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Schrödinger-type identity for Schrödinger free boundary problems

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Abstract
Our aim in this paper is to develop a Schrödinger-type identity for a Schrödinger free boundary problem in $\mathbb{R}^n$. As an application, we establish necessary and sufficient conditions for the product of some distributional functions to satisfy the Schrödinger-type identity. As a consequence, our results significantly improve and generalize previous work.

Keywords: Schrödinger-type identity; Schrödinger transform; Distribution

1 Introduction and main results

Schrödinger-type identities have been studied extensively in the literature (see [1, 12, 13, 18] for the Schrödinger equation, [5, 14] for Schrödinger systems).

In recent years, many exciting phenomena were found by careful experiments on light waves propagating in nonlinear periodic lattices. These phenomena are governed by the following Schrödinger equation:

$$\mathrm{Sch}_\alpha(u) = (-\Delta)^\alpha u + V(x)u - h(x,u) = 0 \tag{1.1}$$

in $\mathbb{R}^n$, where $n \geq 2$, $\alpha \in (0,1)$, $(-\Delta)^\alpha$ stands for the fractional Laplacian, V is a positive continuous potential, $h \in C(\mathbb{R}^2 \times \mathbb{R}, \mathbb{R})$. The fractional Laplacian $(-\Delta)^\alpha$ with $\alpha \in (0,1)$ of a function $\iota \in \mathcal{S}$ is defined by

$$\mathcal{G}\bigl(\bigl((-\Delta)^\alpha\bigr)\iota\bigr)(\xi) = |\xi|^{2\alpha}\mathcal{G}(\iota)(\xi), \quad \forall\alpha \in (0,1),$$

where $\mathcal{S}$ denotes the Schwartz space of rapidly decreasing C^∞ functions in $\mathbb{R}^n$ and

$$\mathcal{F}(\iota)(\xi) = \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^N} e^{-2\pi i\xi\cdot x}\iota(x)\,dx.$$

The Schrödinger transform Sch_α is defined as the following singular integral:

$$\bigl(\mathrm{Sch}_\alpha(f)\bigr)(x) := p.v.\frac{1}{\pi}\int_{\mathbb{R}} \frac{f(y)}{x-y}\,dy = \lim_{\epsilon\to 0}\int_{|y-x|\geq\epsilon}\frac{f(y)}{x-y}\,dy,$$

where $x \in \mathbb{R}$.

The Schrödinger-type identity for Schrödinger free boundary problems

$$\mathrm{Sch}_\alpha(fg) = \mathrm{fSch}_\alpha(g)$$

was first studied in [2–4, 6]. It was proved that the above identity holds if $h, g \in L^2(\mathbb{R})$ satisfy $\operatorname{supp}\hat{f} \subseteq \mathbb{R}_+$ ($\mathbb{R}_+ = [0,\infty)$) and $\operatorname{supp}\hat{g} \subseteq \mathbb{R}_+$ in [20]. In 2015, Wan also obtained more general sufficient conditions by weakening the above condition in [19]. Recently, Lv and Ulker and Huang established the first necessary and sufficient condition in the time domain and a parallel result in the frequency domain for the Poisson inequality in [10, 14].

It is natural that there have been attempts to define the complex signal and prove the Schrödinger-type identity in the multidimensional case.

Definition 1.1 The partial Schrödinger transform $\mathrm{Sch}_{\alpha j}$ of f is given by

$$(\mathrm{Sch}_{\alpha j} f)(x) := p.v.\frac{1}{\pi}\int_{\mathbb{R}} \frac{f(y)}{x_j - y_j}\,dy_j,$$

where $f \in L^p(\mathbb{R}_n)$ and $1 \leq p < \infty$.

The total Schrödinger transform Sch_α of f is defined as follows:

$$\begin{aligned}\big(\mathrm{Sch}_\alpha(f)\big)(x) &:= p.v.\frac{1}{\pi^n}\int_{\mathbb{R}^n} \frac{f(y)}{\prod_{j=1}^n (x_j - y_j)}\,dy \\ &= \lim_{\max \epsilon_j \to 0} \int_{|y_j - x_j| \geq \epsilon_j > 0, j=1,2,\dots,n} \frac{f(y)}{\prod_{j=1}^n (x_j - y_j)}\,dy,\end{aligned}$$

where $f \in L^p(\mathbb{R}_n)$ and $1 \leq p < \infty$. The property

$$\big\|\mathrm{Sch}_\alpha(f)\big\|_{L^p(\mathbb{R}^n)} \leq C_p^n \|f\|_{L^p(\mathbb{R}^n)}$$

was proved in [8]. The iterative nature of it in $L^p(\mathbb{R}^n)$ was shown in [16], where $p > 1$. It was shown that

$$\mathrm{Sch}_\alpha = \prod_{j=1}^n \mathrm{Sch}_{\alpha j}.$$

The operations $\mathrm{Sch}_{\alpha i}$ and $\mathrm{Sch}_{\alpha j}$ commute with each other, where $i, j = 1, 2, \dots, n$.

Now we define the Schrödingerean Fourier transform $\hat{f}$ of f (see [17]) by

$$\hat{f}(x) = \int_{\mathbb{R}^n} f(t) e^{-ix.t}\,dt,$$

where $x \in \mathbb{R}^n$ and $f \in L^1(\mathbb{R}^n)$.

Set

$$D_+ = \left\{ x : x \in \mathbb{R}^n, \operatorname{sgn}(-x) = \prod_{j=1}^n \operatorname{sgn}(-x_j) = 1 \right\},$$

$$D_{-}=\left\{x : x \in \mathbb{R}^{n}, \operatorname{sgn}(-x)=\prod_{j=1}^{n} \operatorname{sgn}(-x_{j})=-1\right\},$$

and

$$D_{0}=\left\{x : x \in \mathbb{R}^{n}, \operatorname{sgn}(-x)=\prod_{j=1}^{n} \operatorname{sgn}(-x_{j})=0\right\}.$$

We denote by $\mathcal{D}_{D_+}(\mathbb{R}^n)$, $\mathcal{D}_{D_-}(\mathbb{R}^n)$ and $\mathcal{D}_{D_0}(\mathbb{R}^n)$ the set of functions in $\mathcal{D}(\mathbb{R}^n)$ that are supported on D_+, D_-, and D_0, respectively.

The Schwartz class $\mathcal{S}(\mathbb{R}^n)$ consists of all functions φ on $\mathbb{R}^n$ such that

$$\sup_{x\in\mathbb{R}^n}\left|x^{\alpha}D^{\beta}\varphi(x)\right|<\infty,$$

where $\alpha, \beta \in \mathbb{Z}_+^n$.

The Schrödingerean Fourier transform $\hat{\varphi}$ is a linear homeomorphism from $S(\mathbb{R}^n)$ onto itself. Meanwhile, the following identity holds:

$$(\mathrm{Sch}_{\alpha}\,\varphi)^{\wedge}(x)=(-i)\operatorname{sgn}(x)\hat{\varphi},$$

where $\varphi \in \mathcal{S}(\mathbb{R}^n)$.

The Schrödingerean Fourier transform $F : \mathbb{S}'(\mathbb{R}^n) \to \mathbb{S}'(\mathbb{R}^n)$ is defined for any $\varphi \in \mathcal{S}(\mathbb{R}^n)$ as follows:

$$\langle\hat{\varrho},\varphi\rangle=\langle\varrho,\hat{\varphi}\rangle,$$

which is a linear isomorphism from $\mathbb{S}'(\mathbb{R}^n)$ onto itself. For the detailed properties of $\mathbb{S}(\mathbb{R}^n)$ and $\mathbb{S}'(\mathbb{R}^n)$, see [3, 7, 15].

