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Small-amplitude nonlinear periodic oscillations in a suspension bridge system

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Abstract

This paper deals with a coupled nonlinear beam-wave system, proposed by Lazer and McKenna, modeling a suspension bridge. By constructing a suitable Nash–Moser-type iteration scheme, the existence of small-amplitude nonlinear periodic oscillations, i.e., the existence of small-amplitude periodic solutions for the suspension bridge system, is obtained, and so is their local uniqueness.

Keywords: Suspension bridge system; Small divisors; Nash–Moser iteration scheme

1 Introduction

In this paper, we consider the following suspension bridge system:

$$m_b u_{tt} - Q u_{xx} - K(w-u)^+ = m_b g + \epsilon h_1(x, \omega t), \tag{1.1}$$

$$m_c w_{tt} + EI w_{xxxx} + K(w-u)^+ = m_c g + \epsilon h_2(x, \omega t), \tag{1.2}$$

with the boundary condition

$$\begin{aligned} &u(0,t) = u(L,t) = 0, \\ &w(0,t) = w(L,t) = 0, \qquad w_{xx}(0,t) = w_{xx}(L,t) = 0, \end{aligned} \tag{1.3}$$

where $x \in (0,\pi)$, $t \in \mathbb{R}$, m_b, m_c, Q and EI are positive constants, the parameter $\epsilon \geq 0$, $(w-u)^+ = \max\{w-u, 0\}$, h_i $(i = 1, 2)$ is a time-periodic external force with period $2\pi/\omega$ and amplitude ϵ.

This model is first proposed by Lazer and McKenna [11] through the observation of the fundamental nonlinearity in suspension bridges of which the stays connecting the supporting cables and the roadbed resist expansion but do not resist compression. The roadbed of length L is modeled by a horizontal vibrating beam with both ends being simply supported; the supporting cable of length L is modeled by a horizontal vibrating string with both ends being fixed; the vertical stays connecting the roadbed to the supporting cable are modeled by one-sided springs with resist expansion but not resist compression; In the model (1.1)–(1.2), $u(x,t)$ and $w(x,t)$ denote the downward deflections of the cable and the roadbed, respectively; m_b and m_c denote the mass densities of the cable and the roadbed, respectively; Q denotes the coefficient of cable tensile strength; EI denotes the roadbed flexural rigidity; K is Hooke's constant of the stays; h_1 and h_2 denote the external aerodynamic forces. In fact, we can write system (1.1) as a Hamiltonian system. Introducing the variables $Z = (q_1, q_2, p_1, p_2)$, with $q_1(x,t) = u(x,t)$, $q_2(x,t) = w(x,t)$, $p_1(x,t) = u_t(x,t)$ and $p_2(x,t) = w_t(x,t)$, and the Hamiltonian

$$H(Z) = \frac{1}{2}\int_0^\pi \bigl[m_b q_1^2 + m_c q_2^2 + Q(\partial_x q_1)^2 + EI(\partial_{xx} q_2)^2 + \bigl(K(q_2 - q_1)^+\bigr)^2\bigr]\,dx$$
$$\qquad - \int_0^\pi \bigl[F_1(q_1,p_1) + F_2(q_2,p_2)\bigr]\,dx,$$

the system (1.1) can be written in the form

$$\frac{\partial}{\partial_t} Z = J\nabla_Z H(Z),$$

where $F_1(q_1,p_1) = m_b g q_1 + \epsilon h_1(x,t) q_1$, $F_2(q_2,p_2) = m_b g q_2 + \epsilon h_1(x,t) q_2$ and J is a 4×4 symplectic matrix.

Theoretically and numerically of periodic oscillations in for suspension bridge model have attracted much attention [5, 9–11]. McKenna and Walter [12] first studied large-amplitude periodic oscillations for a single suspension bridge equation (a beam equation) via Leray–Schauder degree theory. By considering the motion of the cable in suspension bridge, Lazer and MaKenna [11] considered the model (1.1)–(1.2) under boundary condition (1.3). By using a standard IMSL subroutine on a mainframe using high precision, they showed that the approximation system of (1.1)–(1.2) with damped term

$$u'' + \delta_1 u' + a_1 u - K(w-u)^+ = \epsilon g(t),$$
$$w'' + \delta_2 w' + a_2 w + K(w-u)^+ = W_0,$$

has large and small periodic solutions. Recently, when $\omega = 1$, by using Leray–Schauder degree theory, Ding [7] proved that system (1.1)–(1.2) under boundary condition (1.3) has at least two large-amplitude π-periodic solutions, under the assumption that

$$Q \le m_c, \qquad EI \le m_b,$$
$$\lambda_{m,n} = Q(2n+1)^2 - 4m_c m^2 \neq 0, \qquad \mu_{m,n} = EI(2n+1)^4 - 4m_b m^2 \neq 0;$$
$$\lambda_{m,n} + \mu_{m,n} \neq 0, \quad \text{for } m \ge 1, n \ge 1,$$

where $\lambda_{m,n}$ and $\mu_{m,n}$ denote the eigenvalue of wave operator $m_b u_{tt} - Q u_{xx}$ and beam operator $m_c w_{tt} + EI w_{xxxx}$, respectively. Both $\sqrt{\frac{Q}{m_c}}$ and $\sqrt{\frac{Q}{m_b}}$ are rational numbers.

In the present paper, we show the existence of small-amplitude periodic oscillations in problem (1.1)–(1.3). Here, one of our main results is the following.

