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Monotone iterative technique for periodic problem involving Riemann–Liouville fractional derivatives in Banach spaces

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Abstract
In this paper, we use a monotone iterative technique in the presence of lower and upper solutions to discuss the existence and uniqueness of periodic solutions for a class of fractional differential equations in an ordered Banach space E. Under some monotonicity conditions and noncompactness measure conditions of nonlinearity, we obtain the existence of extremal solutions and a unique solution between lower and upper solutions.

Keywords: Monotone iterative technique; Lower and upper solutions; Cone; Fractional differential equations; Measure of noncompactness

1 Introduction

The theory of fractional derivatives equations is an important branch of differential equation theory, which has extensive background in physics, chemistry, control of dynamical systems and realistic mathematical model. It has been found that the differential equations involving fractional derivatives in time are more realistic to describe many phenomena in practical cases than those of integer order in time. Hence, the theory and application of fractional derivatives equations has been rapidly developed in recent years. In particular, the existence of solutions to such problems has been extensively studied by many authors. For details, see the monographs of Miller and Ross [1], Kiryakova [2], Podlubny [3], and Kilbas et al. [4] and the papers by Lakshmikantham and Vatsala [5], Agarwal et al. [6], Darwish et al. [7–10]. Some recent contributions to the theory of fractional differential equations can be seen in [11–23].

In [13], the authors studied periodic boundary value problems for fractional differential equations

$$\begin{cases} D_0^\alpha u(t) = f(t, u(t)), \quad t \in (0, T], \\ t^{1-\alpha} u(t)|_{t=0} = t^{1-\alpha} u(t)|_{t=T}, \end{cases} \tag{1.1}$$

where $0 < T < +\infty$, $f \in C([0, T] \times R)$, and D_0^α is the Riemann–Liouville fractional derivative of order $0 < \alpha \le 1$. Through discussing the properties of well-known Mittag-Leffler function, they established a comparison result for problem (1.1) and obtained the existence and uniqueness of solution for (1.1) by using the monotone iterative method.

However, all of the papers mentioned above are in scalar spaces $\mathbb{R}$. To the best of our knowledge, the work on the periodic solution for fractional differential equations in abstract spaces is yet to be initiated. Motivated by the consideration and [11, 13], in this article, we discuss the periodic boundary value problems for fractional differential equations in an ordered Banach space E

$$\begin{cases} D_0^\alpha u(t) = f(t, u(t)), & t \in (0, T], \\ t^{1-\alpha} u(t)|_{t=0} = t^{1-\alpha} u(t)|_{t=T}, \end{cases} \tag{1.2}$$

where $0 < T < +\infty$, $f \in C([0, T] \times E, E)$, and D_0^α is the Riemann–Liouville fractional derivative of order $0 < \alpha \leq 1$. By combining the theory of measure of noncompactness and the method of lower and upper solutions coupled with the monotone iterative technique, we construct two monotone iterative sequences and prove that the sequences monotonically converge to the minimal and maximal periodic solutions of problem (1.2), respectively, under some monotone conditions and noncompactness measure conditions of f. Our results are more general than those in [11, 13]. Because we consider problem (1.2) in a more general Banach space, therefore, it has more extensive application background. Our main results will be given in Sect. 3. Some preliminaries to discuss problem (1.2) are presented in Sect. 2.

Remark 1.1 We call a function $u(t)$ a classical solution of problem (1.2) if

(i) $u(t)$ is continuous on $(0, T]$, $t^{1-\alpha} u(t)$ is continuous on $[0, T]$, and its fractional integral $I^{1-\alpha} u(t)$ is continuously differentiable for $(0, T]$;

(ii) $u(t)$ satisfies problem (1.2).

2 Preliminaries

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this article.

Let E be an ordered Banach space with the norm $\|\cdot\|$ and the partial order $\leq$, whose positive cone $P = \{u \in E \mid u \geq \theta\}$ is normal with normal constant N. Generally, $C([0, T], E)$ denotes the space of all continuous E-value functions on the interval $[0, T]$. Evidently, $C([0, T], E)$ is an ordered Banach space with the norm $\|u\|_C = \max_{t\in[0,T]} \|u(t)\|$ and the partial order $\leq$ deduced by the positive cone $P_C = \{u \in C([0, T], E) \mid u(t) \geq \theta\}$. P_C is also normal with the same normal constant N. Let $C_r([0, T], E) = \{u \in C((0, T], E) \mid t^r u \in C([0, T], E)\}$, then $C_r([0, T], E)$ is also a Banach space when endowed with the norm $\|u\|_r = \max\{t^r \|u(t)\| : t \in [0, T]\}$. It is easy to verify that $C_r([0, T], E) \subset L^1([0, T], E)$ if $r < 1$, where $L^1([0, T], E)$ denotes the Banach space of all E-value Bochner integrable functions defined on $[0, T]$ with the norm $\|u\|_1 = \int_0^1 \|u(t)\|\, dt$.

Definition 2.1 (see [4]) The Riemann–Liouville fractional integral of order $\alpha > 0$ of a function $y : (0, +\infty) \to E$ is given by

$$I_0^\alpha y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s)\, ds$$

provided the right-hand side is pointwise defined on $(0, +\infty)$.

Definition 2.2 (see [4]) The Riemann–Liouville fractional derivative of order $\alpha > 0$ of a function $y : (0,+\infty) \to E$ is given by

$$D_0^\alpha y(t) = \frac{1}{\Gamma(n-s)}\left(\frac{d}{dt}\right)^n \int_0^t \frac{y(s)}{(t-s)^{\alpha-n+1}}\,ds,$$

where $n = [\alpha] + 1$, provided that the right-hand side is pointwise defined on $(0,+\infty)$.

For problem (1.2), we have the following definitions of upper and lower solutions.

Definition 2.3 A function $v \in C_{1-\alpha}([0,T],E)$ is called a lower solution of problem (1.2) if it satisfies

$$\begin{cases} D_0^\alpha v(t) \le f(t,v(t)), & t \in (0,T], \\ t^{1-\alpha}v(t)|_{t=0} = t^{1-\alpha}v(t)|_{t=T}. \end{cases} \tag{2.1}$$

If all the inequalities of (2.1) are inverse, we call it an upper solution of problem (1.2).