For $\varrho \in \mathcal{S}'(\mathbb{R}^n)$, $\lambda \in \mathcal{S}(\mathbb{R}^n)$, it is easy to check that

$$\langle\tilde{\breve{\varrho}},\lambda\rangle=\langle\tilde{\varrho},\hat{\lambda}\rangle=\langle\varrho,\check{\hat{\lambda}}\rangle=\langle\hat{\varrho},\lambda\rangle=\langle\varrho,\hat{\lambda}\rangle$$

for any $\lambda \in \mathcal{S}(\mathbb{R}^n)$, where

$$\tilde{\lambda}(x)=\lambda(-x),$$

$\tilde{\varrho}$ is the inverse Schrödingerean Fourier transform defined as follows:

$$\langle\breve{\varrho},\lambda\rangle=\langle\varrho,\tilde{\lambda}\rangle.$$

Therefore in the distributional sense, we obtain

$$\tilde{\breve{\varrho}}=\hat{\varrho}.$$

Following the definition in [4], a function λ belongs to the space $\mathcal{D}_{L^p}(\mathbb{R}^n)$, $1 \leq p < \infty$ if and only if

(1) $\lambda \in C^{\infty}(\mathbb{R}^n)$;
(2) $D^k\lambda \in L^p(\mathbb{R}^n)$, $k = 1, 2, \ldots$, where $C^{\infty}(\mathbb{R}^n)$ consists of infinitely differentiable functions,

$$D^k\lambda(x) = \frac{\partial^{|k|}}{\partial x_1^{k_1} \cdots \partial x_n^{k_n}} \lambda(x).$$

In the sequel, we denote by $\mathcal{D}'_{L^p}(\mathbb{R}^n)$ the dual of the corresponding spaces $\mathcal{D}_{L^{p'}}(\mathbb{R}^n)$, where

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

As a consequence, we have

$$\mathcal{D}(\mathbb{R}^n) \subseteq \mathcal{S}(\mathbb{R}^n) \subseteq \mathcal{D}_{L^p}(\mathbb{R}^n) \subseteq L^p(\mathbb{R}^n)$$

and

$$L^p(\mathbb{R}^n) \subseteq \mathcal{D}'_{L^p}(\mathbb{R}^n) \subseteq \mathcal{S}'(\mathbb{R}^n) \subseteq \mathcal{D}'(\mathbb{R}^n).$$

Definition 1.2 Let $f \in \mathcal{D}'_{L^p}(\mathbb{R}^n)$, where $1 < p < \infty$. Then the Schrödinger transform of f is defined as follows:

$$\langle \operatorname{Sch}_\alpha(f), \lambda \rangle = \langle f, (-1)^n \operatorname{Sch}_\alpha \lambda \rangle,$$

where $\lambda \in \mathcal{D}_{L^{p'}}(\mathbb{R}^n)$.

In [10], Huang proved that the total Schrödinger transform is a linear homeomorphism from $\mathcal{D}_{L^p}(\mathbb{R}^n)$ onto itself, and that, if $h \in \mathcal{D}'_{L^p}(\mathbb{R}^n)$ $(1 < p < \infty)$, then $\mathfrak{P}\mathfrak{I}h \in \mathcal{D}'_{L^p}(\mathbb{R}^n)$ and the Schrödinger transform H defined above is a linear isomorphism from $\mathcal{D}'_{L^p}(\mathbb{R}^n)$ onto itself.

Note that, if $\varrho \in L^p(\mathbb{R}^n)$ $(1 < p < \infty)$, then we have

$$\begin{aligned}
\langle (H\varrho)^\wedge, \lambda \rangle &= \langle H\varrho, \hat{\lambda} \rangle \\
&= (-1)^n \langle \varrho, H\hat{\lambda} \rangle \\
&= (-1)^n \langle \check{\varrho}, (H\hat{\lambda})^\wedge \rangle \\
&= (-1)^n \langle \check{\varrho}, (-i)^n \operatorname{sgn}(\cdot)\hat{\hat{\lambda}} \rangle \\
&= \langle \check{\varrho}, (i)^n \operatorname{sgn}(\cdot)\tilde{\lambda} \rangle \\
&= \langle \tilde{\check{\varrho}}, (i)^n \operatorname{sgn}(\cdot)\lambda \rangle \\
&= \langle (-i)^n \operatorname{sgn}(\cdot)\hat{\varrho}, \lambda \rangle,
\end{aligned}$$

where $\lambda \in \mathcal{S}(\mathbb{R}^n)$.

Therefore in the distributional sense

$$(H\varrho)^\wedge(x) = (-i)^n \operatorname{sgn}(\cdot)\hat{\varrho}(x).$$

Define

$$t\Omega = \{tx : x \in \Omega\},$$

where t is a nonzero real number and Ω is a nonempty subset of $\mathbb{R}$. Hence we have

$$\operatorname{supp}\left(u\left(\frac{x}{t}\right)\right) = t\operatorname{supp}(u)$$

for any nonzero real number t.

For a subset $A \subseteq \mathbb{R}$, define

$$A\Omega = \bigcup_{t\in A} t\Omega.$$

2 Schrödinger-type identity for $L^p(\mathbb{R}^n)$ functions

This part is motivated by the need of defining multidimensional complex signals. We define the complex signal of $f \in L^p(\mathbb{R}^n)$ through the total Schrödinger transform Sch_α as $f + i\,\mathrm{Sch}_\alpha(f)$.

In this section we investigate the multidimensional Schrödinger-type identity $\mathrm{Sch}_\alpha(fg) = \mathrm{fSch}_\alpha(g)$ for $f \in \mathcal{S}(\mathbb{R}^n)$ and $g \in L^p(\mathbb{R}^n)$, where $1 < p \leq 2$. In particular, several necessary and sufficient conditions are obtained.

Theorem 2.1 *Suppose that* $f \in \mathcal{S}(\mathbb{R}^n)$; $g \in L^p(\mathbb{R}^n)$ $(1 < p \leq 2)$, *then the Schrödinger transform of the function fg satisfies the Schrödinger-type identity* $\mathrm{Sch}_\alpha(fg) = \mathrm{fSch}_\alpha(g)$ *if and only if*

$$\int_{\mathbb{R}^n} \bigl(\operatorname{sgn}(x) - \operatorname{sgn}(t)\bigr)\hat{f}(x-t)\hat{g}(t)\,\mathrm{d}t = 0. \tag{2.1}$$

Proof According to [10], we use the following equalities:

$$\begin{aligned}
&\mathrm{d}\bar{x} = \varepsilon^2\,\mathrm{d}x,\\
&\mathrm{d}\bar{\Gamma} = \varepsilon\,\mathrm{d}\Gamma \quad \text{on } \mathcal{S}\bigl(\mathbb{R}^n\bigr),\\
&\mathrm{d}\bar{\Gamma} = \varepsilon^2\,\mathrm{d}\Gamma \quad \text{on } L^p\bigl(\mathbb{R}^n\bigr).
\end{aligned}$$

So

$$\begin{aligned}
&\int_0^T \int_{\mathbb{R}^n} (f_\varepsilon \partial_t \lambda + f_\varepsilon g_\varepsilon \cdot \nabla_\varepsilon \lambda)\,\mathrm{d}x\,\mathrm{d}t = 0,\\
&\int_0^T \int_{\mathbb{R}^n} \bigl[f_\varepsilon g_\varepsilon \cdot \partial_t \varrho + f_\varepsilon g_\varepsilon \otimes g_\varepsilon : \omega_\varepsilon(\varrho) + f_\varepsilon \operatorname{div}_\varepsilon \varrho\bigr]\,\mathrm{d}x\,\mathrm{d}t\\
&\quad = \int_0^T \int_{\mathbb{R}^n} \bigl[P\bigl(\bigl|\omega_\varepsilon(g_\varepsilon)\bigr|\bigr)\omega_\varepsilon(g_\varepsilon) : \omega_\varepsilon(\varrho) - f_\varepsilon \cdot \varrho\bigr]\,\mathrm{d}x\,\mathrm{d}t\\
&\qquad + \frac{h(\varepsilon)}{\varepsilon} \int_0^T \int_{\mathcal{S}(\mathbb{R}^n)} g_\varepsilon \cdot \varrho\,\mathrm{d}\Gamma\,\mathrm{d}t + q \int_0^T \int_{L^p(\mathbb{R}^n)} g_\varepsilon \cdot \varrho\,\mathrm{d}\Gamma\,\mathrm{d}t,
\end{aligned}$$

for any $\varrho \in C_0^\infty(0,T;[C^\infty(\bar{\Omega})]^3)$, which leads to

$$\int_0^T \int_\Omega b(f_\varepsilon)\partial_t\lambda + b(f_\varepsilon)g_\varepsilon \cdot \nabla_\varepsilon \lambda + \big[\big(b(f_\varepsilon) - f_\varepsilon b'(f_\varepsilon)\big)\operatorname{div}_\varepsilon g_\varepsilon\big]\lambda \,\mathrm{d}x\,\mathrm{d}t = 0.$$