Theorem 1.1 *Let* $0 < \bar\sigma < \sigma < \tilde\sigma < 1$, $(m_b g, m_c g) \in [0,\chi_0] \times [0,\chi_0]$. *Assume that* $Q, EI > K$, $(h_1,h_2) \in X_{\tilde\sigma} \times Y_{\tilde\sigma}$. *Then there exist* $\epsilon_0 > 0$ *and* $K_0 > 0$ *sufficiently small and a Cantor set* $\mathcal{X}(\nu) = \{\omega : (\nu,\omega) \in \mathcal{D}_\gamma\}$, $\mathcal{D}_\gamma$ *to be specified in* (4.17), *such that, for* $\epsilon \in [0,\epsilon_0]$ *and* $K \in [0,K_0]$, *and every* $(\nu,\omega) \in \mathcal{X}(\nu)$ *there exists a solution of system* (1.1)–(1.2) *under boundary* *condition* (1.3),

$$\begin{aligned}&\big(u(\nu,\omega),w(\nu,\omega)\big)\\&\quad=\big(\bar{u}+\tilde{u}(\nu,\omega),\bar{w}+\tilde{w}(\nu,\omega)\big)\in\mathbb{V}_1\times\mathbb{V}_2\oplus(\mathbb{W}_1\cap X_{\bar{\sigma},s})\times(\mathbb{W}_2\cap Y_{\bar{\sigma},s}),\end{aligned}$$

where the spaces $\mathbb{V}_i$, $\mathbb{W}_i$, $X_{\bar{\sigma},s}$ *and* $Y_{\bar{\sigma},s}$ *are to be specified in Sect.* 2, χ_0 *is a sufficiently small constant.*

Moreover, for every $0<\omega_1<\omega_2<\infty$, *there exists a constant* C *depending on* Q *such that in the rectangular region* $\mathcal{Y}=(\nu_1,\nu_2)\times(\omega_1,\omega_2)$ *we have*

$$\frac{|\mathcal{X}(\nu)\cap\mathcal{Y}|}{|\mathcal{Y}|}\geq 1-C\gamma.$$

For dissipative system

$$m_b u_{tt}-Qu_{xx}+\delta_1 u_t-K(w-u)^+=m_b g+\epsilon h_1(x,\omega t), \tag{1.4}$$

$$m_c w_{tt}+EIw_{xxxx}+\delta_2 w_t+K(w-u)^+=m_c g+\epsilon h_2(x,\omega t), \tag{1.5}$$

where $\delta_1,\delta_2>0$, we have the same result.

Theorem 1.2 *Let* $0<\bar{\sigma}<\sigma<\tilde{\sigma}<1$, $(m_b g,m_c g)\in[0,\chi_0]\times[0,\chi_0]$. *Assume that* $Q,EI>K$, $\delta_1,\delta_2>0$, $(h_1,h_2)\in X_{\tilde{\sigma}}\times Y_{\tilde{\sigma}}$. *Then there exist* $\epsilon_0>0$ *and* $K_0>0$ *sufficiently small and a Cantor set* $\mathcal{X}(\nu)=\{\omega:(\nu,\omega)\in\mathcal{D}_\gamma\}$, $\mathcal{D}_\gamma$ *to be specified in* (4.17), *such that, for* $\epsilon\in[0,\epsilon_0]$ *and* $K\in[0,K_0]$, *and every* $(\nu,\omega)\in\mathcal{X}(\nu)$ *there exists a solution of system* (1.4)–(1.5) *under boundary condition* (1.3)

$$\begin{aligned}&\big(u(\nu,\omega),w(\nu,\omega)\big)\\&\quad=\big(\bar{u}+\tilde{u}(\nu,\omega),\bar{w}+\tilde{w}(\nu,\omega)\big)\in\mathbb{V}_1\times\mathbb{V}_2\oplus(\mathbb{W}_1\cap X_{\bar{\sigma},s})\times(\mathbb{W}_2\cap Y_{\bar{\sigma},s}),\end{aligned}$$

where spaces $\mathbb{V}_i$, $\mathbb{W}_i$, $X_{\bar{\sigma},s}$ *and* $Y_{\bar{\sigma},s}$ *are to be specified in Sect.* 2, χ_0 *is a sufficiently small constant.*

Moreover, for every $0<\omega_1<\omega_2<\infty$, *there exists a constant* C *depending on* Q *such that in the rectangular region* $\mathcal{Y}=(\nu_1,\nu_2)\times(\omega_1,\omega_2)$ *we have*

$$\frac{|\mathcal{X}(\nu)\cap\mathcal{Y}|}{|\mathcal{Y}|}\geq 1-C\gamma.$$

The main difficulty in dealing with problems (1.1)–(1.3) and (1.4)–(1.5) is in the presence of the term $(w-u)^+$, which is piecewise linear. So, we cannot simply apply the contraction map theorem in solving the range equation which is obtained by Lyapunov–Schmidt decomposition. To obtain a $2\pi/\omega$-time-periodic solution, a possible method is to construct a rapidly convergent Nash–Moser iteration. The first pioneering work is due to Moser [13–15]. Rabinowitz [16] showed that a class of nonlinear wave equations with damped term has periodic solutions via constructing a rapidly convergent Nash–Moser iteration. Celletti and Chierchia [4] established a dissipative Nash–Moser theorem via a rapidly convergent Newton iteration, and proved KAM tori smoothly bifurcating into quasi-periodic attractors in dissipative mechanical models. Inspired by the work of [1–4, 6, 8, 16–18], we will construct a suitable rapidly convergent Nash–Moser iteration scheme to prove our results.