Remark 2.1 In what follows, if v and w are lower solution and upper solution of problem (1.2), respectively, we assume that

$$v(t) \le w(t), t \in (0,T]; \qquad t^{1-\alpha}v(t)|_{t=0} \le t^{1-\alpha}w(t)|_{t=0}. \tag{2.2}$$

Let $\mu(\cdot)$ denote the Kuratowski measure of noncompactness of the bounded set. For details of the definition and properties of the measure of noncompactness, see [24]. The following result is important to proving our main results.

Lemma 2.1 *Let $H \subset C_{1-\alpha}([0,T],E)$ be bounded and equicontinuous. Then*

$$\mu(H) = \max_{t\in[0,T]}\bigl\{t^{1-\alpha}\mu\bigl(H(t)\bigr)\bigr\},$$

where $H(t) = \{u(t) \mid u \in H\} \subset E$, $t \in [0,T]$.

Proof By hypotheses, for every $u \in H$ and any $\varepsilon > 0$, there exists $\delta > 0$ such that when $|t_1 - t_2| < \delta$, for any $t_1, t_2 \in [0,T]$, we have

$$\bigl\|t_1^{1-\alpha}u(t_1) - t_2^{1-\alpha}u(t_2)\bigr\| < \varepsilon. \tag{2.3}$$

Let

$$\Delta : 0 = t_0 < t_1 < \cdots < t_n = T$$

be a division of $[0,T]$ such that $\|\Delta\| < \delta$, where $\|\Delta\| = \max\{t_i - t_{i-1}; i = 1,2,\dots,n\}$. Let $B = \bigcup_{i=1}^n t_i^{1-\alpha}H(t_i)$. There is a division $B = \bigcup_{j=1}^m B_j$ such that

$$d(B_j) < \mu(B) + \varepsilon, \quad j = 1,2,\dots,m, \tag{2.4}$$

where $d(B_j)$ denotes the diameter of B_j. Let G be the set of all mappings from $\{1, 2, \dots, n\}$ into $\{1, 2, \dots, m\}$. It is clear that G is a finite set. For any $\beta \in G$, let

$$H_\beta = \{u \in H \mid t_i^{1-\alpha} u(t_i) \in B_{\beta(i)}, i = 1, 2, \dots, n\}.$$

It is clear that $H = \bigcup_{\beta \in G} H_\beta$. For any $u, v \in H_\beta$, and $t \in [0, T]$, we have $t \in [t_{i-1}, t_i]$ for some $i \in \{1, 2, \dots, n\}$, and so, (2.3) and (2.4) imply that

$$\begin{aligned}\|t^{1-\alpha} u(t) - t^{1-\alpha} v(t)\| &\le \|t^{1-\alpha} u(t) - t_i^{1-\alpha} u(t_i)\| + \|t_i^{1-\alpha} u(t_i) - t_i^{1-\alpha} v(t_i)\| \\ &\quad + \|t_i^{1-\alpha} v(t_i) - t^{1-\alpha} v(t)\| \\ &< d(B_{\beta(i)}) + 2\varepsilon \\ &< \mu(B) + 3\varepsilon.\end{aligned}$$

From this and the definition of norm for $C_r([0, T], E)$ it follows that

$$\|u - v\|_{1-\alpha} < \mu(B) + 3\varepsilon;$$

consequently,

$$d(H_\beta) \le \mu(B) + 3\varepsilon, \quad \forall \beta \in G,$$

which implies $\mu(H) \le \mu(B) + 3\varepsilon$. Since ε is arbitrary, we get

$$\begin{aligned}\mu(H) \le \mu(B) &= \max\{\mu(t_i^{1-\alpha} H(t_i)), i = 1, 2, \dots, n\} \\ &= \max\{t_i^{1-\alpha} \mu(H(t_i)), i = 1, 2, \dots, n\} \\ &\le \max_{t \in [0,T]} \{t^{1-\alpha} \mu(H(t))\}.\end{aligned}$$

On the other hand, for any $\varepsilon > 0$, there is a division $H = \bigcup_{l=1}^{k} H_l$ such that

$$d(H_l) < \mu(H) + \varepsilon, \quad l = 1, 2, \dots, k. \tag{2.5}$$

Hence, for $\forall t \in [0, T]$, $\forall x_1, x_2 \in H_l$, $l = 1, 2, \dots, k$, we have

$$\|t^{1-\alpha} x_1(t) - t^{1-\alpha} x_2(t)\| \le \|x_1 - x_2\|_{1-\alpha} < \mu(H) + \varepsilon. \tag{2.6}$$

Since $\{t^{1-\alpha} H(t)\} = \bigcup_{l=1}^{k} t^{1-\alpha} H_l(t)$, together with (2.5) and (2.6) we get

$$d(t^{1-\alpha} H_l(t)) \le \mu(H) + \varepsilon, \quad \forall t \in [0, T],$$

that is,

$$\mu(t^{1-\alpha} H_l(t)) \le \mu(H) + \varepsilon, \quad \forall t \in [0, T],$$

Because ε is arbitrary, we obtain

$$t^{1-\alpha}\mu\big(H_l(t)\big) \le \mu(H), \quad \forall t \in [0,T].$$

Consequently,

$$\max_{t\in[0,T]}\big\{t^{1-\alpha}\mu\big(H_l(t)\big)\big\} \le \mu(H).$$

To sum up, the proof of Lemma 2.1 is complete. □

Lemma 2.2 (see [25]) *Let $B = \{u_n\} \subset C_{1-\alpha}([0,T],E)$ be bounded and countable set. Then $\mu(B(t))$ is Lebesgue integral on $[0,T]$, and*

$$\mu\left(\left\{\int_0^T u_n(t)\,dt \,\middle|\, n\in\mathbb{N}\right\}\right) \le 2\int_0^T \mu\big(B(t)\big)\,dt.$$

Let M be the positive constant. For $h \in C([0,T],E)$, we consider the linear periodic boundary value problems

$$\begin{cases} D_0^\alpha u(t) + Mu(t) = h(t), & t\in(0,T], \\ t^{1-\alpha}u(t)|_{t=0} = t^{1-\alpha}u(t)|_{t=T}. \end{cases} \tag{2.7}$$

By an argument similar to that in [11, Theorem 3.2] or [13, Lemma 1.1], we can obtain the following result.

