L^{(p,q)}_\omega(E)} \ge CS. \tag{2.6}$$

Putting estimates (2.5) and (2.6) together yields (2.3).

On the other hand, we see that (2.4) also holds for the case $q = \infty$ by a similar argument as in the case $0 < q < \infty$, which completes the proof. □

Our argument also rests on the classical Hardy–Littlewood maximal function. For a function $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$, the Hardy–Littlewood maximal function of f is defined by

$$\mathcal{M}f(x) = \sup_{r>0} \fint_{B_r(x)} \bigl|f(y)\bigr|\,dy. \tag{2.7}$$

If f is not defined outside a bounded domain E, then

$$\mathcal{M}_E f = \mathcal{M}(f\chi_E)$$

for the standard characteristic function χ on E.

Finally, for the relation between the boundedness of Hardy–Littlewood maximal operator $\mathcal{M}$ in the weighted Lorentz spaces and the Muckenhoupt class A_p, see [23, Lemma 3.11].

Theorem 2.5 *Let $\omega \in A_p$, $1 < p < \infty$. For any $0 < q \leq \infty$, there exists a positive constant $C = C(d,p,q,[\omega]_p)$ such that*

$$\|\mathcal{M}f\|_{L^{(p,q)}_{\omega}(\mathbb{R}^d)} \leq C\|f\|_{L^{(p,q)}_{\omega}(\mathbb{R}^d)} \tag{2.8}$$

for all $f \in L^{(p,q)}_{\omega}(\mathbb{R}^d)$. Conversely, if (2.8) holds for all $f \in L^{(p,q)}_{\omega}(\mathbb{R}^d)$, then ω is an A_p weight.

3 Approximating the reference problems

In this section, we collect some related facts which are used to approximate weak solution of the variational inequalities (1.1). The following lemma presents a comparison principle, which is needed later to ensure that each of the solution satisfies the admissible test functions for the variational inequalities with the same obstacle condition.

Lemma 3.1 *Let $E \subset \mathbb{R}^d$ be a bounded domain. Suppose that Ψ and $w \in W^{1,2}(E)$ satisfy*

$$\begin{cases} -\operatorname{div}(\mathbf{a}(D\Psi,x)) \leq -\operatorname{div}(\mathbf{a}(Dw,x)) & \text{in } E, \\ \Psi \leq w & \text{on } \partial E, \end{cases} \tag{3.1}$$

where (1.2) is in force. Then, we have $\Psi \leq w$ a.e. on E.

Proof We first rewrite the inequality of (3.1) in the sense of distributions:

$$\int_E \big(\mathbf{a}(D\Psi,x) - \mathbf{a}(Dw,x)\big) D\varphi \, dx \leq 0, \tag{3.2}$$

where $\varphi \in W^{1,2}_0(E)$ and $\varphi(x) \geq 0$ for a.e. $x \in E$. As usual, let us denote $f^+ := \max\{f,0\}$. Note that $\Psi \leq w$ on the boundary ∂E. Then the function $\varphi = (\Psi - w)^+ \in W^{1,2}_0(E)$ is admissible in (3.2), and so we obtain

$$\int_E \big(\mathbf{a}(D\Psi,x) - \mathbf{a}(Dw,x)\big) D\big((\Psi - w)^+\big) \, dx \leq 0,$$

which implies

$$\int_{E\cap\{\Psi>w\}} \big(\mathbf{a}(D\Psi,x) - \mathbf{a}(Dw,x)\big) D(\Psi - w) \, dx \leq 0.$$

Considering the monotonicity (1.3) of the vector field $\mathbf{a}$, it infers that the integral on the left-hand side is nonnegative. Indeed, since $(\Psi - w)^+ = 0$, we have a.e. that $D((\Psi - w)^+) = 0$, whereas on the set where $(\Psi - w)^+ \neq 0$, we can use (1.3) to obtain

$$\int_{E\cap\{\Psi>w\}} \big|D(\Psi - w)\big|^2 \, dx \leq \frac{1}{\lambda} \int_{E\cap\{\Psi>w\}} \big(\mathbf{a}(D\Psi,x) - \mathbf{a}(Dw,x)\big) D(\Psi - w) \, dx \leq 0,$$

which yields

$$\int_E \big|D\big((\Psi - w)^+\big)\big|^2 \, dx \leq 0.$$

This allows us to obtain $\Psi \leq w$ a.e. on E. □

3.1 Interior estimates

We start with an interior estimate to variational inequalities (1.1). To this end, without loss of generality, by a normalized argument of Lemma 2.1 we let

$$Q_6 \subset \Omega \tag{3.3}$$

and assume that

$$\fint_{Q_6} \left|\beta(\mathbf{a}; Q_6)(x)\right|^2 dx \le \delta^2 \quad \text{and} \quad \fint_{Q_6} \left(|Du|^2 + \frac{1}{\delta^2}\left(|\mathbf{f}|^2 + |D\Psi|^2\right)\right) dx \le 1, \tag{3.4}$$

where δ is a constant to be determined later. Under the assumptions (3.3) and (3.4), we compare $u \in \mathcal{A}$ to a weak solution $k \in W^{1,2}(Q_6)$ of

$$\begin{cases} -\operatorname{div}(\mathbf{a}(Dk,x)) = -\operatorname{div}(\mathbf{a}(D\Psi,x)) & \text{in } Q_6, \\ k = u & \text{on } \partial Q_6. \end{cases} \tag{3.5}$$

Lemma 3.2 *Let $u \in \mathcal{A}$ be a weak solution to variational inequalities* (1.1), *and let $k \in W^{1,2}(Q_6)$ be a weak solution of* (3.5). *Under assumptions* (3.3) *and* (3.4), *we have*

$$\fint_{Q_6} \left|D(u-k)\right|^2 dx \le C\delta^2, \tag{3.6}$$

where $C = C(\lambda, \Lambda)$ is a positive constant.

Proof By recalling Lemma 3.1, let $k = u \ge \Psi$ a.e. on ∂Q_6, then $k \ge \Psi$ a.e. in Q_6. We extend k to $\Omega \setminus Q_6$ by u so that $k \in \mathcal{A}$ and $k - u = 0$ in $\Omega \setminus Q_6$. Then, from the variation inequalities (1.1) with $\phi = k$ it follows that

$$\int_{Q_6} \mathbf{a}(Du,x) \cdot D(k-u)\, dx \ge \int_{Q_6} \mathbf{f} \cdot D(k-u)\, dx. \tag{3.7}$$

Let us put (3.7) and (3.5) together and take $\varphi = k - u \in W_0^{1,2}(Q_6)$ as a test function. Then we obtain

$$\int_{Q_6} \left(\mathbf{a}(Dk,x) - \mathbf{a}(Du,x)\right) \cdot D(k-u)\, dx \le \int_{Q_6} \left(\mathbf{a}(D\Psi,x) - \mathbf{f}\right) \cdot D(k-u)\, dx. \tag{3.8}$$

Monotonicity condition (1.3) yields

$$\lambda \int_{Q_6} \left|D(k-u)\right|^2 dx \le \int_{Q_6} \left(\mathbf{a}(Dk,x) - \mathbf{a}(Du,x)\right) \cdot D(k-u)\, dx.$$

Thanks to the structural growth (1.2) and Young's inequality, we have

$$\begin{aligned} &\int_{Q_6} \left(\mathbf{a}(D\Psi,x) - \mathbf{f}\right) \cdot D(k-u)\, dx \\ &\quad \le \frac{\lambda}{2} \int_{Q_6} \left|D(k-u)\right|^2 dx + C(\lambda) \int_{Q_6} \left(\Lambda^2 |D\Psi|^2 + |\mathbf{f}|^2\right) dx. \end{aligned}$$