We organize the paper as follows. In Sect. 2, we recall some basic notations, splitting the problem into the bifurcation equation and the range equation. Section 3, the bifurcation equation is solved via direct calculus of variations. Section 4 is divided into two subsections. The first subsection is to give the strategy of the Nash–Moser algorithm and some KAM estimates. The last subsection is to prove the convergence of the Nash–Moser algorithm and local uniqueness of solutions. In the Appendix, the measure of the Cantor set $\mathcal{D}_\gamma$ is also estimated.

2 Some notations

We start this section by introducing some notations. Consider the following space:

$$\begin{aligned} X_{\sigma,s} := \Big\{ & u(t,x) := \sum_{l\in\mathbb{Z}} u_l(x)e^{ilt} \,\Big|\, u_0 \in \mathbb{H}_0^1\big((0,\pi),\mathbb{R}\big), u_l \in \mathbb{C}\big((0,\pi),\mathbb{R}\big)\ (l\neq 0), u_{-l} = u_l^*, \\ & \|u\|_{\sigma,s}^2 := |u_0|_{\mathbb{H}_0^1}^2 + \sum_{|l|\geq 1}\Big(\max_{x\in(0,\pi)} u_l(x)\Big)^2 |l|^{2s}e^{2\sigma|l|} < \infty \Big\}, \end{aligned}$$

and

$$\begin{aligned} Y_{\sigma,s} := \Big\{ & w(t,x) := \sum_{l\in\mathbb{Z}} w_l(x)e^{ilt} \,\Big|\, w_0 \in \mathbb{H}_0^2\big((0,\pi),\mathbb{R}\big), w_l \in \mathbb{C}\big((0,\pi),\mathbb{R}\big)\ (l\neq 0), w_{-l} = w_l^*, \\ & \|w\|_{\sigma,s}^2 := |w_0|_{\mathbb{H}_0^2}^2 + \sum_{|l|\geq 1}\Big(\max_{x\in(0,\pi)} w_l(x)\Big)^2 |l|^{2s}e^{2\sigma|l|} < \infty \Big\}, \end{aligned}$$

where u_l and w_l denote the $l$the Fourier coefficients.

Obviously, for a nested family of Banach spaces $\{X_{\sigma,s} : \sigma \geq 0, s \geq 0\}$ and $\{Y_{\sigma,s} : \sigma \geq 0, s \geq 0\}$, we have

$$X_{\sigma_2,s} \subset X_{\sigma_1,s}, \qquad Y_{\sigma_2,s} \subset Y_{\sigma_1,s},$$

and

$$\begin{aligned} &\|u\|_{\sigma_1,s} \leq \|u\|_{\sigma_2,s}, \quad \forall 0 \leq \sigma_1 \leq \sigma_2, u \in X_{\sigma,s}, \\ &\|w\|_{\sigma_1,s} \leq \|w\|_{\sigma_2,s}, \quad \forall 0 \leq \sigma_1 \leq \sigma_2, w \in Y_{\sigma,s}. \end{aligned}$$

For $\sigma \geq 0$, $s \geq 0$, the space $X_{\sigma,s}$ is a Banach algebra with respect to multiplication of functions, i.e., if $u_1, u_2 \in X_{\sigma,s}$, then $u_1u_2 \in X_{\sigma,s}$ and there exists a positive constant C, such that

$$\|u_1u_2\|_{\sigma,s} \leq C\|u_1\|_{\sigma,s}\|u_2\|_{\sigma,s}.$$

The space $Y_{\sigma,s}$ is a Banach algebra with respect to multiplication of functions. It is obvious that each function in $X_{\sigma,s}$ or $(Y_{\sigma,s})$ has bounded analytic extension in the complex multi-strip $|Im\vartheta_i| < \sigma$, where $\vartheta_i \in C$, $i = 1, 2$. By the definition of the space $X_{\sigma,s}$, the following inequality holds:

$$\left\|\partial_\vartheta^h u\right\|_{\sigma,s} \le \left\|\partial_\vartheta^k u\right\|_{\sigma,s}, \quad \forall h,k \in N^2 : h_i \le k_i.$$

For more details, see [1].

After a time rescaling, we look for 2π periodic solutions of

$$\omega^2 m_b u_{tt} - Q u_{xx} - K(w-u)^+ = m_b g + \epsilon h_1(x,t), \tag{2.1}$$

$$\omega^2 m_c w_{tt} + EI w_{xxxx} + K(w-u)^+ = m_c g + \epsilon h_2(x,t), \tag{2.2}$$

with the boundary condition

$$\begin{aligned} &u(0,t) = u(\pi,t) = 0, \\ &w(0,t) = w(\pi,t) = 0, \qquad w_{xx}(0,t) = w_{xx}(\pi,t) = 0. \end{aligned} \tag{2.3}$$

Denote the wave operator (d'Alembertian operator) $\mathcal{L}_\omega$ by

$$\mathcal{L}_\omega u = \omega^2 m_b u_{tt} - Q u_{xx},$$

with the Dirichlet boundary condition

$$u(0,t) = u(\pi,t) = 0,$$

and the beam operator $\mathcal{J}_\omega$ by

$$\mathcal{J}_\omega w = \omega^2 m_c w_{tt} + EI w_{xxxx},$$

with the hinged boundary condition

$$w(0,t) = w(\pi,t) = 0, \qquad w_{xx}(0,t) = w_{xx}(\pi,t) = 0.$$

Let $\lambda_{l,j}$ denote the eigenvalues of the wave operator $\mathcal{L}_\omega$ and $\mu_{l,j}$ denote the eigenvalues of the beam operator $\mathcal{J}_\omega$. Then it follows from a direct calculation that

$$\lambda_{l,j} = -\omega^2 m_b l^2 + Qj^2, \qquad \mu_{l,j} = -\omega^2 m_c l^2 + EIj^4,$$

where $l \in \mathbb{Z}$ and $j = 1, 2, \dots$.

