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On a class of stationary loops on $\mathbf{SO}(n)$ and the existence of multiple twisting solutions to a nonlinear elliptic system subject to a hard incompressibility constraint

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Abstract

In this paper we consider the second order nonlinear elliptic system in divergence and variational form

$$\begin{cases} \operatorname{div}[F_\xi(|x|,|\nabla u|^2)\nabla u] = [\operatorname{cof}\nabla u]\nabla\mathscr{P} & \text{in } U, \\ \det\nabla u = 1 & \text{in } U, \\ u = \varphi & \text{on } \partial U, \end{cases}$$

where $F = F(r,\xi)$ is a sufficiently regular Lagrangian satisfying suitable structural properties and $\mathscr{P}$ is an *a priori* unknown Lagrange multiplier. Most notably, for a finite symmetric n-annulus, we prove the existence of an infinite family of monotone twisting solutions to this system in all even dimensions by linking the system to a set of nonlinear isotropic ODEs on the Lie group $\mathbf{SO}(n)$. We prove the existence of multiple closed stationary loops in the geodesic form $\mathbf{Q}(r) = \exp\{f(r)\mathbf{H}\}$ with $\mathbf{H} \in \mathfrak{so}(n)$ to these ODEs that remarkably serve as the twist loops associated with the desired twisting solutions u to the above system. An analysis of curl-free vector fields generated by symmetric matrix fields plays a pivotal role.

Keywords: Nonlinear elliptic systems; Incompressible twists; Geodesics on $\mathbf{SO}(n)$; Multiple stationary loops; Curl-free vector fields; Weighted Dirichlet type Lagrangians; Nonlinear elasticity

1 Introduction

Let $U \subset \mathbb{R}^n$ be a bounded domain with a sufficiently smooth boundary ∂U and consider the variational energy integral

$$\mathbb{F}[u,U] := \int_U F\big(x,|\nabla u|^2\big)\,dx, \tag{1.1}$$

over the space of weakly differentiable incompressible Sobolev maps $\mathscr{A}^p_\varphi(U) = \{u \in W^{1,p}(U,\mathbb{R}^n) : \det\nabla u = 1 \text{ a.e. in } U \text{ and } u \equiv \varphi \text{ on } \partial U\}$ where $\varphi \in \mathscr{C}^1(\partial U)$ is a prescribed boundary map and $p \geq 1$ is fixed. Here $F = F(r,\xi)$ is taken a twice continuously differentiable Lagrangian that is bounded from below and satisfies suitable convexity and growth conditions (see below for a formulation of the assumptions on F), ∇u denotes the gradient of u and $|\nabla u|^2 = \mathrm{tr}\{[\nabla u][\nabla u]^t\}$. The Euler–Lagrange equation associated with $\mathbb{F}[u, U]$ over $\mathscr{A}^p_\varphi(U)$ is given by the second order nonlinear system

$$\mathbf{EL}[u; F, U] := \begin{cases} \mathscr{L}_F[u] = \nabla \mathscr{P} & \text{in } U, \\ \det \nabla u = 1 & \text{in } U, \\ u \equiv \varphi & \text{on } \partial U, \end{cases} \tag{1.2}$$

where the differential operator $\mathscr{L}_F$ has the explicit form

$$\begin{aligned} \mathscr{L}_F[u] &:= (\nabla u)^t \operatorname{div}\bigl[F_\xi\bigl(|x|, |\nabla u|^2\bigr)\nabla u\bigr] \\ &= F_{\xi\xi}\bigl(r, |\nabla u|^2\bigr)(\nabla u)^t \nabla u \nabla\bigl(|\nabla u|^2\bigr) \\ &\quad + F_{r\xi}\bigl(r, |\nabla u|^2\bigr)(\nabla u)^t \nabla u \theta + F_\xi\bigl(r, |\nabla u|^2\bigr)(\nabla u)^t \Delta u. \end{aligned} \tag{1.3}$$

System (1.2) can be formally derived by invoking the Lagrange multiplier method and considering the unconstrained energy (*see*, e.g., [6, 10, 20] for more)

$$\mathbb{G}[u, U] := \int_U \bigl\{F\bigl(|x|, |\nabla u|^2\bigr) - 2\mathscr{P}(x)[\det \nabla u - 1]\bigr\}\, dx, \tag{1.4}$$

where $\mathscr{P}$ is an *a priori* unknown Lagrange multiplier – the hydrostatic pressure – corresponding to the pointwise incompressibility constraint $\det \nabla u = 1$. For the sake of clarity, note that by a [*classical*] solution we hereafter mean a pair $(u, \mathscr{P})$ with u of class $\mathscr{C}^2(U, \mathbb{R}^n) \cap \mathscr{C}(\overline{U}, \mathbb{R}^n)$ and $\mathscr{P}$ of class $\mathscr{C}^1(U) \cap \mathscr{C}(\overline{U})$ such that (1.2) holds in a pointwise sense in U.[a] Now, proceeding forward and arguing either formally and in a distributional sense, or classically, upon assuming further differentiability on $\mathscr{L}_F$, it is seen from (1.2)–(1.3) that $\operatorname{curl} \mathscr{L}_F[u] = \operatorname{curl} \nabla \mathscr{P} \equiv 0$ in U, that is,

$$\begin{aligned} \operatorname{curl} \mathscr{L}_F[u] = \operatorname{curl}\bigl\{& F_{\xi\xi}\bigl(r, |\nabla u|^2\bigr)(\nabla u)^t \nabla u \nabla\bigl(|\nabla u|^2\bigr) + F_{r\xi}\bigl(r, |\nabla u|^2\bigr)(\nabla u)^t \nabla u \theta \\ &+ F_\xi\bigl(r, |\nabla u|^2\bigr)(\nabla u)^t \Delta u\bigr\} \equiv 0. \end{aligned} \tag{1.5}$$

Note, however, that this condition alone, unless U has a particular homology, does not imply that $\mathscr{L}_F[u]$ is a gradient field in U, here, $\nabla \mathscr{P}$. For more on the background formulation and applications of system (1.2)–(1.3), in particular to function theory, mechanics, and nonlinear elasticity, *see* [2, 3, 5, 10, 14, 19] and [1, 4, 7, 11, 12, 15–18] as well as [20–27, 30] and [9, 13, 29, 31, 32] for related results and further applications.

Throughout the paper we specialise to the geometric set up where $U = \mathbb{X}^n = \mathbb{X}^n[a, b] := \{x \in \mathbb{R}^n : a < |x| < b\}$ is a finite symmetric annulus with $b > a > 0$, and $\varphi \equiv x$, i.e., the identity map. In this context by a generalised twist (or *twist* for brevity) we understand a map $u \in \mathscr{C}(\overline{\mathbb{X}}^n, \overline{\mathbb{X}}^n)$ that admits the representation

$$u : (r, \theta) \mapsto \bigl(r, \mathbf{Q}(r)\theta\bigr), \quad r = |x|, \theta = x|x|^{-1}, x \in \overline{\mathbb{X}}^n. \tag{1.6}$$

The curve $\mathbf{Q} \in \mathscr{C}([a, b], \mathbf{SO}(n))$ here is called the twist path associated with u. Moreover, in order to ensure $u \equiv \varphi$ on $\partial U = \partial \mathbb{X}^n$, we set $\mathbf{Q}(a) = \mathbf{Q}(b) = \mathbf{I}_n$. In this event the twist path forms a closed curve in $\mathbf{SO}(n)$, based at $\mathbf{I}_n$, called the twist *loop* that in turn represents an element of the fundamental group $\pi_1(\mathbf{SO}(n)) \cong \mathbb{Z}_2$ ($n \geq 3$) and $\mathbb{Z}$ ($n = 2$). Our aim is to establish the existence of an infinitude of twisting solutions to the nonlinear system (1.2) by appropriately formulating the action of $\mathscr{L}_F$ on sufficiently regular twists u as in (1.6) and solving the resulting PDE. We note that in this setting (*cf.* Proposition 2.1) this action, and subsequently the first equation in (1.2), is given by

$$\begin{aligned}\mathscr{L}_F[u] &= F_{\xi\xi}\big(\mathbf{I}_n + r\mathbf{Q}^t\dot{\mathbf{Q}}\theta \otimes \theta + r\theta \otimes \mathbf{Q}^t\dot{\mathbf{Q}}\theta + r^2|\dot{\mathbf{Q}}\theta|^2\theta \otimes \theta\big)\\ &\quad \times \big(2r|\dot{\mathbf{Q}}\theta|^2\theta + r^2\nabla|\dot{\mathbf{Q}}\theta|^2\big) + F_{r\xi}\big(\theta + r\mathbf{Q}^t\dot{\mathbf{Q}}\theta + r^2|\dot{\mathbf{Q}}\theta|^2\theta\big)\\ &\quad + F_\xi\big[(n+1)\mathbf{Q}^t\dot{\mathbf{Q}} + r\mathbf{Q}^t\ddot{\mathbf{Q}} + \big\{r(n+1)|\dot{\mathbf{Q}}\theta|^2 + r^2\langle\dot{\mathbf{Q}}\theta, \ddot{\mathbf{Q}}\theta\rangle\big\}\mathbf{I}_n\big]\theta = \nabla\mathscr{P}. \qquad (1.7)\end{aligned}$$

We take a close look at (1.5)–(1.7) and formulate the conditions on the twist path $\mathbf{Q} = \mathbf{Q}(r)$ that will then result in the vector field $\mathscr{L}_F[u = r\mathbf{Q}\theta]$ being curl-free in $\mathbb{X}^n$ and in fact being the gradient field $\nabla\mathscr{P}$.

In the course of establishing the existence of multiple twisting solutions $u = r\mathbf{Q}(r)\theta$ to system (1.2), we study three interrelated classes of ODEs for the twist path $\mathbf{Q} = \mathbf{Q}(r)$ (with $a \leq r \leq b$) that are closely linked to one another and formulated solely through the Lagrangian F. The first one is given by

$$\int_{\mathbb{S}^{n-1}} \frac{d}{dr}\big\{r^{n+1}F_\xi\big(r, n + r^2|\dot{\mathbf{Q}}\theta|^2\big)[\dot{\mathbf{Q}}\theta \otimes \mathbf{Q}\theta - \mathbf{Q}\theta \otimes \dot{\mathbf{Q}}\theta]\big\}\, d\mathcal{H}^{n-1}(\theta) = 0, \qquad (1.8)$$

which can be extracted both from the PDE $\mathscr{L}_F[u] = \nabla\mathscr{P}$ in $\mathbb{X}^n$ or directly and more naturally as the Euler–Lagrange equation associated with the restriction of $\mathbb{F}[u, \mathbb{X}^n]$ to the class of twists u in $\mathscr{A}^p_\varphi(\mathbb{X}^n)$ (*cf.* Proposition 2.3 and Proposition 2.4, respectively). Naturally, if u is a twisting solution to (1.2), then its twist path $\mathbf{Q}$ should satisfy (1.8). By discarding the spherical integral in (1.8), one obtains a strengthened version of this ODE, that is,

$$\mathscr{S}[\mathbf{Q}] := \frac{d}{dr}\big\{r^{n+1}F_\xi\big(r, n + r^2|\dot{\mathbf{Q}}\theta|^2\big)[\dot{\mathbf{Q}}\theta \otimes \mathbf{Q}\theta - \mathbf{Q}\theta \otimes \dot{\mathbf{Q}}\theta]\big\} = 0, \quad a < r < b. \qquad (1.9)$$

Evidently every solution $\mathbf{Q} = \mathbf{Q}(r)$ to (1.9) is also a solution to (1.8) but in general not *vice versa*. Indeed it is a complete classification of solutions to (1.9) and their relations to geodesics on the Lie group $\mathbf{SO}(n)$ on the one hand and to twisting solutions u of system (1.2) on the other which will occupy us for parts of the paper. Finally the third ODE with intimate links to (1.9) and the PDE $\mathscr{L}_F[u] = \nabla\mathscr{P}$ is given by

$$\mathscr{M}[\mathbf{Q}] := \frac{1}{r^n}\frac{d}{dr}\big[r^{n+1}F_\xi\big(r, n + r^2|\dot{\mathbf{Q}}\theta|^2\big)\mathbf{Q}^t\dot{\mathbf{Q}}\theta\big] = 0, \quad a < r < b. \qquad (1.10)$$

The connections between these ODEs and their solutions to the nonlinear system (1.2) and its twisting solutions will be discussed at length later on in the paper. In particular it will been shown that all these ODEs have an infinite number of geodesic type solutions in even dimensions in the form $\mathbf{Q}(r) = \exp\{\mathscr{H}(r)\mathbf{H}\}$ for suitable $\mathscr{H} \in \mathscr{C}^2[a, b]$ and skew-symmetric matrix $\mathbf{H}$. The completely integrable case $F(r, \xi) = h(r)\xi$ is of particular interest and will be discussed in full detail in view of a complete and explicit representation of all solutions.

For the sake of future reference, we end by describing the assumptions on the Lagrangian. Here we assume $F \in \mathscr{C}^2([a,b] \times \mathbb{R})$ and that there exist $c_1, c_2 > 0$ and $c_0 \in \mathbb{R}$ such that

$$\left|F_\xi\left(r,\zeta^2\right)\zeta\right| \le c_2|\zeta|^{p-1}, \quad \forall a \le r \le b, \forall \zeta \in \mathbb{R}, \tag{1.11}$$

$$c_0 + c_1|\zeta|^p \le F\left(r,\zeta^2\right) \le c_2|\zeta|^p, \quad \forall a \le r \le b, \forall \zeta \in \mathbb{R}, \tag{1.12}$$

with $1 < p < \infty$. As a result, $\mathbb{F}$ is well-defined, finite, and coercive on $W^{1,p}(U,\mathbb{R}^n)$. As for convexity, we further assume that $F_\xi > 0$, $F_{\xi\xi} \ge 0$ on $[a,b] \times]0,\infty[$ and that the twice continuously differentiable function $\zeta \mapsto F(r,\zeta^2)$ is uniformly convex in ζ for all $a \le r \le b$ and $\zeta \in \mathbb{R}$.

2 Kinematics of generalised twists $u = r\mathbf{Q}(r)\theta$ and a tensorisation of $\mathscr{L}_F[u]$

In this section we take a closer look at the ODEs listed in the previous section and the relationships they bear to the nonlinear system (1.2) and its generalised twist solutions. We first begin by deriving some basic identities needed later on.

Proposition 2.1 *Let $u = r\mathbf{Q}(r)\theta$ be a generalised twist with twist path $\mathbf{Q} \in \mathscr{C}^1(]a,b[, \mathbf{SO}(n)) \cap \mathscr{C}([a,b], \mathbf{SO}(n))$. Then the following hold:*

(i) $\nabla u = \mathbf{Q} + r\dot{\mathbf{Q}}\theta \otimes \theta$,

(ii) $|\nabla u|^2 = n + r^2|\dot{\mathbf{Q}}\theta|^2$,

(iii) $\det \nabla u = \det(\mathbf{Q} + r\dot{\mathbf{Q}}\theta \otimes \theta) = 1$.

If, in addition, $\mathbf{Q} \in \mathscr{C}^2(]a,b[, \mathbf{SO}(n)) \cap \mathscr{C}([a,b], \mathbf{SO}(n))$, then we also have

(iv) $\Delta u = [(n+1)\dot{\mathbf{Q}} + r\ddot{\mathbf{Q}}]\theta$.