Note that (see [19])

$$\begin{aligned}
&\int_{\Omega_\varepsilon}\left(\bar{\rho}_\varepsilon(t)\frac{|\bar{g}_\varepsilon(t)|^2}{2} + \bar{\rho}_\varepsilon(t)\ln\big(\bar{\rho}_\varepsilon(t)\big)\right)\mathrm{d}\bar{x} \\
&\quad + \int_0^t \int_{\Omega_\varepsilon} P\big(|\bar{D}\bar{g}_\varepsilon|\big)\bar{D}\bar{g}_\varepsilon : \bar{D}\bar{g}_\varepsilon \,\mathrm{d}\bar{x}\,\mathrm{d}s + h(\varepsilon)\int_0^t\int_{\Gamma_{1,\varepsilon}} |\bar{g}_\varepsilon|^2 \,\mathrm{d}\bar{\Gamma}\,\mathrm{d}s \\
&\quad + q\int_0^t\int_{\Gamma_{2,\varepsilon}} |\bar{g}_\varepsilon|^2\,\mathrm{d}\bar{\Gamma}\,\mathrm{d}s \\
&= \int_0^t\int_{\Omega_\varepsilon} \bar{\rho}_\varepsilon \bar{\mathbf{f}}_\varepsilon \cdot \bar{g}_\varepsilon \,\mathrm{d}\bar{x}\,\mathrm{d}s + \int_{\Omega_\varepsilon}\left(\frac{|(\bar{\rho}_\varepsilon\bar{g}_\varepsilon)_0|^2}{2\bar{\rho}_{0,\varepsilon}} + \bar{\rho}_{0,\varepsilon}\ln(\bar{\rho}_{0,\varepsilon})\right)\mathrm{d}\bar{x}.
\end{aligned}$$

For any $t \in \langle 0,T\rangle$, this yields

$$\begin{aligned}
&\int_\Omega\left(f_\varepsilon(t)\frac{|g_\varepsilon(t)|^2}{2} + f_\varepsilon(t)\ln\big(f_\varepsilon(t)\big)\right)\mathrm{d}x \\
&\quad + \int_0^t\int_\Omega P\big(\big|\omega_\varepsilon(g_\varepsilon)\big|\big)\big|\omega_\varepsilon(g_\varepsilon)\big|^2\,\mathrm{d}x\,\mathrm{d}s \\
&\quad + \frac{h(\varepsilon)}{\varepsilon}\int_0^t\int_{\mathcal{S}(\mathbb{R}^n)} |g_\varepsilon|^2\,\mathrm{d}\Gamma\,\mathrm{d}s + q\int_0^t\int_{L^p(\mathbb{R}^n)} |g_\varepsilon|^2\,\mathrm{d}\Gamma\,\mathrm{d}s \\
&= \int_0^t\int_\Omega f_\varepsilon \mathbf{g}_\varepsilon \cdot \mathbf{v}_\varepsilon\,\mathrm{d}x\,\mathrm{d}s + \int_\Omega\left(\frac{|(f_\varepsilon g_\varepsilon)_0|^2}{2f_{0,\varepsilon}} + f_{0,\varepsilon}\ln(f_{0,\varepsilon})\right)\mathrm{d}x,
\end{aligned}$$

$$\begin{aligned}
&\|\sqrt{g_\varepsilon}\bar{\partial}\alpha\|_\lambda^2 + \big\|\sqrt{g_\varepsilon}\bar{\partial}_\lambda^*\alpha\big\|_\lambda^2 \\
&= \sum_{|L|=p-1}{}' \sum_{j,k=1}^n \int_{b\Omega} g\frac{\partial^2\rho}{\partial z_j\partial\bar{z}_k}\alpha_{jL}\overline{\alpha}_{kL}e^{-\lambda}\,dS \\
&\quad + \sum_{|K|=p}{}' \sum_{k=1}^n \int_\Omega g\left|\frac{\partial\alpha_K}{\partial\bar{z}_k}\right|^2 e^{-\lambda}\,dV \\
&\quad + \sum_{|L|=p-1}{}' \sum_{j,k=1}^n \int_\Omega\left(g\frac{\partial^2\lambda}{\partial z_j\partial\bar{z}_k} - \frac{\partial^2 g}{\partial z_j\partial\bar{z}_k}\right)\alpha_{jL}\overline{\alpha}_{kL}e^{-\lambda}\,dV \\
&\quad + 2\operatorname{Re}\left\langle \sum_{|L|=p-1}{}' \sum_{j=1}^n \alpha_{jL}\frac{\partial g}{\partial z_j}\,d\bar{z}_L, \bar{\partial}_\lambda^*\alpha\right\rangle_\lambda
\end{aligned}$$

and

$$\begin{aligned}
&2\operatorname{Re}\left\langle \sum_{|L|=p-1}{}' \sum_{j=1}^n \alpha_{jL}\frac{\partial g}{\partial z_j}\,d\bar{z}_L, \bar{\partial}_\lambda^*\alpha\right\rangle_\lambda \\
&\quad \le 2\left|\left\langle \sum_{|L|=p-1}{}' \frac{1}{\sqrt{g_\varepsilon}}e^{-\lambda/2}\sum_{j=1}^n \frac{\partial g}{\partial z_j}\alpha_{jL}\,d\bar{z}_j, \sqrt{g_\varepsilon}e^{-\lambda/2}\bar{\partial}_\lambda^*\alpha\right\rangle\right|
\end{aligned}$$

$$\leq 2\left\| \sum_{|L|=p-1}{}' \frac{1}{\sqrt{g_\varepsilon}} \sum_{j=1}^{n} \frac{\partial g}{\partial z_j} \alpha_{jL}\, d\bar{z}_j \right\|_\lambda \left\| \sqrt{g_\varepsilon}\bar{\partial}_\lambda^* \alpha \right\|_\lambda$$

$$\leq \sum_{|L|=p-1}{}' \varsigma \left\| \frac{1}{\sqrt{g_\varepsilon}} \sum_{j=1}^{n} \frac{\partial g}{\partial z_j} \alpha_{jL} \right\|_\lambda^2 + \frac{1}{\varsigma} \left\| \sqrt{g_\varepsilon}\bar{\partial}_\lambda^* \alpha \right\|_\lambda^2$$

for any $t \in \langle 0, T \rangle$, where

$$g_\varepsilon = \left(f_{1,\varepsilon}, \varepsilon^{-1} f_{2,\varepsilon}, \varepsilon^{-1} f_{3,\varepsilon}\right), \qquad v_\varepsilon = (u_{1,\varepsilon}, \varepsilon u_{2,\varepsilon}, \varepsilon u_{3,\varepsilon}).$$

Since the Schrödingerean Fourier transform is injective from $\mathcal{S}'$ into itself, fg, $\mathrm{Sch}_\alpha(f)g$, $\mathrm{fSch}_\alpha(g) \in L^p(\mathbb{R}^n)$, we have

$$\left(\mathrm{Sch}_\alpha(fg)\right)^\wedge = \left(\mathrm{fSch}_\alpha(g)\right)^\wedge,$$

which is equivalent to

$$(-1)^n \operatorname{sgn}(x) \int_{\mathbb{R}^n} \hat{f}(x-t)\hat{g_\varepsilon}(t)\, dt = \int_{\mathbb{R}^n} (-1)^n \operatorname{sgn}(t) \hat{f}(x-t)\hat{g_\varepsilon}(t)\, dt,$$

where

$$\operatorname{sgn}(x) = \prod_{j}^{n} = 1 \operatorname{sgn}(x_j), \quad x = (x_1, x_2, \ldots, x_n).$$

So

$$\int_{\mathbb{R}^n} \left(\operatorname{sgn}(x) - \operatorname{sgn}(t)\right) \hat{f}(x-t)\hat{g_\varepsilon}(t)\, dt = 0. \qquad \square$$

Let a_j and b_j denote nonnegative real numbers in the rest of the paper, where $j = 1, 2, \ldots, n$.