To find the solution of system (2.1)–(2.2) under boundary condition (2.3), we will introduce the Lyapunov–Schmidt reduction according to the decomposition

$$X_{\sigma,s} \times Y_{\sigma,s} = (\mathbb{V}_1 \times \mathbb{V}_2) \oplus \big((\mathbb{W}_1 \cap X_{\sigma,s}) \times (\mathbb{W}_2 \cap Y_{\sigma,s})\big),$$

where

$$\mathbb{V}_1 := \mathbb{H}_0^1(0,\pi), \qquad \mathbb{W}_1 := \left\{ u = \sum_{l \ne 0} u_l(x) e^{ilt} \in X_{\sigma,s} \right\},$$

$$\mathbb{V}_2 := \mathbb{H}_0^2(0,\pi), \qquad \mathbb{W}_2 := \left\{ w = \sum_{l\neq 0} w_l(x)e^{ilt} \in Y_{\sigma,s} \right\}.$$

Then the solution $(u,w) \in X_{\sigma,s} \times Y_{\sigma,s}$ of system (2.1)–(2.2) can be written as

$$u(x,t) = u_0(x) + \sum_{l\neq 0} u_l(x)e^{ilt}, \qquad w(x,t) = w_0(x) + \sum_{l\neq 0} w_l(x)e^{ilt}.$$

Note that $h_1(x,t)$ and $h_2(x,t)$ are 2π time-periodic forcing terms. So

$$h_1(x,t) = \bar{h}_1(x) + \sum_{l\neq 0} h_{1l}(x)e^{ilt}, \qquad h_2(x,t) = \bar{h}_2(x) + \sum_{l\neq 0} h_{2l}(x)e^{ilt}.$$

Set

$$u(x,t) = \bar{u}(x) + \tilde{u}(x,t), \qquad w(x,t) = \bar{w}(x) + \tilde{w}(x,t)$$

and

$$h_1(x,t) = \bar{h}_1(x) + \tilde{h}_1(x,t), \qquad h_2(x,t) = \bar{h}_2(x) + \tilde{h}_2(x,t).$$

Projecting system (2.1)–(2.2) by P_V and P_W, the bifurcation equation and the range equation are obtained:

$$\begin{aligned} -Q\bar{u}''(x) &= KP_V(\bar{w} - \bar{u} + \tilde{w} - \tilde{u})^+ + \epsilon\bar{h}_1(x), \\ EI\bar{w}''''(x) &= -KP_V(\bar{w} - \bar{u} + \tilde{w} - \tilde{u})^+ + \epsilon\bar{h}_2(x), \end{aligned} \tag{2.4}$$

and

$$\begin{aligned} \mathcal{L}_\omega\tilde{u} &= KP_W(\tilde{w} - \tilde{u} + \bar{w} - \bar{u})^+ + m_c g + \epsilon\tilde{h}_1(x,t), \\ \mathcal{J}_\omega\tilde{w} &= -KP_W(\tilde{w} - \tilde{u} + \bar{w} - \bar{u})^+ + m_b g + \epsilon\tilde{h}_2(x,t). \end{aligned} \tag{2.5}$$

Remark 2.1 To obtain more relaxation conditions on Q and EI, both constants $m_b g$ and $m_c g$ are put in the range equation. In doing so, $EI > K$ is sufficient to solve the bifurcation equation. On the other hand, if we get the bifurcation equation in time t (not x) by splitting system (2.1)–(2.2), $m_b g$ and $m_c g$ will appear in the range equation.

Remark 2.2 We shall find solutions of (2.4)–(2.5) when the ratio $\frac{\nu}{\omega}$ is small (see (4.17)). In this limit $\tilde{w}$ and $\tilde{u}$ tends to 0 and the bifurcation equation reduces to the time-independent equation

$$\begin{aligned} -Q\bar{u}''(x) &= K(\bar{w} - \bar{u})^+ + \epsilon\bar{h}_1(x), \\ EI\bar{w}''''(x) &= -K(\bar{w} - \bar{u})^+ + \epsilon\bar{h}_2(x). \end{aligned}$$

Remark 2.3 For dissipative system (1.4)–(1.5), by Lyapunov–Schmidt reduction, the bifurcation equation is the same as (2.4)–(2.6), and the range equation is

$$\mathcal{L}'_\omega\tilde{u} = K(\tilde{w} - \tilde{u})^+ + m_b g + \epsilon\tilde{h}_1(x,t), \tag{2.6}$$

$$\mathcal{J}'_\omega \tilde{w} = -K(\tilde{w} - \tilde{u})^+ + m_c g + \epsilon \tilde{h}_2(x,t), \tag{2.7}$$

where $\mathcal{L}'_\omega u = \omega^2 m_b u_{tt} - Q u_{xx} + \delta_1 u_t$ and $\mathcal{J}'_\omega w = \omega^2 m_c w_{tt} + EI w_{xxxx} + \delta_2 w_t$.