As a consequence, the action of the second order differential operator $\mathscr{L}_F$ on u can be described as follows:

$$\begin{aligned}\mathscr{L}_F[u] = {} & F_{\xi\xi}\left(\mathbf{I}_n + r\mathbf{Q}^t\dot{\mathbf{Q}}\theta \otimes \theta + r\theta \otimes \mathbf{Q}^t\dot{\mathbf{Q}}\theta + r^2|\dot{\mathbf{Q}}\theta|^2\theta \otimes \theta\right) \\ & \times \left(2r|\dot{\mathbf{Q}}\theta|^2\theta + r^2\nabla|\dot{\mathbf{Q}}\theta|^2\right) + F_{r\xi}\left(\theta + r\mathbf{Q}^t\dot{\mathbf{Q}}\theta + r^2|\dot{\mathbf{Q}}\theta|^2\theta\right) \\ & + F_\xi\left[(n+1)\mathbf{Q}^t\dot{\mathbf{Q}} + r\mathbf{Q}^t\ddot{\mathbf{Q}} + \left\{r(n+1)|\dot{\mathbf{Q}}\theta|^2 + r^2\langle\dot{\mathbf{Q}}\theta, \ddot{\mathbf{Q}}\theta\rangle\right\}\mathbf{I}_n\right]\theta.\end{aligned} \tag{2.1}$$

Here, we have set $F_\xi = F_\xi(r,|\nabla u|^2)$, $F_{r\xi} = F_{r\xi}(r,|\nabla u|^2)$ and $F_{\xi\xi} = F_{\xi\xi}(r,|\nabla u|^2)$.

Proof The first identity is obtained by a straightforward differentiation. To justify the incompressibility condition (iii), using (i) we have

$$\det \nabla u = \det(\mathbf{Q} + r\dot{\mathbf{Q}}\theta \otimes \theta) = 1 + r\left\langle\mathbf{Q}^t\dot{\mathbf{Q}}\theta, \theta\right\rangle = 1, \tag{2.2}$$

where the second equality uses the rank-one affine property of the determinant to write $\det(\mathbf{I}_n + \zeta \otimes \xi) = 1 + \langle\zeta,\xi\rangle$ for every pair of vectors $\zeta, \xi \in \mathbb{R}^n$ and the third equality uses the skew-symmetry of $\mathbf{Q}^t\dot{\mathbf{Q}}$. To prove (ii), we calculate the Hilbert–Schmidt norm $|\nabla u|^2 = \operatorname{tr}\{(\nabla u)(\nabla u)^t\} = \operatorname{tr}\{(\mathbf{Q} + r\dot{\mathbf{Q}}\theta \otimes \theta)(\mathbf{Q}^t + r\theta \otimes \dot{\mathbf{Q}}\theta)\}$ by writing

$$\begin{aligned}|\nabla u|^2 &= \operatorname{tr}\left\{(\mathbf{Q} + r\dot{\mathbf{Q}}\theta \otimes \theta)\left(\mathbf{Q}^t + r\theta \otimes \dot{\mathbf{Q}}\theta\right)\right\} \\ &= \operatorname{tr}\left\{\mathbf{I}_n + r\mathbf{Q}\theta \otimes \dot{\mathbf{Q}}\theta + r\dot{\mathbf{Q}}\theta \otimes \mathbf{Q}\theta + r^2\dot{\mathbf{Q}}\theta \otimes \dot{\mathbf{Q}}\theta\right\}\end{aligned}$$

$$
\begin{aligned}
&= n + 2r\langle \mathbf{Q}\theta, \dot{\mathbf{Q}}\theta\rangle + r^2|\dot{\mathbf{Q}}\theta|^2 \\
&= n + r^2|\dot{\mathbf{Q}}\theta|^2.
\end{aligned}
\tag{2.3}
$$

Next (iv) follows by taking the divergence of ∇u as given by (i). Anticipating the final identity, we first recall that $\mathscr{L}_F[u] = (\nabla u)^t \operatorname{div}[F_\xi(|x|, |\nabla u|^2)\nabla u]$. Now a direct differentiation gives

$$
\begin{aligned}
\operatorname{div}\big[F_\xi\big(r, |\nabla u|^2\big)\nabla u\big] = {} & F_{\xi\xi}\big(r, |\nabla u|^2\big)\nabla u \nabla\big(|\nabla u|^2\big) \\
& + F_{r\xi}\big(r, |\nabla u|^2\big)\nabla u\theta + F_\xi\big(r, |\nabla u|^2\big)\Delta u.
\end{aligned}
\tag{2.4}
$$

Moreover, referring to the description of $|\nabla u|^2$ in (ii) above,

$$
\nabla\big(|\nabla u|^2\big) = \nabla\big(\big[n + r^2|\dot{\mathbf{Q}}\theta|^2\big]\big) = \big(2r|\dot{\mathbf{Q}}\theta|^2\theta + r^2\nabla|\dot{\mathbf{Q}}\theta|^2\big).
$$

Thus upon substitution using all the fragments above, we arrive at

$$
\begin{aligned}
\mathscr{L}_F[u] = {} & \big(\mathbf{Q}^t + r\theta \otimes \dot{\mathbf{Q}}\theta\big)\big\{F_{\xi\xi}\big(r, |\nabla u|^2\big)(\mathbf{Q} + r\dot{\mathbf{Q}}\theta \otimes \theta) \\
& \times \big(2r|\dot{\mathbf{Q}}\theta|^2\theta + r^2\nabla|\dot{\mathbf{Q}}\theta|^2\big) + F_{r\xi}\big(r, |\nabla u|^2\big)(\mathbf{Q}\theta + r\dot{\mathbf{Q}}\theta) \\
& + F_\xi\big(r, |\nabla u|^2\big)\big[(n+1)\dot{\mathbf{Q}} + r\ddot{\mathbf{Q}}\big]\theta\big\}.
\end{aligned}
\tag{2.5}
$$

Factorising $\mathbf{Q}$ and multiplying through the term $(\nabla u)^t = (\mathbf{Q}^t + r\theta \otimes \dot{\mathbf{Q}}\theta)$ give the desired conclusion. □

Theorem 2.1 *Let $u = r\mathbf{Q}(r)\theta$ be a generalised twist with twist path $\mathbf{Q} \in \mathscr{C}^2(]a,b[, \mathbf{SO}(n)) \cap \mathscr{C}([a,b], \mathbf{SO}(n))$. Then*

$$
\begin{aligned}
\mathscr{L}_F[u] \otimes \theta - \theta \otimes \mathscr{L}_F[u] = {} & \frac{1}{r^n}\mathbf{Q}^t \frac{d}{dr}\big\{r^{n+1}F_\xi[\dot{\mathbf{Q}}\theta \otimes \mathbf{Q}\theta - \mathbf{Q}\theta \otimes \dot{\mathbf{Q}}\theta]\big\}\mathbf{Q} \\
& + \nabla F_\xi \otimes \theta - \theta \otimes \nabla F_\xi,
\end{aligned}
\tag{2.6}
$$

where $\mathscr{L}_F[u]$ is as in (2.1) and for brevity $F_\xi = F_\xi(r, n + r^2|\dot{\mathbf{Q}}\theta|^2)$.

Proof For the sake of convenience, let us set $\mathbf{A} = \mathbf{Q}^t\dot{\mathbf{Q}}$. Then action (2.1) of the differential operator $\mathscr{L}_F$ on the twist u can be rewritten as follows:

$$
\begin{aligned}
\mathscr{L}_F[u] = {} & \nabla F_\xi\big(r, n + r^2|\mathbf{A}\theta|^2\big) + \mathscr{A}(r,\theta)\theta \\
& + \frac{1}{r^n}\frac{d}{dr}\big[r^{n+1}F_\xi\big(r, n + r^2|\mathbf{A}\theta|^2\big)\mathbf{A}\big]\theta \\
& + rF_\xi\big(r, n + r^2|\mathbf{A}\theta|^2\big)\mathbf{A}^2\theta.
\end{aligned}
\tag{2.7}
$$

Here, the first term on the right-hand side in the above is given by

$$
\begin{aligned}
\nabla F_\xi\big(r, n + r^2|\mathbf{A}\theta|^2\big) = {} & F_{r\xi}\big(r, n + r^2|\mathbf{A}\theta|^2\big)\theta \\
& + F_{\xi\xi}\big(r, n + r^2|\mathbf{A}\theta|^2\big)\bigg[\frac{d}{dr}\big(r^2|\mathbf{A}\theta|^2\big)\theta - 2r\mathbf{A}^2\theta - 2r|\mathbf{A}\theta|^2\theta\bigg],
\end{aligned}
$$

while we have introduced the scalar-valued function $\mathscr{A} = \mathscr{A}(r,\theta)$ to denote the coefficient of the vector θ in the description of $\mathscr{L}_F[u]$ which is specifically given by the collection of terms

$$\begin{aligned}\mathscr{A}(r,\theta) = r\Big\{ & F_\xi\big(r, n + r^2|\mathbf{A}\theta|^2\big)\big[(n+1)|\mathbf{A}\theta|^2 + r\langle \mathbf{A}\theta, \dot{\mathbf{A}}\theta\rangle\big] \\ & + rF_{r\xi}\big(r, n + r^2|\mathbf{A}\theta|^2\big)|\mathbf{A}\theta|^2 \\ & + rF_{\xi\xi}\big(r, n + r^2|\mathbf{A}\theta|^2\big)|\mathbf{A}\theta|^2 \frac{d}{dr}\big(r^2|\mathbf{A}\theta|^2\big)\Big\}. \end{aligned} \tag{2.8}$$

Now, moving forward by using the formulation (2.7), it is seen upon tensorisation that we have

$$\begin{aligned}\mathscr{L}_F[u]\otimes\theta - \theta\otimes\mathscr{L}_F[u] = {} & \nabla F_\xi \otimes\theta - \theta\otimes\nabla F_\xi \\ & + r\big(F_\xi \mathbf{A}^2\theta\otimes\theta - \theta\otimes F_\xi\mathbf{A}^2\theta\big) \\ & + \frac{1}{r^n}\frac{d}{dr}\big\{r^{n+1}F_\xi[\mathbf{A}\theta\otimes\theta - \theta\otimes\mathbf{A}\theta]\big\}, \end{aligned} \tag{2.9}$$

where in deducing this identity use has been made of the pointwise relation

$$\mathscr{A}(r,\theta)\theta\otimes\theta - \theta\otimes\mathscr{A}(r,\theta)\theta = \mathscr{A}(r,\theta)[\theta\otimes\theta - \theta\otimes\theta] = 0. \tag{2.10}$$

Next referring to the last expression on the right-hand side in (2.9), upon momentarily ignoring the factor $1/r^n$, we can write

$$\begin{aligned} & \frac{d}{dr}\big\{r^{n+1}F_\xi[\mathbf{A}\theta\otimes\theta - \theta\otimes\mathbf{A}\theta]\big\} \\ & \quad = \frac{d}{dr}\big\{r^{n+1}F_\xi\mathbf{Q}^t[\dot{\mathbf{Q}}\theta\otimes\mathbf{Q}\theta - \mathbf{Q}\theta\otimes\dot{\mathbf{Q}}\theta]\mathbf{Q}\big\} \\ & \quad = \mathbf{Q}^t\frac{d}{dr}\big\{r^{n+1}F_\xi[\dot{\mathbf{Q}}\theta\otimes\mathbf{Q}\theta - \mathbf{Q}\theta\otimes\dot{\mathbf{Q}}\theta]\big\}\mathbf{Q} \\ & \qquad + \dot{\mathbf{Q}}^t r^{n+1}F_\xi[\dot{\mathbf{Q}}\theta\otimes\mathbf{Q}\theta - \mathbf{Q}\theta\otimes\dot{\mathbf{Q}}\theta]\mathbf{Q} \\ & \qquad + \mathbf{Q}^t r^{n+1}F_\xi[\dot{\mathbf{Q}}\theta\otimes\mathbf{Q}\theta - \mathbf{Q}\theta\otimes\dot{\mathbf{Q}}\theta]\dot{\mathbf{Q}} \\ & \quad = \mathrm{I} + \mathrm{II} + \mathrm{III}. \end{aligned} \tag{2.11}$$

We can next simplify the sum of the second and third terms in (2.11) by writing

$$\begin{aligned}\frac{(\mathrm{II}+\mathrm{III})}{r^{n+1}} = {} & \dot{\mathbf{Q}}^t F_\xi[\dot{\mathbf{Q}}\theta\otimes\mathbf{Q}\theta - \mathbf{Q}\theta\otimes\dot{\mathbf{Q}}\theta]\mathbf{Q} \\ & + \mathbf{Q}^t F_\xi[\dot{\mathbf{Q}}\theta\otimes\mathbf{Q}\theta - \mathbf{Q}\theta\otimes\dot{\mathbf{Q}}\theta]\dot{\mathbf{Q}} \\ = {} & F_\xi\big\{\dot{\mathbf{Q}}^t\dot{\mathbf{Q}}\theta\otimes\theta - \dot{\mathbf{Q}}^t\mathbf{Q}\theta\otimes\mathbf{Q}^t\dot{\mathbf{Q}}\theta + \mathbf{Q}^t\dot{\mathbf{Q}}\theta\otimes\dot{\mathbf{Q}}^t\mathbf{Q}\theta - \theta\otimes\dot{\mathbf{Q}}^t\dot{\mathbf{Q}}\theta\big\}, \end{aligned} \tag{2.12}$$

and subsequently invoking the orthogonality of $\mathbf{Q}$ and in particular the skew-symmetry of $\dot{\mathbf{Q}}^t\mathbf{Q}$ to set the sum of the middle two terms in (2.12) to zero. Therefore, by returning to (2.11) and taking advantage of the above, we have

$$\frac{d}{dr}\left\{r^{n+1}F_\xi[\mathbf{A}\theta\otimes\theta-\theta\otimes\mathbf{A}\theta]\right\}=\mathbf{Q}^t\frac{d}{dr}\left\{r^{n+1}F_\xi[\dot{\mathbf{Q}}\theta\otimes\mathbf{Q}\theta-\mathbf{Q}\theta\otimes\dot{\mathbf{Q}}\theta]\right\}\mathbf{Q}$$
$$+r^{n+1}F_\xi\left\{\dot{\mathbf{Q}}^t\dot{\mathbf{Q}}\theta\otimes\theta-\theta\otimes\dot{\mathbf{Q}}^t\dot{\mathbf{Q}}\theta\right\}. \tag{2.13}$$

Hence substituting all the above in (2.9) and noting that $-\mathbf{A}^2=\mathbf{A}^t\mathbf{A}=\dot{\mathbf{Q}}^t\dot{\mathbf{Q}}$ (note the skew-symmetry of $\mathbf{A}$), we can write the desired tensor quantity as

$$\begin{aligned}\mathscr{L}_F[u]\otimes\theta-\theta\otimes\mathscr{L}_F[u]&=\nabla F_\xi\otimes\theta-\theta\otimes\nabla F_\xi\\&\quad-rF_\xi\left\{\dot{\mathbf{Q}}^t\dot{\mathbf{Q}}\theta\otimes\theta-\theta\otimes\dot{\mathbf{Q}}^t\dot{\mathbf{Q}}\theta\right\}\\&\quad+\frac{1}{r^n}\mathbf{Q}^t\frac{d}{dr}\left\{r^{n+1}F_\xi[\dot{\mathbf{Q}}\theta\otimes\mathbf{Q}\theta-\mathbf{Q}\theta\otimes\dot{\mathbf{Q}}\theta]\right\}\mathbf{Q}\\&\quad+rF_\xi\left\{\dot{\mathbf{Q}}^t\dot{\mathbf{Q}}\theta\otimes\theta-\theta\otimes\dot{\mathbf{Q}}^t\dot{\mathbf{Q}}\theta\right\},\end{aligned} \tag{2.14}$$

which after cancellations leads at once to the required conclusion. □

Proposition 2.2 *Let $u=r\mathbf{Q}(r)\theta$ be a generalised twist with twist path $\mathbf{Q}\in\mathscr{C}^2(]a,b[,\mathbf{SO}(n))\cap\mathscr{C}([a,b],\mathbf{SO}(n))$. Then, for each $a<r<b$,*

$$\int_{\mathbb{S}^{n-1}}\left(\mathscr{L}_F[u]\otimes\theta-\theta\otimes\mathscr{L}_F[u]\right)d\mathcal{H}^{n-1}(\theta)$$
$$=\frac{1}{r^n}\mathbf{Q}^t\left\{\int_{\mathbb{S}^{n-1}}\frac{d}{dr}\left\{r^{n+1}F_\xi[\dot{\mathbf{Q}}\theta\otimes\mathbf{Q}\theta-\mathbf{Q}\theta\otimes\dot{\mathbf{Q}}\theta]\right\}d\mathcal{H}^{n-1}(\theta)\right\}\mathbf{Q}, \tag{2.15}$$

where $\mathscr{L}_F[u]$ is as in (2.1) and for the sake of brevity $F_\xi=F_\xi(r,n+r^2|\dot{\mathbf{Q}}\theta|^2)$.

