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# Robust exponential attractors for a parabolic–hyperbolic phase-field system

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**Abstract**

In this paper, we construct a robust family of exponential attractors for a parabolic–hyperbolic phase-field system (PHPFS), which describes phase separation in material sciences, e.g., melting and solidification. A consequence of this is the existence of finite fractal dimensional global attractors which are both upper and lower semicontinuous at the parameter $\epsilon = 0$. Hence we establish the convergence of the dynamics of PHPFS to those of the well known Cagilnap phase-field system.

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**Keywords:** Parabolic–hyperbolic phase-field system; Exponential attractors; Continuity

## 1 Introduction

Exponential attractors are compact and positively invariant sets with finite fractal dimension which attract all the trajectories starting from bounded sets at a uniform exponential rate (see [5–7, 14]). The existence of exponential attractors guarantees the existence of a finite fractal dimensional global attractor. Readers may see [4, 8, 13] and references therein for more on the dimension of a global attractor. Thus a finite-dimensional reduction principle can be applied to reduce the infinite-dimensional dynamical system under consideration to a finite-dimensional system of ODEs. The sensitivity of exponential attractors under small perturbations is the main focus in this work. One may see [15] for some recent developments in the construction of exponential attractors.

The phase-field system is a system of equations which couples the temperature $u$ and order parameter $\phi$ also known as "phase-field". It describes phase separations in materials occupying a domain $\Omega \subset \mathbb{R}^d$.

We consider the following parabolic–hyperbolic phase-field system (PHPFS):

$$\begin{cases} \epsilon\phi_{tt} + \phi_t - \Delta\phi + \phi + g(\phi) - u = 0, \\ u_t + \phi_t - \Delta u = 0, \\ \partial_n\phi|_{\partial\Omega} = u|_{\partial\Omega} = 0, \\ \phi(0) = \phi_0, \phi_t(0) = \phi_1, u(0) = u_0, \end{cases} \tag{1.1}$$

in a bounded domain $\Omega \subset \mathbb{R}^d$, $d = 1, 2, 3$ with smooth boundary $\partial\Omega$, where $\epsilon \in (0, 1]$ is a small parameter. Denote the function $G(s) = \int_0^s g(\varsigma)\,d\varsigma$ and assume that $g$ satisfies $g \in$

$\mathcal{C}^2(\mathbb{R})$ and the following conditions hold (cf., e.g., [1, 3]):

$$G(s) \geq -C_1, \qquad C_1 \geq 0, \quad \forall s \in \mathbb{R}, \tag{1.2}$$

$\forall \gamma \in \mathbb{R}, \exists C_2(\gamma) > 0, C_3(\gamma) \geq 0$ such that

$$(s-\gamma)g(s) - C_2 G(s) \geq -C_3, \quad \forall s \in \mathbb{R} \tag{1.3}$$

(where $C_2, C_3$ are bounded when $\gamma$ is bounded),

$$g'(s) \geq -C_4, \qquad C_4 \geq 0, \quad \forall s \in \mathbb{R}, \tag{1.4}$$

$$\left|g''(s)\right| \leq C_5\left(|s|^p + 1\right), \qquad C_5 > 0, \quad \forall s \in \mathbb{R}, \tag{1.5}$$

with $p \geq 0$ when $d = 1, 2$ and $p \in [0, 1]$ when $d = 3$. We note that in space dimension one, no growth assumption on $g$ is needed.

We remark that our results also hold when $\phi$ is subject to a boundary condition of periodic type

$$\begin{cases} u|_{x_i=0} = u|_{x_i=L_i}, \qquad u_{x_i}|_{x_i=0} = u_{x_i}|_{x_i=L_i}, \quad i = 1, \dots, d, \\ \phi|_{x_i=0} = \phi|_{x_i=L_i}, \quad i = 1, \dots, d, \\ \quad \text{for } \phi \text{ and the derivatives of } \phi \text{ of order} \leq 3, \end{cases} \tag{1.6}$$

if $\Omega = \prod_{i=1}^d (0, L_i)$.

We shall construct a robust family of exponential attractors which are both upper and lower semicontinuous at $\epsilon = 0$ with respect to a norm independent of $\epsilon$.

Grasselli and Pata [10] showed a well-posedness result and the existence of the global attractor for the system ($\epsilon > 0$)

$$\begin{cases} \epsilon\phi_{tt} + \phi_t - \Delta\phi + \phi^3 = \gamma(\phi) + \lambda'(\phi)u, \\ u_t + \lambda'(\phi)\phi_t - \Delta u = f. \end{cases}$$

Grasselli and Pata [11] considered the system ($\epsilon > 0$)

$$\begin{cases} \epsilon\phi_{tt} + \phi_t - \Delta\phi + \phi - \lambda'(\phi)u + h(\phi) = \xi, \\ u_t + \lambda'(\phi)\phi_t - \Delta u = 0 \end{cases} \tag{1.7}$$

in 3D, subject to mixed boundary conditions, Neumann on $\phi$ and Dirichlet on $u$. They proved a well-posedness result, the existence of the global attractor and its upper semicontinuity at $\epsilon = 0$, and constructed exponential attractors with respect to a norm depending on $\epsilon$. Also, Grasselli et al. [9] gave a well-posedness result and constructed a robust family of exponential attractors $\mathbb{E}_\epsilon$ for the system

$$\begin{cases} \epsilon\phi_{tt} + \phi_t - \Delta\phi - \lambda'(\phi)u + \chi(\phi) = \xi, \\ u_t + \lambda'(\phi)\phi_t - \Delta u = 0 \end{cases} \tag{1.8}$$

in 3D, subject to Dirichlet boundary conditions on both $\phi$ and $u$, where $\chi(\phi)$ is singular at $\phi = \pm 1$, e.g., $\ln(\frac{1+\phi}{1-\phi})$, $\phi \in (0, 1)$. More precisely, they showed that there exist $c > 0$ and

$\varpi \in (0,1)$, both independent of $\epsilon$, such that

$$\operatorname{dist}^{\mathrm{sym}}_{K,\epsilon}(\mathbb{E}_\epsilon, \mathbb{E}_0) \le c\epsilon^{\varpi}, \quad \forall \epsilon \in [0,1],$$

in the norm $\|(\phi,\phi_t,u)\|^2_{K,\epsilon} = \|\Delta\phi\|^2_{L^2(\Omega)} + \epsilon\|\nabla\phi_t\|^2_{L^2(\Omega)} + \|\phi_t\|^2_{L^2(\Omega)} + \|\Delta u\|^2_{L^2(\Omega)}$, which clearly depends on $\epsilon$.

Finally, we would also like to mention the papers [12, 16, 17] where the convergence to equilibrium of solutions for a parabolic–hyperbolic phase-field model were proven.

This work is organized as follows. In Sect. 1, we give a brief introduction. In Sect. 2, we give some a priori estimates. In Sect. 3, we construct exponential attractors for the system (1.1). Finally, in Sect. 4, we construct a robust family of exponential attractors which are both upper and lower semicontinuous at $\epsilon = 0$ for the system (1.1).

We define the Hilbert space $\mathcal{H}_{r,\epsilon} = H^r \times H^{r-1} \times H^{r-1}_0$, $r \ge 1$, endowed with the norm

$$\|(\varphi,\psi,\nu)\|_{\mathcal{H}_{r,\epsilon}} = \left(\|\varphi\|^2_r + \epsilon\|\psi\|^2_{r-1} + \|\nu\|^2_{r-1}\right)^{1/2},$$

where we understand that $H^0_0 = H^0(\Omega) = L^2(\Omega)$. Hence, we denote $\mathcal{H}_{1,0} = H^1(\Omega) \times L^2(\Omega)$, endowed with the norm $\|(\cdot,\cdot)\|_{\mathcal{H}_{1,0}} = (\|\cdot\|^2_1 + \|\cdot\|^2)^{1/2}$.

