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On positive periodic solutions of second order singular equations

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Abstract

Using the fixed point theorem, we study the existence and multiplicity of positive periodic solutions for the second order differential equations

$$\begin{cases} \ddot{x} + a(t)x = f(x), \\ x(0) = x(T), \quad \dot{x}(0) = \dot{x}(T). \end{cases}$$

For given nonnegative constants $0 < \beta_1 < \beta_2 < \dots < \beta_N$, the function f may be singular at $x = \beta_i$.

Keywords: Periodic solutions; Second order differential equations; Singular; Fixed point theorem

1 Introduction

We say that the differential equation

$$\ddot{x} + a(t)x = \lambda f(t, x) + e(t) \quad (1.1)$$

is singular if the nonlinear term f is singular, which means that a vector-valued function f is only defined on $\mathbb{R} \times \mathbb{R}^N \setminus \Omega$ with $\Omega \subset \mathbb{R}^N$ being nonempty and for any $x_0 \in \Omega$,

$$\lim_{x \rightarrow x_0} f(t, x) = +\infty, \quad \text{uniformly in } t \in \mathbb{R}.$$

During the last two decades, the existence of nontrivial periodic solutions of (1.1) has been studied by many researchers in nonsingular case as well as singular case. Usually, the proof is based on either the method of upper and lower solutions [2, 10, 15], fixed point theorems [6, 7, 17–19], alternative principle of Leray–Schauder [4, 11], or topological degree theory [23, 24]. To our attention, Wang in [22] studied the existence, multiplicity, and nonexistence results for positive solutions of the singular periodic boundary value problem (1.1) in terms of different values of $\lambda \in \mathbb{R}$. But most of these papers are concerned with singularity at $x = x_0$. Later, a similar idea is used to study the singular periodic systems with two parameters,

$$\begin{cases} \ddot{u} + a_1(t)u = \lambda b_1(t)f_1(u, v), \\ \ddot{v} + a_2(t)v = \mu b_2(t)f_2(u, v), \end{cases} \quad (1.2)$$

where $(\lambda, \mu) \in \mathbb{R}_+^2 \setminus \{0, 0\}$. One nice result proved in [20] is that there exist three nonempty subsets of $\mathbb{R}_+^2 \setminus \{0, 0\} : \Gamma, \Delta_1, \Delta_2$ such that $\mathbb{R}_+^2 \setminus \{0, 0\} = \Gamma \cup \Delta_1 \cup \Delta_2$ and (1.2) has at least two positive periodic solutions for $(\lambda, \mu) \in \Delta_1$, one positive periodic solution for $(\lambda, \mu) \in \Gamma$, and no positive periodic solutions for $(\lambda, \mu) \in \Delta_2$. Note that in this paper the word “singularity” is understood in a more general way because we only need that f_1 is singular at the whole line x -axis and f_2 is singular at the whole line y -axis. The proof is based on the following vector version of Krasnosel’skii’s fixed point theorem [14]. In addition, there are many papers concerned with the existence and multiplicity of radial solutions of the elliptic equations with the regular or singular nonlinearities. We refer the readers to [1, 3, 5, 8, 12, 13, 16, 21].

Motivated by these recent developments, in this paper, we investigate the existence and multiplicity of T -periodic solutions of the following problem:

$$\begin{cases} \ddot{x} + a(t)x = f(x), \\ x(0) = x(T), \quad \dot{x}(0) = \dot{x}(T), \end{cases} \quad (1.3)$$

where the function f may be singular at $x = \beta_i$ ($i = 1, \dots, N$). For convenience, we give the following notations:

$$a_* = \min_{t \in [0, T]} a(t), \quad a^* = \max_{t \in [0, T]} a(t),$$

$$f^*(r) = \max\{f(x) : \theta r \leq x \leq r\}, \quad \text{where } r > 0, \theta \in (0, 1).$$

For $\nu = 0$ or $\nu = +\infty$, there exist nonnegative constants f^ν defined as

$$f^\nu = \lim_{x \rightarrow \nu} \frac{f(x)}{x}.$$

The main results can be expressed as follows.