Proof Fix $a<r<b$ and denote the integral on the left-hand side in (2.15) by $\mathscr{I}=\mathscr{I}(r)$. Then, using the description of the integrand as given by (2.6), we can write

$$\begin{aligned}\mathscr{I}(r)&=\int_{\mathbb{S}^{n-1}}\left(\mathscr{L}_F[u]\otimes\theta-\theta\otimes\mathscr{L}_F[u]\right)d\mathcal{H}^{n-1}(\theta)\\&=\int_{\mathbb{S}^{n-1}}\left\{\frac{1}{r^n}\mathbf{Q}^t\frac{d}{dr}\left\{r^{n+1}F_\xi[\dot{\mathbf{Q}}\theta\otimes\mathbf{Q}\theta-\mathbf{Q}\theta\otimes\dot{\mathbf{Q}}\theta]\right\}\mathbf{Q}\right.\\&\quad\left.+\nabla F_\xi\otimes\theta-\theta\otimes\nabla F_\xi\right\}d\mathcal{H}^{n-1}(\theta)\\&=\int_{\mathbb{S}^{n-1}}\frac{1}{r^n}\mathbf{Q}^t\frac{d}{dr}\left\{r^{n+1}F_\xi[\dot{\mathbf{Q}}\theta\otimes\mathbf{Q}\theta-\mathbf{Q}\theta\otimes\dot{\mathbf{Q}}\theta]\right\}\mathbf{Q}\,d\mathcal{H}^{n-1}(\theta),\end{aligned} \tag{2.16}$$

where in concluding the last line we have taken advantage of Lemma 2.1 below with $\mathscr{P}=F_\xi(|x|,|\nabla u|^2)$. This is the required conclusion. □

Proposition 2.3 *Let $u=r\mathbf{Q}(r)\theta$ be a generalised twist in $\mathbb{X}^n$ with twist path $\mathbf{Q}\in\mathscr{C}^2(]a,b[,\mathbf{SO}(n))\cap\mathscr{C}([a,b],\mathbf{SO}(n))$. Then, if $\mathscr{L}_F[u]=\nabla\mathscr{P}$ in $\mathbb{X}^n$, the twist path $\mathbf{Q}$ satisfies, for all $a<r<b$,*

$$\int_{\mathbb{S}^{n-1}}\frac{d}{dr}\left\{r^{n+1}F_\xi\left(r,n+r^2|\dot{\mathbf{Q}}\theta|^2\right)[\dot{\mathbf{Q}}\theta\otimes\mathbf{Q}\theta-\mathbf{Q}\theta\otimes\dot{\mathbf{Q}}\theta]\right\}d\mathcal{H}^{n-1}(\theta)=0. \tag{2.17}$$

Proof From $\mathscr{L}_F[u] = \nabla\mathscr{P}$ it follows upon integration over the sphere and use of Lemma 2.1 below that

$$\begin{aligned}\mathscr{I}(r) &= \int_{\mathbb{S}^{n-1}} \big(\mathscr{L}_F[u]\otimes\theta - \theta\otimes\mathscr{L}_F[u]\big)\,d\mathcal{H}^{n-1}(\theta)\\ &= \int_{\mathbb{S}^{n-1}} [\nabla\mathscr{P}\otimes\theta - \theta\otimes\nabla\mathscr{P}]\,d\mathcal{H}^{n-1}(\theta) = 0, \quad a<r<b.\end{aligned} \tag{2.18}$$

Now a reference to (2.15) and noting the invertibility of $\mathbf{Q}$ give the desired conclusion. The proof is thus complete. □

Lemma 2.1 *Let $\mathscr{P}\in\mathscr{C}^1(U)$ with $U\subset\mathbb{R}^n$ an open neighbourhood of $\mathbb{S}^{n-1}$. Then*

$$\int_{\mathbb{S}^{n-1}} [\nabla\mathscr{P}\otimes\theta - \theta\otimes\nabla\mathscr{P}]\,d\mathcal{H}^{n-1}(\theta) = 0. \tag{2.19}$$

If U contains the closed unit ball $\mathcal{B}$ and $\mathscr{P}\in\mathscr{C}^2(\mathcal{B})$, then the conclusion is easily seen to follow by an application of the divergence theorem on $\mathcal{B}$. Therefore one route to justifying (2.19) is first to extend $\mathscr{P}$ to a neighbourhood of $\mathcal{B}$, e.g., by multiplying $\mathscr{P}$ by a suitable cut-off function $\phi\in\mathscr{C}_c^\infty(\mathbb{R}^n)$ satisfying $\phi\equiv 1$ near $\partial\mathcal{B}$, then mollifying the resulting extension and finally arguing by using the divergence theorem and passing to the limit. A more direct route, however, avoiding any extension and approximation is given below.

Proof Firstly, by restricting to the surface of the unit sphere and splitting the gradient into tangential and normal parts in the usual way, we can write

$$\nabla\mathscr{P} = (\mathbf{I}_n - \theta\otimes\theta)\nabla\mathscr{P} + \langle\nabla\mathscr{P},\theta\rangle\theta = \nabla_T\mathscr{P} + \nabla_N\mathscr{P}. \tag{2.20}$$

It is seen that $\nabla_N\mathscr{P}\otimes\theta - \theta\otimes\nabla_N\mathscr{P} = 0$, and so to establish (2.19) it suffices to justify the integral identity

$$\int_{\mathbb{S}^{n-1}} [\nabla_T\mathscr{P}\otimes\theta - \theta\otimes\nabla_T\mathscr{P}]\,d\mathcal{H}^{n-1}(\theta) = 0. \tag{2.21}$$

Now by a direct differentiation it is evident that $\nabla_T(\mathscr{P}\theta) = \theta\otimes\nabla_T\mathscr{P} + \mathscr{P}\nabla_T\theta$, and so referring to (2.20), we can write

$$\begin{aligned}\nabla_T\mathscr{P}\otimes\theta - \theta\otimes\nabla_T\mathscr{P} &= \big[\nabla_T(\mathscr{P}\theta) - \mathscr{P}\nabla_T\theta\big]^t - \big[\nabla_T(\mathscr{P}\theta) - \mathscr{P}\nabla_T\theta\big]\\ &= \big[\nabla_T(\mathscr{P}\theta)\big]^t - \big[\nabla_T(\mathscr{P}\theta)\big].\end{aligned} \tag{2.22}$$

Here, in deducing the second identity, we have taken into account the symmetry $\nabla_T\theta = [\nabla_T\theta]^t = \mathbf{I}_n - \theta\otimes\theta$. The conclusion now follows by integrating (2.22) and invoking the divergence theorem on the sphere. □

Lemma 2.2 *Let $\mathsf{A}\in\mathscr{C}(\mathbb{R})$ and suppose that $\mathbf{F}\in\mathbb{M}^{n\times n}$ is fixed. Then we have*

$$\int_{\mathbb{S}^{n-1}} \mathsf{A}\big(|\mathbf{F}\theta|^2\big)\big[\mathbf{F}^t\mathbf{F}\theta\otimes\theta - \theta\otimes\mathbf{F}^t\mathbf{F}\theta\big]\,d\mathcal{H}^{n-1}(\theta) = 0. \tag{2.23}$$

Proof This follows by noting that for B taken as a primitive of A upon setting $\mathscr{P}(x) = \mathsf{B}(|\mathbf{F}x|^2)$ we have $\nabla\mathscr{P} = 2\mathsf{A}(|\mathbf{F}x|^2)\mathbf{F}^t\mathbf{F}x$. The conclusion now follows by applying Lemma 2.1. □

Interestingly, the conclusion of Proposition 2.3 and ODE (2.17) can be given a different interpretation and derivation by considering a *restricted* energy functional $\mathbb{E}[\mathbf{Q}, a, b] = \mathbb{F}[r\mathbf{Q}(r)\theta, \mathbb{X}^n]$ written as

$$\mathbb{E}[\mathbf{Q}, a, b] = \int_a^b E(r, \dot{\mathbf{Q}}) r^{n-1}\, dr. \tag{2.24}$$

Here, the integrand $E = E(r, \mathbf{H})$ is given for $a \le r \le b$ and $n \times n$ matrix $\mathbf{H}$ in turn (we are extending the definition from skew-symmetric $\mathbf{H}$ to all matrices as this is needed in the next proposition) by a spherical integral in the form

$$E(r, \mathbf{H}) = \int_{\mathbb{S}^{n-1}} F\big(r, n + r^2|\mathbf{H}\theta|^2\big) r^{n-1}\, d\mathcal{H}^{n-1}(\theta). \tag{2.25}$$

Then upon setting $\mathscr{B}_p = \{\mathbf{Q} \in W^{1,p}(a, b; \mathbf{SO}(n)) : \mathbf{Q}(a) = \mathbf{Q}(b) = \mathbf{I}_n\}$ we can formulate the following statement.

Proposition 2.4 *ODE* (2.17) *is precisely the Euler–Lagrange equation associated with the energy functional* $\mathbb{E}$ *over* $\mathscr{B}_p$.

Proof To see this fix $\mathbf{Q}$ and for $\varepsilon \in \mathbb{R}$ sufficiently small set $\mathbf{Q}_\varepsilon = \mathbf{Q} + \varepsilon(\mathbf{F} - \mathbf{F}^t)\mathbf{Q}$ with $\mathbf{F} \in \mathscr{C}_0^\infty(]a, b[, \mathbb{M}^{n\times n})$. It can be seen that $\mathbf{Q}_\varepsilon$, up to the first order in ε, takes values in $\mathbf{SO}(n)$, that is, $\mathbf{Q}_\varepsilon^t\mathbf{Q}_\varepsilon = \mathbf{I}_n + O(\varepsilon^2) = \mathbf{Q}_\varepsilon\mathbf{Q}_\varepsilon^t$, and so the assertion follows by setting $d\mathbb{E}[\mathbf{Q}_\varepsilon, a, b]/d\varepsilon|_{\varepsilon=0} = 0$, namely,

$$\begin{aligned}
&\frac{d}{d\varepsilon}\mathbb{E}[\mathbf{Q}_\varepsilon, a, b]\bigg|_{\varepsilon=0} \\
&\quad= \frac{d}{d\varepsilon}\int_a^b E(r, \dot{\mathbf{Q}}_\varepsilon) r^{n-1}\, dr\bigg|_{\varepsilon=0} \\
&\quad= \int_a^b \int_{\mathbb{S}^{n-1}} r^{n+1} F_\xi\big(r, n + r^2|\dot{\mathbf{Q}}_\varepsilon\theta|^2\big) \frac{d}{d\varepsilon}|\dot{\mathbf{Q}}_\varepsilon\theta|^2\, dr\, d\mathcal{H}^{n-1}(\theta)\bigg|_{\varepsilon=0}.
\end{aligned}$$

Now it is easily seen that $[d|\dot{\mathbf{Q}}_\varepsilon\theta|^2/d\varepsilon]|_{\varepsilon=0} = 2\langle\dot{\mathbf{Q}}\theta, [(\dot{\mathbf{F}} - \dot{\mathbf{F}}^t)\mathbf{Q} + (\mathbf{F} - \mathbf{F}^t)\dot{\mathbf{Q}}]\theta\rangle$, and so, writing $F_\xi = F_\xi(r, n + r^2|\dot{\mathbf{Q}}\theta|^2)$ for short, the integrand can be written as

$$r^{n+1}F_\xi\big[d|\dot{\mathbf{Q}}_\varepsilon\theta|^2/d\varepsilon\big]|_{\varepsilon=0} = 2r^{n+1}F_\xi\big[\big\langle\dot{\mathbf{Q}}\theta \otimes \mathbf{Q}\theta, \dot{\mathbf{F}} - \dot{\mathbf{F}}^t\big\rangle + \big\langle\dot{\mathbf{Q}}\theta \otimes \dot{\mathbf{Q}}\theta, \mathbf{F} - \mathbf{F}^t\big\rangle\big].$$

The last term is zero by the skew-symmetry of $\mathbf{F} - \mathbf{F}^t$, and so an integration by parts on the remaining terms followed by an application of the fundamental lemma of the calculus of variations now gives by virtue of the arbitrariness of $\mathbf{F}$ the desired conclusion. □

We close this section with a study of the relationships between ODEs (1.8)–(1.10). Recall that here $\mathscr{M} = \mathscr{M}[\mathbf{Q}] = 1/r^n d/dr[r^{n+1}F_\xi\mathbf{Q}^t\dot{\mathbf{Q}}\theta]$ [*cf.* (1.10)].

Theorem 2.2 *Let* $\mathbf{Q} \in \mathscr{C}^2(]a,b[,\mathbf{SO}(n)) \cap \mathscr{C}([a,b],\mathbf{SO}(n))$ *be an arbitrary twist path. Then we have*

$$\begin{aligned}\mathscr{M}[\mathbf{Q}] \otimes \theta - \theta \otimes \mathscr{M}[\mathbf{Q}] = {} & \frac{1}{r^n}\mathbf{Q}^t \frac{d}{dr}\left\{r^{n+1}F_\xi[\dot{\mathbf{Q}}\theta \otimes \mathbf{Q}\theta - \mathbf{Q}\theta \otimes \dot{\mathbf{Q}}\theta]\right\}\mathbf{Q} \\ & + rF_\xi\left\{\dot{\mathbf{Q}}^t\dot{\mathbf{Q}}\theta \otimes \theta - \theta \otimes \dot{\mathbf{Q}}^t\dot{\mathbf{Q}}\theta\right\}. \end{aligned} \tag{2.26}$$

In particular, for $a < r < b$, *we have*

$$\begin{aligned} & \int_{\mathbb{S}^{n-1}} \left(\mathscr{M}[\mathbf{Q}] \otimes \theta - \theta \otimes \mathscr{M}[\mathbf{Q}]\right) d\mathcal{H}^{n-1}(\theta) \\ & \quad = \frac{1}{r^n}\mathbf{Q}^t\left\{\int_{\mathbb{S}^{n-1}} \frac{d}{dr}\left\{r^{n+1}F_\xi[\dot{\mathbf{Q}}\theta \otimes \mathbf{Q}\theta - \mathbf{Q}\theta \otimes \dot{\mathbf{Q}}\theta]\right\} d\mathcal{H}^{n-1}(\theta)\right\}\mathbf{Q}. \end{aligned} \tag{2.27}$$

Proof These follow at once by putting together the earlier results in the section. □

Let us finish off the section by discussing some particular consequences of Theorem 2.1, Proposition 2.2 and Theorem 2.2 as appearing earlier. Here $\mathscr{L}_F = \mathscr{L}_F[u]$ and $\mathscr{M} = \mathscr{M}[\mathbf{Q}]$ are as before and $u = r\mathbf{Q}(r)\theta$ is a generalised twist with $\mathbf{Q} \in \mathscr{C}^2(]a,b[,\mathbf{SO}(n)) \cap \mathscr{C}([a,b],\mathbf{SO}(n))$.