Lemma 2.3 *The linear periodic boundary value problem (2.7) has a unique solution u given by*

$$\begin{aligned} u(t) = {}& \frac{T^{1-\alpha}\Gamma(\alpha)t^{\alpha-1}E_{\alpha,\alpha}(-Mt^\alpha)}{[1-\Gamma(\alpha)E_{\alpha,\alpha}(-MT^\alpha)]}\int_0^T (T-s)^{\alpha-1}E_{\alpha,\alpha}\big(-M(T-s)^\alpha\big)h(s)\,ds \\ & + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}\big(-M(t-s)^\alpha\big)h(s)\,ds, \end{aligned}$$

where $E_{\alpha,\alpha}(x) = \sum_{k=0}^\infty \frac{x^k}{\Gamma((k+1)\alpha)}$ is the Mittag-Leffler function.

Remark 2.2 The well-known two-parameter Mittag-Leffler function

$$E_{\alpha,\beta}(x) = \sum_{k=0}^\infty \frac{x^k}{\Gamma(\alpha k+\beta)}, \quad x\in\mathbb{R}, \alpha,\beta > 0$$

converges uniformly in $\mathbb{R}$.

Remark 2.3 As showed in [16, Lemma 2.1 and Lemma 2.2], for $0 < \alpha \le 1$, we have

$$0 < E_{\alpha,\alpha}(x) = \sum_{k=0}^\infty \frac{x^k}{\Gamma((k+1)\alpha)} < \frac{1}{\Gamma(\alpha)}, \quad \forall x\in\mathbb{R}. \tag{2.8}$$

Hence, $[1-\Gamma(\alpha)E_{\alpha,\alpha}(-MT^\alpha)] > 0$. If $h \ge \theta$, the solution of (2.7) $u \ge \theta$. This comparison result will play a very important role in this paper.

3 Main results

For $v, w \in C_{1-\alpha}([0,T],E)$, we denote

$$[v,w] = \{u \in C_{1-\alpha}([0,T],E) \mid v(t) \le u(t) \le w(t), t \in (0,T],$$
$$t^{1-\alpha}v(t)|_{t=0} \le t^{1-\alpha}u(t)|_{t=0} \le t^{1-\alpha}w(t)|_{t=0}\}.$$

Our main results are as follows.

Theorem 3.1 *Let E be an ordered Banach space, whose positive cone P is normal, $f : [0,T] \times E \to E$ be continuous. Assume that $v_0, w_0 \in C_{1-\alpha}([0,T],E)$ are lower and upper solutions of (1.2) such that (2.2) holds. If the following conditions are satisfied*:

(H1) *There exists a constant $M > 0$ such that*

$$f(t,x_2) - f(t,x_1) \ge -M(x_2 - x_1)$$

for $\forall t \in [0,T]$ and $v_0 \le x_1 \le x_2 \le w_0$.

(H2) *There exists a constant $K > 0$ with*

$$\frac{2KT^\alpha \Gamma(\alpha)}{\Gamma(2\alpha)}\left(\frac{1}{[1-\Gamma(\alpha)E_{\alpha,\alpha}(-MT^\alpha)]} + 1\right) < 1$$

such that

$$\mu(\{f(t,u_n(t)) + Mu_n(t)\}) \le K\mu(\{u_n(t)\})$$

for $\forall t \in [0,T]$, and a monotonous sequence $\{u_n\} \subset [v_0, w_0]$.

Then problem (1.2) has minimal and maximal solutions between v_0 and w_0, which can be obtained by a monotone iterative procedure starting from v_0 and w_0, respectively.

Proof For any $h \in [v_0, w_0]$, consider the linear periodic boundary value problem

$$\begin{cases} D_0^\alpha u(t) + Mu(t) = f(t,h(t)) + Mh(t), & t \in (0,T], \\ t^{1-\alpha}u(t)|_{t=0} = t^{1-\alpha}u(t)|_{t=T}. \end{cases} \tag{3.1}$$

By Lemma 2.3, we obtain that problem (3.1) has unique solution u, which can be expressed as follows:

$$\begin{aligned} u(t) &= \frac{T^{1-\alpha}\Gamma(\alpha)t^{\alpha-1}E_{\alpha,\alpha}(-Mt^\alpha)}{[1-\Gamma(\alpha)E_{\alpha,\alpha}(-MT^\alpha)]}\int_0^T (T-s)^{\alpha-1}E_{\alpha,\alpha}(-M(T-s)^\alpha)(f(t,h(s)) \\ &\quad + Mh(s))\,ds + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-M(t-s)^\alpha)(f(t,h(s)) + Mh(s))\,ds \\ &:= Ah(t). \end{aligned} \tag{3.2}$$

Firstly, we need to show that the operator $A : [v_0, w_0] \to C_{1-\alpha}([0,T],E)$ is well defined, i.e., for $h \in [v_0, w_0]$, $Ah \in C_{1-\alpha}([0,T],E)$. By (H1), for $h \in [v_0, w_0]$, we have

$$f(t,v_0(t)) + Mv_0(t) \le f(t,h(t)) + Mh(t) \le f(t,w_0(t)) + Mw_0(t).$$

We denote

$$F(h)(t) = f\big(t, h(t)\big) + Mh(t), \quad \forall h \in [v_0, w_0].$$

By the normality of the cone P, there exists $L > 0$ such that $\|F(h)\|_{1-\alpha} \le L$, that is,