## 2 A priori estimates

We multiply $(1.1)_1$ by $\phi_t$ and $(1.1)_2$ by $u$, then integrate over $\Omega$. Adding the resulting equations, we obtain

$$\frac{d}{dt}E_1(t) + 2\|\phi_t\|^2 + 2\|\nabla u\|^2 = 0 \tag{2.1}$$

where

$$E_1(t) = \|\nabla\phi\|^2 + \|\phi\|^2 + \epsilon\|\phi_t\|^2 + \|u\|^2 + 2\int_\Omega G(\phi)\,dx.$$

From (1.2), (1.3) and (1.5), we deduce that

$$\int_\Omega G(\phi)\,dx \ge -C_1|\Omega| \quad \text{and} \quad \int_\Omega G(\phi)\,dx \le c\left(\|\phi\|^{p+3}_1 + 1\right).$$

Hence,

$$\|(\phi,\phi_t,u)\|^2_{\mathcal{H}_{1,\epsilon}} - \alpha_1 \le E_1(t) \le \alpha_2\left(\|\phi\|^{p+3}_1 + \epsilon\|\phi_t\|^2 + \|u\|^2 + 1\right), \tag{2.2}$$

for some $\alpha_1, \alpha_2 > 0$ independent of $\epsilon$. Thus integrating (2.1) over $(0,t)$ and accounting for (2.2), we obtain that

$$\int_0^t \left(\|\phi_t(s)\|^2 + \|\nabla u(s)\|^2\right) ds \le E_1(0) + \alpha_1, \quad \forall t \ge 0.$$

Hence by (2.2) again, we get

$$\int_0^\infty \left(\|\phi_t(s)\|^2 + \|\nabla u(s)\|^2\right) ds \le c\left(\|\phi_0\|^{p+3}_1 + \epsilon\|\phi_1\|^2 + \|u_0\|^2 + 1\right). \tag{2.3}$$

Let $(\phi^1, u^1)$ and $(\phi^2, u^2)$ be two solutions of (1.1). Set $\phi = \phi^1 - \phi^2$, $\phi_t = \phi_t^1 - \phi_t^2$ and $u = u^1 - u^2$, then $\phi(0) = 0$, $\phi_t(0) = 0$ and $u(0) = 0$. The pair $(\phi, \phi_t, u)$ is a solution to the problem

$$\begin{cases} \epsilon\phi_{tt} + \phi_t - \Delta\phi + \phi + g(\phi^1) - g(\phi^2) - u = 0, \\ u_t + \phi_t - \Delta u = 0, \\ \phi(0) = \phi_t(0) = u(0) = 0. \end{cases} \tag{2.4}$$

We multiply $(2.4)_1$ and $(2.4)_2$ by $\phi_t$ and $u$, respectively, integrate over $\Omega$, then add the resulting equations to get

$$\frac{1}{2}\frac{d}{dt}\left(\|\nabla\phi\|^2 + \|\phi\|^2 + \epsilon\|\phi_t\|^2 + \|u\|^2\right) + \|\phi_t\|^2 + \|\nabla u\|^2 = -\left(g(\phi^1) - g(\phi^2), \phi_t\right).$$

By Hölder's inequality and (1.5), we have

$$\begin{aligned} \left|\left(g(\phi^1) - g(\phi^2), \phi_t\right)\right| &\le c\left(\left\|\phi^1\right\|_{L^{3p+3}(\Omega)}^{p+1} + \left\|\phi^2\right\|_{L^{3p+3}(\Omega)}^{p+1} + 1\right)\|\phi\|_{L^6(\Omega)}\|\phi_t\| \\ &\le c\left(\left\|\phi^1\right\|_1^{p+1} + \left\|\phi^2\right\|_1^{p+1} + 1\right)\|\phi\|_1\|\phi_t\|. \end{aligned}$$

Therefore, by Young's inequality, we obtain

$$\frac{d}{dt}\left(\|\nabla\phi\|^2 + \|\phi\|^2 + \epsilon\|\phi_t\|^2 + \|u\|^2\right) \le \widetilde{M}(t)\|\phi\|_1^2, \tag{2.5}$$

where

$$\widetilde{M}(t) = \begin{cases} c\sup_{\theta\in[0,1]}\|g'(\theta\phi_1 + (1-\theta)\phi_2)\|_{L^\infty(\Omega)}^2, & \text{if } d = 1, \\ c(\|\phi^1\|_1^{2p+2} + \|\phi^2\|_1^{2p+2} + 1), & \text{if } d = 2, 3. \end{cases}$$

Noting that $t \mapsto \widetilde{M}(t)$ is $L^1(0, T)$, and integrating (2.5) over $(0, t)$, we deduce that

$$\left\|\left(\phi(t), \phi_t(t), u(t)\right)\right\|_{\mathcal{H}_{1,\epsilon}}^2 \le e^{\int_0^t \widetilde{M}(s)\,ds}\left\|\left(\phi(0), \phi_t(0), u(0)\right)\right\|_{\mathcal{H}_{1,\epsilon}}^2, \quad \forall t \ge 0. \tag{2.6}$$

We state a well-posedness result, which is proved in [11, Theorem 3.4].

**Theorem 2.1** *We assume that* (1.2)–(1.5) *hold. If* $(\phi_0, \phi_1, u_0) \in \mathcal{H}_{1,\epsilon}$, *then* (1.1) *possesses a unique solution* $(\phi, u)$ *such that*

$$(\phi, \phi_t, u) \in \mathcal{C}\left([0, T]; \mathcal{H}_{1,\epsilon}\right)$$

*for any* $T > 0$. *Moreover, if* $(\phi_0, \phi_1, u_0) \in \mathcal{H}_{2,\epsilon}$, *then* $(\phi, \phi_t, u) \in \mathcal{C}([0, T]; \mathcal{H}_{2,\epsilon})$.

On account of Theorem 2.1 we can define the semigroup

$$S_\epsilon(t) : \mathcal{H}_{1,\epsilon} \to \mathcal{H}_{1,\epsilon}, \qquad (\phi_0, \phi_1, u_0) \mapsto \left(\phi(t), \phi_t(t), u(t)\right), \quad \forall t \ge 0,$$

where $(\phi(t), \phi_t(t), u(t))$ is the solution to problem (1.1) at time $t$. The semigroup $S_\epsilon(t)$ is strongly continuous (cf. (2.6)).

It is also known from [11] that the semigroup $S_\epsilon(t) : \mathcal{H}_{j,\epsilon} \to \mathcal{H}_{j,\epsilon}$ has bounded absorbing sets $\mathcal{B}_j$ in $\mathcal{H}_{j,\epsilon}$ of the form

$$\mathcal{B}_j = \left\{(\varphi, \psi, \nu) \in \mathcal{H}_{j,\epsilon}, \left\|(\varphi, \psi, \nu)\right\|_{\mathcal{H}_{j,\epsilon}} \le r_j\right\}, \quad j = 1, 2,$$

where $r_j > 0$ is independent of $\epsilon$. In fact, they are exponentially attracting sets.

## 3 Exponential attractors

Now we state sufficient conditions which guarantee the existence of robust exponential attractors, which are continuous with respect to $\epsilon$ (cf. [2, Theorem 5.1]; also [1, 7, 15]).