Theorem 1.1 *Assume that the function f satisfies the following conditions:*

(A) *f may be singular at $x = \beta_i$ ($i = 1, \dots, N$) and continuous on*

$$(0, \beta_1), (\beta_1, \beta_2), \dots, (\beta_{N-1}, \beta_N), (\beta_N, +\infty);$$

(B) *For any interval (β_i, β_{i+1}) , $i = 1, \dots, N - 1$,*

$$(i) \quad f(x) - a^* \beta_i \geq 0, \quad \text{for } x \in (\beta_i, \beta_{i+1}),$$

$$(ii) \quad \text{there exists } \bar{R}_i \in (\beta_i, \beta_{i+1}) \text{ such that}$$

$$\max_{x \in [\theta \bar{R}_i + (1-\theta)\beta_i, \bar{R}_i]} f(x) < \frac{\bar{R}_i - \beta_i}{MT} + a_* \beta_i.$$

Then (1.3) has at least $(2N - 2)$ positive periodic solutions.

Theorem 1.2 *Assume that (A) and (B) hold. In addition, the function f satisfies the following condition:*

(C) There exists $\bar{R} > \beta_N$ such that

- (i) $f(x) - a^* \beta_N \geq 0$, for $x \in (\beta_N, \bar{R})$,
- (ii) $\max_{x \in [\theta \bar{R} + (1-\theta)\beta_N, \bar{R}]} f(x) < \frac{\bar{R} - \beta_N}{MT} + a_* \beta_n$.

Then (1.3) has at least $(2N - 1)$ positive periodic solutions.

Theorem 1.3 Assume that (A) and (B) hold. In addition, the function f satisfies the following condition:

- (D) There exists $\bar{r} \in (0, \beta_1)$ such that $f^*(\bar{r}) < \frac{\bar{r}}{MT}$.

Then (1.3) has at least $(2N - 1)$ positive periodic solutions.

Corollary 1.4 Assume that (A) and (B) hold. If $f^0 = 0$, then (1.3) has at least $(2N - 1)$ positive periodic solutions.

Theorem 1.5 Assume that (A), (B), and (D) hold. If $f^0 = +\infty$, then (1.3) has at least $(2N)$ positive periodic solutions.

Theorem 1.6 Assume that (A), (B), (C), and (D) hold. Then (1.3) has at least $(2N)$ positive periodic solutions.

Theorem 1.7 Assume that (A) and (B) hold. In addition, the function f satisfies the following condition:

- (E) There exists $\hat{R} > \frac{\beta_N}{\theta}$ such that

- (i) $f(x) - a^* \beta_N \geq 0$, for $x \in (\beta_N, \hat{R})$,
- (ii) $\max_{x \in [\theta \hat{R}, \hat{R}]} f(x) < \frac{\hat{R}}{MT} + \min \left\{ a_* - \frac{1}{MT}, 0 \right\} \beta_N$.

If $f^\infty = +\infty$, then (1.3) has at least $(2N)$ positive periodic solutions.

Theorem 1.8 Assume that (A), (B), (C), and (D) hold. If $f^0 = +\infty$, then (1.3) has at least $(2N + 1)$ positive periodic solutions.

Theorem 1.9 Assume that (A), (B), (D), and (E) hold. If $f^\infty = +\infty$, then (1.3) has at least $(2N + 1)$ positive periodic solutions.

Theorem 1.10 Assume that (A), (B), (D), and (E) hold. If $f^0 = +\infty$ and $f^\infty = +\infty$, then (1.3) has at least $(2N + 2)$ positive periodic solutions.

2 Preliminary

The function a satisfies the following assumption:

- (H) The function a is continuous, positive, T -periodic and the linear equation $\ddot{x} + a(t)x = 0$ has a positive Green's function $G(t, s)$, i.e.,

$$G(t, s) > 0 \quad \text{for all } (t, s) \in [0, T] \times [0, T],$$

$$m = \min_{0 \leq s, t \leq T} G(t, s) > 0, \quad M = \max_{0 \leq s, t \leq T} G(t, s), \quad \theta = m/M \in (0, 1).$$