$$\big\|f\big(t, h(t)\big) + Mh(t)\big\| \le Lt^{\alpha-1}. \tag{3.3}$$

By (2.8) and (3.3), we have that

$$\begin{aligned}
\big\|t^{1-\alpha}(Ah)(t)\big\| &= \bigg\|\frac{T^{1-\alpha}\Gamma(\alpha)E_{\alpha,\alpha}(-Mt^\alpha)}{[1-\Gamma(\alpha)E_{\alpha,\alpha}(-MT^\alpha)]}\int_0^T (T-s)^{\alpha-1}E_{\alpha,\alpha}\big(-M(T-s)^\alpha\big)F(h)(s)\,ds \\
&\quad + t^{1-\alpha}\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}\big(-M(t-s)^\alpha\big)F(h)(s)\,ds\bigg\| \\
&\le \frac{T^{1-\alpha}}{\Gamma(\alpha)[1-\Gamma(\alpha)E_{\alpha,\alpha}(-MT^\alpha)]}\int_0^T (T-s)^{\alpha-1}\big\|F(h)(s)\big\|\,ds \\
&\quad + \frac{t^{1-\alpha}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\big\|F(h)(s)\big\|\,ds \\
&\le \frac{T^{1-\alpha}L}{\Gamma(\alpha)[1-\Gamma(\alpha)E_{\alpha,\alpha}(-MT^\alpha)]}\int_0^T (T-s)^{\alpha-1}s^{\alpha-1}\,ds \\
&\quad + \frac{t^{1-\alpha}L}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}s^{\alpha-1}\,ds \\
&= \frac{T^\alpha L\Gamma(\alpha)}{\Gamma(2\alpha)[1-\Gamma(\alpha)E_{\alpha,\alpha}(-MT^\alpha)]} + \frac{L\Gamma(\alpha)t^\alpha}{\Gamma(2\alpha)}.
\end{aligned}$$

That is to say, the integral in (3.2) exists and belongs to $C_{1-\alpha}([0,T],E)$.

By Lemma (2.3), the solution of problem (1.2) is equivalent to the fixed point of the operator A. Now, we complete the proof by four steps.

Step 1. We show that the operator $A : [v_0, w_0] \to C_{1-\alpha}([0,T],E)$ is equicontinuous. For any $u \in [v_0, w_0]$ and $0 \le t_1 \le t_2 \le T$, we have

$$\begin{aligned}
&\big\|t_2^{1-\alpha}Au(t_2) - t_1^{1-\alpha}Au(t_1)\big\| \\
&\quad\le \bigg\|\frac{T^{1-\alpha}\Gamma(\alpha)E_{\alpha,\alpha}(-Mt_2^\alpha)}{[1-\Gamma(\alpha)E_{\alpha,\alpha}(-MT^\alpha)]}\int_0^T (T-s)^{\alpha-1}E_{\alpha,\alpha}\big(-M(T-s)^\alpha\big)F(u)(s)\,ds \\
&\qquad + t_2^{1-\alpha}\int_0^t (t_2-s)^{\alpha-1}E_{\alpha,\alpha}\big(-M(t_2-s)^\alpha\big)F(u)(s)\,ds \\
&\qquad - \frac{T^{1-\alpha}\Gamma(\alpha)E_{\alpha,\alpha}(-Mt_1^\alpha)}{[1-\Gamma(\alpha)E_{\alpha,\alpha}(-MT^\alpha)]}\int_0^T (T-s)^{\alpha-1}E_{\alpha,\alpha}\big(-M(T-s)^\alpha\big)F(u)(s)\,ds \\
&\qquad - t_1^{1-\alpha}\int_0^t (t_1-s)^{\alpha-1}E_{\alpha,\alpha}\big(-M(t_1-s)^\alpha\big)F(u)(s)\,ds\bigg\| \\
&\quad\le \frac{T^{1-\alpha}\Gamma(\alpha)|E_{\alpha,\alpha}(-Mt_2^\alpha) - E_{\alpha,\alpha}(-Mt_1^\alpha)|}{[1-\Gamma(\alpha)E_{\alpha,\alpha}(-MT^\alpha)]} \\
&\qquad \times \int_0^T (T-s)^{\alpha-1}E_{\alpha,\alpha}\big(-M(T-s)^\alpha\big)\big\|F(u)(s)\big\|\,ds
\end{aligned}$$

$$\begin{aligned}&+\left\|t_2^{1-\alpha}\int_0^t (t_2-s)^{\alpha-1}E_{\alpha,\alpha}\bigl(-M(t_2-s)^\alpha\bigr)F(u)(s)\,ds\right.\\&\left.-t_1^{1-\alpha}\int_0^t (t_1-s)^{\alpha-1}E_{\alpha,\alpha}\bigl(-M(t_1-s)^\alpha\bigr)F(u)(s)\,ds\right\|.\end{aligned}$$

For the first term of the above formula, by (2.8) and (3.3), we have

$$\begin{aligned}&\frac{T^{1-\alpha}\Gamma(\alpha)|E_{\alpha,\alpha}(-Mt_2^\alpha)-E_{\alpha,\alpha}(-Mt_1^\alpha)|}{[1-\Gamma(\alpha)E_{\alpha,\alpha}(-MT^\alpha)]}\int_0^T (T-s)^{\alpha-1}E_{\alpha,\alpha}\bigl(-M(T-s)^\alpha\bigr)\bigl\|F(u)(s)\bigr\|\,ds\\&\le\frac{T^{1-\alpha}L}{[1-\Gamma(\alpha)E_{\alpha,\alpha}(-MT^\alpha)]}\int_0^T (T-s)^{\alpha-1}s^{\alpha-1}\,ds\bigl|E_{\alpha,\alpha}\bigl(-Mt_2^\alpha\bigr)-E_{\alpha,\alpha}\bigl(-Mt_1^\alpha\bigr)\bigr|\\&=\frac{T^\alpha(\Gamma(\alpha))^2L}{\Gamma(2\alpha)[1-\Gamma(\alpha)E_{\alpha,\alpha}(-MT^\alpha)]}\bigl|E_{\alpha,\alpha}\bigl(-Mt_2^\alpha\bigr)-E_{\alpha,\alpha}\bigl(-Mt_1^\alpha\bigr)\bigr|.\end{aligned}$$

The function $E_{\alpha,\alpha}(-Mt^\alpha)$ is continuous, so the previous expression has limit zero as $|t_2-t_1|\to 0$.