**Theorem 3.1** ([2]) *Let $E^1$, $E^2$, $V^1$, $V^2$, $W^1$, $W^2$ be Banach spaces such that $W^i \Subset V^i \Subset E^i$, $i = 1, 2$. Set $E_\epsilon = E^1 \times E^2$, $V_\epsilon = V^1 \times V^2$, $W_\epsilon = W^1 \times W^2$ and endow them with the following norms*:

$$\left\|(p,q)\right\|_{E_\epsilon} = \left(\|p\|_{E^1}^2 + \epsilon\|q\|_{E^2}^2\right)^{1/2},$$
$$\left\|(p,q)\right\|_{V_\epsilon} = \left(\|p\|_{V^1}^2 + \epsilon\|q\|_{V^2}^2\right)^{1/2},$$
$$\left\|(p,q)\right\|_{W_\epsilon} = \left(\|p\|_{W^1}^2 + \epsilon\|q\|_{W^2}^2\right)^{1/2},$$

*respectively, where $\epsilon \in [0,1]$, with the convention that $E_0 = E^1$, $V_0 = V^1$, and $W_0 = W^1$. Let $B_\epsilon(r)$ denote a closed ball in $W_\epsilon$ of radius $r > 0$ and centered at zero. Consider a one-parameter family of strongly continuous semigroups $\{S_\epsilon(t)\}_\epsilon$ acting on the phase-space $E_\epsilon$, for each $\epsilon \in [0,1]$. Then assume that there exist $\alpha, \beta, \gamma, \vartheta \in (0,1]$, $\kappa \in (0, \frac{1}{2})$, $\Upsilon_j \ge 0$, and $\varrho > 0$ (all independent of $\epsilon$) such that, setting $B_\epsilon = B_\epsilon(\varrho)$, the following conditions hold*:

1. *There exists a map $\mathcal{L} : B_0 \to V^2$ which is Hölder continuous of exponent $\alpha$. Here $B_0$ is endowed with the metric topology of $E^1$.*
2. *There exists $t^\star > 0$, independent of $\epsilon$, such that*

$$S_\epsilon(t)B_\epsilon \subset B_\epsilon, \quad \forall t \ge t^\star,$$

*and $B_\epsilon$ is uniformly bounded (with respect to $\epsilon$) in the $E_1$-norm. Moreover, setting $S_\epsilon(t^\star) = S_\epsilon$, the map $S_\epsilon$ satisfies, for every $z_1, z_2 \in B_\epsilon$,*

$$S_\epsilon z_1 - S_\epsilon z_2 = L_\epsilon(z_1, z_2) + K_\epsilon(z_1, z_2),$$

*where*

$$\|L_\epsilon z_1 - L_\epsilon z_2\|_{E_\epsilon} \le \kappa\|z_1 - z_2\|_{E_\epsilon},$$
$$\|K_\epsilon z_1 - K_\epsilon z_2\|_{V_\epsilon} \le \Upsilon_1\|z_1 - z_2\|_{E_\epsilon}.$$

3. *For any $z \in B_\epsilon$, there hold*

$$\left\|S_\epsilon^m z - \mathcal{L}S_0^m \Pi_\epsilon z\right\|_{E_1} \le \Upsilon_2^m \epsilon^\beta, \quad \forall m \in \mathbb{N},$$
$$\left\|S_\epsilon(t)z - \mathcal{L}S_0(t)\Pi_\epsilon z\right\|_{E_1} \le \Upsilon_3 \epsilon^\gamma, \quad \forall t \in \left[t^\star, 2t^\star\right].$$

*Here the "lifting" map* $\mathcal{L} : B_0 \to E_\epsilon$ *is defined by*

$$\mathcal{L}x = \begin{cases} (x, \mathcal{L}x), & \text{if } \epsilon > 0, \\ x, & \text{if } \epsilon = 0, \end{cases}$$

*and* $\Pi_\epsilon : B_\epsilon \to B_0$ *is the projection onto the first component when* $\epsilon > 0$, *and the identity map otherwise.*

4. *The map* $z \mapsto S_\epsilon(t)z$ *is Lipschitz continuous on* $B_\epsilon$ *endowed with the metric topology of* $E_\epsilon$, *with a Lipschitz constant independent of* $\epsilon$ *and* $t \in [t^\star, 2t^\star]$.
5. *The map*

$$(t, z) \mapsto S_\epsilon(t)z : \left[t^\star, 2t^\star\right] \times B_\epsilon \to B_\epsilon$$

*is Hölder continuous of exponent* $\vartheta$, *where* $B_\epsilon$ *is endowed with the metric topology of* $E_\epsilon$.

*Then there exists a family of exponential attractors* $\mathcal{E}_\epsilon$ *on* $\mathcal{B}_\epsilon = \overline{B_\epsilon}^{E_\epsilon}$ *with the following properties:*

(i) $\mathcal{E}_\epsilon$ *attracts* $\mathcal{B}_\epsilon$ *with an exponential rate which is uniform with respect to* $\epsilon$, *that is,*

$$\operatorname{dist}_{E_\epsilon}\left(S_\epsilon(t)\mathcal{B}_\epsilon, \mathcal{E}_\epsilon\right) \le M_1 e^{-\omega t}, \quad \forall t \ge 0, \tag{3.1}$$

*for some* $M_1 > 0$ *and some* $\omega > 0$.

(ii) *The fractal dimension of* $\mathcal{E}_\epsilon$ *(denoted as* $\dim_F(\mathcal{E}_\epsilon)$*) is uniformly bounded with respect to* $\epsilon$, *that is,*

$$\dim_F(\mathcal{E}_\epsilon) \le M_2. \tag{3.2}$$

(iii) *The family* $\mathcal{E}_\epsilon$ *is Hölder continuous with respect to* $\epsilon$, *that is, there exist a positive constant* $M_3$ *and* $\tau \in (0, \frac{1}{2}]$ *such that*

$$\operatorname{dist}^{\mathrm{sym}}_{E_\epsilon}(\mathcal{E}_\epsilon, \mathcal{L}\mathcal{E}_0) \le M_3 \epsilon^\tau, \tag{3.3}$$

*for all* $0 < \epsilon \le 1$. *In addition, there exist a positive constant* $M_4$ *and* $\sigma \in (0, \frac{1}{2}]$ *such that*

$$\operatorname{dist}_{E_1}(\mathcal{E}_\epsilon, \mathcal{L}\mathcal{E}_0) \le M_4 \epsilon^\sigma, \tag{3.4}$$

*for all* $0 < \epsilon \le 1$, *and*

$$\lim_{\epsilon \to 0} \operatorname{dist}_{E_1}(\mathcal{L}\mathcal{E}_0, \mathcal{E}_\epsilon) = 0. \tag{3.5}$$

*Here* $\omega$, $\tau$, $\sigma$ *and* $M_j$ *are independent of* $\epsilon$, *and they can be computed explicitly.*

We observe that the solution to the unperturbed problem (i.e., when $\epsilon = 0$ in (1.1)) for the pair $(\phi, u)$ at any time $t$ is given by $(\phi(t), u(t)) = S(t)(\phi_0, u_0)$ and $\phi_t = \mathcal{L}(\phi(t), u(t))$, where

$$\mathcal{L}(\varphi, \vartheta) = -\left(-\Delta\varphi + \varphi - g(\varphi) - \vartheta\right). \tag{3.6}$$

Let $z_1, z_2 \in \mathcal{B}_2$, $z_1 = (\phi_0^1, \phi_1^1, u_0^1)$ and $z_2 = (\phi_0^2, \phi_1^2, u_0^2)$ be initial data for two solutions $(\phi^1, u^1)$ and $(\phi^2, u^2)$ of (1.1), respectively.

We set $(\phi(t), \phi_t(t), u(t)) = S_\epsilon(t)z_1 - S_\epsilon(t)z_2$, $\tilde{\phi}_0 = \phi_0^1 - \phi_0^2$, $\tilde{\phi}_1 = \phi_1^1 - \phi_1^2$ and $\tilde{u}_0 = u_0^1 - u_0^2$. Furthermore, we perform the splitting

$$\big(\phi(t), \phi_t(t), u(t)\big) = \big(\chi(t), \chi_t(t), \vartheta(t)\big) + \big(\Psi(t), \Psi_t(t), \upsilon(t)\big),$$

where $K_\epsilon(z_1, z_2) = (\chi(t), \chi_t(t), \vartheta(t))$ and $L_\epsilon(z_1, z_2) = (\Psi(t), \Psi_t(t), \upsilon(t))$ respectively solve the problems:

$$\begin{cases} \epsilon\chi_{tt} + \chi_t - \Delta\chi_t + \chi + g(\phi_1) - g(\phi_2) - \vartheta = 0, \\ \vartheta_t + \chi_t - \Delta\vartheta = 0, \\ \chi|_{t=0} = 0, \qquad \chi_t|_{t=0} = 0, \qquad \vartheta|_{t=0} = 0 \end{cases} \tag{3.7}$$

and

$$\begin{cases} \epsilon\Psi_{tt} + \Psi_t - \Delta\Psi + \Psi - \upsilon = 0, \\ \upsilon_t + \Psi_t - \Delta\upsilon = 0, \\ \Psi|_{t=0} = \tilde{\phi}_0, \qquad \Psi_t|_{t=0} = \tilde{\phi}_1, \qquad \upsilon|_{t=0} = \tilde{u}_0. \end{cases} \tag{3.8}$$