Corollary 2.1 *Let* $f \in \mathcal{S}(\mathbb{R}^n)$ *and* $g \in L^p(\mathbb{R}^n)$, *where* $1 < p \leq 2$. *If*

$$\operatorname{supp} \hat{f} \subseteq \prod_{j=1}^{n} [-a_j, b_j], \qquad \operatorname{supp} \hat{g_\varepsilon} \subseteq \prod_{j=1}^{n} \mathbb{R} \setminus (-b_j, a_j), \tag{2.2}$$

then the Schrödinger-type identity $\mathrm{Sch}_\alpha(fg) = \mathrm{fSch}_\alpha(g)$ *holds.*

Proof We first prove

$$\int_{\mathbb{R}^n} \left(\operatorname{sgn}(x) - \operatorname{sgn}(t)\right) \hat{f}(x-t)\hat{g_\varepsilon}(t)\, dt = 0$$

from Theorem 2.1.

That is,

$$\int_{D_+} \left(\operatorname{sgn}(x) - \operatorname{sgn}(t)\right) \hat{f}(x-t)\hat{g_\varepsilon}(t)\, dt + \int_{D_-} \left(\operatorname{sgn}(x) - \operatorname{sgn}(t)\right) \hat{f}(x-t)\hat{g_\varepsilon}(t)\, dt = 0.$$

Let $x \in D_+$, if $t \in D_+$, the integrand is vanish so (2.1) holds. If $t \in D_0$, (2.1) holds since the integration is over a set of measure zero. As for the case $t \in D_-$, assume that there exists $t \in D_-$, such that $t \in \operatorname{supp} \hat{f}(x-\cdot)\hat{g}_\varepsilon(\cdot)$, then $t \in \operatorname{supp} \hat{g}_\varepsilon \cap D_-$, $x - t \in \operatorname{supp} \hat{f}$.

Since $D_- \cap D_+ = \emptyset$, there exists $j \in \{1, 2, \dots, n\}$ such that $x_j t_j \leq 0$. We may assume that $x_j > 0$ and $t_j \leq 0$. Thanks to (2.2), we have $t_j \leq -b_j$ and $x_j - t_j \leq b_j$, which is impossible.

By repeating this argument for $x \in D_-$ and $x \in D_0$ (see [6]), we find the same conclusion. □

Lemma 2.1 *Suppose that* $f \in L^p(\mathbb{R}^n)$ *and* $g \in L^q(\mathbb{R}^n)$, *where*

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{r} \leq 1 \quad (1 < p, q \leq 2).$$

Then

$$(f * g)^\wedge = \hat{f}\hat{g}_\varepsilon$$

holds.

Proof Let $y_i = 1$, where $i = 1, 2, \dots, n-1$. Then

$$f(t, 1, \dots, 1) \geq k^b f(t, k, \dots, k),$$

where $k \in (0, 1)$.

So

$$\begin{aligned}
&f\left(t, k^{-1}, \dots, k^{-1}\right) \geq k^b f(t, 1, \dots, 1), \\
&f(t, ky_1, \dots, ky_{n-1}) \leq k^{-b} f(t, y_1, \dots, y_{n-1}), \\
&f(t, k, \dots, k) \leq k^{-b} f(t, 1, \dots, 1),
\end{aligned}$$

where $k \in (0, 1)$, which yields

$$\begin{aligned}
&f_{0^+}^{n-2} w(t) > 0, \qquad f_{0^+}^{n-3} w(t) > 0, \qquad \dots, \qquad f_{0^+}^{1} w(t) > 0, \qquad w(t) > 0, \\
&g\left(t, f_{0^+}^{n-2} w(t), f_{0^+}^{n-3} w(t), \dots, f_{0^+}^{1} w(t), w(t)\right) \\
&\quad \leq g\left(t, f_{0^+}^{n-2} Ae(t), f_{0^+}^{n-3} Ae(t), \dots, f_{0^+}^{1} Ae(t), Ae(t)\right) \\
&\quad \leq g\left(t, f_{0^+}^{n-2} A, f_{0^+}^{n-3} A, \dots, f_{0^+}^{1} A, A\right) \\
&\quad = g\left(t, \frac{A}{(n-2)!} t^{n-2}, \frac{A}{(n-3)!} t^{n-3}, \dots, At, A\right) \\
&\quad \leq g(t, A, A, \dots, A, A) \\
&\quad \leq A^b g(t, 1, 1, \dots, 1, 1) \\
&\quad \leq A^b g(1, 1, 1, \dots, 1, 1),
\end{aligned}$$

and

$$h\left(t, f_{0^+}^{n-2} w(t), f_{0^+}^{n-3} w(t), \dots, f_{0^+}^{1} w(t), w(t)\right)$$

$$
\begin{aligned}
&\le h\left(t, f_{0^+}^{n-2}\frac{1}{A}e(t), f_{0^+}^{n-3}\frac{1}{A}e(t), \dots, f_{0^+}^{1}\frac{1}{A}e(t), \frac{1}{A}e(t)\right)\\
&= h\left(t, \frac{\Gamma(\alpha-n+2)}{A\Gamma(\alpha)}t^{\alpha-1}, \frac{\Gamma(\alpha-n+2)}{A\Gamma(\alpha-1)}t^{\alpha-2}, \dots,\right.\\
&\quad\left.\frac{\Gamma(\alpha-n+2)}{A\Gamma(\alpha-n+3)}t^{\alpha-n+2}, \frac{1}{A}t^{\alpha-n+1}\right)\\
&\le h\left(t, \frac{\zeta}{A}t^{\alpha-1}, \frac{\zeta}{A}t^{\alpha-1}, \dots, \frac{\zeta}{A}t^{\alpha-n+3}, \frac{\zeta}{A}t^{\alpha-n+2}\right)\\
&\le h\left(t, \frac{\zeta}{A}t^{\alpha-1}, \frac{\zeta}{A}t^{\alpha-1}, \dots, \frac{\zeta}{A}t^{\alpha-n+4}, \frac{\zeta}{A}t^{\alpha-n+3}\right)\\
&\le \cdots\\
&\le h\left(t, \frac{\zeta}{A}t^{\alpha-1}, \frac{\zeta}{A}t^{\alpha-1}, \dots, \frac{\zeta}{A}t^{\alpha-1}, \frac{\zeta}{A}t^{\alpha-1}\right)\\
&\le \left(\frac{\zeta}{A}\right)^{-b} t^{-b(\alpha-1)} f(t,1,1,\dots,1,1)\\
&\le \left(\frac{\zeta}{A}\right)^{-b} t^{-b(\alpha-1)} f(0,1,1,\dots,1,1).
\end{aligned}
$$