3 The bifurcation equation

For convenience, we denote $u(x) = \bar{u}(x)$ and $w(x) = \bar{w}(x)$. Consider the following coupled ODEs:

$$-Qu''(x) = K(w-u)^+ + \epsilon h_1(x), \tag{3.1}$$

$$EIw''''(x) = -K(w-u)^+ + \epsilon h_2(x), \tag{3.2}$$

with the boundary condition

$$u(0) = u(\pi) = 0,$$

$$w(0) = w(\pi), \qquad w''(0) = w''(\pi).$$

Define an action functional

$$I(u,w) = \frac{1}{2}\int_0^\pi Q\big|u'\big|^2 + EI\big|w''\big|^2 + K\big[(w-u)^+\big]^2\,dx - \int_0^\pi \bar{F}_1(u,w) + \bar{F}_2(u,w)\,dx,$$

where $I : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$,

$$\bar{F}_1(u,w) = \epsilon h_1(x)u, \qquad \bar{F}_2(u,w) = \epsilon h_2(x)w.$$

Theorem 3.1 *Assume that $Q, EI > K$. Then system (3.1)–(3.2) admits a solution $(u,w) \in \bar{\mathbb{V}} = \mathbb{H}_0^1(0,\pi) \times \mathbb{H}_0^2(0,\pi)$.*

Proof By the direct method of the calculus of variations, it suffices to prove that the action functional $I(u,w)$ is weakly lower semi-continuous and coercive.

We first prove the weakly lower semi-continuity of $I(u,w)$. Note that the first part of $I(u,w)$, $Q|u'|^2 + EI|w''|^2$ is convex, we only need to show that $\frac{1}{2}\int_0^\pi K[(w-u)^+]^2\,dx - \int_0^\pi \bar{F}_1(u,w) + \bar{F}_1(u,w)\,dx$ is weakly lower semi-continuous. Take a sequence $\{(u_j,w_j)\}_{j\in\mathbb{Z}} \in \bar{\mathbb{V}}$ such that $(u_j,w_j) \longrightarrow (u,w)$ weakly. By the Sobolev embedding $\bar{\mathbb{V}} \hookrightarrow \bar{\mathbb{L}} := \mathbb{L}^2(0,\pi) \times \mathbb{L}^2(0,\pi)$, we know that $\{(u_j,w_j)\}_{j\in\mathbb{Z}}$ is strongly convergent in $\bar{\mathbb{L}}$. We derive

$$\frac{1}{2}\int_0^\pi K\big[(w_j-u_j)^+\big]^2\,dx \longrightarrow \frac{1}{2}\int_0^\pi K\big[(w-u)^+\big]^2\,dx.$$

Note that operators $\bar{F}_1(u,w)$ and $\bar{F}_2(u,w)$ are linear on (u,w). So we know that $\frac{1}{2}\int_0^\pi K[(w-u)^+]^2\,dx - \int_0^\pi \bar{F}_1(u,w) + \bar{F}_1(u,w)\,dx$ is weakly semi-continuous.

Next, we prove that the action functional $I(u,w)$ is coercive, i.e.

$$I(u,w) \longrightarrow \infty, \quad \text{as } \big\|(u,w)\big\|_{\bar{V}} \longrightarrow \infty. \tag{3.3}$$

By assumption, there exist a sufficiently small constant $\epsilon_0 > 0$ and positive constants $C_0 := C_0(Q, K, \epsilon_0)$ and $C_1 := C_1(EI, K, \epsilon_0)$ such that

$$\begin{aligned}
I(u,w) &= \frac{1}{2}\int_0^\pi Q\big|u'\big|^2 + EI\big|w''\big|^2 + K\big[(w-u)^+\big]^2\,dx - \int_0^\pi \bar{F}_1(u,w) + \bar{F}_1(u,w)\,dx \\
&\geq \frac{1}{2}\big[(Q-K)\|u\|^2_{\mathbb{H}_0^1} + (EI-K)\|w\|^2_{\mathbb{H}_0^2}\big] - \epsilon\|h_1\|_{\mathbb{H}_0^1}\|u\|^2_{\mathbb{H}_0^1} - \epsilon\|h_2\|_{\mathbb{H}_0^2}\|w\|^2_{\mathbb{H}_0^2} \\
&= \frac{1}{2}\big(Q-K-2\epsilon\|h_1\|_{\mathbb{H}_0^1}\big)\|u\|^2_{\mathbb{H}_0^1} + \frac{1}{2}\big(EI-K-2\epsilon\|h_2\|_{\mathbb{H}_0^2}\big)\|w\|^2_{\mathbb{H}_0^2} \\
&\geq C_0\|u\|^2_{\mathbb{H}_0^1} + C_1\|w\|^2_{\mathbb{H}_0^2},
\end{aligned}$$

which implies that (3.3) holds.

Therefore, we conclude that the action functional $I(u,w)$ is weakly lower semi-continuous and coercive on the Hilbert space $\bar{\mathbb{V}}$, and it possesses the minimum point $(u^*, w^*) \in \bar{\mathbb{V}}$ which is a solution of system (3.1)–(3.2). □

4 The range equation

This section is devoted to solving the range equation (2.5) for system (1.1)–(1.2). Note that the method of solving the range equation (2.6)–(2.7) for system (1.4)–(1.5) is similar, so, we mainly discuss the range equation for the conservation case.