**Proposition 3.1** *There exist $c, c', c_1 > 0$ independent of $\epsilon$ such that*

$$\big\|L_\epsilon(z_1, z_2)\big\|_{\mathcal{H}_{1,\epsilon}} \le ce^{-c_1 t}\|z_1 - z_2\|_{\mathcal{H}_{1,\epsilon}}, \quad \forall t \ge 0, \quad \textit{and} \tag{3.9}$$

$$\big\|K_\epsilon(z_1, z_2)\big\|_{\mathcal{H}_{2,\epsilon}} \le ce^{c' t}\|z_1 - z_2\|^2_{\mathcal{H}_{1,\epsilon}}, \quad \forall t \ge 0. \tag{3.10}$$

*Proof* Firstly, we multiply $(3.8)_1$ by $\Psi_t$ and $(3.8)_2$ by $\upsilon$, integrate over $\Omega$, then add the resulting equations to get

$$\frac{1}{2}\frac{d}{dt}\big(\|\nabla\Psi\|^2 + \|\Psi\|^2 + \epsilon\|\Psi_t\|^2 + \|\upsilon\|^2\big) + \|\Psi_t\|^2 + \|\nabla\upsilon\|^2 = 0. \tag{3.11}$$

Next, we multiply $(3.8)_1$ by $\Psi$ to obtain

$$\frac{1}{2}\frac{d}{dt}\big[\|\Psi\|^2 + 2\epsilon(\Psi, \Psi_t)\big] - \epsilon\|\Psi_t\|^2 + \|\nabla\Psi\|^2 + \|\Psi\|^2 - (\upsilon, \Psi) = 0,$$

and then deduce that

$$\frac{1}{2}\frac{d}{dt}\big[\|\Psi\|^2 + 2\epsilon(\Psi, \Psi_t)\big] + \|\nabla\Psi\|^2 + \frac{1}{2}\|\Psi\|^2 + 2\epsilon(\Psi_t, \Psi) \le 5\epsilon\|\Psi_t\|^2 + c\|\nabla\upsilon\|^2. \tag{3.12}$$

Summing (3.11) and $\kappa$ (3.12), for some $\kappa \in (0,1)$ small enough, we get

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\big(\|\nabla\Psi\|^2 + (1+\kappa)\|\Psi\|^2 + \epsilon\|\Psi_t\|^2 + \|\upsilon\|^2 + 2\kappa\epsilon(\Psi, \Psi_t)\big) + \kappa\|\nabla\Psi\|^2 + \frac{\kappa}{2}\|\Psi\|^2 \\ &\quad + \epsilon(1 - 5\kappa)\|\Psi_t\|^2 + (1 - c\kappa)\|\nabla\upsilon\|^2 + 2\kappa\epsilon(\Psi, \Psi_t) \le 0. \end{aligned}$$

Hence, there exists a $c_1 > 0$ (independent of $\epsilon$) such that

$$\frac{d}{dt}E_2(t) + c_1 E_2(t) \le 0,$$

where $E_2(t) = \|\nabla\Psi\|^2 + (1+\kappa)\|\Psi\|^2 + \epsilon\|\Psi_t\|^2 + \|\upsilon\|^2 + 2\kappa\epsilon(\Psi, \Psi_t)$. Simple integration over $(0,t)$ gives

$$E_2(t) \le e^{-c_1 t}E_2(0), \quad \forall t \ge 0. \tag{3.13}$$

Clearly, by Young's inequality, there exist $b_3, b_4 > 0$ (independent of $\epsilon$) such that

$$b_3\big\|(\Psi, \Psi_t, \upsilon)\big\|^2_{\mathcal{H}_{1,\epsilon}} \le E_2(t) \le b_4\big\|(\Psi, \Psi_t, \upsilon)\big\|^2_{\mathcal{H}_{1,\epsilon}}. \tag{3.14}$$

It follows from (3.13) and (3.14) that

$$\big\|(\Psi, \Psi_t, \upsilon)\big\|^2_{\mathcal{H}_{1,\epsilon}} \le e^{-c_1 t}\big\|(\tilde{\phi}_0, \tilde{\phi}_1, \tilde{u}_0)\big\|^2_{\mathcal{H}_{1,\epsilon}}, \quad \forall t \ge 0.$$

Hence (3.9) follows.

Secondly, we multiply $(3.7)_1$ by $\chi_t$ and $(3.7)_2$ by $\vartheta$, integrate over $\Omega$, then add the resulting equations to get

$$\frac{1}{2}\frac{d}{dt}\big(\|\nabla\chi\|^2 + \|\chi\|^2 + \epsilon\|\chi_t\|^2 + \|\vartheta\|^2\big) + \|\chi_t\|^2 + \|\nabla\vartheta\|^2 = -\big(g\big(\phi^1\big) - g\big(\phi^2\big), \chi_t\big).$$

We have that $\|g(\phi^1) - g(\phi^2)\| \le \|g'(\theta\phi^1 + (1-\theta)\phi^2)\|_{L^\infty(\Omega)}\|\phi\|$, where $\theta \in [0,1]$. It follows that

$$\frac{1}{2}\frac{d}{dt}\big(\|\nabla\chi\|^2 + \|\chi\|^2 + \epsilon\|\chi_t\|^2 + \|\vartheta\|^2\big) + \frac{1}{2}\|\chi_t\|^2 + \|\nabla\vartheta\|^2 \le \|\phi\|^2. \tag{3.15}$$

Integrating (3.15) over $(0,t)$ and then accounting for (2.6), we deduce that

$$\|\chi\|_1^2 + \epsilon\|\chi_t\|^2 + \|\vartheta\|^2 \le ce^{c't}\big\|(\tilde{\phi}_0, \tilde{\phi}_1, \tilde{u}_0)\big\|^2_{\mathcal{H}_{1,\epsilon}}, \quad \forall t \ge 0. \tag{3.16}$$

Next, we multiply $(3.7)_1$ by $-\Delta\chi_t$ and $(3.7)_2$ by $N\vartheta$, integrate over $\Omega$, then add the resulting equations to get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\big(\|\Delta\chi\|^2 + \|\nabla\chi\|^2 + \epsilon\|\nabla\chi_t\|^2 + \|\nabla\vartheta\|^2\big) + \|\nabla\chi_t\|^2 + \|\Delta\vartheta\|^2 \\ &\quad = -\big(\nabla\big(g\big(\phi^1\big) - g\big(\phi^2\big)\big), \nabla\chi_t\big).\end{aligned}$$

We have that $\|\nabla(g(\phi^1) - g(\phi^2))\| \le c\|\phi\|_1$. It follows that

$$\frac{1}{2}\frac{d}{dt}\big(\|\Delta\chi\|^2 + \|\nabla\chi\|^2 + \epsilon\|\nabla\chi_t\|^2 + \|\nabla\vartheta\|^2\big) + \frac{1}{2}\|\nabla\chi_t\|^2 + \|\Delta\vartheta\|^2 \le c\|\phi\|_1^2. \tag{3.17}$$

Integrating (3.17) over $(0,t)$ and taking into account (2.6), we deduce that

$$\|\Delta\chi\|^2 + \|\nabla\chi\|^2 + \epsilon\|\nabla\chi_t\|^2 + \|\nabla\vartheta\|^2 \le ce^{c't}\big\|(\tilde{\phi}_0, \tilde{\phi}_1, \tilde{u}_0)\big\|^2_{\mathcal{H}_{1,\epsilon}}, \quad \forall t \ge 0. \tag{3.18}$$

On account of (3.16) and (3.18), we obtain that

$$\left\|(\chi,\chi_t,\vartheta)\right\|^2_{\mathcal{H}_{2,\epsilon}} \le ce^{c't}\left\|(\tilde{\phi}_0,\tilde{\phi}_1,\tilde{u}_0)\right\|^2_{\mathcal{H}_{1,\epsilon}}, \quad \forall t \ge 0.$$

Hence (3.10) follows. □

We prove the following result.