By combining the last two estimates with (3.8) it follows that

$$\fint_{Q_6} \left|D(k-u)\right|^2 dx \le C(\lambda,\Lambda) \fint_{Q_6} \left(|D\Psi|^2 + |\mathbf{f}|^2\right) dx \le C(\lambda,\Lambda)\delta^2, \tag{3.9}$$

where the last inequality is due to assumption (3.4) and completes this proof. □

Let us take $F = \mathbf{a}(D\Psi, x)$. By the growth (1.2) and a priori assumption (3.4) we get

$$\fint_{Q_6} |F|^2 dx = \fint_{Q_6} \left|\mathbf{a}(D\Psi,x)\right|^2 dx \le \fint_{Q_6} \Lambda^2 |D\Psi|^2 dx \le \Lambda^2\delta^2 = \bar{\delta}^2$$

with $\bar{\delta} = \Lambda\delta$. Next, we consider the local limiting problem

$$\begin{cases} -\operatorname{div}(\bar{\mathbf{a}}(Dv, x_1)) = 0 & \text{in } Q_4, \\ v = k & \text{on } \partial Q_4. \end{cases} \tag{3.10}$$

Employing assumption (3.4) and the standard L^2 estimates (3.5) in (3.10), we deduce

$$\fint_{Q_4} |Dv|^2 dx \le C \fint_{Q_4} |Dk|^2 dx \le C \fint_{Q_6} |Dk|^2 dx \le C \fint_{Q_6} |Du|^2 dx \le C. \tag{3.11}$$

Then, a sufficient regularity to a weak solution of limiting problem (3.10) was derived by the following interior $W^{1,\infty}$-estimate; see Lemma 4.6 in [7].

Lemma 3.3 *Let $v \in W^{1,2}(Q_4)$ be a weak solution of problem (3.10). Then there exists a positive constant $N_1 = N_1(d,\lambda,\Lambda)$ such that*

$$\|Dv\|_{L^\infty(Q_2)} \le N_1. \tag{3.12}$$

In what follows, we also need an approximating lemma; see [7, Sect. 5].

Lemma 3.4 *Let $k \in W^{1,2}(Q_5)$ be a weak solution of (3.5) with $F = \mathbf{a}(D\Psi, x)$, and let $v \in W^{1,2}(Q_4)$ be a weak solution of (3.10). Then, under assumptions (3.3) and (3.4) and Assumption 1.1, we have*

$$\fint_{Q_4} \left|D(k-v)\right|^2 dx \le C\delta^{\sigma_1} \tag{3.13}$$

for some $\sigma_1 = \sigma_1(d,\lambda,\Lambda) > 0$ and $C = C(\lambda,\Lambda)$.

With Lemmas 3.2 and 3.4 in hand, we immediately conclude the following interior comparison estimate.

Lemma 3.5 *Let $u \in \mathcal{A}$ be a weak solution to the variational inequalities (1.1), and let $v \in W^{1,2}(Q_4)$ be a weak solution of (3.10). For any $0 < \varepsilon < 1$, there exists a positive constant δ satisfying Assumption 1.1. Then, under assumptions (3.3) and (3.4), we have*

$$\fint_{Q_4} \left|D(u-v)\right|^2 dx \le \varepsilon^2. \tag{3.14}$$

Proof Note that

$$\fint_{Q_4} \left|D(u-v)\right|^2 dx \le 2\left(\fint_{Q_4} \left|D(u-k)\right|^2 dx + \fint_{Q_4} \left|D(k-v)\right|^2 dx\right).$$

Putting it into Lemmas 3.2 and 3.4, we obtain

$$\fint_{Q_4} \left|D(u-v)\right|^2 dx \le C\left(\delta^2 + \delta^{\sigma_1}\right).$$

Finally, taking δ small enough so that $C(\delta^2 + \delta^{\sigma_1}) < \varepsilon^2$ leads to the conclusion (3.14). □

3.2 Boundary estimates

We are now in a position to study local boundary estimates for variational inequalities (1.1). For this, let us denote

$$\Omega_r = \Omega \cap Q_r, \qquad \partial\Omega_r = (Q_r \cap \partial\Omega) \cup (\partial Q_r \cap \Omega).$$

Since $(\mathbf{a}(\xi,x),\Omega)$ is (δ,R_0)-vanishing of codimension one as in Assumption 1.1, without loss of generality, we let

$$Q_6^+ \subset \Omega_6 \subset Q_6 \cap \{x : x_1 > -12\delta\} \tag{3.15}$$

and

$$\fint_{\Omega_6} \left|\beta(\mathbf{a};\Omega_6)(x)\right|^2 dx \le \delta^2. \tag{3.16}$$

Similarly, by a normalized argument of Lemma 2.1, we further assume

$$\fint_{\Omega_6} \left(|Du|^2 + \frac{1}{\delta^2}\left(|\mathbf{f}|^2 + |D\Psi|^2\right)\right) dx \le 1, \tag{3.17}$$

where $\delta > 0$ is a small constant to be determined later. Under assumptions (3.15)–(3.17), we compare $u \in \mathcal{A}$ to a weak solution $k \in W^{1,2}(\Omega_6)$ of

$$\begin{cases} -\operatorname{div}(\mathbf{a}(Dk,x)) = -\operatorname{div}(\mathbf{a}(D\Psi,x)) & \text{in } \Omega_6, \\ k = u & \text{on } \partial\Omega_6. \end{cases} \tag{3.18}$$

Here our argument follows the line of the proof of interior estimate just as in Lemma 3.2, and we easily get the following local boundary estimate.

Lemma 3.6 *Let $u \in \mathcal{A}$ be a weak solution to variational inequalities* (1.1). *Under assumptions* (3.15)–(3.17), *there exists a weak solution $k \in W^{1,2}(\Omega_6)$ of* (3.18) *satisfying*

$$\fint_{\Omega_6} \left|D(u-k)\right|^2 dx \le C\delta^2,$$

where $C = C(\lambda,\Lambda)$.

Additionally, letting $F = \mathbf{a}(D\Psi, x)$, we get

$$\fint_{\Omega_6} |F|^2\,dx = \fint_{\Omega_6} \bigl|\mathbf{a}(D\Psi, x)\bigr|^2\,dx \le \fint_{\Omega_6} \Lambda^2|D\Psi|^2\,dx \le \Lambda^2\delta^2 = \bar{\delta}^2$$

with $\bar{\delta} = \Lambda\delta$.

Consider a limiting problem in accordance with (3.15):

$$\begin{cases} -\operatorname{div}(\bar{\mathbf{a}}(Dv, x_1)) = 0 & \text{in } Q_4^+, \\ v = 0 & \text{on } Q_4 \cap \{x_1 = 0\}. \end{cases} \tag{3.19}$$

Also, we use the boundary $W^{1,\infty}$-regularity for a weak solution of problem (3.19) from Byun et al. [7]. We extend v from Q_4^+ to Ω_4 by zero extension and get that $v = 0$ on $Q_4 \cap \{x_1 = 0\}$ in the trace sense; for details, see Lemma 5.9 in [7] and the references therein.

Lemma 3.7 *Let $k \in W^{1,2}(\Omega_6)$ be a weak solution of (3.18) with $F = \mathbf{a}(D\Psi, x)$. For any $0 < \varepsilon_1 < 1$, we can choose a small positive constant δ such that Assumption 1.1 holds. Then, under assumptions (3.15)–(3.17), there exists a weak solution $v \in W^{1,2}(Q_4^+)$ of (3.19) with*

$$\fint_{\Omega_4} \bigl|D(k - \bar{v})\bigr|^2\,dx \le \varepsilon_1^2, \qquad \|D\bar{v}\|_{L^\infty(\Omega_2)} \le N_2,$$

where $\bar{v}$ is zero extension of v from Q_4^+ to Ω_4, and the constant $N_2 = N_2(d, \lambda, \Lambda)$.