So

$$
\begin{aligned}
&g\left(t, f_{0^+}^{n-2}w(t), f_{0^+}^{n-3}w(t), \dots, f_{0^+}^{1}w(t), w(t)\right)\\
&\quad\ge g\left(t, f_{0^+}^{n-2}\frac{1}{A}e(t), f_{0^+}^{n-3}\frac{1}{A}e(t), \dots, f_{0^+}^{1}\frac{1}{A}e(t), \frac{1}{A}e(t)\right)\\
&\quad= g\left(t, \frac{\Gamma(\alpha-n+2)}{A\Gamma(\alpha)}t^{\alpha-1}, \frac{\Gamma(\alpha-n+2)}{A\Gamma(\alpha-1)}t^{\alpha-2}, \dots,\right.\\
&\quad\quad\left.\frac{\Gamma(\alpha-n+2)}{A\Gamma(\alpha-n+3)}t^{\alpha-n+2}, \frac{1}{A}t^{\alpha-n+1}\right)\\
&\quad\ge g\left(t, \frac{\zeta}{A}t^{\alpha-1}, \frac{\zeta}{A}t^{\alpha-1}, \dots, \frac{\zeta}{A}t^{\alpha-n+3}, \frac{\zeta}{A}t^{\alpha-n+2}\right)\\
&\quad\ge g\left(t, \frac{\zeta}{A}t^{\alpha-1}, \frac{\zeta}{A}t^{\alpha-1}, \dots, \frac{\zeta}{A}t^{\alpha-n+4}, \frac{\zeta}{A}t^{\alpha-n+3}\right)\\
&\quad\ge \cdots\\
&\quad\ge g\left(t, \frac{\zeta}{A}t^{\alpha-1}, \frac{\zeta}{A}t^{\alpha-1}, \dots, \frac{\zeta}{A}t^{\alpha-1}, \frac{\zeta}{A}t^{\alpha-1}\right)\\
&\quad\ge \left(\frac{\zeta}{A}\right)^{-b} t^{b(\alpha-1)} g(t,1,1,\dots,1,1)\\
&\quad\ge \left(\frac{\zeta}{A}\right)^{-b} t^{b(\alpha-1)} g(0,1,1,\dots,1,1)
\end{aligned}
$$

and

$$
\begin{aligned}
&h\left(t, f_{0^+}^{n-2}w(t), f_{0^+}^{n-3}w(t), \dots, f_{0^+}^{1}w(t), w(t)\right)\\
&\quad\ge h\left(t, f_{0^+}^{n-2}Ae(t), f_{0^+}^{n-3}Ae(t), \dots, f_{0^+}^{1}Ae(t), Ae(t)\right)
\end{aligned}
$$

$$
\begin{aligned}
&\geq h\bigl(t, f_{0^+}^{n-2}A, f_{0^+}^{n-3}A, \ldots, f_{0^+}^{1}A, A\bigr) \\
&= h\biggl(t, \frac{A}{(n-2)!}t^{n-2}, \frac{A}{(n-3)!}t^{n-3}, \ldots, At, A\biggr) \\
&\geq f(t, A, A, \ldots, A, A) \\
&\geq A^{-b} f(t, 1, 1, \ldots, 1, 1) \\
&\geq A^{-b} f(1, 1, 1, \ldots, 1, 1),
\end{aligned}
$$

which yields

$$
\begin{aligned}
&\int_0^1 \int_0^1 H(s,\varsigma) g\bigl(\varsigma, f_{0^+}^{n-2}\nu(\varsigma), \ldots, f_{0^+}^{1}\nu(\varsigma), \nu(\varsigma)\bigr)\,d\varsigma\,ds \\
&\quad \leq \int_0^1 \iota_q\biggl(\frac{s^{\beta-1}}{\Gamma(\beta)} \int_0^1 g\bigl(\varsigma, f_{0^+}^{n-2}\nu(\varsigma), \ldots, f_{0^+}^{1}\nu(\varsigma), \nu(\varsigma)\bigr)\,d\varsigma\biggr)\,ds \\
&\quad \leq \int_0^1 \iota_q\biggl(\frac{s^{\beta-1}A^b g(1,1,\ldots,1)}{\Gamma(\beta)}\biggr)\,ds \\
&\quad \leq \int_0^1 \iota_q\bigl(s^{\beta-1}\bigr)\,ds
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_0^1 \int_0^1 H(s,\varsigma) f\bigl(\varsigma, f_{0^+}^{n-2}w(\varsigma), \ldots, f_{0^+}^{1}w(\varsigma), w(\varsigma)\bigr)\,d\varsigma\,ds \\
&\quad \leq \int_0^1 \iota_q\biggl(\frac{s^{\beta-1}}{\Gamma(\beta)} \int_0^1 f\bigl(\varsigma, f_{0^+}^{n-2}w(\varsigma), \ldots, f_{0^+}^{1}w(\varsigma), w(\varsigma)\bigr)\,d\varsigma\biggr)\,ds \\
&\quad \leq \int_0^1 \iota_q\biggl(\frac{s^{\beta-1}}{\Gamma(\beta)} \int_0^1 \biggl(\frac{\zeta}{A}\biggr)^{-b} \varsigma^{-b(\alpha-1)} f(0,1,1,\ldots,1)\,d\varsigma\biggr)\,ds \\
&\quad \leq \frac{t^{\alpha-n+1}(\zeta^{-b}A^b f(0,1,1,\ldots,1))^{q-1}}{\Gamma(\alpha-n+1)(\Gamma(\beta))^{q-1}} \int_0^1 \iota_q\biggl(s^{\beta-1}\int_0^1 \varsigma^{-b(\alpha-1)}\,d\varsigma\biggr)\,ds.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
&\int_0^1 \iota_q\biggl(\int_0^1 g\bigl(\varsigma, f_{0^+}^{n-2}\nu(\varsigma), \ldots, f_{0^+}^{1}\nu(\varsigma), \nu(\varsigma)\bigr)\,d\varsigma\biggr)\,ds \\
&\quad \geq \int_\xi^1 \iota_q\biggl(\int_\xi^1 g\bigl(\varsigma, f_{0^+}^{n-2}\nu(\varsigma), \ldots, f_{0^+}^{1}\nu(\varsigma), \nu(\varsigma)\bigr)\,d\varsigma\biggr)\,ds \\
&\quad \geq \int_\xi^1 \gamma(s)\iota_q\biggl(\int_\xi^1 \rho(\varsigma) g\bigl(\varsigma, f_{0^+}^{n-2}\nu(\varsigma), \ldots, f_{0^+}^{1}\nu(\varsigma), \nu(\varsigma)\bigr)\,d\varsigma\biggr)\,ds \\
&\quad \geq \int_\xi^1 \gamma(s)\iota_q\biggl(\int_\xi^1 \rho(\varsigma)\biggl(\frac{\zeta}{A}\biggr)^{b} \varsigma^{b(\alpha-1)} g(0,1,1,\ldots,1)\,d\varsigma\biggr)\,ds \\
&\quad = \bigl(\zeta^b A^{-b} g(0,1,1,\ldots,1)\bigr)^{q-1} \int_\xi^1 \gamma(s)\iota_q\biggl(\int_\xi^1 \rho(\varsigma)\varsigma^{b(\alpha-1)}\,d\varsigma\biggr)\,ds \\
&\quad \geq t^{\alpha-n+1}\bigl(\zeta^b A^{-b} g(0,1,1,\ldots,1)\bigr)^{q-1} \\
&\qquad \times \int_\xi^1 \gamma(s)\iota_q\biggl(\int_\xi^1 \rho(\varsigma)\varsigma^{b(\alpha-1)}\,d\varsigma\biggr)\,ds
\end{aligned}
$$

and

$$\begin{aligned}
&\int_0^1 \int_0^1 h\big(\varsigma, f_{0^+}^{n-2} w(\varsigma), \ldots, f_{0^+}^{1} w(\varsigma), w(\varsigma)\big)\, d\varsigma \, ds \\
&\quad \geq \int_\xi^1 \iota_q\left(\int_\xi^1 f\big(\varsigma, f_{0^+}^{n-2} w(\varsigma), \ldots, f_{0^+}^{1} w(\varsigma), w(\varsigma)\big)\, d\varsigma\right) ds \\
&\quad \geq \int_\xi^1 \gamma(s)\iota_q\left(\int_\xi^1 \rho(\varsigma) f\big(\varsigma, f_{0^+}^{n-2} w(\varsigma), \ldots, f_{0^+}^{1} w(\varsigma), w(\varsigma)\big)\, d\varsigma\right) ds \\
&\quad \geq \int_\xi^1 \gamma(s)\iota_q\left(\int_\xi^1 \rho(\varsigma) H(\varsigma,\varsigma) A^{-b} f(1,1,1,\ldots,1)\, d\varsigma\right) ds \\
&\quad \geq \big(A^{-b} f(1,1,1,\ldots,1)\big)^{q-1} \int_\xi^1 \gamma(s)\iota_q\left(\int_\xi^1 \rho(\varsigma) H(\varsigma,\varsigma)\, d\varsigma\right) ds \\
&\quad \geq t^{\alpha-n+1}\big(A^{-b} f(1,1,1,\ldots,1)\big)^{q-1} \int_\xi^1 \gamma(s)\iota_q\left(\int_\xi^1 \rho(\varsigma) H(\varsigma,\varsigma)\, d\varsigma\right) ds,
\end{aligned}$$

which yields

$$T(v,w)(t) \geq \frac{1}{A} t^{\alpha-n+1} = \frac{1}{A} e(t),$$

where $t \in (0,1)$.