**Theorem 3.2** *For every $\epsilon \in (0,1]$, the semigroup $S_\epsilon(t)$ possesses an exponential attractor $\mathcal{E}_\epsilon$ (with dimension independent of $\epsilon$) in $\mathcal{H}_{1,\epsilon}$.*

*Proof* Let $t \in [t^*, 2t^*]$ and set $(\phi(t),\phi_t(t),u(u)) = S_\epsilon(t)z_{01} - S_\epsilon(t)z_{02} = (\phi^1(t),\phi^1_t(t),u^1(t)) - (\phi^2(t),\phi^2_t(t),u^2(t))$. Therefore, the triplet $(\phi(t),\phi_t(t),u(u))$ is a solution to the problem

$$\begin{cases} \epsilon\phi_{tt} + \phi_t - \Delta\phi + \phi + g(\phi_1) - g(\phi_2) - u = 0, \\ u_t + \phi_t - \Delta u = 0, \\ \phi|_{t=0} = \phi^{01} - \phi^{02}, \qquad \phi_t|_{t=0} = \phi^{01}_1 - \phi^{02}_1, \qquad u|_{t=0} = u^{01} - u^{02}. \end{cases} \tag{3.19}$$

On account of (2.6) we obtain

$$\left\|S_\epsilon(t)z_{01} - S_\epsilon(t)z_{02}\right\|_{\mathcal{H}_{1,\epsilon}} \le c\big(t^*\big)\|z_{01} - z_{02}\|_{\mathcal{H}_{1,\epsilon}}, \quad t \le 2t^*, \tag{3.20}$$

where $c(t^*) > 0$ is independent of $\epsilon$. Now, we multiply $(1.1)_1$ and $(1.1)_2$ by $-\Delta\phi_t$ and $-\Delta u$, respectively, integrate over $\Omega$ then add the resulting equations, and deduce

$$\begin{aligned} &\frac{d}{dt}\left(\|\Delta\phi\|^2 + \|\nabla\phi\|^2 + \epsilon\|\nabla\phi_t\|^2 + \|\nabla u\|^2\right) + \|\nabla\phi_t\|^2 + \|\Delta u\|^2 \\ &\quad\le \frac{1}{2}\left\|g'(\phi)\right\|^2_{L^\infty(\Omega)}\|\nabla\phi\|^2 \\ &\quad\le c\|\nabla\phi\|^2. \end{aligned}$$

Integrating over $(0,t)$ and recalling (2.2), we get

$$\begin{aligned} &\|\Delta\phi\|^2 + \|\nabla\phi\|^2 + \epsilon\|\nabla\phi_t\|^2 + \|\nabla u\|^2 + \int_0^t\left(\left\|\nabla\phi_t(s)\right\|^2 + \left\|\Delta u(s)\right\|^2\right)ds \\ &\quad\le c(t+1), \quad \forall t \ge 0. \end{aligned} \tag{3.21}$$

It then follows from (2.3) and (3.21) that

$$\int_0^t\left(\left\|\phi_t(s)\right\|^2_1 + \left\|\Delta u(s)\right\|^2\right)ds \le c(t+1), \quad \forall t \ge 0. \tag{3.22}$$

Next, from $(1.1)_1$, we deduce that

$$\epsilon^2\int_0^t\left\|\phi_{tt}(s)\right\|^2 ds \le \int_0^t\left(\left\|\phi_t(s)\right\|^2 + \left\|\Delta\phi(s)\right\|^2 + \left\|\phi(s)\right\|^2 + \left\|g\big(\phi(s)\big)\right\|^2 + \left\|u(s)\right\|^2\right)ds,$$

then from (2.2), (3.21) and (3.22) it follows that

$$\int_0^t \epsilon \|\phi_{tt}(s)\|^2 \leq \frac{c}{\epsilon}(t+1), \quad \forall t \geq 0. \tag{3.23}$$

Also, from $(1.1)_2$ and (3.22), we deduce that

$$\begin{aligned}\int_0^t \|u_t(s)\|^2\,ds &\leq c\int_0^t \left(\|\phi_t(s)\|^2 + \|\Delta u(s)\|^2\right) ds \\ &\leq c(t+1), \quad \forall t \geq 0.\end{aligned} \tag{3.24}$$

Finally, we have that

$$\begin{aligned}&\|S_\epsilon(t)z_{01} - S_\epsilon(t')z_{02}\|_{\mathcal{H}_{1,\epsilon}} \\ &\quad\leq \|S_\epsilon(t)z_{01} - S_\epsilon(t')z_{01}\|_{\mathcal{H}_{1,\epsilon}} + \|S_\epsilon(t')z_{01} - S_\epsilon(t')z_{02}\|_{\mathcal{H}_{1,\epsilon}}, \quad \forall t,t' \in [t^*, 2t^*].\end{aligned}$$

Indeed, on the one hand, from (3.23) and (3.24), we have

$$\begin{aligned}&\|S_\epsilon(t)z_{01} - S_\epsilon(t')z_{01}\|_{\mathcal{H}_{1,\epsilon}} \\ &\quad\leq c\left(\|\phi(t) - \phi(t')\|_1 + \sqrt{\epsilon}\|\phi_t(t) - \phi_t(t')\| + \|u(t) - u(t')\|\right) \\ &\quad\leq c\int_t^{t'} \left(\|\phi_t(s)\|_1 + \sqrt{\epsilon}\|\phi_{tt}(s)\| + \|u_t(s)\|\right) ds \\ &\quad\leq c(\epsilon, t^*)|t' - t|^{1/2}.\end{aligned}$$

On the other hand, it follows from (3.20) that

$$\|S_\epsilon(t')z_{01} - S_\epsilon(t')z_{02}\|_{\mathcal{H}_{1,\epsilon}} \leq c(t^*)\|z_{01} - z_{02}\|_{\mathcal{H}_{1,\epsilon}}, \quad \forall t' \geq 0. \tag{3.25}$$

Hence, we conclude with

$$\|S_\epsilon(t)z_{01} - S_\epsilon(t')z_{02}\|_{\mathcal{H}_{1,\epsilon}} \leq c(\epsilon, t^*)\left(|t' - t|^{1/2} + \|z_{01} - z_{02}\|_{\mathcal{H}_{1,\epsilon}}\right). \tag{3.26}$$

We now apply Theorem 3.1. We will only need to check Assumptions 2, 4 and 5, for the existence of a family of exponential attractors $\mathcal{E}_\epsilon$ that satisfy (3.1) and (3.2). Assumption 2 follows from estimates (3.9) and (3.10) of Proposition 3.1. Assumptions 4 and 5 follow from (2.6) and (3.26), respectively. This shows the existence of a family of exponential attractors $\mathcal{E}_\epsilon$ in $\mathcal{H}_{1,\epsilon}$ with dimension independent of $\epsilon$. □

## 4 Robust family of exponential attractors

We start by showing the existence of an absorbing set in $\mathcal{H}_{3,\epsilon}$.