By a transitive action according to Lemmas 3.6 and 3.7, the boundary comparison estimate and Lipschitz boundedness for a weak solution of the limiting problem (3.19) are immediate conclusions.

Lemma 3.8 *Let $u \in \mathcal{A}$ be a weak solution to the variational inequalities (1.1). For any $0 < \varepsilon < 1$, we can choose a positive constant δ making Assumption 1.1 true. Then, under assumptions (3.15)–(3.17), there exists a weak solution $v \in W^{1,2}(Q_4^+)$ of (3.19) such that*

$$\fint_{\Omega_4} \bigl|D(u - \bar{v})\bigr|^2\,dx \le \varepsilon^2, \qquad \|D\bar{v}\|_{L^\infty(\Omega_2)} \le N_2, \tag{3.20}$$

where $\bar{v}$ is zero extension of v from Q_4^+ to Ω_4, and $N_2 = N_2(d, \lambda, \Lambda)$ as in Lemma 3.7.

Proof Note that

$$\fint_{\Omega_4} \bigl|D(u - \bar{v})\bigr|^2\,dx \le 2\biggl(\fint_{\Omega_4} \bigl|D(u - k)\bigr|^2\,dx + \fint_{\Omega_4} \bigl|D(k - \bar{v})\bigr|^2\,dx\biggr).$$

Putting it into Lemmas 3.6 and 3.7, we obtain

$$\fint_{\Omega_4} \bigl|D(u - \bar{v})\bigr|^2\,dx \le C\bigl(\delta^2 + \varepsilon_1^2\bigr).$$

Finally, we take δ and ε_1 small enough to arrive at the conclusion (3.20). □

4 Proofs of Theorem 1.4 and Corollary 1.5

In this section, we are mainly devoted to proving the global weighted Lorentz estimate and Lorentz–Morrey estimate for the gradients of weak solution to variational inequalities (1.1). To this end, let us first introduce the following two level sets. For any $\nu > 0$, we set

$$C(\nu) := \left\{x \in \Omega : \left(\mathcal{M}\left(|Du|^2\right)\right)^{\frac{1}{2}} > \nu\right\},$$
$$D(\nu) := \left\{x \in \Omega : \left(\mathcal{M}\left(|Du|^2\right)\right)^{\frac{1}{2}} > \nu\right\} \cup \left\{x \in \Omega : \frac{1}{\delta}\left(\mathcal{M}\left(|\mathbf{f}|^2 + |D\Psi|^2\right)\right)^{\frac{1}{2}} > \nu\right\}.$$

Theorem 4.1 *Let $\omega \in A_s$ with $s > 1$, and let $u \in \mathcal{A}$ be a weak solution of variational inequalities (1.1). For any $\varepsilon > 0$, there exists a positive constant δ such that $(\mathbf{a}(\xi, x), \Omega)$ satisfy (δ, R_0)-vanishing of codimension one as in Assumption 1.1. Then, for any $y \in \Omega$ and small $r > 0$, there exists a constant $T > 1$ such that if*

$$\omega\left(Q_r(y) \cap C(T)\right) \geq \varepsilon\omega\left(Q_r(y)\right), \tag{4.1}$$

then we have $\Omega_r(y) \subset D(1)$.

Proof For fixed $y \in \Omega$ and $r > 0$, we divide the proof into two possible cases.

Case 1 (interior estimate). If $Q_{8r}(y) \subset \Omega$, then we argue it by contradiction. Suppose that $Q_r(y)$ satisfies condition (4.1), but the conclusion is false, which implies that there exists a point $x_0 \in Q_r(y)$ such that, for any $\rho > 0$, we have

$$\fint_{Q_\rho(x_0)} \left(|\mathbf{f}|^2 + |D\Psi|^2\right) dx \leq \delta^2, \qquad \fint_{Q_\rho(x_0)} |Du|^2\, dx \leq 1. \tag{4.2}$$

Then, since $Q_{6r}(y) \subset Q_{7r}(x_0) \subset Q_{8r}(y) \subset \Omega$, setting $\rho = 7r$ it yields that

$$\fint_{Q_{6r}(y)} \left(|\mathbf{f}|^2 + |D\Psi|^2\right) dx \leq \left(\frac{7}{6}\right)^d \fint_{Q_{7r}(x_0)} \left(|\mathbf{f}|^2 + |D\Psi|^2\right) dx \leq \left(\frac{7}{6}\right)^d \delta^2 \tag{4.3}$$

and

$$\fint_{Q_{6r}(y)} |Du|^2\, dx \leq \left(\frac{7}{6}\right)^d \fint_{Q_{7r}(x_0)} |Du|^2\, dx \leq \left(\frac{7}{6}\right)^d. \tag{4.4}$$

We use of Lemmas 3.3 and 3.5 with a suitable scaling and normalization argument, for example, by taking

$$\tilde{u}(x) = \frac{u(rx)}{r\sqrt{(\frac{7}{6})^d}}, \qquad \tilde{\mathbf{f}}(x) = \frac{\mathbf{f}(rx)}{\sqrt{(\frac{7}{6})^d}} \quad \text{and} \quad \tilde{\Psi}(x) = \frac{\Psi(rx)}{r\sqrt{(\frac{7}{6})^d}},$$

which implies that there exists a constant N_1 such that, for any $\varepsilon_1 > 0$, we can select a constant δ satisfying (4.3) and (4.4). Then, after scaling back there exists a weak solution $v \in W^{1,2}(Q_{4r}(y))$ of

$$-\operatorname{div}\left(\bar{\mathbf{a}}(Dv, x_1)\right) = 0 \quad \text{in } Q_{4r}(y) \tag{4.5}$$

with

$$\fint_{Q_{4r}(y)} \left|D(u-v)\right|^2 dx \le \varepsilon_1^2 \quad \text{and} \quad \|Dv\|_{L^\infty(Q_{2r}(y))} \le N_1.$$

Denoting $T_1 = \max\{2N_1, 2^d\} > 1$, this yields

$$\left|\left\{x \in Q_r(y) : \left(\mathcal{M}\left(|Du|^2\right)\right)^{\frac{1}{2}} > T_1\right\}\right| \le \left|\left\{x \in Q_{2r}(y) : \mathcal{M}\left(\left|D(u-v)\right|^2\right) > N_1^2\right\}\right|,$$

which implies that

$$\begin{aligned}\left|\left\{x \in Q_r(y) : \left(\mathcal{M}\left(|Du|^2\right)\right)^{\frac{1}{2}} > T_1\right\}\right| &< \frac{C}{N_1^2} \int_{Q_{4r}(y)} \left|D(u-v)\right|^2 dx \le C\left|Q_{4r}(y)\right|\varepsilon_1^2 \\ &= \varepsilon_2\left|Q_r(y)\right|,\end{aligned}$$

where we take $\varepsilon_2 = C4^d\varepsilon_1^2$ in the last inequality. Applying Lemma 2.2 to this formula, we obtain that, for some $\delta > 0$ and $C > 0$,

$$\omega\left(\left\{x \in Q_r(y) : \left(\mathcal{M}\left(|Du|^2\right)\right)^{\frac{1}{2}} > T_1\right\}\right) < C\varepsilon_2^\delta \omega\left(Q_r(y)\right) = \varepsilon\omega\left(Q_r(y)\right), \tag{4.6}$$

which contradicts with (4.1).