- Differential operators relation $\mathscr{L}_F[u]$ *vs.* $\mathscr{M}[\mathbf{Q}]$:

$$\begin{aligned} & \left(\mathscr{L}_F[u] - \nabla F_\xi\right) \otimes \theta - \theta \otimes \left(\mathscr{L}_F[u] - \nabla F_\xi\right) \\ & \quad = \mathscr{M}[\mathbf{Q}] \otimes \theta - \theta \otimes \mathscr{M}[\mathbf{Q}] \otimes \theta - rF_\xi\left[\dot{\mathbf{Q}}^t\dot{\mathbf{Q}}\theta \otimes \theta - \theta \otimes \dot{\mathbf{Q}}^t\dot{\mathbf{Q}}\theta\right]. \end{aligned} \tag{2.28}$$

- Spherical integrals: For $a < r < b$,

$$\begin{aligned} & \int_{\mathbb{S}^{n-1}} \left(\mathscr{L}_F[u] \otimes \theta - \theta \otimes \mathscr{L}_F[u]\right) d\mathcal{H}^{n-1}(\theta) \\ & \quad = \int_{\mathbb{S}^{n-1}} \left[\mathscr{M}[\mathbf{Q}] \otimes \theta - \theta \otimes \mathscr{M}[\mathbf{Q}]\right] d\mathcal{H}^{n-1}(\theta) \\ & \quad = \frac{1}{r^n}\left\{\mathbf{Q}^t \int_{\mathbb{S}^{n-1}} \frac{d}{dr}\left\{r^{n+1}F_\xi[\dot{\mathbf{Q}}\theta \otimes \mathbf{Q}\theta - \mathbf{Q}\theta \otimes \dot{\mathbf{Q}}\theta]\right\} d\mathcal{H}^{n-1}(\theta)\right\}\mathbf{Q}. \end{aligned} \tag{2.29}$$

- If either of $\mathscr{L}_F[u] = \nabla\mathscr{P}$, $\mathscr{M}[\mathbf{Q}] = 0$ or (1.8) [in particular the strengthened ODE (1.9)] holds, then necessarily all the above integrals vanish.

3 Curl-free vector fields generated by symmetric matrix fields in $\mathbb{X}^n[a,b]$

In this section we present some results on curl-free vector fields as needed later on in relation to the PDE $\mathscr{L}_F[u] = \nabla\mathscr{P}$ and its twisting solutions.

Proposition 3.1 *Let* $\mathsf{A}, \mathsf{B} \in \mathscr{C}^1(]a,b[,\mathbb{M}^{n\times n})$ *be symmetric matrix fields. Consider the vector field* $v \in \mathscr{C}^1(\mathbb{X}^n,\mathbb{R}^n)$ *defined by*

$$v(x) = \mathscr{A}(r,z)x + \mathscr{B}(r,z)\mathsf{B}(r)x, \quad x \in \mathbb{X}^n, \tag{3.1}$$

where $r = |x|$, $z = \langle \mathsf{A}(|x|)x, x\rangle$ *and* $\mathscr{A}, \mathscr{B} \in \mathscr{C}^1(]a,b[\times\mathbb{R},\mathbb{R})$. *Then*

$$\operatorname{curl} v = \mathsf{F}(r,z)x \otimes x - x \otimes \mathsf{F}(r,z)x + 2\mathscr{B}_z(r,z)[\mathsf{B}x \otimes \mathsf{A}x - \mathsf{A}x \otimes \mathsf{B}x], \tag{3.2}$$

where $\mathsf{F}(r,z)$ *is the symmetric matrix field given by*

$$\mathsf{F}(r,z) = -2\mathscr{A}_z(r,z)\mathsf{A} + \frac{1}{r}\big\{\mathscr{B}(r,z)\dot{\mathsf{B}} + \big[\mathscr{B}_r(r,z) + \langle\dot{\mathsf{A}}x,x\rangle\mathscr{B}_z(r,z)\big]\mathsf{B}\big\}. \tag{3.3}$$

A quick remark on notation: here and throughout the proof the dot notation is for the derivatives with respect to r of the matrix fields A and B. Moreover, $\mathscr{A}_r$, $\mathscr{A}_z$ denote the derivatives of $\mathscr{A} = \mathscr{A}(r,z)$ with respect to the first and second variables respectively, with the same notation used for $\mathscr{B}$.

Proof With $v = (v_1,\dots,v_n)$ and $1 \le i < j \le n$, we have $[\operatorname{curl} v]_{ij} = v_{i,j} - v_{j,i}$. Indeed

$$\begin{aligned} v_{i,j} &= \big[\mathscr{A}(r,z)x_i + \mathscr{B}(r,z)\mathsf{B}_{il}x_l\big]_{,j} \\ &= \mathscr{A}_r(r,z)\theta_j x_i + \mathscr{A}_z(r,z)\big[\langle\dot{\mathsf{A}}x,x\rangle\theta_j + 2\mathsf{A}_{jl}x_l\big]x_i + \mathscr{A}(r,z)\delta_{ij} \\ &\quad + \mathscr{B}_r(r,z)\theta_j\mathsf{B}_{il}x_l + \mathscr{B}_z(r,z)\big[\langle\dot{\mathsf{A}}x,x\rangle\theta_j + 2\mathsf{A}_{jl}x_l\big]\mathsf{B}_{il}x_l \\ &\quad + \mathscr{B}(r,z)\dot{\mathsf{B}}_{il}\theta_j x_l + \mathscr{B}(r,z)\mathsf{B}_{ij}, \end{aligned} \tag{3.4}$$

where we are summing over $1 \le l \le n$. A similar computation for $v_{j,i}$ gives

$$\begin{aligned} v_{j,i} &= \mathscr{A}_r(r,z)\theta_i x_j + \mathscr{A}_z(r,z)\big[\langle\dot{\mathsf{A}}x,x\rangle\theta_i + 2\mathsf{A}_{il}x_l\big]x_j + \mathscr{A}(r,z)\delta_{ji} \\ &\quad + \mathscr{B}_r(r,z)\theta_i\mathsf{B}_{jl}x_l + \mathscr{B}_z(r,z)\big[\langle\dot{\mathsf{A}}x,x\rangle\theta_i + 2\mathsf{A}_{il}x_l\big]\mathsf{B}_{jl}x_l \\ &\quad + \mathscr{B}(r,z)\dot{\mathsf{B}}_{jl}\theta_i x_l + \mathscr{B}(r,z)\mathsf{B}_{ji}. \end{aligned} \tag{3.5}$$

After making the appropriate cancellations, using the symmetry of B and writing in tensor notation, we arrive at (3.2)–(3.3). □

Proposition 3.2 *Let* F *be a fixed* $n \times n$ *matrix and put* $\mathbb{S}_{\mathsf{F}}[\theta] := \mathsf{F}\theta \otimes \theta - \theta \otimes \mathsf{F}\theta$. *Then* $\mathbb{S}_{\mathsf{F}}[\theta] \equiv 0$ *for all* $\theta \in \mathbb{S}^{n-1}$ *iff* $\mathsf{F} = \mathsf{f}\mathbf{I}_n$ *for some* $\mathsf{f} \in \mathbb{R}$.

Proof If we first take $\mathsf{F} = \mathsf{f}\mathbf{I}_n$, then $\mathbb{S}_{\mathsf{f}\mathbf{I}_n}[\theta] = \mathsf{f}[\theta \otimes \theta - \theta \otimes \theta] \equiv 0$. Conversely, if $\mathbb{S}_{\mathsf{F}}[\theta] \equiv 0$, then taking $\theta \in \{e_1,\dots,e_n\}$ – the standard basis of $\mathbb{R}^n$ – it is seen that F is diagonal. Proceeding with $\mathsf{F} = \operatorname{diag}(\mathsf{f}_1,\dots,\mathsf{f}_n)$, the condition $\mathbb{S}_{\mathsf{F}}[\theta] \equiv 0$ is equivalent to $\theta_i\theta_j(\mathsf{f}_i - \mathsf{f}_j) \equiv 0$ for all $1 \le i,j \le n$. As such $\mathsf{f}_1 = \cdots = \mathsf{f}_n =: \mathsf{f}$ and the result follows. □

We now look at the consequences of Proposition 3.1 in certain contexts that will be needed or used later on.

- First consider the vector field $v = \langle\mathsf{A}(|x|)x,x\rangle x + \mathsf{B}(|x|)x$ corresponding to taking $\mathscr{A}(r,z) = z$ hence $\mathscr{A}_z(r,z) \equiv 1$ and $\mathscr{B}(r,z) \equiv 1$ giving $\mathscr{B}_r, \mathscr{B}_z \equiv 0$. Then $\operatorname{curl} v = \mathsf{F}x \otimes x - x \otimes \mathsf{F}x$ with $\mathsf{F}(r,z) = \mathsf{F}(r) = -2\mathsf{A} + \dot{\mathsf{B}}/r$, and so it follows from Proposition 3.1 and Proposition 3.2 that v is curl-free in $\mathbb{X}^n$ *iff* $\mathsf{F}(r) = -2\mathsf{A} + \dot{\mathsf{B}}/r = \sigma(r)\mathbf{I}_n$ for some $\sigma \in \mathscr{C}(]a,b[)$.
- Pick v as in the previous example and suppose $\mathsf{A}(|x|) = \mathsf{a}(|x|)\mathsf{S}$, $\mathsf{B}(|x|) = \mathsf{b}(|x|)\mathsf{S}$ with a and b radial functions and S a constant symmetric matrix. Then here

$\mathsf{F} = \mathsf{F}(r) = [-2\mathsf{a} + \dot{\mathsf{b}}/r]\mathsf{S}$, and so we have

$$\operatorname{curl} v \equiv 0 \quad \text{in } \mathbb{X}^n \quad \iff \quad \begin{cases} -2\mathsf{a} + \dot{\mathsf{b}}/r \equiv 0 & \text{on }]a,b[, \\ or \\ \mathsf{S} = \mathsf{f}\mathbf{I}_n, & \mathsf{f} \in \mathbb{R}. \end{cases} \tag{3.6}$$

- Now assume $v = \mathsf{B}(|x|)x$. Then with $\mathscr{A}(r,z) \equiv 0$, $\mathscr{B}(r,z) \equiv 1$ it follows from Proposition 3.1 that $\operatorname{curl} v \equiv 0$ in $\mathbb{X}^n \iff \dot{\mathsf{B}}x \otimes x - x \otimes \dot{\mathsf{B}}x = 0$, which by Proposition 3.2 is true *iff* $\dot{\mathsf{B}} = \sigma \mathbf{I}_n$ with $\sigma \in \mathscr{C}(]a,b[)$. Integration then gives $\mathsf{B} = s(r)\mathbf{I}_n + \mathsf{C}$ with $\dot{s} = \sigma$ and C a constant symmetric matrix. If $\mathsf{B}(r) = \mathsf{b}(r)\mathsf{S}$ with $\mathsf{b} \in \mathscr{C}^1(]a,b[)$ and S constant, then $\operatorname{curl} v \equiv 0$ in $\mathbb{X}^n$ *iff* either B is constant or $\mathsf{S} = \mathsf{f}\mathbf{I}_n$ for some $\mathsf{f} \in \mathbb{R}$ in which case $\mathsf{B}(r) = \mathsf{f}\mathsf{b}(r)\mathbf{I}_n$. In particular the conclusion above holds with $\mathsf{s}(r) = \mathsf{f}\mathsf{b}(r)$ and $\mathsf{C} = 0$.
- Finally, consider the vector field $v = f(r, |\mathbf{H}x|^2)x + g(r, |\mathbf{H}x|^2)\mathbf{H}^2 x$, where $\mathbf{H}$ is a constant skew-symmetric matrix and $f, g \in \mathscr{C}^1(]a,b[\times \mathbb{R}, \mathbb{R})$. Then we are in the setting of Proposition 3.1 with $\mathscr{A}(r,z) = f(r,z)$, $\mathscr{B}(r,z) = g(r,z)$, $\mathsf{B} = \mathbf{H}^2$ and $\mathsf{A} = -\mathsf{B}$. The symmetric vector field $\mathsf{F}(r,z)$ in (3.3) here reduces to $\mathsf{F}(r,z) = [2f_z + g_r/r]\mathbf{H}^2$ as $\dot{\mathsf{A}} = \dot{\mathsf{B}} = 0$, while in (3.2) we have $\mathsf{A}x \otimes \mathsf{B}x - \mathsf{B}x \otimes \mathsf{A}x = 0$. Therefore a further application of Proposition 3.2 gives

$$\operatorname{curl} v \equiv 0 \quad \text{in } \mathbb{X}^n \quad \iff \quad \begin{cases} 2f_z(r,z) + g_r(r,z)/r \equiv 0 & \text{on }]a,b[, \\ or \\ \mathbf{H}^2 = \mathsf{f}\mathbf{I}_n, & \mathsf{f} \in \mathbb{R}. \end{cases} \tag{3.7}$$

4 Explicit solutions for the weighted Dirichlet type Lagrangians $F(r,\xi) = h(r)\xi$

In this section we take on the case $F(r,\xi) = h(r)\xi$ with $h > 0$ in $\mathscr{C}^1[a,b]$. The Euler–Lagrange equation for the restricted functional here takes the form (*cf.* Proposition 2.4)

$$\frac{d}{dr}\left\{ r^{n+1} h(r) \int_{\mathbb{S}^{n-1}} [\dot{\mathbf{Q}}\theta \otimes \mathbf{Q}\theta - \mathbf{Q}\theta \otimes \dot{\mathbf{Q}}\theta] \, d\mathcal{H}^{n-1}(\theta) \right\} = 0. \tag{4.1}$$

Upon directly evaluating the spherical integral by invoking the divergence theorem on the unit ball, it is seen that

$$\int_{\mathbb{S}^{n-1}} [\dot{\mathbf{Q}}\theta \otimes \mathbf{Q}\theta]_{ij} \, d\mathcal{H}^{n-1}(\theta) = \sum_{l,k=1}^{n} \dot{\mathbf{Q}}_{il}\mathbf{Q}_{jk} \int_{\mathbb{S}^{n-1}} \theta_l \theta_k \, d\mathcal{H}^{n-1}(\theta) = \sum_{l,k=1}^{n} \omega_n \dot{\mathbf{Q}}_{il}\mathbf{Q}_{jk}\delta_{lk},$$

where ω_n is the volume of the unit ball $\mathcal{B}$. This therefore by noting the skew-symmetry of $\dot{\mathbf{Q}}\mathbf{Q}^t$ and after suppressing a factor of $2\omega_n$ leads to the ODE

$$\frac{d}{dr}\left\{ r^{n+1} h(r) \dot{\mathbf{Q}}\mathbf{Q}^t \right\} = 0, \quad a < r < b, \tag{4.2}$$

which is exactly the counterpart of (1.10) for the choice of Lagrangian F here. [Note that $d/dr[r^{n+1}F_\xi \dot{\mathbf{Q}}\mathbf{Q}^t] = \mathbf{Q}d/dr[r^{n+1}F_\xi \mathbf{Q}^t\dot{\mathbf{Q}}]\mathbf{Q}^t$ by skew-symmetry and a direct differentiation.] The next proposition characterises all solutions to this equation subject to the endpoint conditions $\mathbf{Q}(a) = \mathbf{Q}(b) = \mathbf{I}_n$.

