Boundary Value Problems
a SpringerOpen Journal

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Well-posedness of a ratio-dependent Lotka–Volterra system with feedback control

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Abstract

The main purpose of this article is to consider a Lotka–Volterra predator–prey system with ratio-dependent functional responses and feedback controls. By using a comparison theorem and constructing a suitable Lyapunov function as well as developing some new analysis techniques, we establish a set of easily verifiable sufficient conditions which guarantee the permanence of the system and the global attractivity of a positive solution for the predator–prey system. Furthermore, some conditions for the existence, uniqueness, and stability of a positive periodic solution for the corresponding periodic system are obtained by using the fixed point theory and some new analysis method. In additional, some numerical solutions of the equations describing the system are given to verify that the obtained criteria are new, general, and easily verifiable.

Keywords: Predator-prey system; Feedback control; Permanence; Attractivity; Periodic solution

1 Introduction

After the pioneering works of Lotka and Volterra, a lot of work has been carried out on the predator–prey model. The most crucial element in these models is the "functional response"—the expression that describes the rate at which the number of prey is consumed by a predator. Modifications were limited to replacing the Malthusian growth function, the predator per capita consumption of prey functions such as Holling type I, II, III functional responses, or density-dependent mortality rates. These functional responses depend only on the prey volume, but soon it became clear that the predator volume can influence this function by direct interference while searching or by pseudo interference [1–3]. A simple way of incorporating predator dependence in the functional response was proposed by Arditi and Ginzburg [4], who first considered this response function as a function of the ratio. Moreover, Jost et al. [5] showed that prey-dependent and ratio-dependent models can fit well with the time series generated by each other. Interestingly, it has been investigated that the ratio-dependent predator–prey models are more appropriate for predator–prey interactions when the predator involves serious hunting processes, like animals searching for animals, etc. [6–8]. It is justified through some basic but different principles that ratio-dependent models are more appropriate for modeling predator–prey interactions [9]. Kesh et al. [10] consider a food web model consisting of two competing prey and one predator population with predator interference. From the

knowledge of saturated equilibria and by constructing the average Lyapunov function, the sufficient condition for permanent coexistence and extinction of species of the system is obtained. Pang and Wang [11] study a class of two-predator-one-prey ecosystem and show that the unique positive equilibrium solution of the system is globally asymptotically stable. Baek et al. [12] study a $x_1(t)$ elliptic system with ratio-dependent functional responses. By employing a comparison argument for the elliptic problem and the fixed-point theory applied to a positive cone on a Banach space, authors examine the positive coexistence of one prey and two competing predators in an interacting system with ratio-dependent functional responses under a hostile environment. Furthermore, Ko and Ahn [13, 14] investigate the stability at all non-negative equilibria and long time behavior of solutions for a ratio-dependent reaction-diffusion system. For more similar work, we refer the reader to [15–21].

On the other hand, one can find that an ecosystem in the real world is continuously distributed by some forces, which can result in changes in the biological parameters such as survival rates. Of practical interest in ecology is the question of whether or not an ecosystem can withstand those disturbances which persist for a finite period of time. In the language of control variables, we refer to the disturbance functions as control variables. This is of significance in the control of ecology balance. One of the methods for the realization of it is to alter the system structurally by introducing feedback control variables. The feedback control mechanism might be implemented by means of some biological control schemes or by harvesting procedure. In fact, during the last decade, the qualitative behavior of the population dynamics with feedback control has been studied extensively. Yin and Li [22] propose a single species model with feedback regulation and distributed time delay. By using the continuation theorem of coincidence degree theory, a criterion which guarantees the existence of a positive periodic solution of the system is obtained. Furthermore, Chen [23] obtains a sufficient condition which guarantees the global attractivity of the positive solution of the system by constructing a suitable Lyapunov functional. Nie et al. [24] consider a non-autonomous predator–prey Lotka–Volterra system with feedback controls. They study whether or not the feedback controls have an influence on the permanence of a positive solution of the general non-autonomous predator–prey Lotka–Volterra type systems, and establish the general criteria on the permanence of the system, which is independent of some feedback controls. In additional, by constructing a suitable Lyapunov function, some sufficient conditions are obtained for the global stability of any positive solution to the system. More work on feedback controls can be found in [25–30].

Commonly, an ecological system, such as that represented by the deterministic Lotka–Volterra model, is not suitable to describe the real behavior of the population dynamics. What we claim as "disturbance functions as control variables", which is mentioned above, is strictly connected to the environmental noise effect. It is necessary to include the effect of environmental variables that can be deterministic, such as the variation of the temperature due to atmospheric conditions, and stochastic, due to the stochastic variability of all the other variables, such as growth rate, resources, etc. [31–36]. Moreover, the study of nonlinear dynamical systems in the presence of external noise has led to the discovery of a number of counterintuitive phenomena, with a constructive role of the noise and high fundamental and practical interests in many scientific areas. The presence of a noise source can change the stability of the ecological system [37]. Ghergu and Radulescu [38] study the existence and non-existence of classical solutions to a general Gierer- Meinhardt

system in which both the activator and the inhibitor have different sources given by general nonlinearities, and regularity and uniqueness of the solution in one dimension are also presented. In 2010, Ghergu and Radulescu [39] consider a class of reaction-diffusion system of Brusselator type, and show that if $f(u)$ has a sublinear growth then no Turing patterns occur, while if $f(u)$ has a superlinear growth then the existence of such patterns is strongly related to the inter-dependence between the parameters a, b and the diffusion coefficients d_1, d_2. Liu et al. [40] establish the existence of at least four positive periodic solutions for a discrete time Lotka–Volterra competitive system with harvesting terms by using Mawhin's continuation theorem of coincidence degree theory. In [41], Giacomoni et al. investigate a quasilinear and singular elliptic system and give some applications to biology. In particular, many mathematical models in biology, chemistry, and population genetics are included and studied in [42]. In recent years many theoretical investigations have been done on noise-induced effects in population dynamics [43–45]. Finally, the noise source can be non-Gaussian and this further enriches the dynamics [46–51].