Case 2 (boundary estimate). Let $Q_{8r}(y) \nsubseteq \Omega$. In this case, there exists a boundary point $y_0 \in \partial\Omega$ such that $y_0 \in Q_{8r}(y)$. As in the interior estimate, we also argue it by contradiction. Suppose that $\Omega_r(y)$ satisfies condition (4.1) such that the conclusion is false, which implies that there is a point $x_0 \in \Omega_r(y)$ such that

$$\fint_{\Omega_\rho(x_0)} \left(|\mathbf{f}|^2 + |D\Psi|^2\right) dx \le \delta^2, \qquad \fint_{\Omega_\rho(x_0)} |Du|^2\, dx \le 1 \tag{4.7}$$

for any $\rho > 0$. Note that $(\mathbf{a}(\xi, x), \Omega)$ is (δ, R_0)-vanishing of codimension one as in Assumption 1.1 and $y_0 \in \partial\Omega \cap Q_{8r}(y)$. There exists a new coordinate system depending only on y_0 and r, whose variables we denote by z, such that, in this new coordinate system, the origin is $o := y_0 + \delta\vec{n}_0$ for some small $\delta > 0$ and an inward unit normal $\vec{n}_0$ to $\partial\Omega$ at y_0. In the z-coordinate system, we rewrite $y = z_0$, $x_0 = z_1$ and have that

$$Q_{42r}^+ \subset \Omega_{42r} \subset Q_{42r} \cap \{z_1 > -84r\delta\} \tag{4.8}$$

and

$$\fint_{Q_{42r}} \left|\beta(\mathbf{a}; Q_{42r})(x)\right|^2 dx \le \delta^2. \tag{4.9}$$

According to the A-type property of Reifenberg domain mentioned in (1.8) and (4.7) by taking $\rho = 42r$ and $x_0 = z_1$, this infers that

$$\fint_{\Omega_{32r}} \left(|\mathbf{f}|^2 + |D\Psi|^2\right) dx \le C\left(\frac{42}{32}\right)^d \fint_{\Omega_{42r}(z_1)} \left(|\mathbf{f}|^2 + |D\Psi|^2\right) dx \le C\left(\frac{42}{32}\right)^d \delta^2 \tag{4.10}$$

and

$$\fint_{\Omega_{32r}} |Du|^2\,dx \le C\left(\frac{42}{32}\right)^d \fint_{\Omega_{42r}(z_1)} |Du|^2\,dx \le C\left(\frac{42}{32}\right)^d. \tag{4.11}$$

On the basis of a suitable scaling and normalization argument and inequalities (4.8)–(4.11), we can check that the hypotheses of Lemma 3.8 are true. Let $T_2 = \max\{2N_2, 2^d\} > 1$. By Lemma 3.8 and a similar argument as in the proof of interior estimate we have

$$\omega\bigl(\bigl\{x \in \Omega_{9r} : \bigl(\mathcal{M}\bigl(|Du|^2\bigr)\bigr)^{\frac{1}{2}} > T_2\bigr\}\bigr) < \varepsilon_3 \omega(\Omega_r),$$

and from $\Omega_r(y) \subset \Omega_{9r}$ it follows that

$$\omega\bigl(\bigl\{x \in \Omega_r(y) : \bigl(\mathcal{M}\bigl(|Du|^2\bigr)\bigr)^{\frac{1}{2}} > T_2\bigr\}\bigr) < C\varepsilon_3 \omega\bigl(\Omega_r(y)\bigr) = \varepsilon\omega\bigl(\Omega_r(y)\bigr) \tag{4.12}$$

for $\varepsilon = C\varepsilon_3$, where the constant C depends only on d, σ, and $[\omega]_s$. This contradicts with (4.1).

Finally, we put estimates (4.6) and (4.12) together and write $T = \max\{T_1, T_2\} > 1$, which completes the proof. □

Proof of Theorem 1.4 We are ready to prove Theorem 1.4 in two steps.

Step 1. First, we prove that

$$\|Du\|_{L^{(p,q)}_\omega(\Omega)} \le C \tag{4.13}$$

under the constraint

$$\frac{1}{\delta}\bigl(\|\mathbf{f}\|_{L^{(p,q)}_\omega(\Omega)} + \|D\Psi\|_{L^{(p,q)}_\omega(\Omega)}\bigr) \le 1, \tag{4.14}$$

where the constant $\delta > 0$ will be specified later.

We first claim that $L^{(p,q)}_\omega(\Omega) \subset L^2(\Omega)$ with $\omega \in A_{p/2}$ for $p > 2$. Indeed, since $\omega \in A_{p/2}$, by using the inverse Hölder inequality to Marcinkiewicz weight function, we may choose small $\sigma > 0$ such that $(p-\sigma)/2 > p/2$ which results in $\omega \in A_{\frac{p-\sigma}{2}}$ and $[\omega]_{\frac{p-\sigma}{2}} \le C[\omega]_{p/2}$; also see Lemma 3.6 in [23]. Then we use embedding inequalities in Proposition 2.3 to obtain

$$\begin{aligned}
\|\mathbf{f}\|_{L^2(\Omega)} &\le \left(\int_\Omega |\mathbf{f}|^{p-\sigma}\omega(x)\,dx\right)^{\frac{1}{p-\sigma}} \left(\int_\Omega \omega^{\frac{-2}{p-\sigma-2}}\,dx\right)^{\frac{p-\sigma-2}{2(p-\sigma)}} \\
&\le \|\mathbf{f}\|_{L^{(p,\infty)}_\omega(\Omega)} \bigl(\omega(\Omega)\bigr)^{\frac{1}{p-\sigma}-\frac{1}{p}} \left(\int_\Omega \omega^{\frac{-2}{p-\sigma-2}}\,dx\right)^{\frac{p-\sigma-2}{2(p-\sigma)}} \\
&\le C[\omega]_{\frac{p-\sigma}{2}}^{\frac{1}{p-\sigma}} \bigl(\omega(\Omega)\bigr)^{-\frac{1}{p}} |\Omega|^{\frac{1}{2}} \|\mathbf{f}\|_{L^{(p,q)}_\omega(\Omega)} \\
&\le C\bigl(\omega(\Omega)\bigr)^{-\frac{1}{p}} |\Omega|^{\frac{1}{2}}\delta,
\end{aligned}$$

where the last inequality comes from (4.14). By a similar argument to this estimate we have

$$\|D\Psi\|_{L^2(\Omega)} \le C\bigl(\omega(\Omega)\bigr)^{-\frac{1}{p}} |\Omega|^{\frac{1}{2}}\delta,$$

Therefore, by the standard L^2-estimate (1.4) it follows that

$$\begin{aligned}\big|C(T)\big| &\le \frac{C}{T^2}\int_\Omega |Du|^2\,dx \le \frac{C}{T^2}\int_\Omega \big(|\mathbf{f}|^2 + |D\Psi|^2\big)\,dx \\ &\le \frac{C|\Omega|}{T^2(\omega(\Omega))^{\frac{2}{p}}}\delta^2,\end{aligned}$$

which implies that, for any $\varepsilon_1 > 0$ there exists small $\delta > 0$ such that

$$\big|C(T)\big| \le \varepsilon_1|Q_1|.$$

According to the relation between the weight measure and Lebesgue measure in Lemma 2.2, we obtain that, for some $\sigma > 0$ and $C > 0$,

$$\omega\big(C(T)\big) \le C\varepsilon_1^\sigma\omega(Q_1) = \varepsilon_2\omega(Q_1) \tag{4.15}$$

with $\varepsilon_2 = C\varepsilon_1^\sigma$. Thanks to Theorem 4.1 and the modified Vitali covering lemma ([8, Lemma 5.4]), we get

$$\omega\big(C(T)\big) \le \gamma_2\varepsilon_2\omega\big(D(1)\big)$$

for some $\gamma_2 = \gamma_2(d,p,[\omega]_{p/2})$. Recalling the definition of sets of $C(\nu)$ and $D(\nu)$, for any $\nu > 0$ we have