For the rest, by the properties of $E_{\alpha,\alpha}(x)$ discussed in [16, Proposition 1], we have

$$\begin{aligned}&\left\|t_2^{1-\alpha}\int_0^{t_2} (t_2-s)^{\alpha-1}E_{\alpha,\alpha}\bigl(-M(t_2-s)^\alpha\bigr)F(u)(s)\,ds\right.\\&\qquad\left.-t_1^{1-\alpha}\int_0^{t_1} (t_1-s)^{\alpha-1}E_{\alpha,\alpha}\bigl(-M(t_1-s)^\alpha\bigr)F(u)(s)\,ds\right\|\\&\le\left\|\int_0^{t_1}\bigl[t_2^{1-\alpha}(t_2-s)^{\alpha-1}E_{\alpha,\alpha}\bigl(-M(t_2-s)^\alpha\bigr)-t_1^{1-\alpha}(t_1-s)^{\alpha-1}E_{\alpha,\alpha}\bigl(-M(t_1-s)^\alpha\bigr)\bigr]\right.\\&\qquad\left.\times F(u)(s)\,ds\right\|\\&\qquad+\left\|t_2^{1-\alpha}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}E_{\alpha,\alpha}\bigl(-M(t_2-s)^\alpha\bigr)F(u)(s)\,ds\right\|\\&\le L\int_0^{t_1}\bigl[t_1^{1-\alpha}(t_1-s)^{\alpha-1}E_{\alpha,\alpha}\bigl(-M(t_1-s)^\alpha\bigr)-t_2^{1-\alpha}(t_2-s)^{\alpha-1}E_{\alpha,\alpha}\bigl(-M(t_2-s)^\alpha\bigr)\bigr]s^{\alpha-1}\,ds\\&\qquad+L\int_{t_1}^{t_2}t_2^{1-\alpha}(t_2-s)^{\alpha-1}E_{\alpha,\alpha}\bigl(-M(t_2-s)^\alpha\bigr)s^{\alpha-1}\,ds\\&=L\int_0^{t_1}t_1^{1-\alpha}(t_1-s)^{\alpha-1}E_{\alpha,\alpha}\bigl(-M(t_1-s)^\alpha\bigr)s^{\alpha-1}\,ds\\&\qquad-L\int_0^{t_2}t_2^{1-\alpha}(t_2-s)^{\alpha-1}E_{\alpha,\alpha}\bigl(-M(t_2-s)^\alpha\bigr)s^{\alpha-1}\,ds\\&\qquad+2L\int_{t_1}^{t_2}t_2^{1-\alpha}(t_2-s)^{\alpha-1}E_{\alpha,\alpha}\bigl(-M(t_2-s)^\alpha\bigr)s^{\alpha-1}\,ds\\&=L\Gamma(\alpha)\bigl(t_1^{1-\alpha}\bigl(I_0^\alpha t_1^{1-\alpha}E_{\alpha,\alpha}\bigl(-Mt_1^\alpha\bigr)\bigr)-t_2^{1-\alpha}\bigl(I_0^\alpha t_2^{1-\alpha}E_{\alpha,\alpha}\bigl(-Mt_2^\alpha\bigr)\bigr)\bigr)\\&\qquad+2L\int_{t_1}^{t_2}t_2^{1-\alpha}(t_2-s)^{\alpha-1}E_{\alpha,\alpha}\bigl(-M(t_2-s)^\alpha\bigr)s^{\alpha-1}\,ds\\&=L\Gamma(\alpha)\bigl(t_1^\alpha E_{\alpha,2\alpha}\bigl(-Mt_1^\alpha\bigr)-t_2^\alpha E_{\alpha,2\alpha}\bigl(-Mt_2^\alpha\bigr)\bigr)\\&\qquad+2L\int_{t_1}^{t_2}t_2^{1-\alpha}(t_2-s)^{\alpha-1}E_{\alpha,\alpha}\bigl(-M(t_2-s)^\alpha\bigr)s^{\alpha-1}\,ds\end{aligned}$$

$$
\begin{aligned}
&= -\frac{L\Gamma(\alpha)}{M}\big(E_{\alpha,\alpha}\big(-Mt_1^{\alpha}\big) - E_{\alpha,\alpha}\big(-Mt_2^{\alpha}\big)\big) \\
&\quad + 2L\int_{t_1}^{t_2} t_2^{1-\alpha}(t_2-s)^{\alpha-1}E_{\alpha,\alpha}\big(-M(t_2-s)^{\alpha}\big)s^{\alpha-1}\,ds \\
&\le -\frac{L\Gamma(\alpha)}{M}\big(E_{\alpha,\alpha}\big(-Mt_1^{\alpha}\big) - E_{\alpha,\alpha}\big(-Mt_2^{\alpha}\big)\big) + \frac{2Lt_2^{1-\alpha}t_1^{\alpha-1}}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}\,ds \\
&= -\frac{L\Gamma(\alpha)}{M}\big(E_{\alpha,\alpha}\big(-Mt_1^{\alpha}\big) - E_{\alpha,\alpha}\big(-Mt_2^{\alpha}\big)\big) + \frac{2Lt_2^{1-\alpha}t_1^{\alpha-1}}{\Gamma(\alpha)}(t_2-t_1)^{\alpha}.
\end{aligned}
$$

Obviously, the previous expression also tends to zero for $|t_2 - t_1| \to 0$. That is to say, $A : [v_0, w_0] \to C_{1-\alpha}([0,T],E)$ is equicontinuous.

Step 2. We show that $v_0 \le Av_0, Aw_0 \le w_o$, and $Au_1 \le Au_2$ for any $u_1, u_2 \in [v_0, w_0]$ with $u_1 \le u_2$.