However, as far as we know, no work has been done for the one-predator and two-prey system with ratio-dependent functional responses and feedback controls. So, in this paper, we will consider the following system:

$$
\begin{cases}
\dot{x}_1(t) = x_1(t)[r_1(t) - a_{11}(t)x_1(t) - \frac{a_{12}(t)x_2(t)}{b_{12}(t)x_2(t)+x_1(t)} - \frac{a_{13}(t)x_3(t)}{b_{13}(t)x_3(t)+x_1(t)} - d_1(t)u_1(t)], \\
\dot{x}_2(t) = x_2(t)[-r_2(t) + \frac{a_{21}(t)x_1(t)}{b_{12}(t)x_2(t)+x_1(t)} - a_{23}(t)x_3(t) + d_2(t)u_2(t)], \\
\dot{x}_3(t) = x_3(t)[-r_3(t) + \frac{a_{31}(t)x_1(t)}{b_{13}(t)x_3(t)+x_1(t)} - a_{32}(t)x_2(t) + d_3(t)u_3(t)], \\
\dot{u}_1(t) = e_1(t) - f_1(t)u_1(t) + q_1(t)x_1(t), \\
\dot{u}_2(t) = e_2(t) - f_2(t)u_2(t) - q_2(t)x_2(t), \\
\dot{u}_3(t) = e_3(t) - f_3(t)u_3(t) - q_3(t)x_3(t),
\end{cases}
\tag{1.1}
$$

where $x_1(t)$, $x_2(t)$ and $x_3(t)$ stand for the densities of one prey and two competing predators, respectively, and $u_i(t)$ ($i = 1, 2, 3$) are the indirect control variables. The given coefficients $a_{ij}(t)$, $b_{ij}(t)$, $d_i(t)$, $e_i(t)$, $f_i(t)$, $q_i(t)$, and $r_i(t)$ are positive continuous bounded functions of t for $i, j = 1, 2, 3$. System (1.1) describes the interaction between prey and predator species which is based on a ratio-dependent functional response. System (1.1) is the so-called food web system with two competing predators and one prey. Here, $r_1(t)$ is the intrinsic growth rate in the absence of predators $r_2(t)$ and $x_3(t)$; the parameters $a_{12}(t)$ and $a_{13}(t)$ are the capturing (or catching efficiency) rates of the two predators; $a_{21}(t)$ and $a_{31}(t)$ are the conversion rates (or maximum growth rates); $b_{12}(t)$ and $b_{13}(t)$ are the interference coefficients of predator species; $r_2(t)$ and $r_3(t)$ are the death rates of the two predator species $x_2(t)$ and $x_3(t)$; $d_i(t)$, $e_i(t)$, $q_i(t)$, and $f_i(t)$ ($i = 1, 2, 3$) are the controls parameters. When there are no feedback controls, system (1.1) is reduced to the following Lotka and Volterra model:

$$
\begin{cases}
\dot{x}_1(t) = x_1(t)[a_1(t) - a_{11}(t)x_1(t) - \frac{a_{12}(t)x_2(t)}{b_{12}(t)x_2(t)+x_1(t)} - \frac{a_{13}(t)x_3(t)}{b_{13}(t)x_3(t)+x_1(t)}], \\
\dot{x}_2(t) = x_2(t)[-a_2(t) + \frac{a_{21}(t)x_1(t)}{b_{12}(t)x_2(t)+x_1(t)} - a_{23}(t)x_3(t)], \\
\dot{x}_3(t) = x_3(t)[-a_3(t) + \frac{a_{31}(t)x_1(t)}{b_{13}(t)x_3(t)+x_1(t)} - a_{32}(t)x_2(t)],
\end{cases}
\tag{1.2}
$$

where the meanings of the parameters of system (1.2) are the same as those of (1.1). Lu et al. [52] show that this system is permanent and globally asymptotically stable under some

appropriate conditions by constructing a suitable Lyapunov function. Comparing systems (1.1) and (1.2), one could see that we introduce the control variables $u_i(t)$ ($i = 1, 2, 3$) so as to implement a feedback control mechanism.

This paper is organized as follows. In Sect. 2, we provide the conditions for permanence to system (1.1) by using a comparison theorem and developing some new analysis techniques. In Sect. 3, by constructing a non-negative Lyapunov function, we shall derive sufficient conditions for the global attractivity of positive solution for the predator–prey system (1.1). In Sect. 4, some conditions for the existence, uniqueness, and stability of a positive periodic solution for the corresponding periodic system are obtained by using the fixed point theory and some new analysis method. Some numerical solutions of the equations describing the system are given in Sect. 5 to verify that the obtained criteria are verifiable.

2 Permanence

In order to establish a permanence result for system (1.1), we need some preparations. Due to the biological interpretation of the system, it is reasonable to consider only positive solution of (1.1), in other words, to take admissible initial conditions $x_i(t_0) > 0$, $u_i(t_0) > 0$ ($i = 1, 2, 3$). Firstly, we introduce the following notations and definitions. Given a function $g(t)$ defined on $[t_0, +\infty)$, we set

$$g^m = \sup\{g(t) : t_0 < t < +\infty\}, \qquad g^l = \inf\{g(t) : t_0 < t < +\infty\}.$$

Definition 2.1 System (1.1) is called permanent if there exist positive constants M_i, N_i, m_i, n_i ($i = 1, 2, 3$), and T such that $m_i \le x_i(t) \le M_i$, $n_i \le u_i(t) \le N_i$ as $t > T$ for any positive solution $(x_1(t), x_2(t), x_3(t), u_1(t), u_2(t), u_3(t))$ of system (1.1) with positive initials.

For system (1.1), we let

$$M_1^* = \frac{r_1^m}{a_{11}^l}, \qquad N_1^* = \frac{e_1^m + q_1^m M_1}{f_1^l},$$

$$m_1^* = \frac{r_1^l b_{12}^l b_{13}^l - a_{12}^m b_{13}^l - a_{13}^m b_{13}^l - d_1^m b_{12}^l b_{13}^l N_1}{a_{11}^m b_{12}^l b_{13}^l},$$

$$N_3^* = \frac{e_3^m}{f_3^l}, \qquad M_3^* = M_1 \frac{d_3^m N_3 + a_{31}^m - r_3^l}{r_3^l b_{13}^l - d_3^m b_{13}^l N_3},$$

$$N_2^* = \frac{e_2^m}{f_2^l}, \qquad M_2^* = M_1 \frac{d_2^m N_2 + a_{21}^m - r_2^l}{r_2^l b_{12}^l - d_2^m b_{12}^l N_2},$$

$$m_2^* = \frac{m_1(-a_{23}^m M_3 + a_{21}^l - r_2^m)}{r_2^m b_{12}^m + a_{23}^m b_{12}^m M_3}, \qquad m_3^* = \frac{m_1(-a_{23}^m M_2 + a_{31}^l - r_3^m)}{r_3^m b_{13}^m + a_{32}^m b_{13}^m M_2},$$

$$n_1^* = \frac{e_1^l + q_1^l m_1}{f_1^m}, \qquad n_2^* = \frac{e_2^l - q_2^m M_2}{f_2^m}, \qquad n_3^* = \frac{e_3^l - q_3^m M_3}{f_3^m},$$

where M_i, N_i, m_i, n_i are some appropriate positive constants such that

$$0 < m_i < m_i^* < M_i^* < M_i, \qquad 0 < n_i < n_i^* < N_i^* < N_i, \quad i = 1, 2, 3.$$