$$\begin{aligned}&\omega\big(\big\{x\in\Omega : \big(\mathcal{M}\big(|Du|^2\big)\big)^{\frac{1}{2}} > T\big\}\big) \\ &\quad\le \varepsilon_3\omega(\big\{x\in\Omega : \big(\mathcal{M}\big(|Du|^2\big)\big)^{\frac{1}{2}} > 1\big\} \cup \left\{x\in\Omega : \frac{1}{\delta}\big(\mathcal{M}\big(|\mathbf{f}|^2 + |D\Psi|^2\big)^{\frac{1}{2}} > 1\right\}) \\ &\quad\le \varepsilon_3\big(\omega\big(\big\{x\in\Omega : \big(\mathcal{M}\big(|Du|^2\big)\big)^{\frac{1}{2}} > 1\big\}\big) + \omega\big(\big\{x\in\Omega : \big(\mathcal{M}\big(|\mathbf{f}|^2 + |D\Psi|^2\big)\big)^{\frac{1}{2}} > \delta\big\}\big)\big)\end{aligned}$$

with $\varepsilon_3 = \gamma_2\varepsilon_2$. Note that $(a+b)^\beta \le C(a^\beta + b^\beta)$ for $\beta, a, b > 0$ with $C = \max\{1, 2^{\beta-1}\}$, which yields

$$\begin{aligned}&\omega\big(\big\{x\in\Omega : \big(\mathcal{M}\big(|Du|^2\big)\big)^{\frac{1}{2}} > T\big\}\big)^{\frac{q}{p}} \\ &\quad\le C\varepsilon_3^{q/p}\big(\omega\big(\big\{x\in\Omega : \big(\mathcal{M}\big(|Du|^2\big)\big)^{\frac{1}{2}} > 1\big\}\big)^{\frac{q}{p}} \\ &\qquad + \omega\big(\big\{x\in\Omega : \big(\mathcal{M}\big(|\mathbf{f}|^2 + |D\Psi|^2\big)\big)^{\frac{1}{2}} > \delta\big\}\big)^{\frac{q}{p}}\big).\end{aligned}$$

We iterate this estimate with finite times $k \ge 2$ to find

$$\begin{aligned}&\omega\big(\big\{x\in\Omega : \big(\mathcal{M}\big(|Du|^2\big)\big)^{\frac{1}{2}} > T^k\big\}\big)^{\frac{q}{p}} \\ &\quad\le \varepsilon^k\omega\big(\big\{x\in\Omega : \big(\mathcal{M}\big(|Du|^2\big)\big)^{\frac{1}{2}} > 1\big\}\big)^{\frac{q}{p}} \\ &\qquad + \sum_{i=1}^{k}\varepsilon^i\omega\big(\big\{x\in\Omega : \big(\mathcal{M}\big(|\mathbf{f}|^2 + |D\Psi|^2\big)\big)^{\frac{1}{2}} > \delta T^{k-i}\big\}\big)^{\frac{q}{p}},\end{aligned}$$

where $\varepsilon = C\varepsilon_3^{q/p}$. Putting Lemma 2.4, Theorem 2.5, and Proposition 2.3 together, for $2 < p < \infty$ and $0 < q < \infty$, we obtain that

$$\begin{aligned}
&\big\| \big(\mathcal{M}\big(|Du|^2\big)\big)^{\frac{1}{2}} \big\|^q_{L^{(p,q)}_\omega(\Omega)} \\
&\quad \le C\bigg(\omega(\Omega)^{\frac{q}{p}} + \sum_{k\ge 1} T^{kq}\omega\big(\big\{x\in\Omega : \big(\mathcal{M}\big(|Du|^2\big)\big)^{\frac{1}{2}} > T^k\big\}\big)^{\frac{q}{p}}\bigg) \\
&\quad \le C\omega(\Omega)^{\frac{q}{p}}\bigg(1 + \sum_{k\ge 1}\big(T^q\varepsilon\big)^k\bigg) \\
&\qquad + C\sum_{i\ge 1}\big(T^q\varepsilon\big)^i \sum_{k=i}^{\infty} T^{(k-i)q}\omega\big(\big\{x\in\Omega : \big(\mathcal{M}\big(|\mathbf{f}|^2 + |D\Psi|^2\big)\big)^{\frac{1}{2}} > \delta T^{k-i}\big\}\big)^{\frac{q}{p}} \\
&\quad \le C\omega(\Omega)^{\frac{q}{p}}\bigg(1 + \sum_{k\ge 1}\big(T^q\varepsilon\big)^k\bigg) + C\sum_{i\ge 1}\big(T^q\varepsilon\big)^i \big\| \big(\mathcal{M}\big(|\mathbf{f}|^2 + |D\Psi|^2\big)\big)^{\frac{1}{2}} \big\|^q_{L^{(p,q)}_\omega(\Omega)} \\
&\quad \le C\omega(\Omega)^{\frac{q}{p}}\bigg(1 + \sum_{k\ge 1}\big(T^q\varepsilon\big)^k\bigg) + C\sum_{i\ge 1}\big(T^q\varepsilon\big)^i \big(\|\mathbf{f}\|^q_{L^{(p,q)}_\omega(\Omega)} + \|D\Psi\|^q_{L^{(p,q)}_\omega(\Omega)}\big) \\
&\quad \le C\omega(\Omega)^{\frac{q}{p}}\bigg(1 + \sum_{k\ge 1}\big(T^q\varepsilon\big)^k\bigg) + C\delta^q\sum_{i\ge 1}\big(T^q\varepsilon\big)^i.
\end{aligned}$$

Now taking $\varepsilon > 0$ sufficiently small so that $T^q\varepsilon < 1$, we conclude that

$$\|Du\|^q_{L^{(p,q)}_\omega(\Omega)} \le \big\| \big(\mathcal{M}\big(|Du|^2\big)\big)^{\frac{1}{2}} \big\|^q_{L^{(p,q)}_\omega(\Omega)} \le C. \tag{4.16}$$

On the other hand, we employ a similar argument to the proof procedure of (4.16) and obtain the estimate

$$\|Du\|_{L^{(p,\infty)}_\omega(\Omega)} \le C.$$

Putting the two estimates together, we complete the proof of the gradient estimate (4.13).

Step 2. Define

$$\begin{aligned}
&\widetilde{u} = \frac{\delta u}{\|\mathbf{f}\|_{L^{(p,q)}_\omega(\Omega)} + \|D\Psi\|_{L^{(p,q)}_\omega(\Omega)}}, \\
&\widetilde{\mathbf{f}} = \frac{\delta \mathbf{f}}{\|\mathbf{f}\|_{L^{(p,q)}_\omega(\Omega)} + \|D\Psi\|_{L^{(p,q)}_\omega(\Omega)}} \quad \text{and} \quad \widetilde{\Psi} = \frac{\delta \Psi}{\|\mathbf{f}\|_{L^{(p,q)}_\omega(\Omega)} + \|D\Psi\|_{L^{(p,q)}_\omega(\Omega)}}.
\end{aligned}$$

It is easy to check that

$$\frac{1}{\delta}\big(\|\widetilde{\mathbf{f}}\|_{L^{(p,q)}_\omega(\Omega)} + \|D\widetilde{\Psi}\|_{L^{(p,q)}_\omega(\Omega)}\big) \le 1.$$

Then we obtain

$$\|D\widetilde{u}\|_{L^{(p,q)}_\omega(\Omega)} \le C,$$

which implies

$$\|Du\|_{L_\omega^{(p,q)}(\Omega)} \le C\big(\|\mathbf{f}\|_{L_\omega^{(p,q)}(\Omega)} + \|D\Psi\|_{L_\omega^{(p,q)}(\Omega)}\big) \tag{4.17}$$

for any $\omega \in A_{p/2}$, $2 < p < \infty$, and $0 < q \le \infty$. This completes the proof. □