Let

$$
\sigma(t) := D^{\alpha}v_0(t) + Mv_0(t),
$$

then, by Definition 2.3, we have $\sigma(t) \le F(v_0)(t)$. Hence,

$$
\begin{aligned}
v_0(t) &= \frac{T^{1-\alpha}\Gamma(\alpha)t^{\alpha-1}E_{\alpha,\alpha}(-Mt^{\alpha})}{[1-\Gamma(\alpha)E_{\alpha,\alpha}(-MT^{\alpha})]}\int_0^T (T-s)^{\alpha-1}E_{\alpha,\alpha}\big(-M(T-s)^{\alpha}\big)\sigma(s)\,ds \\
&\quad + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}\big(-M(t-s)^{\alpha}\big)\sigma(s)\,ds \\
&\le \frac{T^{1-\alpha}\Gamma(\alpha)t^{\alpha-1}E_{\alpha,\alpha}(-Mt^{\alpha})}{[1-\Gamma(\alpha)E_{\alpha,\alpha}(-MT^{\alpha})]}\int_0^T (T-s)^{\alpha-1}E_{\alpha,\alpha}\big(-M(T-s)^{\alpha}\big)F(v_0)(s)\,ds \\
&\quad + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}\big(-M(t-s)^{\alpha}\big)F(v_0)(s)\,ds \\
&= Av_0(t),
\end{aligned}
$$

namely $v_0 \le Av_0$. Similarly, we can show that $Aw_0 \le w_0$. For any $u_1, u_2 \in [v_0, w_0]$ with $u_1 \le u_2$, by assumption (H1),

$$
f\big(t, u_1(t)\big) + Mu_1(t) \le f\big(t, u_2(t)\big) + Mu_2(t),
$$

which implies that $Au_1 \le Au_2$.

Step 3. From Step 2 we know that A maps $[v_0, w_0]$ into itself, and $A : [v_0, w_0] \to [v_0, w_0]$ is a continuously increasing operator. We can now define the sequences

$$
v_n = Av_{n-1}, \qquad w_n = Aw_{n-1}, \quad n = 1, 2, \ldots. \tag{3.4}
$$

Then from the monotonicity of A it follows that

$$
v_0 \le v_1 \le v_2 \le \cdots \le v_n \le \cdots \le w_n \le \cdots \le w_2 \le w_1 \le w_0. \tag{3.5}
$$

Obviously, $\{v_n\}, \{w_n\} \subset [v_0, w_0]$ are equicontinuous. Next, we show that $\{v_n\}$ and $\{w_n\}$ are convergent in $C_{1-\alpha}([0,T],E)$.

From (H2), Lemma 2.1, and Lemma 2.2, for any $t \in [0, T]$, we have that

$$\begin{aligned}
&t^{1-\alpha}\mu\{v_n(t)\} \\
&= t^{1-\alpha}\mu\{Av_{n-1}(t)\} \\
&= t^{1-\alpha}\mu\bigg(\bigg\{\frac{T^{1-\alpha}\Gamma(\alpha)t^{\alpha-1}E_{\alpha,\alpha}(-Mt^\alpha)}{[1-\Gamma(\alpha)E_{\alpha,\alpha}(-MT^\alpha)]}\int_0^T (T-s)^{\alpha-1}E_{\alpha,\alpha}\big(-M(T-s)^\alpha\big)F(v_{n-1})(s)\,ds \\
&\quad + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}\big(-M(t-s)^\alpha\big)F(v_{n-1})(s)\,ds\bigg\}\bigg) \\
&\le \mu\bigg(\bigg\{\frac{T^{1-\alpha}\Gamma(\alpha)E_{\alpha,\alpha}(-Mt^\alpha)}{[1-\Gamma(\alpha)E_{\alpha,\alpha}(-MT^\alpha)]}\int_0^T (T-s)^{\alpha-1}E_{\alpha,\alpha}\big(-M(T-s)^\alpha\big)F(v_{n-1})(s)\,ds\bigg\}\bigg) \\
&\quad + t^{1-\alpha}\mu\bigg(\bigg\{\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}\big(-M(t-s)^\alpha\big)F(v_{n-1})(s)\,ds\bigg\}\bigg) \\
&\le \frac{2T^{1-\alpha}\Gamma(\alpha)E_{\alpha,\alpha}(-Mt^\alpha)}{[1-\Gamma(\alpha)E_{\alpha,\alpha}(-MT^\alpha)]}\int_0^T (T-s)^{\alpha-1}E_{\alpha,\alpha}\big(-M(T-s)^\alpha\big)\mu\big(\big\{F(v_{n-1})(s)\big\}\big)\,ds \\
&\quad + 2t^{1-\alpha}\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}\big(-M(t-s)^\alpha\big)\mu\big(\big\{F(v_{n-1})(s)\big\}\big)\,ds \\
&\le \frac{2KT^{1-\alpha}E_{\alpha,\alpha}(-Mt^\alpha)}{[1-\Gamma(\alpha)E_{\alpha,\alpha}(-MT^\alpha)]}\int_0^T (T-s)^{\alpha-1}\mu\big(\big\{v_{n-1}(s)\big\}\big)\,ds \\
&\quad + \frac{2Kt^{1-\alpha}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\mu\big(\big\{v_{n-1}(s)\big\}\big)\,ds \\
&\le \frac{2KT^{1-\alpha}\Gamma(\alpha)}{[1-\Gamma(\alpha)E_{\alpha,\alpha}(-MT^\alpha)]}\int_0^T (T-s)^{\alpha-1}s^{\alpha-1}\,ds\mu\big(\{v_{n-1}\}\big) \\
&\quad + \frac{2Kt^{1-\alpha}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}s^{\alpha-1}\,ds\mu\big(\{v_{n-1}\}\big) \\
&= \frac{2KT^\alpha\Gamma(\alpha)}{\Gamma(2\alpha)}\bigg(\frac{1}{[1-\Gamma(\alpha)E_{\alpha,\alpha}(-MT^\alpha)]}+1\bigg)\mu\big(\{v_n\}\big).
\end{aligned}$$

Since $\{v_n\}$ is equicontinuous, using Lemma (2.1), we have

$$\mu\big(\{v_n\}\big) = \max_{t\in[0,T]}\big\{t^{1-\alpha}\mu\big(\big\{v_n(t)\big\}\big)\big\} \le \frac{2KT^\alpha\Gamma(\alpha)}{\Gamma(2\alpha)}\bigg(\frac{1}{[1-\Gamma(\alpha)E_{\alpha,\alpha}(-MT^\alpha)]}+1\bigg)\mu\big(\{v_n\}\big).$$

While

$$\frac{2KT^\alpha\Gamma(\alpha)}{\Gamma(2\alpha)}\bigg(\frac{1}{[1-\Gamma(\alpha)E_{\alpha,\alpha}(-MT^\alpha)]}+1\bigg) < 1,$$

hence $\mu(\{v_n\}) = 0$. So, $\{v_n\}$ are relatively compact in $C_{1-\alpha}([0,T],E)$. Hence, $\{v_n\}$ has a convergent subsequence in $C_{1-\alpha}([0,T],E)$. Combining this with the monotonicity (3.5), we easily prove that $\{v_n\}$ itself is convergent in $C_{1-\alpha}([0,T],E)$.