Proposition 4.1 *The general solution $\mathbf{Q} = \mathbf{Q}(r)$ to (4.2) subject to $\mathbf{Q}(a) = \mathbf{I}_n$ has the form $\mathbf{Q}(r) = \exp\{\mathscr{H}(r)\mathbf{H}\}$ where $\mathscr{H} \in \mathscr{C}^2[a,b]$ is given by the integral*

$$\mathscr{H}(r) = \frac{\mathsf{H}(r)}{\mathsf{H}(b)}, \qquad \mathsf{H}(r) = \int_a^r \frac{ds}{s^{n+1}h(s)}, \tag{4.3}$$

and $\mathbf{H}$ is an arbitrary skew-symmetric matrix. Additionally, $\mathbf{Q}(b) = \mathbf{I}_n$ iff $\mathbf{H}$ has the form

$$\mathbf{H} = \begin{cases} \mathbf{P}\,\mathrm{diag}(2m_1\pi\mathbf{J}, \dots, 2m_{k-1}\pi\mathbf{J}, 0)\mathbf{P}^t, & n = 2k-1, \\ \mathbf{P}\,\mathrm{diag}(2m_1\pi\mathbf{J}, \dots, 2m_{k-1}\mathbf{J}, 2m_k\pi\mathbf{J})\mathbf{P}^t, & n = 2k. \end{cases} \tag{4.4}$$

Here, $m_1, \dots, m_k \in \mathbb{Z}$, $\mathbf{P} \in \mathbf{O}(n)$ is arbitrary and $\mathbf{J} = \sqrt{-\mathbf{I}_2}$ is the 2×2 skew-symmetric rotation matrix in (4.5).

Proof Integrating (4.2) twice yields $\mathbf{Q}(r) = \exp\{\mathscr{H}(r)\mathbf{H}\}$ with $\mathbf{H}$ being $n \times n$ skew-symmetric and $\mathscr{H} = \mathscr{H}(r) \in \mathscr{C}^2[a,b]$ – the profile of $\mathbf{Q}$ – being as in (4.3). Now $\mathbf{Q}(a) = \mathbf{I}_n$ is immediately seen to be satisfied by virtue of $\mathscr{H}(a) = 0$. For $\mathbf{Q}(b) = \mathbf{I}_n$, we first consider the orthogonal diagonalisation of $\mathbf{H}$. Here, depending on n being even or odd, we can write $\mathbf{H} = \mathbf{P}\,\mathrm{diag}(c_1\mathbf{J}, \dots, c_k\mathbf{J})\mathbf{P}^t$ when $n = 2k$ and $\mathbf{H} = \mathbf{P}\,\mathrm{diag}(c_1\mathbf{J}, \dots, c_{k-1}\mathbf{J}, c_k)\mathbf{P}^t$ when $n = 2k-1$ (with $c_k = 0$ for n odd). Moreover, $\mathbf{P} \in \mathbf{O}(n)$ and the 2×2 matrices $\mathbf{J}$ and $\mathcal{R}$ are given respectively by

$$\mathbf{J} = \mathcal{R}[\pi/2] = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \mathcal{R}[t] = \exp\{t\mathbf{J}\} = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}. \tag{4.5}$$

Note that the string of scalars $c_1, \dots, c_k \subset \mathbb{R}$ describe the spectrum of $\mathbf{H}$. Specifically, when n is even, $\pm ic_j$ with $1 \le j \le k$, and when n is odd, $\pm ic_j$, 0 with $1 \le j \le k-1$ are the eigenvalues of $\mathbf{H}$. Now $\mathbf{Q}(b) = \mathbf{I}_n \iff \exp\{\mathbf{H}\} = \mathbf{I}_n$ since $\mathscr{H}(b) = 1$. From this it follows at once that $c_j = 2m_j\pi$ where $m_j \in \mathbb{Z}$ for all $1 \le j \le k$ as described. □

Moving forward we now turn our attention to the counterpart of (1.9) (the stronger form of (4.1) without the spherical integral) that has the formulation

$$\frac{d}{dr}\left\{r^{n+1}h(r)[\dot{\mathbf{Q}}\theta \otimes \mathbf{Q}\theta - \mathbf{Q}\theta \otimes \dot{\mathbf{Q}}\theta]\right\} = 0, \quad a < r < b. \tag{4.6}$$

By the reasoning given at the start of the section (*see* also below) solutions $\mathbf{Q} = \mathbf{Q}(r)$ to this with $\mathbf{Q}(a) = \mathbf{I}_n$ must come from amongst those characterised in the first part of Proposition 4.1. A full description of these is given below.

Proposition 4.2 *For a twist path $\mathbf{Q} \in \mathscr{C}^2(]a,b[, \mathbf{SO}(n)) \cap \mathscr{C}([a,b], \mathbf{SO}(n))$ with $\mathbf{Q}(a) = \mathbf{I}_n$, the following assertions are equivalent*:

(i) *$\mathbf{Q}$ is a solution to (4.6).*

(ii) *$\mathbf{Q}$ solves (4.2) and $\dot{\mathbf{Q}}^t\dot{\mathbf{Q}}\theta \otimes \theta - \theta \otimes \dot{\mathbf{Q}}^t\dot{\mathbf{Q}}\theta = 0$.*

In either case $\mathbf{Q}(r) = \exp\{\mathscr{H}(r)\mathbf{H}\}$, where $\mathscr{H}$ is as in (4.3) and $\mathbf{H}^2 = -c^2\mathbf{I}_n$.

Proof For the implication (i) $\Longrightarrow$ (ii), assume $\mathbf{Q}$ solves (4.6). Then integration over the sphere gives (4.1) and hence (4.2). As seen earlier, solutions here subject to the endpoint condition $\mathbf{Q}(a) = \mathbf{I}_n$ are given by $\mathbf{Q}(r) = \exp\{\mathscr{H}(r)\mathbf{H}\}$ with $\mathscr{H} = \mathscr{H}(r)$ as in (4.3) and $\mathbf{H}$ skew-symmetric. Substituting this into (4.6) and noting $\dot{\mathbf{Q}} = \dot{\mathscr{H}}\mathbf{H}\mathbf{Q}$ and that $\mathbf{Q}$ and $\mathbf{H}$ commute gives

$$\begin{aligned}
0 &= \frac{d}{dr}\{r^{n+1}h(r)\dot{\mathscr{H}}(r)[\mathbf{H}\mathbf{Q}\theta \otimes \mathbf{Q}\theta - \mathbf{Q}\theta \otimes \mathbf{H}\mathbf{Q}\theta]\} \\
&= \frac{d}{dr}\{r^{n+1}h(r)\dot{\mathscr{H}}\}[\mathbf{H}\mathbf{Q}\theta \otimes \mathbf{Q}\theta - \mathbf{Q}\theta \otimes \mathbf{H}\mathbf{Q}\theta] \\
&\quad + r^{n+1}h(r)\dot{\mathscr{H}}^2\big[\mathbf{H}^2\mathbf{Q}\theta \otimes \mathbf{Q}\theta - \mathbf{Q}\theta \otimes \mathbf{H}^2\mathbf{Q}\theta\big] \\
&\iff \quad \mathbf{Q}\big[\mathbf{H}^2\theta \otimes \theta - \theta \otimes \mathbf{H}^2\theta\big]\mathbf{Q}^t = 0,
\end{aligned} \tag{4.7}$$

where we have used (4.2). Now, as $\dot{\mathscr{H}}^2[\mathbf{H}^2\theta \otimes \theta - \theta \otimes \mathbf{H}^2\theta] = 0$ is precisely the condition $\dot{\mathbf{Q}}^t\dot{\mathbf{Q}}\theta \otimes \theta - \theta \otimes \dot{\mathbf{Q}}^t\dot{\mathbf{Q}}\theta = 0$, this justifies the first part.

For the reverse implication (ii) $\Longrightarrow$ (i), we first suppose that $\mathbf{Q}$ solves (4.2). Then tensorising as in (2.13) gives

$$\begin{aligned}
0 &= \frac{1}{r^n}\left\{\frac{d}{dr}\big[r^{n+1}h(r)\mathbf{Q}^t\dot{\mathbf{Q}}\big]\theta \otimes \theta - \theta \otimes \frac{d}{dr}\big[r^{n+1}h(r)\mathbf{Q}^t\dot{\mathbf{Q}}\big]\theta\right\} \\
&= \frac{1}{r^n}\frac{d}{dr}\{r^{n+1}h(r)\big[\mathbf{Q}^t\dot{\mathbf{Q}}\theta \otimes \theta - \theta \otimes \mathbf{Q}^t\dot{\mathbf{Q}}\theta\big]\} \\
&= \frac{1}{r^n}\mathbf{Q}^t\frac{d}{dr}\{r^{n+1}h(r)[\dot{\mathbf{Q}}\theta \otimes \mathbf{Q}\theta - \mathbf{Q}\theta \otimes \dot{\mathbf{Q}}\theta]\}\mathbf{Q} \\
&\quad + rh(r)\big[\dot{\mathbf{Q}}^t\dot{\mathbf{Q}}\theta \otimes \theta - \theta \otimes \dot{\mathbf{Q}}^t\dot{\mathbf{Q}}\theta\big].
\end{aligned} \tag{4.8}$$

Now, taking advantage of the additional assumption $\dot{\mathbf{Q}}^t\dot{\mathbf{Q}}\theta \otimes \theta - \theta \otimes \dot{\mathbf{Q}}^t\dot{\mathbf{Q}}\theta = 0$ and the invertibility of $\mathbf{Q}$, we conclude. For the final assertion, it suffices to note that (4.7) $\iff$ $\mathbf{H}^2 = -c^2\mathbf{I}_n$. □

Having satisfactorily resolved ODEs (4.2) and (4.6), we next move to the PDE $\mathscr{L}_F[u] = \nabla\mathscr{P}$ and system (1.2) for suitable $(u, \mathscr{P})$ with $u = r\mathbf{Q}(r)\theta$. By Proposition 2.3, (4.1) and Proposition 4.1, it is plain that the twist path here must take the form $\mathbf{Q}(r) = \exp\{\mathscr{H}(r)\mathbf{H}\}$, and so $u = r\exp\{\mathscr{H}(r)\mathbf{H}\}\theta$. Now the action of $\mathscr{L}_F$ on u, upon invoking ODE (4.2), is first seen to reduce to

$$\begin{aligned}
\mathscr{L}_F[u] &= (\nabla u)^t \operatorname{div}\big[h\big(|x|\big)\nabla u\big] = (\nabla u)^t\big[\dot{h}(r)\nabla u\theta + h(r)\Delta u\big] \\
&= \big(\mathbf{Q}^t + r\theta \otimes \dot{\mathbf{Q}}\theta\big)\big\{\dot{h}(r)(\mathbf{Q}\theta + r\dot{\mathbf{Q}}\theta) + h(r)\big[(n+1)\dot{\mathbf{Q}}\theta + r\ddot{\mathbf{Q}}\theta\big]\big\} \\
&= \dot{h}(r)\big[1 + r^2|\dot{\mathbf{Q}}\theta|^2\big]\theta + h(r)\big[(n+1)r|\dot{\mathbf{Q}}\theta|^2\theta + r^2\langle\dot{\mathbf{Q}}\theta, \ddot{\mathbf{Q}}\theta\rangle\theta\big] \\
&\quad + \mathbf{Q}^t\big[(n+1)h\dot{\mathbf{Q}}\mathbf{Q}^t + r\dot{h}\dot{\mathbf{Q}}\mathbf{Q}^t + rh\ddot{\mathbf{Q}}\mathbf{Q}^t\big]\mathbf{Q}\theta \\
&= \dot{h}(r)\big[1 + r^2|\dot{\mathbf{Q}}\theta|^2\big]\theta + h(r)\big[(n+1)r|\dot{\mathbf{Q}}\theta|^2 + r^2\langle\dot{\mathbf{Q}}\theta, \ddot{\mathbf{Q}}\theta\rangle\big]\theta \\
&\quad - rh(r)\dot{\mathbf{Q}}^t\dot{\mathbf{Q}}\theta.
\end{aligned} \tag{4.9}$$

Therefore, by making the substitution $\mathbf{Q}(r) = \exp\{\mathscr{H}(r)\mathbf{H}\}$ and noting that by differentiation $\dot{\mathbf{Q}} = \dot{\mathscr{H}}\mathbf{H}\mathbf{Q}$, $\ddot{\mathbf{Q}} = \ddot{\mathscr{H}}\mathbf{H}\mathbf{Q} + \dot{\mathscr{H}}^2\mathbf{H}^2\mathbf{Q}$, $\dot{\mathbf{Q}}^t\dot{\mathbf{Q}} = -\dot{\mathscr{H}}^2\mathbf{H}^2$ and $\langle\dot{\mathbf{Q}}\theta, \ddot{\mathbf{Q}}\theta\rangle = \dot{\mathscr{H}}\ddot{\mathscr{H}}\langle\mathbf{H}\mathbf{Q}\theta,$

$\mathbf{HQ}\theta\rangle + \dot{\mathscr{H}}^3\langle \mathbf{HQ}\theta, \mathbf{H}^2\mathbf{Q}\theta\rangle = \dot{\mathscr{H}}\ddot{\mathscr{H}}|\mathbf{H}\theta|^2$, the action (4.9) simplifies to

$$\begin{aligned} \mathscr{L}_F[u] &= \nabla h(|x|) + r\big[(n+1)h(r) + r\dot{h}(r)\big]\dot{\mathscr{H}}^2|\mathbf{H}\theta|^2\theta \\ &\quad + r^2h(r)\dot{\mathscr{H}}\ddot{\mathscr{H}}|\mathbf{H}\theta|^2\theta + rh(r)\dot{\mathscr{H}}^2\mathbf{H}^2\theta. \end{aligned} \tag{4.10}$$

We are now in a position to apply Proposition 3.1 to the vector field $\mathscr{L}_F[u] - \nabla h$. In the context of (3.6), here we have $\mathsf{S} = -\mathbf{H}^2$, and the scalar functions a, b are in turn

$$\begin{aligned} \mathsf{a}(r) &= \frac{\dot{\mathscr{H}}(r)}{r^2}\big\{(n+1)h(r)\dot{\mathscr{H}}(r) + r\dot{h}(r)\dot{\mathscr{H}}(r) + rh(r)\ddot{\mathscr{H}}(r)\big\}, \\ \mathsf{b}(r) &= -h(r)\dot{\mathscr{H}}^2(r). \end{aligned}$$

A further use of ODE (4.2) gives $\mathsf{a} = 0$, and so Proposition 3.1 and (3.6) imply that subject to $-2\mathsf{a} + \dot{\mathsf{b}}/r = \dot{\mathscr{H}}^2/r^2[r\dot{h}(r) + 2(n+1)h(r)] \not\equiv 0$ on $]a,b[$ we have

$$\begin{aligned} \mathscr{L}_F[u] = \nabla\mathscr{P} \quad &\Longrightarrow \quad \operatorname{curl}\big(\mathscr{L}_F[u] - \nabla h\big) = 0 \\ &\Longrightarrow \quad \mathbf{H}^2 = -c^2\mathbf{I}_n \quad \Longrightarrow \quad \mathscr{L}_F[u] = \nabla\mathscr{P} \end{aligned} \tag{4.11}$$

for some $c \in \mathbb{R}$. Note that the final implication in (4.11) follows by substituting $\mathbf{H}^2 = -c^2\mathbf{I}_n$ in (4.10) and observing that, as a result, $\mathscr{L}_F[u]$ is a gradient field. So it follows that for n even $c_1^2 = \cdots = c_k^2 = c^2$ and for n odd $c_1 = \cdots = c_k = 0$ due to the presence of (at least one) zero eigenvalue for $\mathbf{H}$. In particular for n odd this gives $\mathbf{Q}(r) \equiv \mathbf{I}_n$. Lastly, to satisfy the endpoint condition $\mathbf{Q}(b) = \mathbf{I}_n$, we note that for n odd we already have $\mathbf{Q}(b) = \mathbf{I}_n$ and for n even we write

$$\begin{aligned} \mathbf{Q}(b) &= \exp\big\{\mathscr{H}(b)\mathbf{H}\big\} = \exp\big\{\mathbf{P}\operatorname{diag}(c\mathbf{J},\dots,c\mathbf{J})\mathbf{P}^t\big\} \\ &= \mathbf{P}\operatorname{diag}\big(\mathcal{R}[c],\dots,\mathcal{R}[c]\big)\mathbf{P}^t = \mathbf{I}_n \quad \Longleftrightarrow \quad c = 2\pi m, \quad m \in \mathbb{Z}. \end{aligned} \tag{4.12}$$

In conclusion, we have shown that subject to $r\dot{h}(r) + 2(n+1)h(r) \not\equiv 0$, the twist $u = r\mathbf{Q}(r)\theta$ is a solution to system (1.2) *iff* $\mathbf{Q}(r) = \mathbf{P}\exp\{2m\pi\mathscr{H}(r)\mathbf{J}_n\}\mathbf{P}^t$ for some $m \in \mathbb{Z}$ when n is even and $\mathbf{Q} \equiv \mathbf{I}_n$ when n is odd.[b] Combining this with Proposition 4.2, we have proved the following statement.