As a special setting, we take $\omega(x) = 1$ and $\Psi(x) = 0$ a.e. $x \in \Omega$. Theorem 1.4 leads to a global Lorentz estimate for the gradients of weak solution to the following Dirichlet problem (1.1), which is useful in the coming section:

$$\begin{cases} \operatorname{div}(\mathbf{a}(Du,x)) = \operatorname{div}\mathbf{f} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.18}$$

Corollary 4.2 *For $2 < p < \infty$, $0 < q \le \infty$, and $R_0 > 0$, there exists a positive constant $\delta = \delta(d,p,q,\lambda,\Lambda)$ such that $(\mathbf{a}(\xi,x),\Omega)$ satisfies (δ,R_0)-vanishing of codimension one as in Assumption 1.1. If $\mathbf{f} \in L^{(p,q)}(\Omega)$, then, for each weak solution $u \in W_0^{1,2}(\Omega)$ of Dirichlet problem (4.18) with $(H1)$, we have $Du \in L^{(p,q)}(\Omega)$ with the estimate*

$$\|Du\|_{L^{(p,q)}(\Omega)} \le C\|\mathbf{f}\|_{L^{(p,q)}(\Omega)}, \tag{4.19}$$

where the constant C is independent of u and $\mathbf{f}$.

Proof of Corollary 1.5 First, we extend each of weak solution u, the given datum $\mathbf{f}$ and Ψ by a zero extension outside Ω, respectively. Let us now take a special weight function $\omega(x) = (\mathcal{M}(\chi_{B_\rho(y)}))^\sigma$ for $y \in \Omega$, $0 < \rho \le \operatorname{diam}(\Omega)$, and $\sigma \in (\frac{d-\theta}{d}, 1)$. In accordance with the definition of A_s weight, we get that $\omega(x) \in A_1 \subset A_s$ for $1 < s < \infty$. Then, by Theorem 1.4 we have

$$\begin{aligned} \|Du\|^q_{L^{(p,q)}(B_\rho(y)\cap\Omega)} &= p\int_0^\infty \left(\alpha^p \int_{\{x\in B_\rho(y)\cap\Omega:|Du|>\alpha\}} dx\right)^{\frac{q}{p}} \frac{d\alpha}{\alpha} \\ &\le p\int_0^\infty \left(\alpha^p \int_{\{x\in\Omega:|Du|>\alpha\}} \big(\mathcal{M}(\chi_{B_\rho(y)})\big)^\sigma dx\right)^{\frac{q}{p}} \frac{d\alpha}{\alpha} \\ &= p\int_0^\infty \big(\alpha^p \omega\big(\{x\in\Omega : |Du| > \alpha\}\big)\big)^{\frac{q}{p}} \frac{d\alpha}{\alpha} \\ &= \|Du\|^q_{L_\omega^{(p,q)}(\Omega)} \le C\big(\|\mathbf{f}\|^q_{L_\omega^{(p,q)}(\Omega)} + \|D\Psi\|_{L_\omega^{(p,q)}(\Omega)}\big). \end{aligned} \tag{4.20}$$

Thanks to special weight function $\omega(x) = (\mathcal{M}(\chi_{B_\rho(y)}))^\sigma$, it suffices to prove the following inequalities

$$\|\mathbf{f}\|_{L_\omega^{(p,q)}(\Omega)} \le C\rho^{\frac{(d-\theta)}{p}}\|\mathbf{f}\|_{L^{p,q;\theta}(\Omega)} \quad\text{and}\quad \|D\Psi\|_{L_\omega^{(p,q)}(\Omega)} \le C\rho^{\frac{(d-\theta)}{p}}\|D\Psi\|_{L^{p,q;\theta}(\Omega)}. \tag{4.21}$$

In fact, we employ the dyadic decomposition $\mathbb{R}^d = B_{2\rho}(y) \cup (\bigcup_{k=1}^\infty B_{2^{k+1}\rho}(y)\backslash B_{2^k\rho}(y))$ and get

$$\|\mathbf{f}\|^q_{L_\omega^{(p,q)}(\mathbb{R}^d)} = p\int_0^\infty \big(\alpha^p \omega\big(\{x\in\mathbb{R}^d : |\mathbf{f}| > \alpha\}\big)\big)^{\frac{q}{p}} \frac{d\alpha}{\alpha}$$

$$
\begin{aligned}
&= p\int_0^\infty \left(\alpha^p \int_{\{x\in\mathbb{R}^d:|\mathbf{f}|>\alpha\}} \omega(x)\,dx\right)^{\frac{q}{p}} \frac{d\alpha}{\alpha} \\
&= p\int_0^\infty \left(\alpha^p \left(\int_{B_{2\rho}(y)} \chi_{\{|\mathbf{f}|>\alpha\}} \big(\mathcal{M}(\chi_{B_\rho(y)})\big)^\sigma\,dx \right.\right. \\
&\quad \left.\left. + \sum_{k=1}^\infty \int_{B_{2^{k+1}\rho}(y)\setminus B_{2^k\rho}(y)} \chi_{\{|\mathbf{f}|>\alpha\}} \big(\mathcal{M}(\chi_{B_\rho(y)})\big)^\sigma\,dx\right)\right)^{\frac{q}{p}} \frac{d\alpha}{\alpha} \\
&\le C\left(p\int_0^\infty \left(\alpha^p \int_{\{x\in B_{2\rho}(y):|\mathbf{f}|>\alpha\}} \big(\mathcal{M}(\chi_{B_\rho(y)})\big)^\sigma\,dx\right)^{\frac{q}{p}} \frac{d\alpha}{\alpha} \right. \\
&\quad \left. + p\int_0^\infty \left(\alpha^p \sum_{k=1}^\infty \int_{\{x\in B_{2^{k+1}\rho}(y)\setminus B_{2^k\rho}(y)):|\mathbf{f}|>\alpha\}} \big(\mathcal{M}(\chi_{B_\rho(y)})\big)^\sigma\,dx\right)^{\frac{q}{p}} \frac{d\alpha}{\alpha}\right) \\
&:= C(I_0 + I_1),
\end{aligned}
\tag{4.22}
$$

where $C = \max\{2^{\frac{q}{p}-1}, 1\}$. To estimate I_0, by considering $(\mathcal{M}(\chi_{B_\rho(y)}))^\sigma \le 1$ a.e. $x \in \mathbb{R}^d$ we get

$$
\begin{aligned}
I_0 &\le p\int_0^\infty \left(\alpha^p \int_{\{x\in B_{2\rho}(y):|\mathbf{f}|>\alpha\}} dx\right)^{\frac{q}{p}} \frac{d\alpha}{\alpha} \\
&= p\int_0^\infty \left(\alpha^p \big|\{x\in B_{2\rho}(y):|\mathbf{f}|>\alpha\}\big|\right)^{\frac{q}{p}} \frac{d\alpha}{\alpha} \\
&= \|\mathbf{f}\|^q_{L^{(p,q)}(B_{2\rho}(y))} \le \rho^{\frac{(d-\theta)q}{p}} \|\mathbf{f}\|^q_{L^{p,q;\theta}(\Omega)}.
\end{aligned}
$$

To estimate I_1, note that $x \in B_{2^{k+1}\rho}(y)\setminus B_{2^k\rho}(y)$ and $r > 2^{k+1}\rho - \rho > 2^k\rho$. Then $B_r(x)\cap B_\rho(y) \neq \emptyset$ for all $k \ge 1$, and we deduce

$$
\big(|\chi_{B_\rho(y)}|\big)_{B_r(x)} = ⨍_{B_r(x)} |\chi_{B_\rho(y)}|\,dz = \frac{|B_r(x)\cap B_\rho(y)|}{|B_r(x)|} \le \frac{1}{2^{kd}},
$$

which yields

$$
\begin{aligned}
I_1 &\le p\int_0^\infty \left(\alpha^p \sum_{k=1}^\infty \frac{1}{2^{kd\sigma}} \int_{\{x\in B_{2^{k+1}\rho}(y)\setminus B_{2^k\rho}(y)):|\mathbf{f}|>\alpha\}} dx\right)^{\frac{q}{p}} \frac{d\alpha}{\alpha} \\
&\le p\int_0^\infty \left(\alpha^p \sum_{k=1}^\infty \frac{1}{2^{kd\sigma}} \big|\{x\in B_{2^{k+1}\rho}(y):|\mathbf{f}|>\alpha\}\big|\right)^{\frac{q}{p}} \frac{d\alpha}{\alpha}.
\end{aligned}
\tag{4.23}
$$

We further separately consider the cases $q > p$ and $0 < q \le p$.