4.1 The Nash–Moser algorithm

For convenience, we denote $u(x,t) = \tilde{u}(x,t)$, $w(x,t) = \tilde{w}(x,t)$ and $P_W(\tilde{w} - \tilde{u} + \bar{w} - \bar{u})^+ = (\tilde{w} - \tilde{u})^+$. Consider the range equation

$$\mathcal{L}_\omega u = K(w-u)^+ + m_b g + \epsilon\tilde{h}_1(x,t), \tag{4.1}$$

$$\mathcal{J}_\omega w = -K(w-u)^+ + m_c g + \epsilon\tilde{h}_2(x,t), \tag{4.2}$$

with the boundary condition

$$\begin{aligned}
&u(0,t) = u(\pi,t) = 0, \\
&w(0,t) = w(\pi,t) = 0, \qquad w_{xx}(0,t) = w_{xx}(\pi,t) = 0.
\end{aligned}$$

Define a sequence of subspaces

$$\begin{aligned}
&W_1^{(i)} = \Big\{u = \sum_{1\leq|l|\leq N_i} u_{l,j}\varphi_j(x)e^{ilt}\Big\}, \qquad W_2^{(i)} = \Big\{w = \sum_{1\leq|l|\leq N_i} w_{l,j}\psi_j(x)e^{ilt}\Big\}, \\
&\big(W_1^{(i)}\big)^\perp = \Big\{u = \sum_{|l|\geq N_i} u_{l,j}\varphi_j(x)e^{ilt}\Big\}, \qquad \big(W_2^{(i)}\big)^\perp = \Big\{w = \sum_{|l|\geq N_i} w_{l,j}\psi_j(x)e^{ilt}\Big\}.
\end{aligned}$$

Then

$$W_1 = W_1^{(i)} \otimes \big(W_1^{(i)}\big)^\perp, \qquad W_2 = W_2^{(i)} \otimes \big(W_2^{(i)}\big)^\perp, \quad i \in \mathbb{N}.$$

Let S_i denote the smooth projections on $W_1^{(i)}$ and $W_2^{(i)}$. For all $\sigma, \sigma' \geq 0$, the following smoothing properties hold:

$$\begin{aligned} &\|S_i u\|_{\sigma+\sigma'} \leq N_i^{\sigma'} \|u\|_\sigma, \\ &N_i^{\sigma'} \left\|(I - S_i)u\right\|_\sigma \leq \|u\|_{\sigma+\sigma'}, \end{aligned} \tag{4.3}$$

where θ satisfies assumption (4.53) below and $N_i = e^{\theta^i}$ for all $i \in \mathbb{N}$.

For convenience, we write in short

$$\begin{aligned} X_\sigma &= X_{\sigma,s}, & \|u\|_\sigma &= \|u\|_{\sigma,s}, \\ Y_\sigma &= Y_{\sigma,s}, & \|w\|_\sigma &= \|w\|_{\sigma,s}. \end{aligned}$$

Note that $(\mathcal{L}_\omega + C)^{-1} 1 = \frac{1}{C}$ (or for $\mathcal{J}_\omega$) holds in a suitable big space, for a constant $C \neq 0$. We introduce a fixed positive constant η_i ($i \in \mathbb{N}$) and operators

$$\begin{aligned} \tilde{\mathcal{L}}_\omega &= \omega^2 m_b \partial_{tt} - Q\partial_{xx} + \eta_i, \\ \tilde{\mathcal{J}}_\omega &= \omega^2 m_c \partial_{tt} + EI\partial_{xxxx} + \eta_i. \end{aligned}$$

Remark 4.1 Another reason of introducing the parameters η_i is to get a more exact domain where $m_c g$ and $m_b g$ exist. Moreover, the optimal value of parameters η_i is obtained from the estimate in (4.57).

The range equations (4.1)–(4.2) become

$$\tilde{\mathcal{L}}_\omega u = K(w - u)^+ + \eta_i u + m_c g + \epsilon \tilde{h}_1(x,t), \tag{4.4}$$

$$\tilde{\mathcal{J}}_\omega w = -K(w - u)^+ + \eta_i w + m_b g + \epsilon \tilde{h}_2(x,t). \tag{4.5}$$

Summing up (4.4)–(4.5) yields

$$\tilde{\mathcal{L}}_\omega u + \tilde{\mathcal{J}}_\omega w = \eta_i (u + w) + (m_b + m_c) g + \epsilon (\tilde{h}_1 + \tilde{h}_2). \tag{4.6}$$

Set

$$A_1 = \tilde{\mathcal{J}}_\omega^{-1} u, \qquad A_2 = \tilde{\mathcal{L}}_\omega^{-1} w, \qquad A_3 = \tilde{\mathcal{L}}_\omega^{-1} A_1 + \tilde{\mathcal{J}}_\omega^{-1} A_2. \tag{4.7}$$

By Eq. (4.6), it follows that

$$A_1 + A_2 = \eta_i A_3 + \frac{1}{\eta_i^2}(m_b + m_c) g + \epsilon \tilde{\mathcal{L}}_\omega^{-1} \tilde{\mathcal{J}}_\omega^{-1} (\tilde{h}_1 + \tilde{h}_2). \tag{4.8}$$

Combining the second equation in (4.2) with (4.8), we get

$$\mathcal{J}_\omega \tilde{\mathcal{L}}_\omega A_2 = -K(f_1)^+ + m_b g + \epsilon \tilde{h}_2(x,t), \tag{4.9}$$

where

$$f_1 = \left((\tilde{\mathcal{L}}_\omega + \tilde{\mathcal{J}}_\omega) A_2 - \eta_i \tilde{\mathcal{J}}_\omega A_3 - \frac{1}{\eta_i}(m_b + m_c) g - \epsilon \tilde{\mathcal{L}}_\omega^{-1} (\tilde{h}_1 + \tilde{h}_2) \right)^+.$$

Let

$$\Lambda = \mathcal{J}_\omega\tilde{\mathcal{L}}_\omega(\tilde{\mathcal{L}}_\omega + \tilde{\mathcal{J}}_\omega)^{-1}, \tag{4.10}$$

and

$$A_4 = (\tilde{\mathcal{L}}_\omega + \tilde{\mathcal{J}}_\omega)A_2. \tag{4.11}$$