Theorem 2.1 *Assume that system* (1.1) *satisfies the following conditions*:

(H_1)

$$r_1^l b_{12}^l b_{13}^l > a_{12}^m b_{13}^l + a_{13}^m b_{13}^l + d_1^m b_{12}^l b_{13}^l N_1;$$

(H_2)

$$\frac{d_3^m N_3 + a_{31}^m - r_3^l}{r_3^l b_{13}^l - d_3^m b_{13}^l N_3} > 0;$$

(H_3)

$$\frac{d_2^m N_2 + a_{21}^m - r_2^l}{r_2^l b_{12}^l - d_2^m b_{12}^l N_2} > 0;$$

(H_4)

$$-a_{23}^m M_3 + a_{21}^l > r_2^m;$$

(H_5)

$$-a_{23}^m M_2 + a_{31}^l > r_3^m;$$

(H_6)

$$e_2^l > q_2^m M_2;$$

(H_7)

$$e_3^l > q_3^m M_3.$$

Then system (1.1) *is permanent.*

Proof From the first equation of system (1.1), we have

$$\dot{x}_1(t) \le x_1(t)\big[r_1(t) - a_{11}(t)x_1(t)\big] \le x_1(t)\big[r_1^m - a_{11}^l x_1(t)\big] = a_{11}^l x_1(t)\big[M_1^* - x_1(t)\big],$$

thus, it holds that

$$\dot{x}_1(t)|_{x_1(t)=M_i^*} \le a_{11}^l M_1\big[M_1^* - M_1\big] < 0.$$

In view of the comparison theorem, one has

(1) When $0 < x_1(t_0) < M_1$, if $t \ge t_0$, then $x_1(t) \le M_1$.

(2) When $x_1(t_0) \ge M_1$, for a sufficiently large t, one has $x_1(t) \le M_1$. Otherwise, if $x_1(t) > M_1$, then there exists $\alpha > 0$ such that $x_1(t) \ge M_1^* + \alpha$. Moreover, we have

$$\dot{x}_1(t)|_{x_1(t)>M_1} \le x_1(t)\big[r_1(t) - a_{11}(t)x_1(t)\big] \le a_{11}^l x_1(t)\big[M_1^* - x_1(t)\big] < -a_{11}^l \alpha x_1(t),$$

thus, it holds that $x_1(t) < x_1(t_0)\exp(-a_{11}^l \alpha t) \to 0$ as $t \to +\infty$. This is a contradiction, so there exist sufficiently large $T_1 \geq t_0 \geq 0$ such that

$$x_1(t) \leq M_1 \quad \text{as } t > T_1. \tag{2.1}$$

By the sixth equation of system (1.1), we can get

$$\dot{u}_3 \leq e_3^m - f_3^l u_3(t) = f_3^l\big[N_3^* - u_3(t)\big].$$

Thus, employing the comparison theorem and analysis similar to the one used above, it holds that there exist sufficiently large $T_6 \geq t_0 \geq 0$ such that

$$u_3(t) \leq N_3 \quad \text{as } t > T_6. \tag{2.2}$$

Similarly, from the fifth equation of model (1.1), one has that there exist sufficiently large $T_5 \geq t_0 \geq 0$ such that

$$u_2(t) \leq N_2 \quad \text{as } t > T_5. \tag{2.3}$$

From the third equation of system (1.1), and combining (2.1) and (2.2), the following holds:

$$
\begin{aligned}
\dot{x}_3 &\leq x_3(t)\left[-r_3^l + \frac{a_{31}^m M_1}{b_{13}(t)x_3(t) + M_1} + d_3^m N_3\right] \\
&\leq x_3(t)\left[-r_3^l + \frac{a_{31}^m M_1}{b_{13}^l x_3(t) + M_1} + d_3^m N_3\right] \\
&= x_3(t)\left[\frac{-r_3^l(b_{13}^l x_3(t) + M_1) + a_{31}^m M_1 + d_3^m N_3(b_{13}^l x_3(t) + M_1)}{b_{13}^l x_3(t) + M_1}\right] \\
&= x_3(t)\frac{r_3^l b_{13}^l - d_3^m N_3 b_{13}^l}{b_{13}^l x_3(t) + M_1}\left[-x_3(t) + \frac{M_1(d_3^m N_3 + a_{31}^m - r_3^l)}{r_3^l b_{13}^l - d_3^m N_3 b_{13}^l}\right] \\
&= x_3(t)\frac{r_3^l b_{13}^l - d_3^m N_3 b_{13}^l}{b_{13}^l x_3(t) + M_1}\big[-x_3(t) + M_3\big].
\end{aligned}
$$

In view of the comparison theorem and the same analysis as above, we can obtain that

(3) When $0 < x_3(t_0) < M_3$, if $t \geq t_0$, then $x_3(t) \leq M_3$;

(4) When $x_3(t_0) \geq M_3$, for a sufficiently large t, one has $x_3(t) \leq M_3$. So it holds that there exist sufficiently large $T_3 \geq t_0 \geq 0$ such that

$$x_3(t) \leq M_3 \quad \text{as } t > T_3. \tag{2.4}$$

Similarly, from the second equation of system (1.1), and combining (2.1) and (2.3), it holds that there exist sufficiently large $T_3 \geq t_0 \geq 0$ such that

$$x_2(t) \leq M_2 \quad \text{as } t > T_2. \tag{2.5}$$

By the fourth equation of system (1.1), we can obtain

$$\dot{u}_1 \le e_1^m - f_1^l u_1(t) + q_1^m M_1 = f_1^l \big[N_1^* - u_1(t) \big].$$