Proposition 4.3 *Assume that $u = r\mathbf{Q}(r)\theta$ is a generalised twist with twist path $\mathbf{Q} \in \mathscr{C}^2(]a,b[,\mathbf{SO}(n)) \cap \mathscr{C}([a,b],\mathbf{SO}(n))$ verifying $\mathbf{Q}(a) = \mathbf{Q}(b) = \mathbf{I}_n$. Consider*

(i) *u satisfies $\mathscr{L}_F[u] = \nabla\mathscr{P}$.*
(ii) *$\mathbf{Q}$ is a solution to (1.9).*
(iii) *$\mathbf{Q}$ satisfies (1.10) and $\dot{\mathbf{Q}}^t\dot{\mathbf{Q}}\theta \otimes \theta - \theta \otimes \dot{\mathbf{Q}}^t\dot{\mathbf{Q}}\theta = 0$.*

Then (ii) $\equiv$ (iii) $\Longrightarrow$ (i). *If $r\dot{h} + 2(n+1)h \not\equiv 0$ the above are all equivalent.*

5 General Lagrangians $F = F(r,\xi)$: ODEs (1.9)–(1.10) *vs* the PDE $\mathscr{L}_F[u] = \nabla\mathscr{P}$

Motivated by the results and the explicit description of solutions for the weighted Dirichlet Lagrangians in the previous section, in seeking solutions to system (1.2) as well as ODEs (1.9)–(1.10), here we focus on geodesic type twist paths $\mathbf{Q}(r) = \exp\{\mathscr{G}(r)\mathbf{H}\}$ for suitable $\mathscr{G} \in \mathscr{C}^2[a,b]$ and $\mathbf{H} = \mathbf{P}\mathbf{J}_n\mathbf{P}^t$ with $\mathbf{P} \in \mathbf{O}(n)$, $\mathbf{J}_n = \operatorname{diag}(\mathbf{J},\dots,\mathbf{J})$ for n even and $\mathbf{J}_n = \operatorname{diag}(\mathbf{J},\dots,\mathbf{J},0)$

for n odd. It is seen that here $\dot{\mathbf{Q}} = \dot{\mathscr{G}}\mathbf{H}\mathbf{Q}$, $\mathbf{Q}^t\dot{\mathbf{Q}} = \dot{\mathscr{G}}\mathbf{H}$ (note $\mathbf{Q}\mathbf{H} = \mathbf{H}\mathbf{Q}$), and so (1.10) reduces to

$$\mathscr{M}[\mathbf{Q}] = \frac{1}{r^n}\frac{d}{dr}\left\{r^{n+1}F_\xi\left(r, n + r^2\dot{\mathscr{G}}^2|\mathbf{H}\theta|^2\right)\dot{\mathscr{G}}\mathbf{H}\theta\right\} = 0. \tag{5.1}$$

Likewise $\dot{\mathbf{Q}}\theta \otimes \mathbf{Q}\theta = \dot{\mathscr{G}}\mathbf{H}\mathbf{Q}\theta \otimes \mathbf{Q}\theta$ and $\ddot{\mathbf{Q}}\theta \otimes \mathbf{Q}\theta = \ddot{\mathscr{G}}\mathbf{H}\mathbf{Q}\theta \otimes \mathbf{Q}\theta + \dot{\mathscr{G}}^2\mathbf{H}^2\mathbf{Q}\theta \otimes \mathbf{Q}\theta$, and so (1.9) reduces to

$$\begin{aligned}\mathscr{S}[\mathbf{Q}] &= \frac{d}{dr}\left\{r^{n+1}F_\xi\left(r, n + r^2\dot{\mathscr{G}}^2|\mathbf{H}\theta|^2\right)\dot{\mathscr{G}}\right\}[\mathbf{H}\mathbf{Q}\theta \otimes \mathbf{Q}\theta - \mathbf{Q}\theta \otimes \mathbf{H}\mathbf{Q}\theta] \\ &\quad + r^{n+1}F_\xi\left(r, n + r^2\dot{\mathscr{G}}^2|\mathbf{H}\theta|^2\right)\dot{\mathscr{G}}^2\left[\mathbf{H}^2\mathbf{Q}\theta \otimes \mathbf{Q}\theta - \mathbf{Q}\theta \otimes \mathbf{H}^2\mathbf{Q}\theta\right] = 0.\end{aligned} \tag{5.2}$$

Thus, upon taking into account the endpoint conditions $\mathbf{Q}(a) = \mathbf{Q}(b) = \mathbf{I}_n$, it follows that for n even the twist path $\mathbf{Q}(r) = \exp\{\mathscr{G}(r)\mathbf{H}\}$ satisfies ODEs (1.9)–(1.10) *iff* $\mathscr{G}$ is a solution to the boundary value problem (with $m \in \mathbb{Z}$)[c]

$$\mathbf{BVP}[\mathscr{G}; F, m] = \begin{cases} \frac{d}{dr}[r^{n+1}F_\xi(r, n + r^2\dot{\mathscr{G}}^2)\dot{\mathscr{G}}] = 0, \\ \mathscr{G}(a) = 0, \\ \mathscr{G}(b) = 2m\pi. \end{cases} \tag{5.3}$$

This therefore leads to the existence of an infinite family of solutions to ODEs (1.9)–(1.10) for n even, of course, subject to justifying the solvability of the boundary value problem $\mathbf{BVP}[\mathscr{G}; F, m]$ above. Setting the existence question aside for now (*cf.* Theorem 5.2 below), let us move forward and justify the above choice of $\mathbf{Q}$ by taking a twist path $\mathbf{Q} \in \mathscr{C}^1([a,b], \mathbf{SO}(n))$ and considering for $\theta \in \mathbb{S}^{n-1}$ the integral

$$I(\mathbf{Q}, \theta) = \int_a^b |\dot{\mathbf{Q}}\theta|\, dr. \tag{5.4}$$

Evidently, this integral represents the length of the curve $\gamma \in \mathscr{C}^1([a,b], \mathbb{S}^{n-1})$ given by $\gamma(r) = \mathbf{Q}(r)\theta$. Now in the event $\mathbf{Q}(r) = \exp\{\mathscr{G}(r)\mathbf{H}\}$ for some function $\mathscr{G} \in \mathscr{C}^1[a,b]$ and some skew-symmetric matrix $\mathbf{H} = \mathbf{P}\mathbf{J}_n\mathbf{P}^t$, by virtue of $\dot{\mathbf{Q}} = \dot{\mathscr{G}}\mathbf{H}\mathbf{Q}$, we have $|\dot{\mathbf{Q}}\theta| = |\dot{\mathscr{G}}\theta^\star|$ where $\theta^\star = \mathbf{H}\theta$. In particular when n is even, $|\theta^\star| = |\theta| = 1$, and so $|\dot{\mathbf{Q}}\theta| = |\dot{\mathscr{G}}|$ and integral (5.4) is independent of θ. The following theorem gives a complete characterisation of solutions $\mathbf{Q} = \mathbf{Q}(r)$ to ODEs (1.9)–(1.10) when additionally the integral $I(\mathbf{Q}, \theta)$ is independent of θ. Note that this last condition is implied by $\dot{\mathbf{Q}}^t\dot{\mathbf{Q}}\theta \otimes \theta - \theta \otimes \dot{\mathbf{Q}}^t\dot{\mathbf{Q}}\theta \equiv 0$.

Theorem 5.1 *Let $\mathbf{Q} \in \mathscr{C}^1([a,b], \mathbf{SO}(n)) \cap \mathscr{C}^2(]a,b[, \mathbf{SO}(n))$ satisfy the endpoint conditions $\mathbf{Q}(a) = \mathbf{Q}(b) = \mathbf{I}_n$. Assume that $\mathbf{Q}$ satisfies either of ODEs (1.9) or (1.10) and that the integral $I(\mathbf{Q}, \theta)$ in (5.4) is independent of θ. Then, depending on the dimension n being even or odd, $\mathbf{Q}$ has the following form*:

- (*n even*): *For some $m \in \mathbb{Z}$ and $\mathbf{P} \in \mathbf{O}(n)$, we have*

$$\begin{aligned}\mathbf{Q}(r) &= \exp\left\{\mathscr{G}(r;m)\mathbf{H}\right\}, \quad a \le r \le b, \\ &= \mathbf{P}\,\mathrm{diag}\left(\mathcal{R}[\mathscr{G}](r;m), \ldots, \mathcal{R}[\mathscr{G}](r;m)\right)\mathbf{P}^t,\end{aligned} \tag{5.5}$$

where $\mathbf{H} = \mathbf{P}\mathbf{J}_n\mathbf{P}^t$, $\mathbf{J}_n = \mathrm{diag}(\mathbf{J},\dots,\mathbf{J})$ *and* $\mathscr{G} = \mathscr{G}(r;m) \in \mathscr{C}^2[a,b]$ *is the unique solution to the boundary value problem* (5.3).

- (n *odd*): $\mathbf{Q}(r) \equiv \mathbf{I}_n$ *on* $[a,b]$.

Conversely, if $\mathbf{Q}$ *has the form above, then the integral* $I(\mathbf{Q},\theta)$ *is independent of* θ *and* $\mathbf{Q}$ *is a solution to ODEs* (1.9) *and* (1.10).

Referring to the discussion at the start of the section, in either of the cases above, $\mathbf{Q}$ is a solution to ODEs (1.9)–(1.10). Moreover, by an easy inspection, the integral $I(\mathbf{Q},\theta)$ is seen to be independent of θ; indeed, for n even, noting that $\dot{\mathscr{G}}$ does not change sign (in fact, as a result of $F_\xi > 0$ and (5.3), $\dot{\mathscr{G}}(r;m)$ has the same sign as $m \in \mathbb{Z}$), it is seen that

$$\begin{aligned} I(\mathbf{Q},\theta) = I\big(\exp\{\mathscr{G}(r;m)\mathbf{H}\},\theta\big) &= \int_a^b \big|\dot{\mathscr{G}}(r;m)\mathbf{H}\mathbf{Q}\theta\big|\,dr \\ &= \int_a^b \big|\dot{\mathscr{G}}(r;m)\big|\big|\theta^\star\big|\,dr = \left|\int_a^b \dot{\mathscr{G}}(r;m)\,dr\right| = 2\pi|m|, \end{aligned} \tag{5.6}$$

while for n odd, $I(\mathbf{Q},\theta) \equiv 0$.[d] This therefore immediately gives the converse part of the theorem and justifies the existence of infinitely many solutions to either ODEs (1.9) and (1.10) for n even. Hence in the following we focus entirely on the direct implication in the theorem.

Proof As $I(\mathbf{Q},\theta) = 0$ implies $|\dot{\mathbf{Q}}\theta| = 0$ and therefore $\mathbf{Q} \equiv \mathbf{I}_n$, we hereafter assume $I(\mathbf{Q},\theta) > 0$. We now consider the two cases (1.9) and (1.10) separately.

(*Part 1*) First assume that $\mathbf{Q}$ is a solution to (1.10) and set

$$\mathscr{F}(r,\theta) = \int_a^r \big|\dot{\mathbf{Q}}(s)\theta\big|\,ds, \quad a \le r \le b, |\theta| = 1. \tag{5.7}$$

We aim to show that $\mathscr{F}$ satisfies the ODE in (5.3) for each fixed θ. Towards this end, we first note that (1.10) $\Longrightarrow \langle \mathscr{M}[\mathbf{Q}], \mathbf{Q}^t\dot{\mathbf{Q}}\theta\rangle = 0$, and so

$$\frac{d}{dr}\big\{r^{n+1}F_\xi\big(r,n+r^2|\dot{\mathbf{Q}}\theta|^2\big)|\dot{\mathbf{Q}}\theta|^2\big\} - r^{n+1}F_\xi\big(r,n+r^2|\dot{\mathbf{Q}}\theta|^2\big)\langle\dot{\mathbf{Q}}\theta,\ddot{\mathbf{Q}}\theta\rangle = 0, \tag{5.8}$$

where we have used the identity $\langle \mathbf{Q}^t\dot{\mathbf{Q}}\theta, (\dot{\mathbf{Q}}^t\dot{\mathbf{Q}} + \mathbf{Q}^t\ddot{\mathbf{Q}})\theta\rangle = \langle \mathbf{Q}^t\dot{\mathbf{Q}}\theta, \mathbf{Q}^t\ddot{\mathbf{Q}}\theta\rangle$ that follows from $\langle \mathbf{Q}^t\dot{\mathbf{Q}}\theta, \dot{\mathbf{Q}}^t\dot{\mathbf{Q}}\theta\rangle = 0$ and the skew-symmetry of $\dot{\mathbf{Q}}\mathbf{Q}^t$. Proceeding directly therefore we can write

$$\begin{aligned} \text{LHS of (5.8)} &= \frac{d}{dr}\big\{r^{n+1}F_\xi\big(r,n+r^2|\dot{\mathbf{Q}}\theta|^2\big)|\dot{\mathbf{Q}}\theta|\big\}|\dot{\mathbf{Q}}\theta| \\ &\quad + r^{n+1}F_\xi\big(r,n+r^2|\dot{\mathbf{Q}}\theta|^2\big)|\dot{\mathbf{Q}}\theta|\frac{\langle\dot{\mathbf{Q}}\theta,\ddot{\mathbf{Q}}\theta\rangle}{|\dot{\mathbf{Q}}\theta|} \\ &\quad - r^{n+1}F_\xi\big(r,n+r^2|\dot{\mathbf{Q}}\theta|^2\big)\langle\dot{\mathbf{Q}}\theta,\ddot{\mathbf{Q}}\theta\rangle \\ &= \frac{d}{dr}\big\{r^{n+1}F_\xi\big(r,n+r^2|\dot{\mathbf{Q}}\theta|^2\big)|\dot{\mathbf{Q}}\theta|\big\}|\dot{\mathbf{Q}}\theta| = 0. \end{aligned} \tag{5.9}$$

Note that this argument shows that, as a function of r, $r^{n+1}F_\xi(r,n+r^2|\dot{\mathbf{Q}}\theta|^2)|\dot{\mathbf{Q}}\theta|$ is a positive constant on any interval on which $|\dot{\mathbf{Q}}\theta|$ is non-zero, and so a basic continuity argument implies that either $|\dot{\mathbf{Q}}\theta| \equiv 0$ on $[a,b]$ or $|\dot{\mathbf{Q}}\theta| > 0$ on $[a,b]$. Furthermore, it also shows that $\mathscr{F}(r,\theta)$ is a [*non-zero*] solution to the ODE in (5.3) for every fixed $|\theta| = 1$ as claimed.