Using a similar argument to that for $\{w_n\}$, we can prove that $\{w_n\}$ is also convergent in $C_{1-\alpha}([0,T],E)$. Then there are $\underline{u}, \bar{u} \in C_{1-\alpha}([0,T],E)$ such that

$$\underline{u} = \lim_{n\to\infty} v_n, \qquad \bar{u} = \lim_{n\to\infty} w_n.$$

Letting $n \to \infty$ in (3.4), we see that

$$\underline{u} = A\underline{u}, \qquad \bar{u} = A\bar{u}.$$

Therefore, $\underline{u}, \bar{u} \in C_{1-\alpha}([0,T],E)$ are fixed points of A.

Step 4. We prove the minimal and maximal property of $\underline{u}$, $\bar{u}$. Assume that $\tilde{u}$ is a fixed point of A in $[v_0, w_0]$, then we have

$$v_0(t) \le \tilde{u}(t) \le w_0(t), \quad t \in (0,T],$$
$$t^{1-\alpha} v_0(t)|_{t=0} \le t^{1-\alpha}\tilde{u}(t)|_{t=0} \le t^{1-\alpha} w_0(t)|_{t=0}.$$

By the monotonicity of A, it is easy to see that

$$v_1(t) = (Av_0)(t) \le (A\tilde{u})(t) = \tilde{u}(t) \le (Aw_0)(t) = w_1(t), \quad t \in (0,T],$$
$$t^{1-\alpha} v_1(t)|_{t=0} \le t^{1-\alpha}\tilde{u}(t)|_{t=0} \le t^{1-\alpha} w_1(t)|_{t=0}.$$

Furthermore, we have

$$v_n \le \tilde{u} \le w_n, \quad n = 1, 2, \ldots. \tag{3.6}$$

Letting $n \to \infty$ in (3.6), we obtain $\underline{u} \le \tilde{u} \le \bar{u}$. So $\underline{u}$, $\bar{u}$ are the minimal and maximal fixed points of A in $[v_0, w_0]$, and therefore, they are the minimal and maximal solutions of problem (1.2) in $[v_0, w_0]$, respectively.

This completes the proof of Theorem 3.1. □

Remark 3.1 When $E = \mathbb{R}$, we do not need condition (H3), $\{v_n\}$ and $\{w_n\}$ defined in (3.4) are convergent in $C_{1-\alpha}([0,T],\mathbb{R})$ automatically. Therefore, Theorem 3.1 improves the main results in [13].

Next, we discuss the uniqueness of the solution to problem (1.2) in $[v_o, w_o]$.

Theorem 3.2 *Let E be an ordered Banach space, whose positive cone P is normal, $f : [0,T] \times E \to E$ be continuous. Assume that $v_0, w_0 \in C_{1-\alpha}([0,T],E)$ are lower and upper solutions of* (1.2) *such that* (2.2) *holds. If conditions* (H1), (H2) *and the following condition are satisfied*:

(H3) *There exists a constant $C > 0$ with*

$$\frac{N(M+C)T^\alpha \Gamma(\alpha)}{\Gamma(2\alpha)}\left(\frac{1}{[1-\Gamma(\alpha)E_{\alpha,\alpha}(-MT^\alpha)]} + 1\right) < 1$$

such that

$$f(t,x_2) - f(t,x_1) \le C(x_2 - x_1),$$

for $\forall t \in [0,T]$, and $v_0 \le x_1 \le x_2 \le w_0$, where N is a normal constant.

Then problem (1.2) *has a unique solution between v_0 and w_0, which can be obtained by a monotone iterative procedure starting from v_0 or w_0.*

Proof From the proof of Theorem 3.1, we know that the iterative sequences $\{v_n\}$ and $\{w_n\}$ defined by (3.4) satisfy (3.5). Now, we show that there exists a unique $u^\star \in C_{1-\alpha}([0,T],E)$ such that $u^\star = Au^\star$. For $t \in [0,T]$, by (H3), we have