Case 1. When $q > p$, Minkowski's integral inequality allows us to get the estimate from (4.23):

$$
\begin{aligned}
I_1 &\le \left(p\sum_{k=1}^\infty \frac{1}{2^{kd\sigma}} \left(\int_0^\infty \left(\alpha^p \big|\{x\in B_{2^{k+1}\rho}(y):|\mathbf{f}|>\alpha\}\big|\right)^{\frac{q}{p}} \frac{d\alpha}{\alpha}\right)^{\frac{p}{q}}\right)^{\frac{q}{p}} \\
&\le C\left(\sum_{k=1}^\infty \frac{1}{2^{kd\sigma}} \|\mathbf{f}\|^p_{L^{(p,q)}(B_{2^{k+1}\rho}(y))}\right)^{\frac{q}{p}}
\end{aligned}
$$

$$\le C\left(\sum_{k=1}^{\infty}\frac{1}{2^{kd\sigma}}\left(2^{k+1}\rho\right)^{d-\theta}\|\mathbf{f}\|^{p}_{L^{p,q;\theta}(B_{2^{k+1}\rho}(y))}\right)^{\frac{q}{p}}$$
$$\le C\|\mathbf{f}\|^{q}_{L^{p,q;\theta}(\Omega)}\left(\sum_{k=1}^{\infty}2^{k(d-\theta-d\sigma)}2^{d-\theta}\rho^{d-\theta}\right)^{\frac{q}{p}},$$

where $C=p^{\frac{q}{p}-1}$. Since $\sigma\in(\frac{d-\theta}{d},1)$, we get $d-\theta-d\sigma<0$ and $I_1\le C\rho^{\frac{(d-\theta)q}{p}}\|\mathbf{f}\|^{q}_{L^{p,q;\theta}(\Omega)}$.

Case 2. When $0<q\le p$, we begin with an elementary inequality $(\sum_{k=1}^{\infty}a_k)^{\frac{q}{p}}\le\sum_{k=1}^{\infty}a_k^{\frac{q}{p}}$ for all nonnegative sequences $\{a_i\}$. Putting it into the right-hand side of (4.23), we get

$$I_1\le\sum_{k=1}^{\infty}\left(\frac{1}{2^{kd\sigma}}\right)^{\frac{q}{p}}p\int_0^{\infty}\left(\alpha^p\left|\left\{x\in B_{2^{k+1}\rho}(y):|\mathbf{f}|>\alpha\right\}\right|\right)^{\frac{q}{p}}\frac{d\alpha}{\alpha}$$
$$=\sum_{k=1}^{\infty}\left(\frac{1}{2^{kd\sigma}}\right)^{\frac{q}{p}}\|\mathbf{f}\|^{q}_{L^{(p,q)}(B_{2^{k+1}\rho}(y))}$$
$$\le\sum_{k=1}^{\infty}\left(\frac{1}{2^{kd\sigma}}\right)^{\frac{q}{p}}\left(2^{k+1}\rho\right)^{\frac{(d-\theta)q}{p}}\|\mathbf{f}\|^{q}_{L^{p,q;\theta}(B_{2^{k+1}\rho}(y))}$$
$$=\sum_{k=1}^{\infty}2^{k(d-\theta-d\sigma)\frac{q}{p}}2^{\frac{(d-\theta)q}{p}}\rho^{\frac{(d-\theta)q}{p}}\|\mathbf{f}\|^{q}_{L^{p,q;\theta}(\Omega)}.$$

The remainder is similar to the argument of Case 1, and we have

$$I_1\le C\rho^{\frac{(d-\theta)q}{p}}\|\mathbf{f}\|^{q}_{L^{p,q;\theta}(\Omega)}.$$

Inserting I_0 and I_1 into (4.22), we get (4.21). As in the proof of (4.21), we also obtain

$$\|D\Psi\|_{L^{(p,q)}_{\omega}(\Omega)}\le C\rho^{\frac{(d-\theta)}{p}}\|D\Psi\|_{L^{p,q;\theta}(\Omega)}. \tag{4.24}$$

Finally, putting (4.21) into (4.20), by a finite covering principle on a bounded domain Ω we get

$$\|Du\|_{\mathcal{L}^{p,q;\theta}(\Omega)}\le C\left(\|\mathbf{f}\|_{\mathcal{L}^{p,q;\theta}(\Omega)}+\|D\Psi\|_{\mathcal{L}^{p,q;\theta}(\Omega)}\right), \tag{4.25}$$

which completes the proof. □

Finally, we immediately conclude from estimate (4.25) a higher integrability of the gradient and even Hölder continuity for a weak solution for variational inequalities (1.1) with appropriate high values of p, q, and θ. More precisely, we have the following:

Corollary 4.3 *Under the hypotheses of Corollary 1.5 with $p=q\in(2,\infty)$, let $u\in\mathcal{A}$ be a weak solution to the nonlinear variational inequalities (1.1). Then we have*:

(i) $u\in\mathcal{L}^{\frac{dp}{d-p};\frac{d\theta}{d-p}}(\Omega)\subset\mathcal{L}^{p;\theta+p}(\Omega)$ *if* $p+\theta<d$;
(ii) $u\in\mathcal{L}^{p';\theta'}(\Omega)$ *for any* $p'<\infty$ *and any* $0<\theta'<d$ *if* $p+\theta=d$;
(iii) $u\in C^{0,1-\frac{d-\theta}{p}}(\Omega)$ *if* $p+\theta>d$.

We should mention that global Hölder regularity for weak solution with the same exponent as above was obtained for linear elliptic equations [8] with measurable coefficients in a bounded Reifenberg domain. In contrast, Corollary 4.3 shows a global Hölder regularity for weak solution to nonlinear variational inequalities with measurable nonlinearities over a bounded Reifenberg domain.

5 Proof of Theorem 1.6

In this section, we focus on proving the global Lorentz estimate for the gradient of weak solution to Dirichlet problem (1.13) with controlled growth. Now let us return to hypotheses (H1) and (H2). By taking into account the ellipticity (1.2), the controlled growth (1.15), and $\psi \in L^{(p,q)}(\Omega)$ for $p \geq \frac{2d}{d+2}$ and $q > 0$, a higher integrability for the gradient of weak solution to (1.13) holds by the reverse Hölder inequality from the Gehring–Giaquinta–Modica lemma (cf. [17, Proposition 1.1, Chapter V]). Here, we would like to point out that the higher integrability is global since the Reifenberg flat domain is a A-type domain (see (1.8)) based on Ladyzhenskaya and Ural'tseva's work, which implies that a reverse Hölder inequality automatically holds for the Reifenberg flat domain. In summary, we have the following:

Lemma 5.1 *Let $u \in W_0^{1,2}(\Omega)$ be a weak solution to Dirichlet problem (1.13) in a Reifenberg flat domain with (H1) and (H2). If $\psi \in L^{(p,q)}(\Omega)$ for $p \geq \frac{2d}{d+2}$ and $q > 0$, then there exists an exponent $p_0 > 2$ such that $Du \in L^{p_0}(\Omega)$ and*

$$\|Du\|_{L^{p_0}(\Omega)} \leq C, \tag{5.1}$$

where p_0 depends on $d, p, q, \lambda, \Lambda, \mu, |\Omega|$, whereas the constant C depends only on $\|Du\|_{L^2(\Omega)}$ and $\|\psi\|_{L^{(p,q)}(\Omega)}$.