Then we rewrite (4.9) as

$$\Lambda A_4 + K(A_4 - f_2)^+ = m_b g + \epsilon\tilde{h}_2(x,t), \tag{4.12}$$

where

$$f_2 = \eta_i\tilde{\mathcal{J}}_\omega A_3 + \frac{1}{\eta_i}(m_b + m_c)g + \epsilon\tilde{\mathcal{L}}_\omega^{-1}(\tilde{h}_1 + \tilde{h}_2).$$

For convenience, we rewrite (4.12) as

$$\Lambda a + K(a - f_2)^+ = m_c g + \epsilon\tilde{h}_2(x,t). \tag{4.13}$$

Then from

$$w - u = \tilde{\mathcal{L}}_\omega w - \tilde{\mathcal{J}}_\omega u = a - f_2, \tag{4.14}$$

the solution of system (4.1)–(4.2) can be written as

$$u = \mathcal{L}_\omega^{-1}\bigl[K(a - f_2)^+ + m_b g + \epsilon\tilde{h}_1\bigr], \tag{4.15}$$

$$w = \mathcal{J}_\omega^{-1}\bigl[-K(a - f_2)^+ + m_c g + \epsilon\tilde{h}_2\bigr]. \tag{4.16}$$

Remark 4.2 Outline of the strategy of the Nash–Moser algorithm:

Our target is to construct the approximation solution $a_\infty = \sum_{i=0}^{\infty} a_i$ and the approximation parameters $(m_c g)_\infty = \sum_{i=0}^{\infty}(m_c g)_i$ and $(m_b g)_\infty = \sum_{i=0}^{\infty}(m_b g)_i$ of Eq. (4.13) by the Nash–Moser algorithm. Then, by (4.15)–(4.16), the solution in the range equations (4.1)–(4.2) is

$$u_\infty = \mathcal{L}_\omega^{-1}\bigl[K\bigl(a_\infty - f_2^{(\infty)}\bigr)^+ + (m_b g)_\infty + \epsilon\tilde{h}_1\bigr],$$

$$w_\infty = \mathcal{J}_\omega^{-1}\bigl[-K\bigl(a_\infty - f_2^{(\infty)}\bigr)^+ + (m_c g)_\infty + \epsilon\tilde{h}_2\bigr],$$

where

$$f_2^{(\infty)} = \frac{K}{2} + \epsilon\tilde{L}_\omega(\tilde{h}_1 + \tilde{h}_2).$$

In fact, if we choose suitable initial approximation (u_0, w_0), then, by (4.7) and (4.11), the initial approximation a_0 of (4.13) can be obtained. Furthermore, by (4.15)–(4.16), the corresponding first step approximation solution of (4.1)–(4.2) is

$$u_1 = \mathcal{L}_\omega^{-1}\bigl[K\bigl(a_0 + a_1 - f_2^{(0)}\bigr)^+ + (m_b g)_0 + (m_b g)_1 + \epsilon\tilde{h}_1\bigr] \in W_1^{(1)},$$

$$w_1 = \mathcal{J}_\omega^{-1}\big[-K\big(a_0 + a_1 - f_2^{(0)}\big)^+ + (m_c g)_0 + (m_c g)_1 + \epsilon\tilde{h}_2\big] \in W_2^{(1)}.$$

To obtain the ith step approximation solution (u_i, w_i), we first need to get the ith step approximation solution $\sum_{k=0}^{i} a_k$ and the ith step approximation parameters $\sum_{k=0}^{i}(m_b g)_k$ and $\sum_{k=0}^{i}(m_c g)_k$. Lemmas 4.3–4.4 show how to get the ith approximation step a_i. In the process of proving convergence of Nash–Moser algorithm, the optimal ith approximation step value of parameters $(m_b g)_i$ and $(m_c g)_i$ can be determined in Lemmas 4.5–4.6. Then, by (4.15)–(4.16), the corresponding ith step approximation solution of (4.1)–(4.2) is

$$\begin{aligned} u_i &= \mathcal{L}_\omega^{-1}\left[K\left(\sum_{k=0}^{i} a_k - f_2^{(i-1)}\right)^+ + \sum_{k=0}^{i}(m_c g)_k + \epsilon S_{i-1}\tilde{h}_1\right] \in W_1^{(i)}, \\ w_i &= \mathcal{J}_\omega^{-1}\left[-K\left(\sum_{k=0}^{i} a_k - f_2^{(i-1)}\right)^+ + \sum_{k=0}^{i}(m_b g)_k + \epsilon S_{i-1}\tilde{h}_2\right] \in W_2^{(i)}, \end{aligned}$$

where

$$f_2^{(i-1)} = \eta_{i-1}\tilde{\mathcal{J}}_\omega A_3^{(i-1)} + \frac{(m_b g)_{i-1} + (m_c g)_{i-1}}{\eta_{i-1}} + \epsilon\tilde{\mathcal{L}}_\omega^{-1} S_{i-1}(\tilde{h}_1 + \tilde{h}_2).$$

Since there are errors (denoted by E_i) in constructing each approximation step, the convergence of Nash–Moser algorithm remains to be treated. We will prove it in Lemma 4.6.