**Proposition 4.1** *The semigroup $S_\epsilon(t)$ possesses an exponentially attracting bounded absorbing set $\mathcal{B}_3$ in $\mathcal{H}_{3,\epsilon}$.*

*Proof* Let $B \subset \mathcal{H}_{3,\epsilon}$ be a bounded set, and let $(\phi_0, \phi_1, u_0) \in B$. Hence, since $\mathcal{H}_{3,\epsilon} \subset \mathcal{H}_{2,\epsilon}$, there exists a $t(B) > 0$ such that $(\phi(t), \phi_t(t), u(t)) \in \mathcal{B}_2$, $\forall t \geq t(B)$. That is,

$$\|\phi(t)\|_2^2 + \epsilon\|\phi_t(t)\|_1^2 + \|u(t)\|_1^2 \leq r_2, \quad \forall t \geq t(B). \tag{4.1}$$

The following estimates hold true:

$$\begin{aligned}\big(\Delta g(\phi), \Delta\phi_t\big) &\leq \|g'(\phi)\|_{L^\infty(\Omega)}\|\Delta\phi\|\|\Delta\phi_t\| + \|g''(\phi)\|_{L^\infty(\Omega)}\|\nabla\phi\|_{L^4\Omega}^2\|\Delta\phi_t\| \\ &\leq c\big(\|g'(\phi)\|_{L^\infty(\Omega)}^2\|\Delta\phi\|^2 + \|g''(\phi)\|_{L^\infty(\Omega)}^2\|\nabla\phi\|_1^4\big) + \frac{1}{2}\|\Delta\phi_t\|^2,\end{aligned} \tag{4.2}$$

$$\begin{aligned}\big(g(\phi), \Delta^2\phi\big) &\leq \|g'(\phi)\|_{L^\infty(\Omega)}\|\nabla\phi\|\|\nabla\Delta\phi\| \\ &\leq \|g'(\phi)\|_{L^\infty(\Omega)}^2\|\nabla\phi\|^2 + \frac{1}{4}\|\nabla\Delta\phi\|^2,\end{aligned} \tag{4.3}$$

$$\big(u, \Delta^2\phi\big) \leq \|\nabla u\|^2 + \frac{1}{4}\|\nabla\Delta\phi\|^2, \tag{4.4}$$

$$\epsilon(\Delta\phi, \Delta\phi_t) \leq \frac{1}{2}\|\Delta\phi\|^2 + \epsilon\|\Delta\phi_t\|^2. \tag{4.5}$$

Multiply $(1.1)_1$ by $\Delta^2\phi_t$ and $\kappa\Delta^2\phi$ with $0 < \kappa \leq \frac{1}{8}$, then multiply $(1.1)_2$ by $\Delta^2 u$, and integrate over $\Omega$. Adding the resulting equations gives, on account of (4.2)–(4.5),

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\big[\|\nabla\Delta\phi\|^2 + (1+\kappa)\|\Delta\phi\|^2 + \epsilon\|\Delta\phi_t\|^2 + \|\Delta u\|^2 + 2\kappa\epsilon(\Delta\phi, \Delta\phi_t)\big] \\ &\quad + \frac{\kappa}{2}\|\nabla\Delta\phi|^2 + \frac{\kappa}{2}\|\Delta\phi\|^2 + \epsilon\left(\frac{1}{2} - 2\kappa\right)\|\Delta\phi_t\|^2 + \epsilon\kappa(\Delta\phi, \Delta\phi_t) \\ &\leq c\big(\|g'(\phi)\|_{L^\infty(\Omega)}^2\|\Delta\phi\|^2 + \|g''(\phi)\|_{L^\infty(\Omega)}^2\|\nabla\phi\|_1^4 + \|\nabla u\|^2\big).\end{aligned}$$

Hence from (4.1), there exists a constant $\varpi_1 > 0$ independent of $\epsilon$ such that

$$\frac{d}{dt}E_3(t) + \varpi_1 E_3(t) \leq c(r_2), \tag{4.6}$$

where

$$E_3(t) = \|\nabla\Delta\phi\|^2 + (1+\varpi)\|\Delta\phi\|^2 + \epsilon\|\Delta\phi_t\|^2 + \|\Delta u\|^2 + 2\varpi\epsilon(\Delta\phi, \Delta\phi_t).$$

Clearly, by Hölder's and Young's inequalities, there exist constants $\varpi_2, \varpi_3 > 0$, independent of $\epsilon$ such that

$$\begin{aligned}&\varpi_2\big(\|\nabla\Delta\phi\|^2 + \|\Delta\phi\|^2 + \epsilon\|\Delta\phi_t\|^2 + \|\Delta u\|^2\big) \\ &\quad \leq E_3(t) \\ &\quad \leq \varpi_3\big(\|\nabla\Delta\phi\|^2 + \|\Delta\phi\|^2 + \epsilon\|\Delta\phi_t\|^2 + \|\Delta u\|^2\big).\end{aligned} \tag{4.7}$$

Applying the generalized Gronwall's lemma to (4.6) and using (4.7), we obtain

$$\big\|\big(\phi(t), \phi_t(t), u(t)\big)\big\|_{\mathcal{H}_{3,\epsilon}}^2 \leq c(B)e^{-\varpi_1 t} + c(r_2), \quad \forall t \geq 0. \tag{4.8}$$

Hence, we have that

$$\mathcal{B}_3 = \left\{(\varphi, \psi, \nu) \in \mathcal{H}_{3,\epsilon}, \left\|(\varphi, \psi, \nu)\right\|_{\mathcal{H}_{3,\epsilon}} \le \sqrt{2c(r_2)/\varpi_1} = r_3\right\}$$

is an exponentially attracting absorbing set for $S_\epsilon(t)$ on $\mathcal{H}_{3.,\epsilon}$. □

We prove the following result.

**Proposition 4.2** *For every* $\epsilon \in (0,1]$, *there exists a* $c > 0$, *independent of* $\epsilon$, *such that for any* $\mathrm{z} \in \mathcal{B}_3$,

$$\left\|S_\epsilon(t)\mathrm{z}\right\|_{\mathcal{H}_{2,0}} \le c, \quad \forall t \ge 1. \tag{4.9}$$

*Proof* Let $\mathrm{z}_0 = (\phi_0, \phi_1, u_0) \in \mathcal{B}_3$. We set $(\phi(t), \phi_t(t), u(t)) = S_\epsilon(t)(\phi_0, \phi_1, u_0)$, $\forall t \ge 0$. There exists a $c > 0$, independent of $\epsilon$, such that

$$\left\|\phi(t)\right\|_3^2 + \epsilon\left\|\phi_t(t)\right\|_2^2 + \left\|u(t)\right\|_2^2 \le c, \quad \forall t \ge 0. \tag{4.10}$$

Multiplying the first equation of (1.1) by $\varGamma\phi_t$, where $\varGamma = I - \Delta$, then integrating over $\Omega$, we obtain

$$\frac{\epsilon}{2}\frac{d}{dt}\|\phi_t\|_1^2 + \|\phi_t\|_1^2 + (-\Delta\phi, \varGamma\phi_t) + (\phi, \varGamma\phi_t) + \big(g(\phi), \varGamma\phi_t\big) - (u, \varGamma\phi_t) = 0.$$

Hence, we deduce due to (4.10), that

$$\epsilon\frac{d}{dt}\|\phi_t\|_1^2 + \|\phi_t\|_1^2 \le c. \tag{4.11}$$

First, we multiply (4.11) by $e^{ct/\epsilon}$ and integrate between $\tau$ and $t+1$, for any $\tau \le t+1$. This yields

$$\epsilon\left\|\phi_t(t+1)\right\|_1^2 e^{c(t+1)/\epsilon} \le c\epsilon\left\|\phi_t(\tau)\right\|_1^2 e^{cs/\epsilon} + c\epsilon\left(e^{c(t+1)/\epsilon} - e^{c\tau/\epsilon}\right). \tag{4.12}$$

Now, integrating (4.12) between $t$ and $t+1$ with respect to $\tau$, we deduce

$$\left\|\phi_t(t)\right\|_1^2 \le c, \quad \forall t \ge 1, \tag{4.13}$$

hence the estimate (4.9) holds. □

The following estimate holds for difference of two solutions.

**Proposition 4.3** *There exist* $t_\star > 0$, $c$ *and* $c' > 0$ *all independent of* $\epsilon$ *such that*

$$\left\|S_\epsilon(t)(\phi_0, \phi_1, u_0) - \mathcal{L}S(t)(\phi_0, u_0)\right\|_{\mathcal{H}_{1,\epsilon}}^2 \le c\sqrt[4]{\epsilon}e^{c't}, \quad \forall t \ge t_\star, \tag{4.14}$$

*for any* $(\phi_0, \phi_1, u_0) \in \mathcal{B}_3$, *and*

$$\left\|S_\epsilon(t)(\phi_0, \phi_1, u_0) - \mathcal{L}S(t)(\phi_0, u_0)\right\|_{\mathcal{H}_{1,0}}^2 \le c\sqrt[4]{\epsilon}e^{c't}, \quad \forall t \ge t_\star, \tag{4.15}$$

*for any* $(\phi_0, \phi_1, u_0) \in S_\epsilon(1)\mathcal{B}_3$*, and any* $\epsilon \in (0,1]$*, where* $\mathcal{L}(\psi(t), \upsilon(t)) = (\psi(t), \mathcal{L}(\psi(t), \upsilon(t)), \upsilon(t))$.