Moreover, employing the comparison theorem and analysis similar to the one used above, it holds that there exist sufficiently large $T_4 \ge t_0 \ge 0$ such that

$$u_1(t) \le N_1 \quad \text{as } t > T_4. \tag{2.6}$$

On the other hand, by the same analysis process, we have

$$\dot{x}_1(t) \ge x_1(t) \left[r_1^l - a_{11}^m x_1(t) - \frac{a_{12}^m}{b_{12}^l} - \frac{a_{13}^m}{b_{13}^l} - d_1^m N_1 \right]$$

$$= x_1(t) a_{11}^m \left[-x_1(t) + \frac{r_1^l b_{12}^l b_{13}^l - a_{12}^m b_{13}^l - a_{13}^m b_{12}^l - b_{12}^l b_{13}^l d_1^m N_1}{a_{11}^m b_{12}^l b_{13}^l} \right]$$

$$= x_1(t) a_{11}^m \big[-x_1(t) + m_1^* \big].$$

Using the comparison theorem and the same analysis as above, it holds that

(5) When $m_1 < x_1(t_0)$, if $t \ge t_0$, then $m_1 \le x_1(t)$;

(6) When $0 < x_1(t_0) \le m_1$, for a sufficiently large t, one has $m_1 \le x_1(t)$. Otherwise, if $x_1(t) < m_1$, then there exists $\beta > 0$ such that $x_1(t) \le m_1^* - \beta$. Moreover, we have

$$\dot{x}_1(t)|_{x_1(t) < m_1} \ge a_{11}^m x_1(t) \big[-x_1(t) + m_1^* \big] > a_{11}^m \beta x_1(t),$$

thus, it holds that $x_1(t) > x_1(t_0) \exp(a_{11}^m \beta t) \to +\infty$ as $t \to +\infty$. This is a contradiction, so there exist sufficiently large $T_1' \ge t_0 \ge 0$ such that

$$x_1(t) \ge m_1 \quad \text{as } t > T_1'. \tag{2.7}$$

Similarly, we can obtain that there exist five sufficiently large positive constants T_i' ($i = 2, \ldots, 6$) such that

$$x_i(t) \ge m_i, \quad \text{as } t > T_i', i = 2, \ldots, 6. \tag{2.8}$$

From (2.1)–(2.8), and setting $T = \max_{1 \le i \le 6} \{T_i, T_i'\}$, we have $m_i \le x_i(t) \le M_i, n_i \le u_i(t) \le N_i$ as $t > T$ for any positive solution $(x_1(t), x_2(t), x_3(t), u_1(t), u_2(t), u_3(t))$ of system (1.1) with positive initials. This ends the proof of Theorem 2.1. □

3 Global attractivity

In this section, the global attractivity of system (1.1) is studied. To get the sufficient conditions for global attractivity of system (1.1), the following definition and lemma are firstly given.

Definition 3.1 System (1.1) is said to be globally attractive if there exists a positive solution $X(t) = (x_1(t), x_2(t), x_3(t), u_1(t), u_2(x), u_3(x))$ of system (1.1) such that

$$\lim_{t \to +\infty} \big| x_i(t) - y_i(t) \big| = 0, \qquad \lim_{t \to +\infty} \big| u_i(t) - v_i(t) \big| = 0,$$

for any other positive solution $Y(t) = (y_1(t), y_2(t), y_3(t), v_1(t), v_2(t), v_3(t))$ of system (1.1).

Lemma 3.1 ([53]) *If the function $f(t) : R^+ \to R$ is uniformly continuous, and the limit $\lim_{t \to \infty} \int_0^t f(s)\,ds$ exists and is finite, then $\lim_{t \to +\infty} f(t) = 0$.*

Theorem 3.1 *Assume that system* (1.1) *satisfies* (H_1)–(H_7) *and the following conditions*:
(H_8)

$$a_{11}^l - \frac{(a_{12}^m + b_{12}^m a_{21}^m)M_2}{(b_{12}^l m_2 + m_1)^2} - \frac{(a_{13}^m + b_{13}^m a_{31}^m)M_3}{(b_{13}^l m_3 + m_1)^2} - q_1^m > 0;$$

(H_9)

$$-a_{32}^m - \frac{a_{12}^m M_1}{(b_{12}^l m_2 + m_1)^2} + \frac{a_{21}^l b_{12}^l m_1}{(b_{12}^m M_2 + M_1)^2} - q_2^m > 0;$$

(H_{10})

$$-a_{23}^m - \frac{a_{13}^m M_1}{(b_{13}^l m_3 + m_1)^2} + \frac{a_{31}^l b_{13}^l m_1}{(b_{13}^m M_3 + M_1)^2} - q_3^m > 0;$$

(H_{11})

$$f_i^l > d_i^m \quad (i = 1, 2, 3).$$

Then system (1.1) *is globally attractive.*

Proof Let $X(t) = (x_1(t), x_2(t), x_3(t), u_1(t), u_2(t), u_3(t))$ be a positive solution of system (1.1) and $Y(t) = (y_1(t), y_2(t), y_3(t), v_1(t), v_2(t), v_3(t))$ be any positive solution of system (1.1) with initial conditions $x_i(t_0) > 0, u_i(t_0) > 0, i = 1, 2, 3$, then from Theorem 2.1, there exist positive constants M_i, N_i, m_i, n_i, and T such that $m_i \le x_i(t) \le M_i, n_i \le u_i(t) \le N_i$ for all $t > T$.