Then we prove that $T : Q_e \times Q_e \to Q_e$ is a mixed monotone operator. We have

$$\begin{aligned}
&\int_0^1 \iota_q\left(\int_0^1 g\big(\varsigma, f_{0^+}^{n-2} v_1(\varsigma), \ldots, f_{0^+}^{1} v_1(\varsigma), v_1(\varsigma)\big)\, d\varsigma\right) ds \\
&\quad \leq \int_0^1 \iota_q\left(\int_0^1 H(s,\varsigma) g\big(\varsigma, f_{0^+}^{n-2} v_2(\varsigma), \ldots, f_{0^+}^{1} v_2(\varsigma), v_2(\varsigma)\big)\, d\varsigma\right) ds.
\end{aligned}$$

Thus $T(v,w)(t)$ is nondecreasing in v for any $w \in Q_e$.

Let $w_1, w_2 \in Q_e$ and $w_1 \geq w_2$. Then

$$\begin{aligned}
&\int_0^1 P(s,\varsigma)\iota_q\left(\int_0^1 Q(s,\varsigma) f\big(\varsigma, f_{0^+}^{n-2} w_1(\varsigma), \ldots, f_{0^+}^{1} w_1(\varsigma), w_1(\varsigma)\big)\, d\varsigma\right) ds \\
&\quad \leq \int_0^1 P(s,\varsigma)\iota_q\left(\int_0^1 Q(s,\varsigma) f\big(\varsigma, f_{0^+}^{n-2} w_2(\varsigma), \ldots, f_{0^+}^{1} w_2(\varsigma), w_2(\varsigma)\big)\, d\varsigma\right) ds,
\end{aligned}$$

i.e.,

$$T(v,w_1)(t) \leq T(v,w_2)(t), \quad w \in Q_e.$$

Therefore $T(v,w)(t)$ is nonincreasing in w for any $v \in Q_e$.

We shall show that the operator T has a fixed point.

It follows that

$$\begin{aligned}
&\int_0^1 P(s,\varsigma)\iota_q\left(\int_0^1 Q(s,\varsigma) g\big(\varsigma, f_{0^+}^{n-2} tv(\varsigma), \ldots, f_{0^+}^{1} tv(\varsigma), tv(\varsigma)\big)\, d\varsigma\right) ds \\
&\quad = \int_0^1 P(s,\varsigma)\iota_q\left(\int_0^1 Q(s,\varsigma) g\big(\varsigma, t f_{0^+}^{n-2} v(\varsigma), \ldots, t f_{0^+}^{1} v(\varsigma), tv(\varsigma)\big)\, d\varsigma\right) ds
\end{aligned}$$

$$\begin{aligned} &\geq \int_0^1 P(s,\varsigma)\iota_q\left(\int_0^1 Q(s,\varsigma)t^b g\big(\varsigma, f_{0^+}^{n-2}v(\varsigma),\ldots,f_{0^+}^1 v(\varsigma),v(\varsigma)\big)\,d\varsigma\right)ds \\ &\geq t^b \int_0^1 P(s,\varsigma)\iota_q\left(\int_0^1 Q(s,\varsigma)g\big(\varsigma, f_{0^+}^{n-2}v(\varsigma),\ldots,f_{0^+}^1 v(\varsigma),v(\varsigma)\big)\,d\varsigma\right)ds \end{aligned}$$

and

$$\begin{aligned} &\int_0^1 P(s,\varsigma)\iota_q\left(\int_0^1 Q(s,\varsigma)f\big(\varsigma, f_{0^+}^{n-2}t^{-1}w(\varsigma),\ldots,f_{0^+}^1 t^{-1}w(\varsigma),t^{-1}w(\varsigma)\big)\,d\varsigma\right)ds \\ &\quad= \int_0^1 P(s,\varsigma)\iota_q\left(\int_0^1 Q(s,\varsigma)f\big(\varsigma, t^{-1}f_{0^+}^{n-2}w(\varsigma),\ldots,t^{-1}f_{0^+}^1 w(\varsigma),t^{-1}w(\varsigma)\big)\,d\varsigma\right)ds \\ &\quad\geq \int_0^1 P(s,\varsigma)\iota_q\left(\int_0^1 Q(s,\varsigma)t^b f\big(\varsigma, f_{0^+}^{n-2}w(\varsigma),\ldots,f_{0^+}^1 w(\varsigma),w(\varsigma)\big)\,d\varsigma\right)ds \\ &\quad\geq t^b\int_0^1 P(s,\varsigma)\iota_q\left(\int_0^1 Q(s,\varsigma)f\big(\varsigma, f_{0^+}^{n-2}w(\varsigma),\ldots,f_{0^+}^1 w(\varsigma),w(\varsigma)\big)\,d\varsigma\right)ds, \end{aligned}$$

we obtain

$$T\left(tx,\frac{1}{t}y\right)\geq t^b T(x,y),\quad x,y\in Q_e, t\in(0,1), b\in(0,1).$$

Therefore

$$\begin{aligned} \frac{\Gamma(\alpha-n+2)}{A\Gamma(\alpha)}t^{\alpha-1} &= \frac{1}{A}f_{0^+}^{n-2}e(t)\leq u(t) \\ &\leq Mf_{0^+}^{n-2}e(t)=\frac{A\Gamma(\alpha-n+2)}{\Gamma(\alpha)}t^{\alpha-1},\quad t\in(0,1). \end{aligned}$$

Since $f\in L^p(\mathbb{R}^n)$ and $g\in L^q(\mathbb{R}^n)$ (see [11]), we have

$$f*g\in L^r\big(\mathbb{R}^n\big),$$

which shows that there exist functions $g_n\in\mathcal{S}(\mathbb{R}^n)$ such that

$$\|g-g_n\|_q\to 0$$

as $n\to\infty$,

$$(f*g_n)^\wedge=\hat{f}\hat{g}_{\varepsilon n},$$

and

$$(fg_n)^\wedge=\hat{f}*\hat{g}_{\varepsilon n}.$$

Thus in the distributional sense

$$\lim_{n\to\infty}(f*g_n)^\wedge(x)=(f*g)^\wedge(x).$$

On the other hand

$$\left|\langle\hat{f}(\hat{g}_{\varepsilon n}-\hat{g}_\varepsilon,\lambda)\rangle\right| = \left|\langle(\hat{g}_{\varepsilon n}-\hat{g}_\varepsilon,\hat{f}\lambda)\rangle\right| \leq \|\hat{f}\lambda\|_q\|\hat{g}_{\varepsilon n}-\hat{g}_\varepsilon\|_{q'} \to 0$$

as $n \to \infty$.

Hence the result

$$(f * g)^\wedge = \hat{f}\hat{g}_\varepsilon$$

is obtained. □

We define

$$S_n = \{\sigma_k : \{1,2,\ldots,n\} \to \{+1,-1\}\},$$
$$Q_{\sigma_k} = \{y = (y_1,y_2,\ldots,y_n) \in \mathbb{R}^n : y_j\sigma_k(j) > 0\}$$

and

$$-Q_{\sigma_k} = \{\xi \in \mathbb{R}^n : -\xi \in Q_{\sigma_k}\}, \qquad \operatorname{sgn}(\xi) = \prod_{j=1}^n \operatorname{sgn}(\xi_j),$$

where $j = 1,2,\ldots,n$.