From (H), it is clear that a function $x(t)$ is a T -periodic solution of (1.3) if and only if

$$x(t) = \int_0^T G(t,s)f(x(s))ds.$$

Define the operator $A : K \rightarrow E$ by

$$Ax(t) = \int_0^T G(t,s)f(x(s))ds, \quad 0 \leq t \leq T.$$

Let E denote the Banach space $C[0, T]$ with the usual max-norm and define a subcone K by

$$K = \{x(t) \in C[0, T] : x(t) \geq \theta \|x\|\}.$$

For $r > 0$, let

$$K_r := \{x(t) \in K : \|x(t)\| < r\},$$

$$\partial K_r := \{x(t) \in K : \|x(t)\| = r\}.$$

Therefore, the solution of (1.3) is equivalent to the fixed point of the operator A . The discussion is based on the following well-known fixed point theorem.

Lemma 2.1 ([9, Theorem 2.3.4]) *Let E be a Banach space and $K \subset E$ be a cone in E . Assume that Ω_1, Ω_2 are open subsets of E with $0 \in \Omega_1, \overline{\Omega_1} \subset \Omega_2$, and let $T : K \cap (\overline{\Omega_2} \setminus \Omega_1) \rightarrow K$ be a completely continuous operator such that either*

- (i) $\|Tu\| \leq \|u\|, u \in K \cap \partial\Omega_1$ and $\|Tu\| \geq \|u\|, u \in K \cap \partial\Omega_2$; or
- (ii) $\|Tu\| \geq \|u\|, u \in K \cap \partial\Omega_1$ and $\|Tu\| \leq \|u\|, u \in K \cap \partial\Omega_2$.

Then T has a fixed point in $K \cap (\overline{\Omega_2} \setminus \Omega_1)$.

3 Proof of the main results

First, we consider the existence of solutions of (1.3) on $(0, \beta_1)$.

Lemma 3.1 *Assume that (H), (A), and (D) hold. Then (1.3) has at least one positive periodic solution.*

Proof Since $f(\cdot)$ is singular at $x = \beta_1^-$, there exists sufficiently small $\delta < (1 - \theta)\beta_1$ such that $f(x) \geq \rho x$, for $0 < |x - \beta_1| < \delta_1$, where ρ satisfies $\rho m|\Omega_2|\theta \geq 1$, Ω_2 is given as follows. Choose $R \in (\beta_1 - \delta, \beta_1)$. Then, for $x \in \partial K_R$, one has $x(t) \geq \theta \|x\| \geq \theta R$. If $\theta R \geq \beta_1 - \delta$, then $\frac{\beta_1 - \delta}{\theta} \leq R \leq \beta_1$. Further, we have $\delta > (1 - \theta)\beta_1$, which yields a contradiction. Now we only consider the case $\theta R < \beta_1 - \delta$. Let $[0, T] = \Omega_1 \cup \Omega_2$, where

$$\Omega_1 = \{t \in [0, T] : \theta R \leq x(t) < \beta_1 - \delta\},$$

$$\Omega_2 = \{t \in [0, T] : \beta_1 - \delta \leq x(t) \leq R\}.$$

Since $\|x\| = R$ and x is continuous, Ω_2 is nonempty and $|\Omega_2| > 0$. Therefore, we have

$$\begin{aligned}\|Ax\| &= \max_{t \in [0, T]} \int_0^T G(t, s) f(x(s)) \, ds \\ &\geq m \left[\int_{\Omega_1} f(x(s)) \, ds + \int_{\Omega_2} f(x(s)) \, ds \right] \\ &\geq m \int_{\Omega_2} f(x(s)) \, ds \geq \rho m |\Omega_2| \theta \|x\| \geq \|x\|.\end{aligned}$$

For any $x \in \partial K_{\bar{r}}$, we have

$$\|Ax\| = \max_{t \in [0, T]} \int_0^T G(t, s) f(x(s)) \, ds \leq \int_0^T G(t, s) \frac{\bar{r}}{MT} \, ds < \|x\|.$$

From [22], it is clear that $A : \overline{K_R} \setminus K_{\bar{r}} \rightarrow K$, and A is completely continuous on $\overline{K_R} \setminus K_{\bar{r}}$. Therefore, by Lemma 2.1, (1.3) has at least a positive solution $x(t) \in \overline{K_R} \setminus K_{\bar{r}}$. \square

Lemma 3.2 Assume that (H) and (A) hold. If $f^0 = 0$, then (1.3) has at least one positive periodic solution.