Set the Lyapunov function

$$V(t) = \sum_{i=1}^{3} \left[\left| \ln x_i(t) - \ln y_i(t) \right| + \left| u_i(t) - v_i(t) \right| \right].$$

We compute the upper right derivative of $V(t)$ along with the solution of system (1.1) only using H-assumptions and simple calculations:

$$\begin{aligned}
D^+ V(t) &\le \sum_{i=1}^{3} D^+ \left[\left| \ln x_i(t) - \ln y_i(t) \right| + \left| u_i(t) - v_i(t) \right| \right] \\
&= \sum_{i=1}^{3} \operatorname{sgn}\{x_i(t) - y_i(t)\} \left(\frac{\dot{x}_i(t)}{x_i(t)} - \frac{\dot{y}_i(t)}{y_i(t)} \right) + \sum_{i=1}^{3} \operatorname{sgn}\{u_i(t) - v_i(t)\} (\dot{u}_i(t) - \dot{v}_i(t)) \\
&= \operatorname{sgn}\{x_1(t) - y_1(t)\} \left[-a_{11}(t)(x_1(t) - y_1(t)) - a_{12}(t) \left(\frac{x_2(t)}{(b_{12}(t)x_2(t) + x_1(t))} \right. \right. \\
&\quad \left. \left. - \frac{y_2(t)}{(b_{12}(t)y_2(t) + y_1(t))} \right) \right.
\end{aligned}$$

$$- a_{13}(t)\left(\frac{x_3(t)}{(b_{13}(t)x_3(t) + x_1(t))} - \frac{y_3(t)}{(b_{13}(t)y_3(t) + y_1(t))}\right)$$

$$- d_1(t)\big(u_1(t) - v_1(t)\big)\bigg] + \mathrm{sgn}\{x_2(t) - y_2(t)\}\bigg[-a_{23}(t)\big(x_3(t) - y_3(t)\big)$$

$$+ a_{21}(t)\left(\frac{x_1(t)}{b_{12}(t)x_2(t) + x_1(t)} - \frac{y_1(t)}{b_{12}(t)y_2(t) + y_1(t)}\right) + d_2(t)\big(u_2(t) - v_2(t)\big)\bigg]$$

$$+ \mathrm{sgn}\{x_3(t) - y_3(t)\}$$

$$\times \bigg[-a_{32}\big(x_2(t) - y_2(t)\big) + a_{31}(t)\left(\frac{x_1(t)}{b_{13}(t)x_3(t) + x_1(t)} - \frac{y_1(t)}{b_{13}(t)y_3(t) + y_1(t)}\right)$$

$$+ d_3(t)\big(u_3(t) - v_3(t)\big)\bigg]$$

$$+ \mathrm{sgn}\{u_1(t) - v_1(t)\}\big[-f_1(t)\big(u_1(t) - v_1(t)\big) + q_1(t)\big(x_1(t) - y_1(t)\big)\big]$$

$$+ \mathrm{sgn}\{u_2(t) - v_2(t)\}\big[-f_2(t)\big(u_2(t) - v_2(t)\big) - q_2(t)\big(x_2(t) - y_2(t)\big)\big]$$

$$+ \mathrm{sgn}\{u_3(t) - v_3(t)\}\big[-f_3(t)\big(u_3(t) - v_3(t)\big) - q_3(t)\big(x_3(t) - y_3(t)\big)\big]$$

$$= \mathrm{sgn}\{x_1(t) - y_1(t)\}\bigg[-a_{11}(t)\big(x_1(t) - y_1(t)\big)$$

$$- a_{12}(t)\left(\frac{y_1(t)(x_2(t) - y_2(t)) - y_2(t)(x_1(t) - y_1(t))}{(b_{12}(t)x_2(t) + x_1(t))(b_{12}(t)y_2(t) + y_1(t))}\right)$$

$$- a_{13}(t)\left(\frac{y_1(t)(x_3(t) - y_3(t)) - y_3(t)(x_1(t) - y_1(t))}{(b_{13}(t)x_3(t) + x_1(t))(b_{13}(t)y_3(t) + x_1(t))}\right) - d_1(t)\big(u_1(t) - v_1(t)\big)\bigg]$$

$$+ \mathrm{sgn}\{x_2(t) - y_2(t)\}\bigg[a_{21}(t)b_{12}(t)\left(\frac{y_2(t)(x_1(t) - y_1(t)) - y_1(t)(x_2(t) - y_2(t))}{(b_{12}(t)x_2(t) + x_1(t))(b_{12}(t)y_2(t) + y_1(t))}\right)$$

$$- a_{23}(t)\big(x_3(t) - y_3(t)\big) + d_2(t)\big(u_2(t) - v_2(t)\big)\bigg]$$

$$+ \mathrm{sgn}\{x_3(t) - y_3(t)\}\bigg[a_{31}(t)b_{13}(t)\left(\frac{y_3(t)(x_1(t) - y_1(t)) - y_1(t)(x_3(t) - y_3(t))}{(b_{13}(t)x_3(t) + x_1(t))(b_{13}(t)y_3(t) + y_1(t))}\right)$$

$$- a_{23}(t)\big(x_2(t) - y_2(t)\big) + d_3(t)\big(u_3(t) - v_3(t)\big)\bigg]$$

$$+ \mathrm{sgn}\{u_1(t) - v_1(t)\}\big[-f_1(t)\big(u_1(t) - v_1(t)\big) + q_1(t)\big(x_1(t) - y_1(t)\big)\big]$$

$$+ \mathrm{sgn}\{u_2(t) - v_2(t)\}\big[-f_2(t)\big(u_2(t) - v_2(t)\big) - q_2(t)\big(x_2(t) - y_2(t)\big)\big]$$

$$+ \mathrm{sgn}\{u_3(t) - v_3(t)\}\big[-f_3(t)\big(u_3(t) - v_3(t)\big) - q_3(t)\big(x_3(t) - y_3(t)\big)\big]$$

$$\leq \big|x_1(t) - y_1(t)\big|\bigg[-a_{11}(t) + \frac{(a_{12}(t) + b_{12}(t)a_{21}(t))y_2(t)}{(b_{12}(t)x_2(t) + x_1(t))(b_{12}(t)y_2(t) + y_1(t))}$$

$$+ \frac{(a_{13}(t) + b_{13}(t)a_{31}(t))y_3(t)}{(b_{13}(t)x_3(t) + x_1(t))(b_{13}(t)y_3(t) + y_1(t))} + q_1(t)\bigg]$$

$$+ \big|x_2(t) - y_2(t)\big|\bigg[a_{32}(t) + \frac{(a_{12}(t) - b_{12}(t)a_{21}(t))y_1(t)}{(b_{12}(t)x_2(t) + x_1(t))(b_{12}(t)y_2(t) + y_1(t))} + q_2(t)\bigg]$$

$$+ \big|x_3(t) - y_3(t)\big|\bigg[a_{23}(t) + \frac{(a_{13}(t) - b_{13}(t)a_{31}(t))y_1(t)}{(b_{13}(t)x_2(t) + x_1(t))(b_{13}(t)y_2(t) + y_1(t))} + q_3(t)\bigg]$$