It is well known that, for $f \in L^p(E)$ with $p > 1$ defined in a bounded domain E of $\mathbb{R}^d$, there exists a vector field $\mathbf{F} \in L^{p^*}(E)$ such that

$$f(x) = \operatorname{div} \mathbf{F}(x) \quad \text{a.e. } x \in E \quad \text{and} \quad \|\mathbf{F}\|_{L^{p^*}(E)} \leq C\|f\|_{L^p(E)}; \tag{5.2}$$

for instance, see [25, Lemma 3.1]. Following a very similar argument to [25, Lemma 3.1], we also obtain that, for a give function in weighted Lorentz space, it is written as the divergence of a suitable weighted Lorentz regular vector field. More precisely, we have the following:

Lemma 5.2 *For $p \in (1, \infty)$ and $q \in (0, \infty]$, let $f \in L^{(p,q)}(E)$ defined in a bounded domain E of $\mathbb{R}^d$. There exists a vector field $\mathbf{F} \in L^{(p^*,q)}(E)$ such that*

$$f(x) = \operatorname{div} \mathbf{F}(x) \quad \text{a.e. } x \in E \text{ with } \|\mathbf{F}\|_{L^{(p^*,q)}(E)} \leq C\|f\|_{L^{(p,q)}(E)},$$

where $C = C(d, p, q, |E|)$.

Proof Indeed, let us extend $f(x)$ as zero outside E and set

$$\mathcal{N}f(x) = \int_E \Gamma(x-y) f(y)\, dy$$

be the Newtonian potential of f with the fundamental solution $\Gamma(x-y)$ of Laplace operator in $\mathbb{R}^d$. By using the sharp gradient regularity in Lorentz spaces (cf. [24, Theorem 13]) it is clear that $\Delta(\mathcal{N}f(x)) = f(x)$ a.e. $x \in E$ and

$$f \in L^{(p,q)}(E) \quad \Longrightarrow \quad D\big(\mathcal{N}f(x)\big) \in L^{(p^*,q)}(E).$$

Denote

$$\mathbf{F}(x) := D\big(\mathcal{N}f(x)\big),$$

which implies that $f(x) = \operatorname{div}(\operatorname{grad}(\mathcal{N}f(x))) = \operatorname{div}\mathbf{F}(x)$ a.e. $x \in E$. The proof is complete. □

Proof of Theorem 1.6 Set

$$f(x) := -B(x, Du) \quad \text{a.e. } x \in \Omega. \tag{5.3}$$

Thanks to a higher gradient integrability from Lemma 5.1, we get

$$|Du| \in L^{p_0}(\Omega) \equiv L^{(p_0,p_0)}(\Omega) \quad \text{for some } p_0 > 2.$$

Let us set $m_0 = \frac{p_0+2}{2}$ and obtain $2 < m_0 < p_0$. It follows from Proposition 2.3 that

$$|Du| \in L^{(p_0,p_0)}(\Omega) \subset L^{(m_0,\frac{d+2}{d}q)}(\Omega)$$

for any $q \in (0,\infty]$, which is equivalent to

$$|Du|^{\frac{d+2}{d}} \in L^{(\frac{dm_0}{d+2},q)}(\Omega). \tag{5.4}$$

Noting that $\psi \in L^{(p,q)}(\Omega)$ in (1.14), it is clear that $f(x) \in L^{(\min\{\frac{dm_0}{d+2},p\},q)}(\Omega)$ as a consequence of (5.3) and (H2). Then, we use Lemma 5.2 to show that there exists a vector field

$$\mathbf{F} \in L^{(\min\{(\frac{dm_0}{d+2})^*,p^*\},q)}(\Omega)$$

with $f(x) = \operatorname{div}\mathbf{F}(x)$ a.e. $x \in \Omega$. Then, the Dirichlet problem (1.13) can be written as

$$\begin{cases} \operatorname{div}\mathbf{a}(x, Du) = \operatorname{div}\mathbf{F}(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.5}$$

By Corollary 4.2 we have

$$|Du| \in L^{(\min\{(\frac{dm_0}{d+2})^*,p^*\},q)}(\Omega). \tag{5.6}$$

If $(\frac{dm_0}{d+2})^* \geq p^*$ or $\frac{dm_0}{d+2} \geq d$, then it immediately completes the proof of Theorem 1.6. Therefore, the remainder essential situation is to consider

$$p_1 := \left(\frac{dm_0}{d+2}\right)^* < p^* \quad \text{and} \quad \frac{dm_0}{d+2} < d. \tag{5.7}$$

A straightforward calculation based on the Sobolev conjugate of $\frac{dm_0}{d+2}$ gives

$$p_1 = \frac{dm_0}{d+2-m_0} > m_0$$

because of $m_0 > 2$, which, as a consequence of (5.6) and (5.7), reads

$$|Du| \in L^{(p_1,q)}(\Omega).$$

Again, we set $m_1 = \frac{p_1+m_0}{2}$ and obtain $m_0 < m_1 < p_1$. By Lemma 2.3 we have

$$|Du| \in L^{(p_1,q)}(\Omega) \subset L^{(m_1,\frac{d+2}{d}q)}(\Omega).$$

To proceed just as before successively, we derive

$$2 < m_0 < m_1 < \cdots < m_k < \cdots \quad \text{for } k = 0,1,2,\ldots.$$

Keeping Lemma 5.2 in mind, we iterate the procedure finitely many times until

$$p_k := \left(\frac{dm_{k-1}}{d+2}\right)^* \geq p^* \quad \text{or} \quad \frac{dm_{k-1}}{d+2} \geq d. \tag{5.8}$$

A calculation implies that there exists a large enough integer k such that (5.8) holds. Indeed, by defining

$$\tau = \frac{d+1-m_0/2}{d+2-m_0} \tag{5.9}$$

we know that $\tau > 1$ due to $m_0 > 2$. Therefore this yields

$$\begin{aligned} m_k &= \frac{p_k + m_{k-1}}{2} = \frac{1}{2}\left(\frac{d}{d+2-m_{k-1}} + 1\right)m_{k-1} \\ &> \frac{1}{2}\left(\frac{d}{d+2-m_0} + 1\right)m_{k-1} = \tau m_{k-1}, \end{aligned}$$

and a successive procedure yields

$$m_k > \tau m_{k-1} > \cdots > \tau^k m_0 > 2\tau^k \quad \text{for all } k = 0,1,2,\ldots, \tag{5.10}$$

which implies

$$m_k \longrightarrow \infty \quad \text{as } k \to \infty$$

because of $\tau > 1$. Finally, by (5.8) we have

$$|Du| \in L^{(p^*,q)}(\Omega)$$

for $p \geq \frac{2d}{d+2}$ and $q \in (0,\infty]$. This completes the proof of Theorem 1.6. □

Acknowledgements
The authors sincerely thank the reviewers for their useful comments to the original manuscript.

Funding
This paper is supported by the National Natural Science Foundation of China (No. 11371050).

Availability of data and materials
Data sharing not applicable to this article.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors contributed equally to the writing of this paper. The authors read and approved the final manuscript.

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Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 12 March 2018 Accepted: 8 July 2018 Published online: 20 July 2018

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