Next we see that $\mathscr{F}(a,\theta) = 0$ and $\mathscr{F}(b,\theta) = I[\mathbf{Q},\theta]$ which is independent of θ by assumption. Given that solutions of (5.3) are extremisers of the energy functional

$$\Lambda : \mathscr{G} \mapsto \int_a^b F\big(r, n + r^2\dot{\mathscr{G}}^2\big) r^{n-1}\, dr, \tag{5.10}$$

it follows from standard convexity arguments that these solutions are the unique minimisers of this energy functional with respect to their own Dirichlet boundary conditions (note that this functional is *strictly* convex given the assumptions F). Since $\mathscr{F}$ has been shown to be independent of θ at its end-points, it follows that $\mathscr{F}(r,\theta) \equiv \mathscr{F}(r)$ is independent of θ for all $a \le r \le b$. Now, since $F_\xi > 0$, all solutions of (5.3) are monotone and hence invertible. Denoting $\mathscr{F}^{-1}(s) = r(s)$ put $\mathbf{K}(s) = \mathbf{Q}(r(s))$ for $\mathbf{K} \in \mathscr{C}^2(]0,l[,\mathbf{SO}(n)) \cap \mathscr{C}([0,l],\mathbf{SO}(n))$ where $l = \mathscr{F}(b)$. Thus $\mathbf{Q}(r) = \mathbf{K}(\mathscr{F}(r))$ and $\dot{\mathbf{Q}} = \mathbf{K}'\dot{\mathscr{F}}$ (with prime denoting d/ds). Returning to ODE (1.10), we have, after a change of variables,

$$\frac{d}{ds}\Big\{r^{n+1}F_\xi\big(r,n+r^2\dot{\mathscr{F}}^2\big)\dot{\mathscr{F}}\mathbf{K}^t\mathbf{K}'\Big\} = 0, \quad \mathbf{K}' = \frac{d}{ds}\mathbf{K}. \tag{5.11}$$

Since $r^{n+1}F_\xi(r,n+r^2\dot{\mathscr{F}}^2)\dot{\mathscr{F}} \equiv c$, as $\mathscr{F}$ solves (5.3), the above equation is then seen to be equivalent to

$$c\frac{d}{ds}\big(\mathbf{K}^t\mathbf{K}'\big) = 0, \quad 0 < s < l. \tag{5.12}$$

This ODE has solutions $\mathbf{K}(s) = \exp\{s\mathbf{L}\}$ with $\mathbf{L}$ skew-symmetric. As such with $s(r) = \mathscr{F}(r)$, we have $\mathbf{Q}(r) = \exp\{\mathscr{G}(r)\mathbf{L}\}$ where $\mathscr{G}$ solves (5.3). To ensure that the integral $I[\mathbf{Q},\theta]$ is independent of θ, we need $|\mathbf{L}\theta|$ to be independent of θ. For this we orthogonally diagonalise $\mathbf{L}$ by writing $\mathbf{L} = \mathbf{P}\,\mathrm{diag}(c_1\mathbf{J},\dots,c_k\mathbf{J})\mathbf{P}^t$ when $n = 2k$ is even and $\mathbf{L} = \mathbf{P}\,\mathrm{diag}(c_1\mathbf{J},\dots,c_{k-1}\mathbf{J},0)\mathbf{P}^t$ when $n = 2k-1$ is odd with $\mathbf{J}$ as defined in (4.5). It is then easily seen that $|\mathbf{L}\theta|$ is independent of θ *iff* $|c_1| = \cdots = |c_k| =: |c|$, that is, $\mathbf{L} = 0$ when n is odd and $\mathbf{L} = c\mathbf{P}\mathbf{J}_n\mathbf{P}^t$ when n is even. Finally, for n even, since $\mathscr{G}(a) = 0$, we have $\mathbf{Q}(a) = \mathbf{I}_n$, and then $\mathbf{Q}(b) = \exp\{c\mathscr{G}(b)\mathbf{P}\mathbf{J}_n\mathbf{P}^t\} = \mathbf{I}_n$ follows by setting $c = 1$ and $l = \mathscr{G}(b) = 2\pi m$. This therefore completes the first part of the proof.

(*Part 2*) For the second part of the proof, assume that $\mathbf{Q}$ is a solution to (1.9). Then, writing $F_\xi = F_\xi(r,n+r^2|\dot{\mathbf{Q}}\theta|^2)$ for brevity, we observe that

$$\begin{aligned}\frac{d}{dr}\big\{r^{n+1}F_\xi\dot{\mathbf{Q}}\theta\big\} &= \frac{d}{dr}\big\{r^{n+1}F_\xi[\dot{\mathbf{Q}}\theta\otimes\mathbf{Q}\theta - \mathbf{Q}\theta\otimes\dot{\mathbf{Q}}\theta]\mathbf{Q}\theta\big\}\\ &= \frac{d}{dr}\big\{r^{n+1}F_\xi[\dot{\mathbf{Q}}\theta\otimes\mathbf{Q}\theta - \mathbf{Q}\theta\otimes\dot{\mathbf{Q}}\theta]\big\}\mathbf{Q}\theta - r^{n+1}F_\xi|\dot{\mathbf{Q}}\theta|^2\mathbf{Q}\theta.\end{aligned} \tag{5.13}$$

The first term is zero since $\mathbf{Q}$ by assumption is a solution to (1.9), and so

$$\frac{d}{dr}\big\{r^{n+1}F_\xi\big(r,n+r^2|\dot{\mathbf{Q}}\theta|^2\big)\dot{\mathbf{Q}}\theta\big\} + r^{n+1}F_\xi\big(r,n+r^2|\dot{\mathbf{Q}}\theta|^2\big)|\dot{\mathbf{Q}}\theta|^2\mathbf{Q}\theta = 0. \tag{5.14}$$

Let $\mathscr{F} = \mathscr{F}(r,\theta)$ be as defined in (5.7). As in *Part 1* we proceed by showing that $\mathscr{F}$ is a solution to (5.3) for each fixed $|\theta| = 1$. Indeed here we can write

$$\begin{aligned}\frac{d}{dr}\left\{r^{n+1}F_\xi\left(r,n+r^2|\dot{\mathbf{Q}}\theta|^2\right)|\dot{\mathbf{Q}}\theta|\right\} &= \frac{d}{dr}\left[r^{n+1}F_\xi\left(r,n+r^2|\dot{\mathbf{Q}}\theta|^2\right)\right]|\dot{\mathbf{Q}}\theta| \\ &\quad + r^{n+1}F_\xi\left(r,n+r^2|\dot{\mathbf{Q}}\theta|^2\right)\frac{\langle\ddot{\mathbf{Q}}\theta,\dot{\mathbf{Q}}\theta\rangle}{|\dot{\mathbf{Q}}\theta|} \\ &= -r^{n+1}F_\xi\left(r,n+r^2|\dot{\mathbf{Q}}\theta|^2\right)\langle\dot{\mathbf{Q}}\theta,\mathbf{Q}\theta\rangle|\dot{\mathbf{Q}}\theta| = 0,\end{aligned}$$

where we have used $\langle$LHS of (5.14)$,\dot{\mathbf{Q}}\theta\rangle = 0$ and $\langle\dot{\mathbf{Q}}\theta,\mathbf{Q}\theta\rangle = 0$ by virtue of $\mathbf{Q}^t\dot{\mathbf{Q}}$ being skew-symmetric. It therefore follows from this that again $\mathscr{F}(r,\theta) \equiv \mathscr{F}(r)$. Next, given that $\mathscr{F}$ solves (5.3), we have $r^{n+1}F_\xi(r,n+r^2|\dot{\mathbf{Q}}\theta|^2)|\dot{\mathbf{Q}}\theta| \equiv c$, and so (5.14) can be written as

$$\frac{d}{dr}\left\{r^{n+1}F_\xi\left(r,n+r^2|\dot{\mathbf{Q}}\theta|^2\right)\dot{\mathbf{Q}}\theta\right\} + c|\dot{\mathbf{Q}}\theta|\mathbf{Q}\theta = 0. \tag{5.15}$$

Now, invoking the invertibility of $\mathscr{F}$ on $[a,b]$ and setting $r(s) = \mathscr{F}^{-1}(s)$, write $\mathbf{K}(s) = \mathbf{Q}(r(s))$ for $\mathbf{K} \in \mathscr{C}^2(]0,l[,\mathbf{SO}(n)) \cap \mathscr{C}^1([0,l],\mathbf{SO}(n))$ where $l = \mathscr{F}(b)$. Thus from (5.15) upon changing variables it follows that

$$c\left(\frac{d}{ds}\mathbf{K}' + \left|\mathbf{K}'\theta\right|^2\mathbf{K}\right)\theta = 0, \quad 0 < s < l, \tag{5.16}$$

which is easily seen to be the geodesic equation on $\mathbb{S}^{n-1}$ for the curve $s \mapsto \mathbf{K}(s)\theta$. Referring to Lemma 7.1 (*see* the Appendix), it is seen that for n odd, $\mathbf{K}(s) \equiv \mathbf{I}_n$ (i.e., $\mathbf{Q} \equiv \mathbf{I}_n$) and for n even, $\mathbf{K}(s) = \mathbf{P}\,\mathrm{diag}(\mathcal{R}[\mathcal{H}(s,m,l)],\dots,\mathcal{R}[\mathcal{H}(s,m,l)])\mathbf{P}^t$, where $\mathcal{H}(s,m,l) = (2m\pi)s/l$ for $m \in \mathbb{Z}$. This therefore upon changing variables gives $\mathbf{Q}(r) = \mathbf{P}\,\mathrm{diag}(\mathcal{R}[\mathscr{G}](r),\dots,\mathcal{R}[\mathscr{G}](r))\mathbf{P}^t$, where $\mathscr{G}$ is a solution to (5.3) with $\mathscr{G}(a) = 0$ and $\mathscr{G}(b) = 2m\pi$. The proof is thus complete. □

Theorem 5.2 *For each $m \in \mathbb{Z}$, the boundary value problem* $\mathbf{BVP}[\mathscr{G};F,m]$ *as given by* (5.3) *has a unique solution $\mathscr{G} \in \mathscr{C}^2[a,b]$.*

Proof It is easily seen that (5.3) is the Euler–Lagrange equation associated with the energy functional

$$\Lambda[\mathscr{G};a,b] := \int_a^b F\left(r,n+r^2\dot{\mathscr{G}}^2\right)r^{n-1}\,dr, \tag{5.17}$$

over the Dirichlet space $\mathscr{B}^p_m(a,b) = \{\mathscr{G} \in W^{1,p}(a,b) : \mathscr{G}(a) = 0, \mathscr{G}(b) = 2m\pi\}$. The existence of a minimiser follows by an application of the direct methods of the calculus of variations and the $\mathscr{C}^2$ regularity of the minimiser from the Tonelli–Hilbert–Weierstrass differentiability theorem (*see*, e.g., [8], pp. 55–61). A basic convexity argument upon noting the uniform convexity of the function $\xi \mapsto F(r,n+r^2\xi^2)$ on $[a,b]$ for $\xi \in \mathbb{R}$ shows that firstly any solution to the Euler–Lagrange equation (5.3) is a minimiser of Λ with respect to its own boundary conditions and secondly that minimisers of Λ over $\mathscr{B}^p_m(a,b)$ are unique. □

Proposition 5.1 *Let $u = r\mathbf{Q}(r)\theta$ be a generalised twist with twist path $\mathbf{Q}(r) = \exp\{\mathscr{G}(r)\mathbf{H}\}$, where $\mathbf{H}$ is a constant skew-symmetric matrix and $\mathscr{G} \in \mathscr{C}^2[a,b]$. Then with $\theta = x|x|^{-1}$ and $\theta^\star = \mathbf{H}\theta$ the following hold*:

(i) $\nabla u = \mathbf{Q}(\mathbf{I}_n + r\dot{\mathscr{G}}\theta^\star \otimes \theta)$,

(ii) $|\nabla u|^2 = n + r^2\dot{\mathscr{G}}^2|\theta^\star|^2$,

(iii) $\det \nabla u = \det[\mathbf{Q}(\mathbf{I}_n + r\dot{\mathscr{G}}\theta^\star \otimes \theta)] = 1$.

As a result, the action of the differential operator $\mathscr{L}_F$ on u can be described as

$$\begin{aligned}
&\mathscr{L}_F[u] \\
&\quad = F_{\xi\xi}\big(r, n + r^2\dot{\mathscr{G}}^2|\theta^\star|^2\big) \\
&\qquad \times \big\{\big(\mathbf{I}_n + r\dot{\mathscr{G}}\theta \otimes \theta^\star + r\dot{\mathscr{G}}\theta^\star \otimes \theta + r^2\dot{\mathscr{G}}^2|\theta^\star|^2\theta \otimes \theta\big)\big(2r\dot{\mathscr{G}}^2|\theta^\star|^2\theta + r^2\nabla\big[\dot{\mathscr{G}}^2|\theta^\star|^2\big]\big)\big\} \\
&\qquad + F_{r\xi}\big(r, n + r^2\dot{\mathscr{G}}^2|\theta^\star|^2\big)\big(\theta + r\dot{\mathscr{G}}\theta^\star + r^2\dot{\mathscr{G}}^2|\theta^\star|^2\theta\big) + F_\xi\big(r, n + r^2\dot{\mathscr{G}}^2|\theta^\star|^2\big) \\
&\qquad \times \big[(n+1)\dot{\mathscr{G}}\theta^\star + r\big(\ddot{\mathscr{G}}\theta^\star + \dot{\mathscr{G}}^2\mathbf{H}\theta^\star\big) + (n+1)r\dot{\mathscr{G}}^2|\theta^\star|^2\theta + r^2\dot{\mathscr{G}}\ddot{\mathscr{G}}|\theta^\star|^2\theta\big].
\end{aligned} \tag{5.18}$$

Proof Noting $\mathbf{QH} = \mathbf{HQ}$ and $\langle\theta^\star, \theta\rangle = 0$, by virtue of $\mathbf{H}$ being skew-symmetric, we have $\dot{\mathbf{Q}} = \dot{\mathscr{G}}\mathbf{HQ}$, $\ddot{\mathbf{Q}} = (\ddot{\mathscr{G}}\mathbf{H} + \dot{\mathscr{G}}^2\mathbf{H}^2)\mathbf{Q}$ and $|\dot{\mathbf{Q}}\theta|^2 = \dot{\mathscr{G}}^2\langle\mathbf{H}\theta, \mathbf{H}\theta\rangle = \dot{\mathscr{G}}^2|\theta^\star|^2$. The first three identities now follow immediately from those in Proposition 2.1. For the last identity, referring to Proposition 2.1, we can write

$$\begin{aligned}
\mathscr{L}_F[u] = \mathscr{L}_F\big[r\exp\{\mathscr{G}(r)\mathbf{H}\}\theta\big] &= \big(\mathbf{I}_n + r\dot{\mathscr{G}}\theta \otimes \theta^\star\big) \\
&\quad \times \big\{F_{\xi\xi}\big(r, n + r^2\dot{\mathscr{G}}^2|\theta^\star|^2\big)\big(\mathbf{I}_n + r\dot{\mathscr{G}}\theta^\star \otimes \theta\big)\big(2r\dot{\mathscr{G}}^2|\theta^\star|^2\theta + r^2\nabla\big[\dot{\mathscr{G}}^2|\theta^\star|^2\big]\big) \\
&\quad + F_{r\xi}\big(r, n + r^2\dot{\mathscr{G}}^2|\theta^\star|^2\big)\big(\theta + r\dot{\mathscr{G}}\theta^\star\big) \\
&\quad + F_\xi\big(r, n + r^2\dot{\mathscr{G}}^2|\theta^\star|^2\big)\big[(n+1)\dot{\mathscr{G}}\theta^\star + r\big(\ddot{\mathscr{G}}\theta^\star + \dot{\mathscr{G}}^2\mathbf{H}\theta^\star\big)\big]\big\}.
\end{aligned}$$

The conclusion follows by a straightforward calculation. □

Theorem 5.3 *For $n \geq 2$ even and $m \in \mathbb{Z}$ put $\mathbf{Q}(r;m) = \exp\{\mathscr{G}(r;m)\mathbf{H}\}$ (with $a \leq r \leq b$) where $\mathbf{H}$ is the skew-symmetric matrix $\mathbf{H} = \mathbf{PJ}_n\mathbf{P}^t$ and $\mathscr{G} = \mathscr{G}(r;m)$ is the unique solution to (5.3). Then the vector field $v = \mathscr{L}_F[r\mathbf{Q}(r;m)\theta]$ is a gradient field. In particular the generalised twist $u = r\mathbf{Q}(r;m)\theta$ is a solution to system (1.2). Thus system (1.2) admits a countably infinite family of monotone twisting solutions.*