$$+ \big|u_1(t) - v_1(t)\big|\big[-f_1(t) + d_1(t)\big] + \big|u_2(t) - v_2(t)\big|\big[-f_2(t) + d_2(t)\big]$$

$$+ \left|u_3(t) - v_3(t)\right|\left[-f_3(t) + d_3(t)\right]$$

$$\leq -\left|x_1(t) - y_1(t)\right|\left[a_{11}^l - \frac{(a_{12}^m + b_{12}^m a_{21}^m)M_2}{(b_{12}^l m_2 + m_1)^2} - \frac{(a_{13}^m + b_{13}^m a_{31}^m)M_3}{(b_{13}^l m_3 + m_1)^2} - q_1^m\right]$$

$$- \left|x_2(t) - y_2(t)\right|\left[-a_{32}^m - \frac{a_{12}^m M_1}{(b_{12}^l m_2 + m_1)^2} + \frac{a_{21}^l b_{12}^l m_1}{(b_{12}^m M_2 + M_1)^2} - q_2^m\right]$$

$$- \left|x_3(t) - y_3(t)\right|\left[-a_{23}^m - \frac{a_{13}^m M_1}{(b_{13}^l m_3 + m_1)^2} + \frac{a_{31}^l b_{13}^l m_1}{(b_{13}^m M_3 + M_1)^2} - q_3^m\right]$$

$$- \left|u_1(t) - v_1(t)\right|\left[f_1^l - d_1^m\right] - \left|u_2(t) - v_2(t)\right|\left[f_2^l - d_2^m\right]$$

$$- \left|u_3(t) - v_3(t)\right|\left[f_3^l - d_3^m\right].$$

In view of conditions (H_8)–(H_{11}), one has

$$\alpha = \min\left\{a_{11}^l - \frac{(a_{12}^m + b_{12}^m a_{21}^m)M_2}{(b_{12}^l m_2 + m_1)^2} - \frac{(a_{13}^m + b_{13}^m a_{31}^m)M_3}{(b_{13}^l m_3 + m_1)^2} - q_1^m, \right.$$

$$-a_{32}^m - \frac{a_{12}^m M_1}{(b_{12}^l m_2 + m_1)^2} + \frac{a_{21}^l b_{12}^l m_1}{(b_{12}^m M_2 + M_1)^2} - q_2^m,$$

$$\left. -a_{23}^m - \frac{a_{13}^m M_1}{(b_{13}^l m_3 + m_1)^2} + \frac{a_{31}^l b_{13}^l m_1}{(b_{13}^m M_3 + M_1)^2} - q_3^m, f_1^l - d_1^m, f_2^l - d_2^m, f_3^l - d_3^m\right\} > 0.$$

Thus,

$$D^+ V(t) \leq -\alpha \sum_{i=1}^{3}\left[\left|x_i(t) - y_i(t)\right| + \left|u_i(t) - v_i(t)\right|\right]. \tag{3.1}$$

Integrating (3.1) from T to t ($T \geq t_0$), one has

$$V(t) + \alpha \int_T^t \left\{\sum_{i=1}^{3}\left[\left|x_i(t) - y_i(t)\right| + \left|u_i(t) - v_i(t)\right|\right]\right\} ds \leq V(T) < +\infty.$$

Therefore,

$$\int_T^t \left\{\sum_{i=1}^{3}\left[\left|x_i(t) - y_i(t)\right| + \left|u_i(t) - v_i(t)\right|\right]\right\} ds \leq \frac{V(T)}{\alpha} < +\infty. \tag{3.2}$$

By (3.2), we have

$$\sum_{i=1}^{3}\left[\left|x_i(t) - y_i(t)\right| + \left|u_i(t) - v_i(t)\right|\right] \in L^1(T, +\infty).$$

From the uniform permanence of system (1.1), $\sum_{i=1}^{3}[|x_i(t) - y_i(t)| + |u_i(t) - v_i(t)|]$ is uniformly continuous. By Lemma 3.1, we can obtain that

$$\lim_{t \to +\infty}\left|x_i(t) - y_i(t)\right| = 0, \qquad \lim_{t \to +\infty}\left|u_i(t) - v_i(t)\right| = 0 \quad (i = 1, 2, 3).$$

This ends the proof of Theorem 3.1. $\qquad\square$

4 Periodic solution

Assuming that the coefficients of system (1.1) are positive continuous, ω-periodic functions, then system (1.1) is changed to an ω-periodic system. In this section, we shall obtain conditions for the existence, uniqueness, and stability of a positive periodic solution for system (1.1) by using the fixed point theory and some new analysis method. For convenience, we give firstly the following lemma.

Lemma 4.1 ([54]) *Let $S \subset R_n$ be convex and compact. If the mapping $T : S \to S$ is continuous, then there exists a fixed point, i.e., there exists $x^* \in S$ such that $T(x^*) = x^*$.*

Theorem 4.1 *Assume that system (1.1) is an ω-periodic system and satisfies conditions (H_1)–(H_{11}), then system (1.1) has a positive unique ω-periodic solution, which is globally asymptotically stable.*

Proof According to the existence and uniqueness theorem of solutions of differential equations, we can obtain a Poincaré mapping $T : R_+^6 \to R_+^6$ defined as follows:

$$T(X_0) = X(t, \omega, t_0, X_0),$$

where $X(t, \omega, t_0, X_0) = (x_1(t), x_2(t), x_3(t), u_1(t), u_2(t), u_3(t))$ is a positive solution of system (1.1) with the initial conditions $X_0 = (x_1(t_0), x_2(t_0), x_3(t_0), u_1(t_0), u_2(t_0), u_3(t_0))$. And define
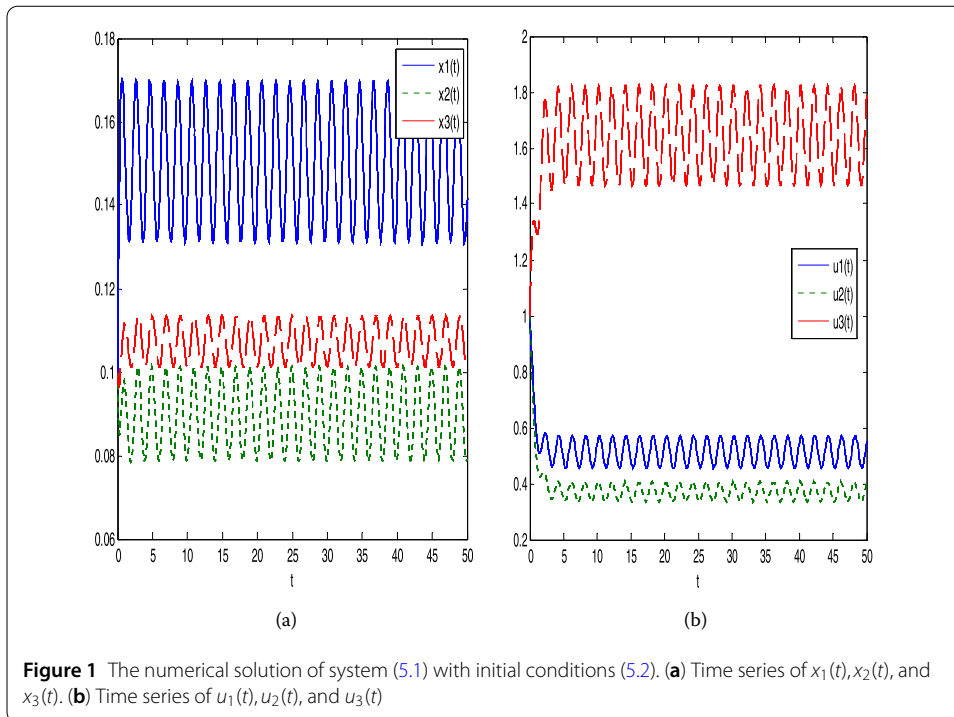
$$S = \big\{(x_1, x_2, x_3, u_1, u_2, u_3) \in R_+^6 | m_i \leq x_i \leq M_i, n_i \leq u_i \leq N_i, i = 1, 2, 3\big\},$$

then it is obvious that $S \subset R_+^6$ is a convex and compact set. By Theorem 2.1 and the continuity of solution of system (1.1) with respect to the initial conditions, the mapping $T : S \to S$ is continuous. Furthermore, it is not difficult to show that system (1.1) has a positive unique ω-periodic solution, which is globally asymptotically stable by using Lemma 4.1, Theorem 2.1, and Theorem 3.1. □

5 Numerical simulation

In this section, we give some numerical simulations supporting our theoretical analysis. As an example, we consider the following Lotka–Volterra predator–prey system with ratio-dependent functional responses and feedback controls:

$$\begin{cases}
\dot{x}_1(t) = x_1(t)[5 + \cos \pi t - [6.25 + 1.25 \sin \pi t]x_1(t) - [0.4 + 0.1 \sin \pi t]x_2(t) \\
\qquad - \frac{[0.75 + 0.25 \sin \pi t]x_3(t)}{[1.5 + 0.5 \sin \pi t]x_3(t) + x_1(t)} - (0.015 + 0.005 \sin \pi t)u_1(t)], \\
\dot{x}_2(t) = x_2(t)[2.5 + 0.5 \cos \pi t - [0.13 + 0.1 \sin \pi t]x_1(t) - [7 + \sin \pi t]x_2(t) \\
\qquad - \frac{[1 + 0.5 \sin \pi t]x_3(t)}{[1.8 + 0.2 \sin \pi t]x_3(t) + x_2(t)} - (0.5 + 0.1 \sin \pi t)u_2(t)], \\
\dot{x}_3(t) = x_3(t)[-[4.25 + 0.25 \cos \pi t] + \frac{[7.5 + 0.5 \sin \pi t]x_1(t)}{[1.5 + 0.5 \sin \pi t]x_3(t) + x_1(t)} \\
\qquad + \frac{[5.5 + 0.5 \sin \pi t]x_2(t)}{[1.8 + 0.2 \sin \pi t]x_3(t) + x_2(t)} + (0.03 + 0.01 \sin \pi t)u_3(t)], \\
\dot{u}_1(t) = [0.8 + 0.2 \cos \pi t] - [1.6 + 0.1 \sin \pi t]u_1(t) + [0.15 + 0.05 \sin \pi t]x_1(t), \\
\dot{u}_2(t) = [0.5 + 0.1 \cos \pi t] - [1.4 + 0.2 \sin \pi t]u_2(t) + [0.30 + 0.1 \sin \pi t]x_2(t), \\
\dot{u}_3(t) = [2 + 0.5 \cos \pi t] - [1.2 + 0.2 \sin \pi t]u_3(t) - [0.15 + 0.01 \sin \pi t]x_3(t).
\end{cases} \tag{5.1}$$

Figure 1 The numerical solution of system (5.1) with initial conditions (5.2). (**a**) Time series of $x_1(t), x_2(t)$, and $x_3(t)$. (**b**) Time series of $u_1(t), u_2(t)$, and $u_3(t)$

It is easy to show that system (5.1) satisfies the conditions of Theorem 4.1. It follows from Theorem 2.1, Theorem 3.1, and Theorem 4.1 that the predator–prey system (5.1) is permanent and globally attractive and that it has a unique and stable periodic solution which is globally asymptotically stable. By employing the software package MATLAB 7.1, we can solve the numerical solutions of Eqs. (5.1) which are shown in Fig. 1, Fig. 2, and Fig. 3. Figure 1 shows the permanence of system (5.1) with the initial conditions

$$
\begin{aligned}
&x_1(t_0) = 0.1, \qquad x_2(t_0) = 0.1, \qquad x_3(t_0) = 0.1, \\
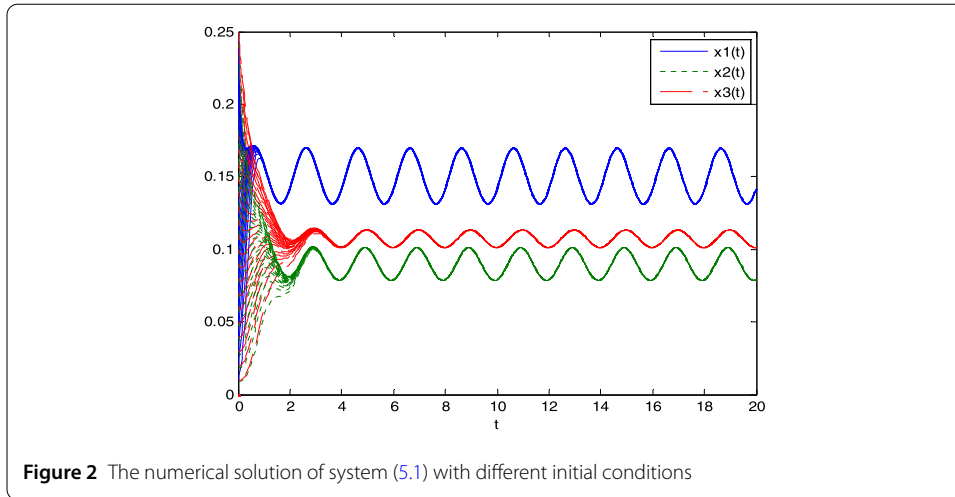&u_1(t_0) = 1, \qquad u_2(t_0) = 1, \qquad u_3(t_0) = 1.
\end{aligned}
\tag{5.2}
$$