Proof Since $f^0 = 0$, there exists sufficiently small $\bar{r} > 0$ such that $f(x) \leq \epsilon x$ for $x \in [0, \bar{r}]$, where $\epsilon MT < 1$. Then, for any $x \in \partial K_{\bar{r}}$, we have

$$\|Ax\| = \max_{t \in [0, T]} \int_0^T G(t, s) f(x(s)) \, ds \leq \epsilon MT \|x\| < \|x\|.$$

The remainder is similar to the proof of Lemma 3.1, so we omit it. \square

Lemma 3.3 Assume that (H), (A), and (D) hold. If $f^0 = +\infty$, then (1.3) has at least two positive periodic solutions.

Remark $f^0 = +\infty$ implies that f may be singular at $x = 0$.

Proof Since $f(\cdot)$ is singular at $x = \beta_1^-$, from the proof of Lemma 3.1, there exists $R \in (\beta_1 - \delta, \beta_1) > \bar{r}$ such that $\|Ax\| > \|x\|$ for $x \in \partial K_R$.

Since $f^0 = +\infty$, there exists sufficiently small $r < \bar{r}$ such that $f(x) \geq \varrho x$ for $x \in [0, r]$, where $\varrho m \theta T > 1$. Then, for any $x \in \partial K_r$, we have

$$\|Ax\| = \max_{t \in [0, T]} \int_0^T G(t, s) f(x(s)) \, ds \geq \varrho m \theta T \|x\| < \|x\|.$$

For any $x \in \partial K_{\bar{r}}$, we have

$$\|Ax\| = \max_{t \in [0, T]} \int_0^T G(t, s) f(x(s)) \, ds \leq \epsilon MT \|x\| < \|x\|.$$

Further, from (A) and the Arzela–Ascoli theorem it follows that now we verify that $A : \overline{K_R} \setminus K_r \rightarrow K$, and A is completely continuous.

Therefore, by Lemma 2.1, (1.3) has at least two positive solutions $x_1(t) \in \overline{K_R} \setminus K_{\tilde{r}}$ and $x_2(t) \in \overline{K_{\tilde{r}}} \setminus K_r$. \square

Second, we give some existence results of solutions of (1.3) on $(\beta_N, +\infty)$.

Lemma 3.4 *Assume that (H), (A), and (C) hold. Then (1.3) has at least one positive periodic solution.*

Remark Since $f(\cdot)$ is singular at $x = \beta_N^+$, there exists sufficiently small $\delta < \frac{1-\theta}{\theta}$ such that $f(x) \geq \rho x$ for $0 < x - \beta_N < \delta$. Choose $r \in (\beta_N, \beta_N + \delta)$. Then, for $x \in \partial K_r$, one has $x(t) \geq \theta \|x\| \geq \theta r$. If $\theta r \geq \beta_N$, then $\frac{\beta_N}{\theta} \leq r \leq \beta_N + \delta$, which yields a contradiction. Therefore $\theta r < \beta_N$, which implies that A is not continuous on $\overline{K_R} \setminus K_r$, where $R > \frac{\beta_N}{\theta}$.

Proof Let $\omega = u - \beta_N$. Then (1.3) is equivalent to the problem

$$\begin{cases} \ddot{\omega} + a(t)\omega = f(\omega + \beta_N) - a(t)\beta_N = F(\omega), \\ \omega(0) = \omega(T), \quad \dot{\omega}(0) = \dot{\omega}(T). \end{cases} \quad (3.1)$$

From (A) and (C), it follows that

- (a₁) $\lim_{\omega \rightarrow 0^+} F(\omega) = +\infty$;
- (a₂) let $\tilde{r} = \bar{R} - \beta_N$, $F(\omega) \geq 0$, for $\omega \in (0, \tilde{r})$;
- (a₃) there exists sufficiently small $\bar{r} < \tilde{r}$ such that

$$F(\omega) > L\|\omega\|, \quad \text{for } \omega \in (0, \bar{r}),$$

where $LmT > 1$.