*Proof* Let $(\phi_0, \phi_1, u_0) \in \mathcal{B}_3$. We set $(\phi^\epsilon(t), \phi_t^\epsilon(t), u^\epsilon(t)) = S_\epsilon(t)(\phi_0, \phi_1, u_0)$, and $(\phi(t), \phi_t(t), u(t)) = \mathcal{L}S(t)(\phi_0, u_0)$.

We have that

$$\|\phi^\epsilon(t)\|_3^2 + \epsilon\|\phi_t^\epsilon(t)\|_2^2 + \|u^\epsilon(t)\|_2^2 \le c, \quad \forall t \ge 0, \tag{4.16}$$

$$\|\phi(t)\|_3^2 + \|u(t)\|_2^2 \le c, \quad \forall t \ge 0. \tag{4.17}$$

We set $P = \phi^\epsilon - \phi$ and $R = u^\epsilon - u$, then the pair $(P, R)$ solves the problem:

$$\begin{cases} \epsilon P_{tt} + P_t - \Delta P + P + g(\phi^\epsilon) - g(\phi) - R = -\epsilon\phi_{tt}, \\ R_t + P_t - \Delta R = 0, \\ P|_{t=0} = 0, \qquad P_t|_{t=0} = \phi_1 - \mathcal{L}(\phi_0, u_0), \qquad R|_{t=0} = 0. \end{cases} \tag{4.18}$$

We multiply $(4.18)_1$ and $(4.18)_1$ by $P_t$ and $R$, respectively, then integrate over $\Omega$. Adding the resulting equations, we obtain

$$\frac{1}{2}\frac{d}{dt}\left(\|P\|_1^2 + \epsilon\|P_t\|^2 + \|R\|^2\right) + \|P_t\| + \|\nabla R\|^2 = -\left(g(\phi^\epsilon) - g(\phi), P_t\right) - \epsilon(\phi_{tt}, P_t).$$

We deduce that

$$\frac{d}{dt}\left(\|P\|_1^2 + \epsilon\|P_t\|^2 + \|R\|^2\right) + \|P_t\|^2 + \|\nabla R\|^2 \le c'\|P\|^2 + 2\epsilon^2\|\phi_{tt}\|^2. \tag{4.19}$$

The following holds true:

$$\int_0^t \|\phi_{tt}(s)\|^2\, ds \le ce^{\nu t}, \quad \forall t \ge 0. \tag{4.20}$$

We integrate (4.19) over $(0, t)$, and on account of (4.20) we obtain

$$\|P(t)\|_1^2 + \epsilon\|P_t(t)\|^2 + \|R(t)\|^2 \le c\left(\epsilon\|\phi_1 - \mathcal{L}(\phi_0, u_0)\|^2 + \epsilon^2\right)e^{c't}, \quad \forall t \ge 0. \tag{4.21}$$

Similarly, we multiply $(4.18)_1$ and $(4.18)_1$ by $-\Delta P_t$ and $-\Delta R$, respectively, then integrate over $\Omega$. Adding the resulting equations and proceeding like in the proof of estimate (4.21) above, we obtain

$$\|P(t)\|_2^2 + \epsilon\|\nabla P_t(t)\|^2 + \|R(t)\|_1^2 \le c\left(\epsilon\|\phi_1 - \mathcal{L}(\phi_0, u_0)\|_1^2 + \epsilon^2\right)e^{c't}, \quad \forall t \ge 0. \tag{4.22}$$

Now, integrating (4.19) between 0 and $t$, we obtain

$$\int_0^t \left(\|P_t(s)\|^2 + \|R(s)\|_1^2\right) ds \le c\left(\epsilon\|\phi_1 - \mathcal{L}(\phi_0, u_0)\|^2 + \epsilon^2\right)e^{c't}, \quad \forall t \ge 0, \tag{4.23}$$

due to (4.20) and (4.21). Next, we multiply $(4.18)_1$ by $P_t$ and integrate over $\Omega$ to deduce

$$\frac{d}{dt}\epsilon\|P_t\|^2+\|P_t\|^2\le c\big(\|P\|_2^2+\|R\|^2+\epsilon^2\|\phi_{tt}\|^2\big). \tag{4.24}$$

We multiply (4.24) by $t$ to get

$$\frac{d}{dt}\big(\epsilon t\|P_t\|^2e^{t/\epsilon}\big)\le\epsilon\|P_t\|^2e^{t/\epsilon}+\big[ct\big(\|P\|^2+\|R\|^2+\epsilon^2\|\phi_{tt}\|^2\big)\big]e^{t/\epsilon}. \tag{4.25}$$

Integrating (4.25) between 0 and $t$, due to (4.20), (4.21), (4.22) and (4.23), we obtain

$$\begin{aligned}\epsilon t\big\|P_t(t)\big\|^2&\le\epsilon\int_0^t\big\|P_t(s)\big\|^2\,ds+c\epsilon t\big(\epsilon\big\|\phi_1-\mathcal{L}(\phi_0,u_0)\big\|_1^2+\epsilon^2\big)e^{c't}\\&\quad+c\epsilon^2t\int_0^t\big\|\phi_{tt}(s)\big\|^2\,ds\\&\le c\epsilon\big(\epsilon^2+\epsilon\big\|\phi_1-\mathcal{L}(\phi_0,u_0)\big\|^2\big)e^{c't}+ct\epsilon\big(\epsilon^2+\epsilon\big\|\phi_1-\mathcal{L}(\phi_0,u_0)\big\|_1^2\big)e^{c't}\\&\quad+ct\epsilon^2e^{c't},\quad\forall t\ge0.\end{aligned}$$

Hence

$$\begin{aligned}\epsilon\big\|P_t(t)\big\|^2&\le c\epsilon t^{-1}\big(\epsilon^2+\epsilon\big\|\phi_1-\mathcal{L}(\phi_0,u_0)\big\|^2\big)e^{c't}\\&\quad+c\epsilon\big(\epsilon+\epsilon\big\|\phi_1-\mathcal{L}(\phi_0,u_0)\big\|_1^2\big)e^{c't},\quad\forall t\ge0.\end{aligned}$$

Therefore, we have

$$\epsilon\big\|P_t(\sqrt{\epsilon})\big\|^2\le c\sqrt{\epsilon}\big(\epsilon^2+\epsilon\big\|\phi_1-\mathcal{L}(\phi_0,u_0)\big\|^2\big)+c\epsilon\big(\epsilon+\epsilon\big\|\phi_1-\mathcal{L}(\phi_0,u_0)\big\|_1^2\big). \tag{4.26}$$

Using the interpolation inequality, (4.22) and (4.23), we deduce

$$\begin{aligned}\big\|P(t)\big\|_1^2&\le c\big\|P(t)\big\|\big\|P(t)\big\|_2\\&\le c\sqrt{t}\big(\epsilon^2+\epsilon\big\|\phi_1-\mathcal{L}(\phi_0,u_0)\big\|_1^2\big)e^{c't},\quad\forall t\ge0.\end{aligned}$$

Therefore,

$$\big\|P(\sqrt{\epsilon})\big\|_1^2\le c\sqrt[4]{\epsilon}\big(\epsilon^2+\epsilon\big\|\phi_1-\mathcal{L}(\phi_0,u_0)\big\|_1^2\big). \tag{4.27}$$

From $(4.18)_2$ and (4.23), we deduce

$$\begin{aligned}\int_0^t\big\|R_t(s)\big\|_{-1}^2\,ds&\le c\int_0^t\big(\big\|P_t(s)\big\|^2+\big\|\nabla R(s)\big\|^2\big)\,ds\\&\le c\big(\epsilon^2+\epsilon\big\|\phi_1-\mathcal{L}(\phi_0,u_0)\big\|^2\big)e^{c't},\quad\forall t\ge0.\end{aligned} \tag{4.28}$$