It follows that if $\xi \in Q_{\sigma_k}$ and $\eta \in -Q_{\sigma_k}$, then $\operatorname{sgn}(\xi) = \operatorname{sgn}(\eta)$ when n is an even, and $\operatorname{sgn}(\xi) = -\operatorname{sgn}(\eta)$ when n is an odd.

With these notations we have the following.

Theorem 2.2 *Let n be an odd and $f \in \mathcal{S}(\mathbb{R}^n)$, $g \in L^p(\mathbb{R}^n)$ $(1 < p \leq 2)$ satisfy* $\operatorname{supp}\hat{f}$, $\operatorname{supp}\hat{g}_\varepsilon \subseteq Q_{\sigma_k} \cup -Q_{\sigma_k}$ *with $a_j\sigma_k(j)$, $-b_j\sigma_k(j) \in \operatorname{supp}\hat{f}$ $(j = 1,2,\ldots,n)$ and*

$$\operatorname{supp}\hat{f} \subseteq \{\xi \in Q_{\sigma_k} : \sigma_k(j)\xi_j \leq a_j\} \cup \{\xi \in -Q_{\sigma_k} : -\sigma_k(j)\xi_j \leq b_j\}.$$

Then $g \in L^p(\mathbb{R}^n)$ satisfies the Schrödinger-type identity $\operatorname{Sch}_\alpha(fg) = \operatorname{fSch}_\alpha(g)$ *if and only if*

$$\operatorname{supp}\hat{f} \subseteq \left\{\xi \in Q_{\sigma_k} : \sum_{j=1}^n \frac{\sigma_k(j)\xi_j}{b_j} \geq 1\right\} \cup \left\{\xi \in \sum_{j=1}^n \frac{-\sigma_k(j)\xi_j}{a_j} \geq 1\right\}.$$

Proof Suppose that Q_{σ_k} is the first octant in $\mathbb{R}^n$, that is to say, all the $\sigma_k(j) = 1$, where $j = 1,2,\ldots,n$.

Let

$$\begin{aligned}
&\hat{f}\big(s, g_{0^+}^{n-2}\operatorname{sgn}(s), g_{0^+}^{n-3}\operatorname{sgn}(s),\ldots,g_{0^+}^1\operatorname{sgn}(s),\operatorname{sgn}(s)\big) \\
&\quad = \begin{cases} f(s,g_{0^+}^{n-2}m(s),g_{0^+}^{n-3}m(s),\ldots,g_{0^+}^1 m(s),m(s)), & \operatorname{sgn}(s) < m(s), \\ f(s,g_{0^+}^{n-2}\operatorname{sgn}(s),g_{0^+}^{n-3}\operatorname{sgn}(s),\ldots,g_{0^+}^1\operatorname{sgn}(s),\operatorname{sgn}(s)), & m(s) \leq \operatorname{sgn}(s) \leq n(s), \\ f(s,g_{0^+}^{n-2}n(s),g_{0^+}^{n-3}n(s),\ldots,g_{0^+}^1 n(s),n(s)), & \operatorname{sgn}(s) > n(s). \end{cases}
\end{aligned} \tag{2.3}$$

Consider the fractional differential equation

$$\begin{aligned} &Q_{0^+}^{\beta}\iota_p\big(Q_{0^+}^{\alpha-n+2}n(s)\big) + \hat{f}\big(s, g_{0^+}^{n-2}\operatorname{sgn}(s), g_{0^+}^{n-3}\operatorname{sgn}(s), \ldots, g_{0^+}^{1}\operatorname{sgn}(s), \operatorname{sgn}(s)\big) = 0, \\ &0 < t < 1, \\ &\nu(0) = 0, \qquad \nu(1) = a\nu(\xi), \qquad Q_{0^+}^{\alpha-n+2}\nu(0) = Q_{0^+}^{\alpha-n+2}\nu(1) = 0. \end{aligned} \tag{2.4}$$

Set $\Omega_2 = \{\nu \in E_2 : \|\nu\| \le M_1\iota_q(M_2L_2)\}$, then Ω_2 is a closed, bounded and convex set, where

$$L_2 := \sup_{t\in[0,1],\nu\in\Omega_2} \big|\hat{f}\big(s, g_{0^+}^{n-2}\operatorname{sgn}(s), \ldots, g_{0^+}^{1}\operatorname{sgn}(s), \operatorname{sgn}(s)\big)\big| + 1.$$

The operator $A : \Omega_2 \to E_2$ is defined by

$$A\operatorname{sgn}(s) = \int_0^1 P(s,\varsigma)\iota_q\biggl(\int_0^1 Q(s,\varsigma)\hat{f}\big(\varsigma, g_{0^+}^{n-2}\nu(\varsigma), \ldots, g_{0^+}^{1}\nu(\varsigma), \nu(\varsigma)\big)\,d\varsigma\biggr)\,ds.$$

Now, we show that A is a completely continuous operator. It follows that

$$\begin{aligned} &\big|(A\nu)(s)\big| \\ &\quad = \biggl|\int_0^1 P(s,\varsigma)\iota_q\biggl(\int_0^1 Q(s,\varsigma)\hat{f}\big(\varsigma, g_{0^+}^{n-2}\nu(\varsigma), \ldots, g_{0^+}^{1}\nu(\varsigma), \nu(\varsigma)\big)\,d\varsigma\biggr)\,ds\biggr| \\ &\quad \le \int_0^1 P(s,\varsigma)\iota_q\biggl(\int_0^1 Q(s,\varsigma)\big|\hat{f}\big(\varsigma, g_{0^+}^{n-2}\nu(\varsigma), \ldots, g_{0^+}^{1}\nu(\varsigma), \nu(\varsigma)\big)\big|\,d\varsigma\biggr)\,ds \\ &\quad \le L_2^{q-1}\int_0^1 P(s,\varsigma)\iota_q\biggl(\int_0^1 Q(s,\varsigma)\,d\varsigma\biggr)\,ds \\ &\quad \le L_2^{q-1}\int_0^1 P(s,s)\iota_q\biggl(\int_0^1 Q(\varsigma,\varsigma)\,d\varsigma\biggr)\,ds \\ &\quad < +\infty, \end{aligned}$$

which yields

$$P(t_1,s) - P(t_2,s) < \frac{\varepsilon}{L_2^{q-1}\iota_q(\int_0^1 Q(\varsigma,\varsigma)\,d\varsigma)}.$$

So

$$\begin{aligned} &\big|A\nu(t_2) - A\nu(t_1)\big| \\ &\quad \le \int_0^1 \big|P(t_2,s) - P(t_1,s)\big|\iota_q\biggl(\int_0^1 Q(s,\varsigma)\hat{f}\big(\varsigma, g_{0^+}^{n-2}\nu(\varsigma), \ldots, g_{0^+}^{1}\nu(\varsigma), \nu(\varsigma)\big)\,d\varsigma\biggr)\,ds \\ &\quad \le L_2^{q-1}\int_0^1 \big|P(t_2,s) - P(t_1,s)\big|\iota_q\biggl(\int_0^1 Q(\varsigma,\varsigma)\,d\varsigma\biggr)\,ds \\ &\quad \le L_2^{q-1}\iota_q\biggl(\int_0^1 Q(\varsigma,\varsigma)\,d\varsigma\biggr)\int_0^1 \big|P(t_2,s) - P(t_1,s)\big|\,ds \\ &\quad < \varepsilon \end{aligned}$$

for any $\nu \in \Omega_2$.