Define the operator $A : K \rightarrow E$ by

$$Tx(t) = \int_0^T G(t,s) [f(\omega(s) + \beta_N) - a(t)\beta_N] ds, \quad 0 \leq t \leq T.$$

Then, for any $\omega \in \partial K_{\tilde{r}}$, from (ii) of (C), we have

$$\begin{aligned} \max_{\omega \in [\theta\tilde{r}, \tilde{r}]} [f(\omega(s) + \beta_N) - a(t)\beta_N] &\leq \max_{\omega \in [\theta\tilde{r}, \tilde{r}]} [f(\omega(s) + \beta_N) - a_*\beta_N] \\ &= \max_{x \in [\theta\tilde{R} + (1-\theta)\beta_N, \tilde{R}]} f(x) - a_*\beta_N < \frac{\tilde{r}}{MT}. \end{aligned}$$

Further, we have

$$\begin{aligned} \|T\omega\| &= \max_{t \in [0,T]} T\omega(t) = \max_{t \in [0,T]} \int_0^T G(t,s) [f(\omega(s) + \beta_N) - a(t)\beta_N] ds \\ &\leq MT \max_{\omega \in [\theta\tilde{r}, \tilde{r}]} [f(\omega(s) + \beta_N) - a(t)\beta_N] < \tilde{r} = \|\omega\|. \end{aligned}$$

For $\omega \in \partial K_{\tilde{r}}$, we have

$$\begin{aligned} \|T\omega\| &= \max_{t \in [0,T]} T\omega(t) = \max_{t \in [0,T]} \int_0^T G(t,s) [f(\omega(s) + \beta_N) - a(t)\beta_N] ds \\ &\geq mTL\|\omega\| > \|\omega\|. \end{aligned}$$

From (f_5) and the Arzela–Ascoli theorem it follows that T is completely continuous on $\overline{K_{\tilde{r}}}\setminus K_{\tilde{r}}$. Therefore, by Lemma 2.1, (3.1) has at least a positive solution $\omega(t) \in \overline{K_{\tilde{r}}}\setminus K_{\tilde{r}}$. Namely, (1.3) has at least a positive solution $x(t) = \omega(t) + \beta_N$ satisfying $\|x(t)\| < \beta_N + \tilde{r} = \bar{R}$. \square

Corollary 3.5 *Assume that (H) and (A) hold. In addition,*

(f_1) $f(\cdot)$ is continuous on $(\beta_N, +\infty)$ and $f(x) \geq a^\beta_N$ on $(\beta_N, +\infty)$.*

If $f^\infty = 0$, then (1.3) has at least one positive periodic solution.

Proof If (f_1) holds, then $F(\omega) \geq$ and (i) of (C) hold. From (f_1) and $f^\infty = 0$ it follows that $\lim_{\omega \rightarrow +\infty} \frac{F(\omega)}{\omega} = 0$. Then there exists sufficiently large $R > 0$ such that $F(\omega) \leq \epsilon\omega$ for $\omega \geq R$, where $\epsilon MT < 1$. Choose $\tilde{r} > \max\{\frac{R}{\theta}, \bar{r}\}$, where \bar{r} is given in (a_3) . Then, for any $\omega \in \partial K_{\tilde{r}}$, one has $\omega(t) \geq \theta\|\omega\| > R$, and further we have

$$\|T\omega\| = \max_{t \in [0, T]} \int_0^T G(t, s) F(\omega(s)) ds \leq \epsilon MT \|\omega\| < \|\omega\|.$$

Combining Lemma 3.4 and Lemma 2.1, we have that (1.3) has at least one positive periodic solution. \square

Lemma 3.6 *Assume that (H), (A), and (E) hold. If $f^\infty = +\infty$, then (1.3) has at least two positive periodic solutions.*

Proof It is easy to see that (E) implies (C). Then, from Lemma 3.3, we can obtain a solution $x_1(t)$ satisfying $\|x_1(t)\| < \bar{R}$.