From Fig. 1, it is not difficult to find that

$$
0.07 \le x_i(t) \le 0.18, \quad i = 1, 2, 3,
$$

and

$$
0.2 \le u_i(t) \le 2, \quad i = 1, 2, 3 \text{ as } t > 10.
$$

From Fig. 2, one can find that $\lim_{t\to+\infty} |x_i(t) - y_i(t)| = 0$, $i = 1, 2, 3$, for any two solutions $X(t) = (x_1(t), x_2(t), x_3(t))$ and $Y(t) = (y_1(t), y_2(t), y_3(t))$ of system (5.1) with different initial conditions, which shows that the periodic predator–prey system (5.1) has a unique periodic solution which is globally asymptotically stable. Figure 3 shows the dynamic behavior of system (5.1).

Figure 2 The numerical solution of system (5.1) with different initial conditions

Figure 3 The dynamic behavior of system (5.1). (**a**) Phase portrait of $x_1(t)$ and $x_2(t)$. (**b**) Phase portrait of $x_1(t)$ and $x_3(t)$. (**c**) Phase portrait of $x_2(t)$ and $x_3(t)$. (**d**) Phase portrait of $x_1(t), x_2(t)$, and $x_3(t)$
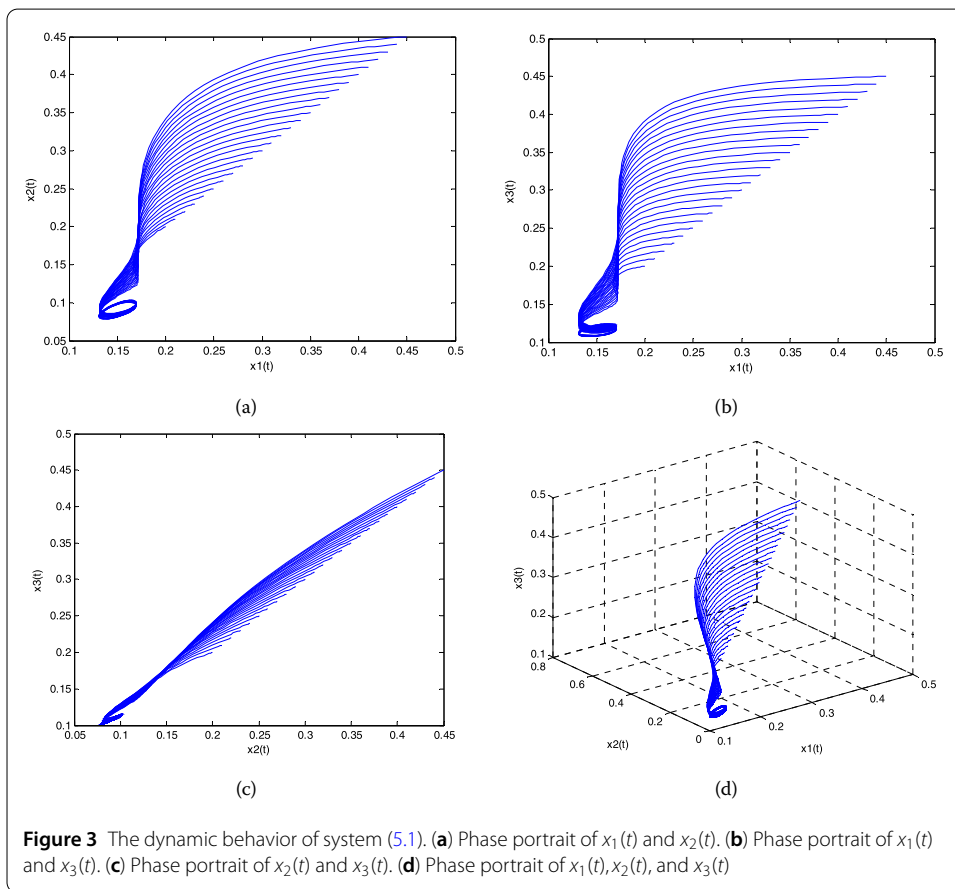
6 Conclusion

This paper presents the use of Lyapunov stability theorem and comparison theorem as well as fixed point theory for a system of nonlinear differential equations. This method is a powerful tool for solving nonlinear differential equations in mathematical physics, chemistry, engineering, etc. The technique constructing a suitable Lyapunov function and Poincaré mapping provides a new efficient method to handle the nonlinear structure.

We have dealt with the problem of positive solution and periodic solution for a Lotka–Volterra predator–prey system with ratio-dependent functional responses and feedback controls. The general sufficient conditions have been obtained to ensure the permanence of the system and the global asymptotic stability of a positive solution for the predator–prey system. Furthermore, some conditions for the existence, uniqueness, and stability of a positive periodic solution for the corresponding periodic system are obtained. In addition, some numerical solutions of the equations describing the system are given to illustrate our results. In particular, the sufficient conditions that we obtained are very simple, which provides flexibility for the application and analysis of the Lotka–Volterra predator–prey system.

Remark Obviously, model (1.1) is the extension of model (1.2). Adding delay term to the proposed model (1.1) is our next research work.

Availability of data and materials
Not applicable.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
CW carried out the study of permanence and global attractivity and participated in the design of the study, YZ carried out the study of periodic solution and participated in the design of the study, YL drafted the manuscript and participated in the design of the study. RL carried out the numerical simulation and helped to draft the manuscript. All authors read and approved the final manuscript.

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