We prove that the fractional differential equation has at least one positive solution. Suppose that $d(s)$ is a solution of (2.4) (see [9]), then

$$d(0) = 0, \qquad d(1) = ad(\xi), \qquad Q_{0^+}^{\alpha-n+2}d(0) = Q_{0^+}^{\alpha-n+2}d(1) = 0.$$

So

$$\begin{aligned}
&f\bigl(s, g_{0^+}^{n-2}n(s), g_{0^+}^{n-3}n(s), \ldots, g_{0^+}^{1}n(s), n(s)\bigr)\\
&\quad \le \hat{f}\bigl(s, g_{0^+}^{n-2}d(s), g_{0^+}^{n-3}d(s), \ldots, g_{0^+}^{1}d(s), d(s)\bigr)\\
&\quad \le f\bigl(s, g_{0^+}^{n-2}m(s), g_{0^+}^{n-3}m(s), \ldots, g_{0^+}^{1}m(s), m(s)\bigr).
\end{aligned}$$

So

$$\begin{aligned}
&f\bigl(s, g_{0^+}^{n-2}q(s), g_{0^+}^{n-3}q(s), \ldots, g_{0^+}^{1}q(s), q(s)\bigr)\\
&\quad \le \hat{f}\bigl(s, g_{0^+}^{n-2}d(s), g_{0^+}^{n-3}d(s), \ldots, g_{0^+}^{1}d(s), d(s)\bigr)\\
&\quad \le f\bigl(s, g_{0^+}^{n-2}p(s), g_{0^+}^{n-3}p(s), \ldots, g_{0^+}^{1}p(s), p(s)\bigr),
\end{aligned}$$

which yields

$$\begin{aligned}
Q_{0^+}^{\beta}\iota_p\bigl(Q_{0^+}^{\alpha-n+2}n(s)\bigr) &= Q_{0^+}^{\beta}\iota_p\bigl(Q_{0^+}^{\alpha-n+2}(Fp)(s)\bigr)\\
&= f\bigl(s, g_{0^+}^{n-2}p(s), g_{0^+}^{n-3}p(s), \ldots, g_{0^+}^{1}p(s), p(s)\bigr).
\end{aligned}$$

From the above discussions, we have

$$\begin{aligned}
&Q_{0^+}^{\beta}\iota_p\bigl(Q_{0^+}^{\alpha-n+2}n(s)\bigr) - Q_{0^+}^{\beta}\iota_p\bigl(Q_{0^+}^{\alpha-n+2}d(s)\bigr)\\
&\quad = f\bigl(s, g_{0^+}^{n-2}p(s), g_{0^+}^{n-3}p(s), \ldots, g_{0^+}^{1}p(s), p(s)\bigr)\\
&\qquad - \hat{f}\bigl(s, g_{0^+}^{n-2}d(s), g_{0^+}^{n-3}d(s), \ldots, g_{0^+}^{1}d(s), d(s)\bigr)\\
&\quad \ge 0,
\end{aligned}$$

where $t \in [0,1]$.

If we let $z(s) = \iota_p(Q_{0^+}^{\alpha-n+2}n(s)) - \iota_p(Q_{0^+}^{\alpha-n+2}d(s))$, then $z(0) = z(1) = 0$. By Lemma 2.1, we have $z(s) \le 0$.

Hence,

$$\iota_p\bigl(Q_{0^+}^{\alpha-n+2}n(s)\bigr) \le \iota_p\bigl(Q_{0^+}^{\alpha-n+2}d(s)\bigr),$$

where $s \in [0,1]$.

Since ι_p is monotone increasing,

$$Q_{0^+}^{\alpha-n+2}n(s) \le Q_{0^+}^{\alpha-n+2}d(s),$$

that is,

$$Q_{0^+}^{\alpha-n+2}(n-d)(s) \le 0.$$

By the assumption that $\operatorname{supp}\hat{f}$, $\operatorname{supp}\hat{g_\varepsilon} \subseteq \sigma_k(j) \cup -\sigma_k(j)$, we obtain

$$\int_{Q_{\sigma_k}} \hat{f}(x-t)\hat{g_\varepsilon}(s)\,ds = 0,$$

where $x \in -Q_{\sigma_k}$, and

$$\int_{-Q_{\sigma_k}} \hat{f}(x-t)\hat{g_\varepsilon}(s)\,ds = 0,$$

where $x \in Q_{\sigma_k}$.

So

$$\operatorname{supp}\hat{g_\varepsilon}\chi Q_{\sigma_k} \subseteq \left\{\xi \in Q_{\sigma_k} : \sum_{j=1}^{n} \frac{\xi_j}{b_j} \geq 1\right\} \tag{2.5}$$

as the other case can be obtained in a similar way.

Let $\lambda = \hat{f}\chi_{-Q_{\sigma_k}}$ and $\varrho = \hat{g_\varepsilon}\chi Q_{\sigma_k}$. We decompose ϱ into

$$\varrho = \varrho_1 + \varrho_2$$

with $\operatorname{supp}\varrho_1 \subseteq \prod_{j=1}^{n}[0,b_j]$ and $\operatorname{supp}\varrho_2 \subseteq \overline{Q_{\sigma_k} \setminus \prod_{j=1}^{n}(0,b_j)}$.

By (2.5) we obtain

$$(\varrho_1 * \lambda)(x) = -(\varrho_2 * \lambda)(x),$$

where $x \in -Q_{\sigma_k}$.

Meanwhile

$$\operatorname{supp}(\varrho_2 * \lambda) \subseteq \operatorname{supp}\varrho_2 + \operatorname{supp}\lambda \subseteq \overline{Q_{\sigma_k} \setminus \prod_{j=1}^{n}(0,b_j)} + \prod_{j=1}^{n}[-b_j,0] \subseteq \mathbb{R}^n \setminus (-Q_{\sigma_k}) \tag{2.6}$$

and

$$\operatorname{supp}(\varrho_1 * \lambda) \subseteq \operatorname{supp}\varrho_1 + \operatorname{supp}\lambda \subseteq \prod_{j=1}^{n}[0,b_j] + \prod_{j=1}^{n}[-b_j,0] \subseteq \prod_{j=1}^{n}[-b_j,b_j]. \tag{2.7}$$

By (2.6) it is clear that

$$\operatorname{supp}(\varrho_1 * \lambda) \subseteq \mathbb{R}^n \setminus (-)Q_{\sigma_k}.$$

This together with (2.7) implies that

$$\operatorname{conv}\operatorname{supp}(\varrho_1 * \lambda) \subseteq \left\{\xi \in \mathbb{R}^n : -b_j \leq \xi_j \leq b_j, \sum_{j=1}^{n} \frac{\xi_j}{a_j} \geq 1-n\right\}. \tag{2.8}$$

We claim that, for any $\xi \in \operatorname{supp} \varrho_1$,

$$\sum_{j=1}^{n} \frac{\xi_j}{a_j} \geq 1$$

holds.

If it is invalid, then there is $\xi^1 \in \operatorname{conv} \operatorname{supp} \varrho_1$ satisfying

$$\sum_{j=1}^{n} \frac{\xi_j}{b_j} < 1.$$

Note that $\xi^2 = b \in \operatorname{supp} \lambda$ satisfies

$$\sum_{j=1}^{n} \frac{\xi_j}{b_j} = -n.$$

Since

$$\operatorname{conv} \operatorname{supp}(\varrho_1 * \lambda) = \operatorname{conv} \operatorname{supp} \varrho_1 + \operatorname{conv} \operatorname{supp} \lambda,$$

there exists some point $\xi \in \operatorname{conv} \operatorname{supp}(\varrho_1 * \lambda)$ such that

$$\sum_{j=1}^{n} \frac{\xi_j}{a_j} < 1 - n.$$

This contradicts (2.8). We conclude that

$$\operatorname{supp} \varrho_1 = \operatorname{supp} \hat{g}_\varepsilon \chi Q_{\sigma_k} \subseteq \left\{ \xi \in Q_{\sigma_k} : \sum_{j=1}^{n} \frac{\xi_j}{b_j} \geq 1 \right\}.$$

This completes the proof. □

3 Conclusions

This paper was mainly devoted to developing the Schrödinger-type identity for a Schrödinger free boundary problem in $\mathbb{R}^n$. As an application, we established necessary and sufficient conditions for the product of some distributional functions to satisfy the Schrödinger-type identity. As a consequence, our results significantly improved and generalized previous work.

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