For any $x(t) \in \partial K_{\hat{R}}$, we get

$$\begin{aligned} \|Ax\| &= \max_{t \in [0, T]} Ax(t) = \max_{t \in [0, T]} \int_0^T G(t, s) f(x(s)) ds \\ &\leq MT \max_{x \in [\theta \hat{R}, \hat{R}]} f(x(s)) < \hat{R} = \|x\|. \end{aligned}$$

By the definition of $f^\infty = \infty$, there exists $\tilde{R} > \hat{R}$ such that $f(x) \geq \mu x$ for $x \geq \tilde{R}$, where μ satisfies $\mu m T \theta > 1$.

Choose $R = \frac{\tilde{R}}{\theta}$. Then, for $x \in \partial K_R$, one has $x(t) \geq \theta\|x\| \geq \tilde{R}$, and

$$\begin{aligned} \|Ax\| &= \max_{t \in [0, T]} \int_0^T G(t, s) f(x(s)) ds \\ &\geq \mu m T \theta \|x\| > \|x\|. \end{aligned}$$

From (A) and the Arzela–Ascoli theorem it follows that A is completely continuous on $\overline{K_R}\setminus K_{\hat{R}}$. Therefore, by Lemma 2.1, (1.3) has another positive solution $x_2(t) \in \overline{K_R}\setminus K_{\hat{R}}$. \square

Finally, we are studying the existence of solutions of (1.3) on (β_i, β_{i+1}) , $i = 1, 2, \dots, N-1$.

Lemma 3.7 *Assume that (H), (A), and (B) hold. Then (1.3) has at least two positive periodic solutions.*

Proof Let $\omega = u - \beta_i$. Then (1.3) is equivalent to the problem

$$\begin{cases} \ddot{\omega} + a(t)\omega = f(\omega + \beta_i) - a(t)\beta_i = F_i(\omega), \\ \omega(0) = \omega(T), \quad \dot{\omega}(0) = \dot{\omega}(T). \end{cases} \quad (3.2)$$

From (A) and (B), it follows that

(F₁) $F_i(\cdot)$ is nonnegative and continuous on $(0, \beta_{i+1} - \beta_i)$;

(F₂) $F_i(\cdot)$ is singular at $x = 0^+$, $(\beta_{i+1} - \beta_i)^-$;

(F₃) Let $\bar{r}_i = \bar{R}_i - \beta_i \in (0, \beta_{i+1} - \beta_i)$ such that $F_i^*(\bar{r}_i) < \frac{\bar{r}_i}{MT}$.

Then from Lemma 3.3 it follows that (1.3) has at least two positive periodic solutions. \square

Now we give the proof of the main results.

Proof of Theorem 1.1 The number of the interval (β_i, β_{i+1}) is $N - 1$, then we can obtain the result from Lemma 3.7. \square

Proof of Theorem 1.2 From Lemmas 3.4 and 3.7 the result follows. \square

Proof of Theorem 1.3 From Lemmas 3.1 and 3.7 the result follows. \square

Proof of Corollary 1.4 From Lemmas 3.2 and 3.7 the result follows. \square

Proof of Theorem 1.5 From Lemmas 3.3 and 3.7 the result follows. \square

Proof of Theorem 1.6 From Lemmas 3.1, 3.4, and 3.7 the result follows. \square

Proof of Theorem 1.7 . From Lemmas 3.6 and 3.7 the result follows. \square

Proof of Theorem 1.8 From Lemmas 3.3, 3.4, and 3.7 the result follows. \square

Proof of Theorem 1.9 From Lemmas 3.1, 3.6, and 3.7 the result follows. \square

Proof of Theorem 1.10 From Lemmas 3.3, 3.6, and 3.7 the result follows. \square

Now we just give an example to illustrate Theorem 1.10.