Again, by interpolation inequality, (4.22) and (4.28), we have

$$\begin{aligned}\big\|R(t)\big\|^2&\le c\big\|R(t)\big\|_{-1}\big\|R(t)\big\|_1\\&\le c\sqrt{t}\big(\epsilon^2+\epsilon\big\|\phi_1-\mathcal{L}(\phi_0,u_0)\big\|_1^2\big)e^{c't},\quad\forall t\ge0,\end{aligned}$$

so that

$$\|R(\sqrt{\epsilon})\|^2 \le c\sqrt[4]{\epsilon}\big(\epsilon^2 + \epsilon\|\phi_1 - \mathcal{L}(\phi_0, u_0)\|_1^2\big). \tag{4.29}$$

We now apply Gronwall's lemma to (4.19) between $\sqrt{\epsilon}$ and $t + \sqrt{\epsilon}$. We find

$$\big(\|P\|_1^2 + \epsilon\|P_t\|^2 + \|R\|^2\big)(t + \sqrt{\epsilon}) \le c\big[\big(\|P\|_1^2 + \epsilon\|P_t\|^2 + \|R\|^2\big)(\sqrt{\epsilon}) + \epsilon^2\big]e^{c't}, \tag{4.30}$$

for every $t \ge 0$.

Due to (4.26), (4.27) and (4.29), from (4.30) it follows that

$$\big(\|P\|_1^2 + \epsilon\|P_t\|^2 + \|R\|^2\big)(t + \sqrt{\epsilon}) \le c\sqrt[4]{\epsilon}\big(\epsilon + \epsilon\|\phi_1 - \mathcal{L}(\phi_0, u_0)\|_1^2\big)e^{c't}, \quad \forall t \ge 0. \tag{4.31}$$

Again, integrating (4.19) between $s$ and $t$, we arrive at the following estimate:

$$\|P(t)\|_1^2 + \epsilon\|P_t(t)\|^2 + \|R(t)\|^2 \le c\big(\|P(s)\|_1^2 + \epsilon\|P_t(s)\|^2 + \|R(s)\|^2 + \epsilon^2\big)e^{c't},$$

for any given $s \ge 0$ and any $t > s$. Let $t_\star > 0$, independent of $\epsilon$, be such that $t_\star > \sqrt{\epsilon}$. This latter estimate, with $s = \sqrt{\epsilon}$, in combination with (4.31) gives

$$\|P(t)\|_1^2 + \epsilon\|P_t(t)\|^2 + \|R(t)\|^2 \le c\sqrt[4]{\epsilon}\big(\epsilon + \epsilon\|\phi_1 - \mathcal{L}(\phi_0, u_0)\|_1^2\big)e^{c't}, \quad \forall t > \sqrt{\epsilon}. \tag{4.32}$$

Finally, estimate (4.14) follows from (4.32) while estimate (4.15) follows from (4.9) and (4.32). □

We have the following corollary of Proposition 4.3.

**Corollary 4.1**

$$\|\Pi_\epsilon S_\epsilon(t)(\phi_0, \phi_1, u_0) - S(t)(\phi_0, u_0)\|_{\mathcal{H}_{1,0}}^2 \le c\sqrt[4]{\epsilon}e^{c't}, \quad \forall t \ge t_\star, \tag{4.33}$$

*where* $\Pi_\epsilon(X \times Y \times Z) = X \times Z$, *i.e.*, $\|\phi^\epsilon(t) - \phi(t)\|_1^2 + \|u^\epsilon(t) - u(t)\|^2 \le c\sqrt[4]{\epsilon}e^{c't}, \forall t \ge t_\star$.

The semigroup $S(t)$ for the variable $(\phi, u)$ alone possesses an exponential attractor $\mathcal{E}_0$ on $\mathcal{H}_{1,0}$, see Theorem 9.14 in [11]. We set $\widetilde{\mathcal{B}}_3 = S_\epsilon(t^*)\mathcal{B}_3$, where $t^* > 0$ is independent of $\epsilon$.

**Theorem 4.1** *There exist* $\varpi_1, \varpi_2 \in (0, \frac{1}{2}]$ *and* $M_1, M_2 > 0$, *all independent of* $\epsilon$, *and a family of exponential attractors* $\mathcal{E}_\epsilon$ *enjoying all the properties of Theorem* 3.2 *and such that*

$$\operatorname{dist}^{\mathrm{sym}}_{\mathcal{H}_{1,\epsilon}}(\mathcal{E}_\epsilon, \mathcal{E}) \le M_1\epsilon^{\varpi_1}, \tag{4.34}$$

$$\operatorname{dist}_{\mathcal{H}_{1,0}}(\mathcal{E}_\epsilon, \mathcal{E}) \le M_2\varepsilon^{\varpi_2}, \quad \textit{and} \tag{4.35}$$

$$\lim_{\epsilon\to 0}\operatorname{dist}_{\mathcal{H}_{1,0}}(\mathcal{E}, \mathcal{E}_\epsilon) = 0, \tag{4.36}$$

*where* $\mathcal{E} = \mathcal{L}\mathcal{E}_0 = \{(\varphi, \mathcal{L}(\varphi, \vartheta), \vartheta), (\varphi, \vartheta) \in \mathcal{E}_0\}$.

*Proof* On account of Theorem 3.1, we let $E_\epsilon = \mathcal{H}_{1,\epsilon}$, $V_\epsilon = \mathcal{H}_{2,\epsilon}$, $W_\epsilon = \mathcal{H}_{3,\epsilon}$, $B_\epsilon = \widetilde{\mathcal{B}}_4$ and we check all Assumptions 1–5. To verify Assumption 1, using the interpolation inequality, there exists a constant $c$ such that for some $\theta \in [0,1]$ we have

$$\begin{aligned} \big\| \mathcal{L}(\varphi,\vartheta) - \mathcal{L}(\psi,\nu)\big\| &\le \big\|\Delta(\varphi-\psi)\big\| + \|\varphi-\psi\| + \big\|g(\varphi)-g(\psi)\big\| + \|\vartheta-\nu\| \\ &\le c\big(\|\varphi-\psi\|^{1/2} + \|\varphi-\psi\|_3^{1/2}\big)\|\varphi-\psi\|_1^{1/2} + \|\vartheta-\nu\| \\ &\le c\big(\|\varphi-\psi\|_1^{1/2} + \|\vartheta-\nu\|^{1/2}\big), \end{aligned} \tag{4.37}$$

for any $(\varphi,\vartheta)$ and $(\psi,\nu)$ in $\mathcal{B}$.

Assumptions 2, 4 and 5 were checked in Theorem 3.2. Assumption 3 follows from (4.14) and (4.15). This shows the existence of exponential attractors in $\mathcal{H}_{1,0}$ that satisfy (4.34), (4.35) and (4.36). □

We also state the following theorem, which is a direct consequence of Corollary 4.33.

**Theorem 4.2** *For every $\epsilon \in (0,1]$, there exists a constant $M_1 > 0$ independent of $\epsilon$ such that the family of exponential attractors $\mathcal{E}_\epsilon$ of the semigroup $S_\epsilon(t)$ on $\mathcal{H}_{1,\epsilon}$ satisfies*

$$\operatorname{dist}^{\mathrm{sym}}_{\mathcal{H}_{1,0}}(\Pi_\epsilon \mathcal{E}_\epsilon, \mathcal{E}_0) \le M_1 \sqrt[4]{\epsilon}. \tag{4.38}$$

## 5 Conclusion

In this work, we considered a parabolic–hyperbolic phase-field system, a model which describes phase separation in material sciences. An example is melting and solidification processes. We constructed a robust family of exponential attractors, which are both upper and lower semicontinuous at the parameter $\epsilon = 0$. A consequence of this is the existence of fractal dimensional global attractor and, moreover, the dynamics of the global attractor converges to that of the well known Cagilnap phase-field system. Most interestingly, estimates were obtained in norms which are independent of the parameter $\epsilon$.

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**Abbreviations**
PHPFS, Parabolic–Hyperbolic Phase-Field System.

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## Publisher's Note



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