$$\begin{aligned}
&t^{1-\alpha}\bigl(w_n(t)-v_n(t)\bigr)\\
&\quad= t^{1-\alpha}\bigl(Aw_n(t)-Av_n(t)\bigr)\\
&\quad= \frac{T^{1-\alpha}\Gamma(\alpha)E_{\alpha,\alpha}(-Mt^\alpha)}{[1-\Gamma(\alpha)E_{\alpha,\alpha}(-MT^\alpha)]}\int_0^T (T-s)^{\alpha-1}E_{\alpha,\alpha}\bigl(-M(T-s)^\alpha\bigr)F(w_{n-1})(s)\,ds\\
&\qquad+ t^{1-\alpha}\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}\bigl(-M(t-s)^\alpha\bigr)F(w_{n-1})(s)\,ds\\
&\qquad- \frac{T^{1-\alpha}\Gamma(\alpha)E_{\alpha,\alpha}(-Mt^\alpha)}{[1-\Gamma(\alpha)E_{\alpha,\alpha}(-MT^\alpha)]}\int_0^T (T-s)^{\alpha-1}E_{\alpha,\alpha}\bigl(-M(T-s)^\alpha\bigr)F(v_{n-1})(s)\,ds\\
&\qquad- t^{1-\alpha}\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}\bigl(-M(t-s)^\alpha\bigr)F(v_{n-1})(s)\,ds\\
&\quad= \frac{T^{1-\alpha}\Gamma(\alpha)E_{\alpha,\alpha}(-Mt^\alpha)}{[1-\Gamma(\alpha)E_{\alpha,\alpha}(-MT^\alpha)]}\\
&\qquad\times \int_0^T (T-s)^{\alpha-1}E_{\alpha,\alpha}\bigl(-M(T-s)^\alpha\bigr)\bigl(F(w_{n-1})(s)-F(v_{n-1})(s)\bigr)\,ds\\
&\qquad+ t^{1-\alpha}\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}\bigl(-M(t-s)^\alpha\bigr)\bigl(F(w_{n-1})(s)-F(v_{n-1})(s)\bigr)\,ds\\
&\quad\le \frac{(M+C)T^{1-\alpha}\Gamma(\alpha)E_{\alpha,\alpha}(-Mt^\alpha)}{[1-\Gamma(\alpha)E_{\alpha,\alpha}(-MT^\alpha)]}\\
&\qquad\times \int_0^T (T-s)^{\alpha-1}E_{\alpha,\alpha}\bigl(-M(T-s)^\alpha\bigr)\bigl(w_{n-1}(s)-v_{n-1}(s)\bigr)\,ds\\
&\qquad+ (M+C)t^{1-\alpha}\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}\bigl(-M(t-s)^\alpha\bigr)\bigl(w_{n-1}(s)-v_{n-1}(s)\bigr)\,ds\\
&\quad\le \frac{(M+C)T^{1-\alpha}}{\Gamma(\alpha)[1-\Gamma(\alpha)E_{\alpha,\alpha}(-MT^\alpha)]}\int_0^T (T-s)^{\alpha-1}s^{\alpha-1}\bigl[s^{1-\alpha}\bigl(w_{n-1}(s)-v_{n-1}(s)\bigr)\bigr]\,ds\\
&\qquad+ \frac{(M+C)t^{1-\alpha}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}s^{\alpha-1}\bigl[s^{1-\alpha}\bigl(w_{n-1}(s)-v_{n-1}(s)\bigr)\bigr]\,ds.
\end{aligned}$$

From the normality of the cone P, it follows that

$$\begin{aligned}
&\bigl\|t^{1-\alpha}\bigl(w_n(t)-v_n(t)\bigr)\bigr\|\\
&\quad\le \frac{N(M+C)T^{1-\alpha}}{\Gamma(\alpha)[1-\Gamma(\alpha)E_{\alpha,\alpha}(-MT^\alpha)]}\int_0^T (T-s)^{\alpha-1}s^{\alpha-1}\bigl\|s^{1-\alpha}\bigl(w_{n-1}(s)-v_{n-1}(s)\bigr)\bigr\|\,ds\\
&\qquad+ \frac{N(M+C)t^{1-\alpha}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}s^{\alpha-1}\bigl\|s^{1-\alpha}\bigl(w_{n-1}(s)-v_{n-1}(s)\bigr)\bigr\|\,ds\\
&\quad\le \frac{N(M+C)T^{1-\alpha}}{\Gamma(\alpha)[1-\Gamma(\alpha)E_{\alpha,\alpha}(-MT^\alpha)]}\int_0^T (T-s)^{\alpha-1}s^{\alpha-1}\,ds\|w_{n-1}-v_{n-1}\|_{1-\alpha}\\
&\qquad+ \frac{N(M+C)t^{1-\alpha}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}s^{\alpha-1}\,ds\|w_{n-1}-v_{n-1}\|_{1-\alpha}\\
&\quad= \frac{N(M+C)T^{\alpha}\Gamma(\alpha)}{\Gamma(2\alpha)[1-\Gamma(\alpha)E_{\alpha,\alpha}(-MT^\alpha)]}\|w_{n-1}-v_{n-1}\|_{1-\alpha}
\end{aligned}$$

$$+\frac{N(M+C)t^{\alpha}\Gamma(\alpha)}{\Gamma(2\alpha)}\|w_{n-1}-v_{n-1}\|_{1-\alpha}$$
$$\le \frac{N(M+C)T^{\alpha}\Gamma(\alpha)}{\Gamma(2\alpha)}\left(\frac{1}{[1-\Gamma(\alpha)E_{\alpha,\alpha}(-MT^{\alpha})]}+1\right)\|w_{n-1}-v_{n-1}\|_{1-\alpha}.$$

Therefore,

$$\|w_n-v_n\|_{1-\alpha}\le \frac{N(M+C)T^{\alpha}\Gamma(\alpha)}{\Gamma(2\alpha)}\left(\frac{1}{[1-\Gamma(\alpha)E_{\alpha,\alpha}(-MT^{\alpha})]}+1\right)\|w_{n-1}-v_{n-1}\|_{1-\alpha}.$$

Again using the above inequality, we get

$$\|w_n-v_n\|_{1-\alpha}\le \left[\frac{N(M+C)T^{\alpha}\Gamma(\alpha)}{\Gamma(2\alpha)}\left(\frac{1}{[1-\Gamma(\alpha)E_{\alpha,\alpha}(-MT^{\alpha})]}+1\right)\right]^n\|w_0-v_0\|_{1-\alpha},$$

which implies that for $n\to\infty$ we have $\|w_n-v_n\|_{1-\alpha}\to 0$. Then there exists a unique $u^{\star}\in C_{1-\alpha}([0,T],E)$ such that

$$\lim_{n\to\infty} w_n=\lim_{n\to\infty} v_n=u^{\star}.$$

So let $n\to\infty$ in (3.4), we have $u^{\star}=Au^{\star}$, which means that $u^{\star}$ is a unique solution of problem (1.2).

This completes the proof of Theorem 3.2. □

4 Conclusion

By using the method of lower and upper solutions coupled with the monotone iterative technique, combining the theory of measure of noncompactness, we present some monotone conditions and noncompactness measure conditions of f such that problem (1.2) has minimal and maximal periodic solutions. In addition, we investigate the uniqueness of the solution for this problem. Our results are more general than those in [11, 13], because we consider problem (1.2) in a more general Banach space, it has more extensive application background. Our main results improve the main results in [11, 13].

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