Example For convenience, we consider the following periodic boundary value problem:

$$\begin{cases} \ddot{x} + \mu x = f(x), \\ x(0) = x(T), \quad \dot{x}(0) = \dot{x}(T), \end{cases} \quad (3.3)$$

where $a(t) = \mu$ is a constant such that (H) holds, and $f(x)$ is expressed by

$$f(x) = \begin{cases} C_{01}x^{v_{01}} + \frac{C_{02}}{(x-\beta_1)^{v_{02}}}, & x \in (0, \beta_1); \\ \frac{C_{11}}{(x-\beta_1)^{v_{11}}} + \frac{C_{12}}{(x-\beta_2)^{v_{12}}} + \beta_1\mu, & x \in (\beta_1, \beta_2); \\ \frac{C_{21}}{(x-\beta_2)^{v_{21}}} + \frac{C_{22}}{(x-\beta_3)^{v_{22}}} + \beta_2\mu, & x \in (\beta_2, \beta_3); \\ \frac{C_{31}}{(x-\beta_3)^{v_{31}}} + C_{32}x^{v_{32}} + \beta_3\mu, & x \in (\beta_3, +\infty), \end{cases}$$

where

(I) C_{ij} are positive constants; ($i = 0, 1, 2, 3, j = 1, 2$);

(II) $a^* = a_* = \mu$;

(III) $\nu_{01} < 1$, $\nu_{11}, \nu_{21}, \nu_{31} > 0$, $\nu_{32} > 1$, $\nu_{02}, \nu_{12}, \nu_{22}$ are even.

It is obvious that (A) holds. First we show that assumption (B) holds.

(i) $f(x) - a^* \beta_i \geq 0$ obviously holds for $x \in (\beta_i, \beta_{i+1})$,

(ii) $\max_{x \in [\theta \bar{R}_i + (1-\theta)\beta_i, \bar{R}_i]} f(x) < \frac{\bar{R}_i - \beta_i}{MT} + a_* \beta_i$

\Updownarrow

$$\max_{x \in [\theta \bar{R}_i + (1-\theta)\beta_i, \bar{R}_i]} \left[\frac{C_{i1}}{(x - \beta_i)^{\nu_{i1}}} + \frac{C_{i2}}{(x - \beta_{i+1})^{\nu_{i2}}} \right] < \frac{\bar{R}_i - \beta_i}{MT}.$$

If C_{ij} ($i = 1, 2, j = 1, 2$) is sufficiently small, there exists $\bar{R}_i \in (\beta_i, \beta_{i+1})$ such that the above inequality holds.

Second, if C_{0j} ($j = 1, 2$) is sufficiently small, there exists $\bar{r} \in (0, \beta_1)$ such that $f^*(\bar{r}) < \frac{\bar{r}}{MT}$, namely (D) holds.

Third, it is clear that $f(x) - a^* \beta_N \geq 0$ for $x \in (\beta_N, \hat{R})$. In addition, if C_{3j} ($j = 1, 2$) is sufficiently small, there exists $\hat{R} > \frac{\beta_N}{\theta}$ such that

$$\max_{x \in [\theta \hat{R}, \hat{R}]} f(x) < \frac{\bar{R}}{MT} + \min \left\{ a_* - \frac{1}{MT}, 0 \right\} \beta_N,$$

namely (E) holds.

Finally, we can verify that

$$f^0 = \lim_{x \rightarrow 0} \frac{f(x)}{x} = \lim_{x \rightarrow 0} \frac{C_{01}x^{\nu_{01}} + \frac{C_{02}}{(x - \beta_1)^{\nu_{02}}}}{x} = +\infty,$$

$$f^\infty = \lim_{x \rightarrow \infty} \frac{f(x)}{x} = \lim_{x \rightarrow \infty} \frac{\frac{C_{31}}{(x - \beta_3)^{\nu_{31}}} + C_{32}x^{\nu_{32}} + \beta_3\mu}{x} = +\infty.$$

Therefore, by Theorem 1.10, (3.3) has at least $(2N + 2)$ positive periodic solutions.

4 Conclusion

Some sufficient conditions are given to illustrate the existence and multiplicity of positive periodic solutions for the second order differential equations

$$\begin{cases} \ddot{x} + a(t)x = f(x), \\ x(0) = x(T), \quad \dot{x}(0) = \dot{x}(T). \end{cases}$$

For given nonnegative constants $0 < \beta_1 < \beta_2 < \cdots < \beta_N$, the function f may be singular at $x = \beta_i$.

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The authors declare that they have no competing interests.

Authors' contributions

All the authors contributed equally and significantly in writing this article. All the authors read and approved the final manuscript.

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