Proof By referring to the formulation of $\mathscr{L}_F[u = r\mathbf{Q}(r)\theta]$ with the twist path $\mathbf{Q}(r) = \exp\{\mathscr{G}(r)\mathbf{H}\}$ and $\mathbf{H} = \mathbf{PJ}_n\mathbf{P}^t$ hence $\dot{\mathbf{Q}} = \dot{\mathscr{G}}\mathbf{HQ}$, $\ddot{\mathbf{Q}} = (\ddot{\mathscr{G}}\mathbf{H} - \dot{\mathscr{G}}^2\mathbf{I}_n)\mathbf{Q}$ and $|\theta^\star|^2 = 1$ where $\theta^\star = \mathbf{H}\theta$, we can write

$$\begin{aligned}
\mathscr{L}_F[u] = \mathscr{L}_F\big[r\exp\{\mathscr{G}(r)\mathbf{H}\}\theta\big] &= F_{\xi\xi}\big(r, n + r^2\dot{\mathscr{G}}^2\big) \\
&\quad \times \big(2r\dot{\mathscr{G}}^2 + 2r^2\dot{\mathscr{G}}\ddot{\mathscr{G}}\big)\big(\theta + r\dot{\mathscr{G}}\mathbf{H}\theta + r^2\dot{\mathscr{G}}^2\theta\big) \\
&\quad + F_{r\xi}\big(r, n + r^2\dot{\mathscr{G}}^2\big)\big(\theta + r\dot{\mathscr{G}}\mathbf{H}\theta + r^2\dot{\mathscr{G}}^2\theta\big) + F_\xi\big(r, n + r^2\dot{\mathscr{G}}^2\big) \\
&\quad \times \big\{\big[(n+1)\dot{\mathscr{G}} + r\ddot{\mathscr{G}}\big]\mathbf{H}\theta + \big[(n+1)r\dot{\mathscr{G}}^2 - r\dot{\mathscr{G}}^2 + r^2\dot{\mathscr{G}}\ddot{\mathscr{G}}\big]\theta\big\}.
\end{aligned} \tag{5.19}$$

Now a straightforward calculation and rearrangement of terms enable us to write the above action in the convenient form

$$\mathscr{L}_F\big[u = r\exp\{\mathscr{G}(r)\mathbf{H}\}\theta\big] = \mathscr{A}(r)\theta + \mathscr{B}(r)\mathbf{H}\theta, \tag{5.20}$$

where

$$\begin{aligned}\mathscr{A}(r) &= \big[F_{\xi\xi}\big(r, n + r^2\dot{\mathscr{G}}^2\big)\big(2r\dot{\mathscr{G}}^2 + 2r^2\dot{\mathscr{G}}\ddot{\mathscr{G}}\big) + F_{r\xi}\big(r, n + r^2\dot{\mathscr{G}}^2\big)\big]\big(1 + r^2\dot{\mathscr{G}}^2\big)\\ &\quad + F_\xi\big(r, n + r^2\dot{\mathscr{G}}^2\big)\big[(n+1)r\dot{\mathscr{G}}^2 - r\dot{\mathscr{G}}^2 + r^2\dot{\mathscr{G}}\ddot{\mathscr{G}}\big]\\ &= \frac{\dot{\mathscr{G}}}{r^{n-1}}\frac{d}{dr}\big[r^{n+1}F_\xi\big(r, n + r^2\dot{\mathscr{G}}^2\big)\dot{\mathscr{G}}\big] + F_{\xi\xi}\big(r, n + r^2\dot{\mathscr{G}}^2\big)\big(2r\dot{\mathscr{G}}^2 + 2r^2\dot{\mathscr{G}}\ddot{\mathscr{G}}\big)\\ &\quad + F_{r\xi}\big(r, n + r^2\dot{\mathscr{G}}^2\big) - rF_\xi\big(r, n + r^2\dot{\mathscr{G}}^2\big)\dot{\mathscr{G}}^2,\end{aligned} \tag{5.21}$$

and

$$\begin{aligned}\mathscr{B}(r) &= rF_{\xi\xi}\big(r, n + r^2\dot{\mathscr{G}}^2\big)\dot{\mathscr{G}}\big(2r\dot{\mathscr{G}}^2 + 2r^2\dot{\mathscr{G}}\ddot{\mathscr{G}}\big) + rF_{r\xi}\big(r, n + r^2\dot{\mathscr{G}}^2\big)\dot{\mathscr{G}}\\ &\quad + F_\xi\big(r, n + r^2\dot{\mathscr{G}}^2\big)\big[(n+1)\dot{\mathscr{G}} + r\ddot{\mathscr{G}}\big]\\ &= \frac{1}{r^n}\frac{d}{dr}\big[r^{n+1}F_\xi\big(r, n + r^2\dot{\mathscr{G}}^2\big)\dot{\mathscr{G}}\big].\end{aligned} \tag{5.22}$$

Thus, as the function $\mathscr{G}$ is chosen as a solution of (5.3), this immediately gives $\mathscr{B}(r) \equiv 0$, and so we have

$$\begin{aligned}\mathscr{L}_F[u] &= F_{\xi\xi}\big(r, n + r^2\dot{\mathscr{G}}^2\big)\big(2r\dot{\mathscr{G}}^2 + 2r^2\dot{\mathscr{G}}\ddot{\mathscr{G}}\big)\theta\\ &\quad + F_{r\xi}\big(r, n + r^2\dot{\mathscr{G}}^2\big)\theta - rF_\xi\big(r, n + r^2\dot{\mathscr{G}}^2\big)\dot{\mathscr{G}}^2\theta\\ &= \nabla F_\xi\big(r, n + r^2\dot{\mathscr{G}}^2\big) - rF_\xi\big(r, n + r^2\dot{\mathscr{G}}^2\big)\dot{\mathscr{G}}^2\theta.\end{aligned} \tag{5.23}$$

We thus conclude that $\mathscr{L}_F[u]$ is a gradient field, and so as a result the twist $u = r\exp\{\mathscr{G}(r)\mathbf{H}\}\theta$ is a solution to system (1.2) with $\mathscr{P} = F_\xi(r, n + r^2\dot{\mathscr{G}}^2) - G(r)$, where $\nabla G = rF_\xi(r, n + r^2\dot{\mathscr{G}}^2)\dot{\mathscr{G}}^2\theta$. □

6 Conclusions

We have proved the existence of an infinite scale of topologically distinct twisting solutions to a nonlinear elliptic system in divergence and variational form in a finite symmetric annulus subject to suitable boundary conditions and a hard incompressibility constraint. This is done by connecting the system to a set of nonlinear isotropic types ODEs on the compact Lie group $\mathbf{SO}(n)$ and proving the existence of multiple closed stationary loops in the geodesic form to these ODEs. The resulting stationary loops then remarkably serve as the twist loops associated with the sought twisting solutions to the system. Particular attention is paid to a totally integrable case where a complete and explicit description of all the infinite scale of twisting solutions is given.

Appendix

In this appendix we give the proof of a result that was used earlier in the paper in the course of the proof of Theorem 5.1. We believe that this characterisation of twist paths in relation to geodesics on the sphere and $\mathbf{SO}(n)$ is of independent interest.

Lemma 7.1 *Let* $\mathbf{K} \in \mathscr{C}^1([0,l],\mathbf{SO}(n)) \cap \mathscr{C}^2(]0,l[,\mathbf{SO}(n))$ *for some* $0 < l < \infty$ *satisfy the endpoint conditions* $\mathbf{K}(0) = \mathbf{K}(l) = \mathbf{I}_n$. *Then the curve* $s \mapsto \mathbf{K}(s)\theta$ *with* $0 \le s \le l$ *is a geodesic on the sphere* $\mathbb{S}^{n-1}$ *for every* $\theta \in \mathbb{S}^{n-1}$ *iff depending on the dimension* n *being even or odd*, $\mathbf{K}$ *takes one of the following forms*:

- (n *even*) *For some* $m \in \mathbb{Z}$ *and* $\mathbf{P}$ *in* $\mathbf{O}(n)$, *we have*

$$\begin{aligned} \mathbf{K}(s) = \mathbf{K}(s;m) &= \exp\{\mathcal{H}(s;m,l)\mathbf{H}\} \\ &= \mathbf{P}\,\mathrm{diag}\big(\mathcal{R}[\mathcal{H}](s),\dots,\mathcal{R}[\mathcal{H}](s)\big)\mathbf{P}^t, \quad 0 \le s \le l, \end{aligned} \tag{7.1}$$

  where $\mathcal{H} = \mathcal{H}(s;m,l) := (2m\pi)s/l$ *and* $\mathcal{R}$ *is given by* (4.5). *Here the skew-symmetric matrix* $\mathbf{H}$ *is a square root of* $-\mathbf{I}_n$ *and has the form* $\mathbf{H} = \mathbf{P}\mathbf{J}_n\mathbf{P}^t$ *with* $\mathbf{J}_n = \mathrm{diag}(\mathbf{J},\mathbf{J},\dots,\mathbf{J})$ *and* $\mathbf{J} = \mathcal{R}[\pi/2]$.
- (n *odd*) $\mathbf{K}(s) \equiv \mathbf{I}_n$ *for all* $s \in [0,l]$.

Proof On a round sphere geodesics are segments of great circles and satisfy the geodesic equation $\gamma'' + |\gamma'|^2\gamma = 0 : \gamma \in \mathscr{C}^2([0,\ell],\mathbb{S}^{n-1})$. Hence, if $s \mapsto \mathbf{K}(s)\theta$ is a geodesic for every θ, then $\mathbf{K}$ satisfies

$$\big(\mathbf{K}'' + \big|\mathbf{K}'\theta\big|^2\mathbf{K}\big)\theta = 0, \quad 0 < s < l, ' = \frac{d}{ds}, \tag{7.2}$$

for every $|\theta| = 1$. It is now seen that if $\mathbf{K}$ satisfies (7.2) then $|\mathbf{K}'\theta|^2$ is constant in θ and s. Indeed, differentiating with respect to s yields

$$\frac{1}{2}\frac{d}{ds}\big|\mathbf{K}'\theta\big|^2 = \langle\mathbf{K}''\theta,\mathbf{K}'\theta\rangle = \langle -\big|\mathbf{K}'\theta\big|^2\mathbf{K}\theta,\mathbf{K}'\theta\rangle = -\big|\mathbf{K}'\theta\big|^2\langle\mathbf{K}'^t\mathbf{K}\theta,\theta\rangle = 0$$

by the skew-symmetry of $\mathbf{K}'^t\mathbf{K}$. Thus $|\mathbf{K}'\theta|^2 = f(\theta)$ for some $f \in \mathscr{C}(\mathbb{S}^{n-1})$, and so rearranging (7.2) we obtain $\mathbf{K}^t\mathbf{K}''\theta = -f(\theta)\theta$. For fixed s this asserts that $-f(\theta)$ is an eigenvalue of $\mathbf{K}^t\mathbf{K}''$. However, as $\mathbf{K}^t\mathbf{K}''$ has at most n eigenvalues, it follows, by the continuity of f, that f must be constant, say $f(\theta) = |\mathbf{K}'\theta|^2 = t^2$. Thus summarising we conclude that if $\mathbf{K}(s)\theta$ is a geodesic for all θ, then $\mathbf{K}$ must satisfy the second order ODE:

$$\mathbf{K}'' + t^2\mathbf{K} = 0, \quad 0 < s < l. \tag{7.3}$$

Now the general solution to this ODE, taking into account $\mathbf{K}(0) = \mathbf{I}_n$, is given by $\mathbf{K}(s) = \exp\{s\mathbf{A}\}$ where $\mathbf{A}$ is a constant skew-symmetric matrix. Indeed by differentiating $\mathbf{K}$ it is seen that $\mathbf{A}$ satisfies $(\mathbf{A}^2 + t^2\mathbf{I}_n)\mathbf{K} = 0$, and so, in view of $\mathbf{K}$ being invertible, $\mathbf{A}^2 = -t^2\mathbf{I}_n$. Now, upon diagonalising, we can write $\mathbf{A} = \mathbf{P}\,\mathrm{diag}(t_1\mathbf{J},\dots,t_{k-1}\mathbf{J},t_k\mathbf{J})\mathbf{P}^t$ when $n = 2k$ and $\mathbf{A} = \mathbf{P}\,\mathrm{diag}(t_1\mathbf{J},\dots,t_{k-1}\mathbf{J},t_k)\mathbf{P}^t$ when $n = 2k-1$, where $\mathbf{P} \in \mathbf{O}(n)$, $(t_j)_{j=1}^k \subset \mathbb{R}$ and $t_k = 0$ when $n = 2k-1$. Therefore, by returning to the identity $\mathbf{A}^2 = -t^2\mathbf{I}_n$, it is seen that:

- when $n = 2k$, we have

$$\begin{aligned}\mathbf{A}^2 &= -\mathbf{P}\,\mathrm{diag}\big(t_1^2\mathbf{I}_2,\dots,t_k^2\mathbf{I}_2\big)\mathbf{P}^t\\ &= -t^2\mathbf{I}_n \quad\Longrightarrow\quad |t_1| = |t_2| = \cdots = |t_k| = |t|.\end{aligned} \tag{7.4}$$

- when $n = 2k-1$, we have

$$\begin{aligned}\mathbf{A}^2 &= -\mathbf{P}\,\mathrm{diag}\big(t_1^2\mathbf{I}_2,\dots,t_{k-1}^2\mathbf{I}_2,t_k\big)\mathbf{P}^t\\ &= -t^2\mathbf{I}_n \quad\Longrightarrow\quad |t_1| = |t_2| = \cdots = |t_k| = |t| = 0.\end{aligned} \tag{7.5}$$

We now finish off by choosing $\mathbf{A}$ so that the endpoint condition $\mathbf{K}(\ell) = \mathbf{I}_n$ is satisfied and this is done by writing $\exp\{\ell\mathbf{A}\}$ in a block diagonal form and then comparing with $\mathbf{I}_n$ (*cf.* also [28] Theorem 2.1). □

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Endnotes

[a] A solution to system (1.2) for any $n \geq 2$ is the identity map $u \equiv x$. This follows by noting $\mathscr{L}_F[u \equiv x] = (\nabla u)^t \operatorname{div}[F_\xi(|x|,|\nabla u|^2)\nabla u] = \nabla F_\xi(|x|,n) = \nabla \mathscr{P}$ with the choice of the hydrostatic pressure $\mathscr{P}(x) = F_\xi(|x|,n)$ modulo an additive constant.

[b] When $n = 2k$, (4.11) gives $c_1,\dots,c_k \in \{\pm c\}$. By adjusting $\mathbf{P} \in \mathbf{O}(n)$, we can arrange and assume without loss of generality that $c_1 = \cdots = c_k$. Note also that $r\dot{h} + 2(n+1)h \equiv 0 \iff h(r) = \alpha r^{-2(n+1)}$; thus, referring to (4.10), $\mathscr{L}_F[u] - \nabla h = h(r)\dot{\mathscr{H}}^2\mathbf{H}^2x = -\nabla|\mathbf{H}x|^2/(2\alpha \mathrm{H}(b)^2)$ regardless of the choice of $c_1,\dots,c_k$. (*See* [20, 21] for further extensions of these results.)

[c] Note that for n even $\mathbf{H} = \sqrt{-\mathbf{I}_n}$.

[d] It is useful to contrast this with Proposition 4.1 and the further restriction on $\mathbf{H}$ as a result of (5.4).

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