Fix the following "nonresonant" set:

$$\begin{aligned} \mathcal{D}_\gamma := \bigg\{&(\nu,\omega) \in (\nu',\nu'') \times (\gamma,+\infty) : \big|\omega\sqrt{m_b + m_c}\,l - \sqrt{EIj^4 + Qj^2}\big| \geq \frac{\gamma}{|l|^{\kappa+1}}, \\ &\left|1 + K\left(\frac{1}{2\omega\sqrt{m_b}\,l(\omega\sqrt{m_b}\,l - \sqrt{Q}j)} + \frac{1}{2\omega\sqrt{m_c}\,l(\omega\sqrt{m_c}\,l - \sqrt{EI}j^2)}\right)\right| \geq \frac{\gamma}{|l|^{\kappa+1}}, \\ &|\omega\sqrt{m_b}\,l - \sqrt{Q}j| \geq \frac{\gamma}{|l|^{\kappa+1}}, \big|\omega\sqrt{m_c}\,l - \sqrt{EI}j^2\big| \geq \frac{\gamma}{|l|^{\kappa+1}}, \frac{\nu}{\omega} \leq C\gamma^2, l, j \geq 1\bigg\}, \end{aligned} \tag{4.17}$$

where $\kappa \in (0,+\infty)$, (ν',ν'') denotes a neighborhood of ν_0, for some $\nu_0 \in [0,\nu''']$, and

$$\nu = \max\left\{\frac{1}{\sqrt{m_b}}, \frac{1}{\sqrt{m_c}}\right\}.$$

Remark 4.3 In what follows, for each iteration step $i \in \mathbb{N}$, the nonresonant conditions

$$\begin{aligned} &\big|\sqrt{\omega^2(m_b + m_c)l^2 - 2\eta_i} - \sqrt{EIj^4 + Qj^2}\big| \geq \frac{\gamma}{|l|^{\kappa+1}}, \\ &\big|\sqrt{\omega^2 m_b l^2 - \eta_i} - \sqrt{Q}j\big| \geq \frac{\gamma}{|l|^{\kappa+1}}, \qquad \big|\sqrt{\omega^2 m_c l^2 - \eta_i} - \sqrt{Q}j^2\big| \geq \frac{\gamma}{|l|^{\kappa+1}}, \end{aligned}$$

and

$$\begin{aligned} &\left|1 + K\left[\frac{1}{-\omega^2 m_b l + Qj^2 + \eta_i} + \frac{1}{-\omega^2 m_c l^2 + EIj^4} + \frac{\eta_i}{(-\omega^2 m_b l + Qj^2 + \eta_i)(-\omega^2 m_c l^2 + EIj^4)}\right]\right| \\ &\quad \geq \frac{\gamma}{|l|^{\kappa+1}} \end{aligned}$$

are also needed. But we find that if $\omega \in [\gamma, +\infty]\backslash\mathcal{D}_\gamma$, then, for $i \in \mathbb{N}$, $|l| < N_i$ and $\eta_i < e^{-N_i}$, one derives

$$\begin{aligned}
&\left|\sqrt{\omega^2(m_b+m_c)l^2 - 2\eta_i} - \sqrt{EIj^4+Qj^2}\right| \\
&\quad = \left|\omega\sqrt{m_b+m_c}l - \sqrt{EIj^4+Qj^2} + \sqrt{\omega^2(m_b+m_c)l^2 - 2\eta_i} - \omega\sqrt{m_b+m_c}l\right| \\
&\quad \geq \left|\omega\sqrt{m_b+m_c}l - \sqrt{EIj^4+Qj^2}\right| - \frac{2\eta_i}{\sqrt{\omega^2(m_b+m_c)l^2-2\eta_i} + \omega\sqrt{m_b+m_c}l} \\
&\quad \geq \frac{\gamma}{|l|^{\kappa+1}} - \frac{C(\gamma)\eta_i}{2|l|} \\
&\quad \geq \frac{\gamma}{2|l|^{\kappa+1}}.
\end{aligned}$$

In a similar manner, we obtain

$$\left|\sqrt{\omega^2 m_b l^2 - \eta_i} - \sqrt{Q}j\right| \geq \frac{\gamma}{|l|^{\kappa+1}}, \qquad \left|\sqrt{\omega^2 m_c l^2 - \eta_i} - \sqrt{Q}j^2\right| \geq \frac{\gamma}{|l|^{\kappa+1}}.$$

Finally, for $i \in \mathbb{N}$, $|l| < N_i$ and $\eta_i < e^{-N_i}$, we have

$$\begin{aligned}
&\left|1 + K\left[\frac{1}{-\omega^2 m_b l + Qj^2 + \eta_i} + \frac{1}{-\omega^2 m_c l^2 + EIj^4} + \frac{\eta_i}{(-\omega^2 m_b l + Qj^2 + \eta_i)(-\omega^2 m_c l^2 + EIj^4)}\right]\right| \\
&\quad \geq \left|1 + K\left(\frac{1}{-\omega^2 m_b l + Qj^2} + \frac{1}{-\omega^2 m_c l^2 + EIj^4}\right)\right| \\
&\qquad - K\left|\frac{1}{-\omega^2 m_b l + Qj^2 + \eta_i} - \frac{1}{-\omega^2 m_b l + Qj^2}\right| \\
&\qquad - \left|\frac{K\eta_i}{(-\omega^2 m_b l + Qj^2 + \eta_i)(-\omega^2 m_c l^2 + EIj^4)}\right| \\
&\quad \geq \left|1 + K\left(\frac{1}{2\omega\sqrt{m_b}l(\omega\sqrt{m_b}l - \sqrt{Q}j)} + \frac{1}{2\omega\sqrt{m_c}l(\omega\sqrt{m_c}l - \sqrt{EI}j^2)}\right)\right| \\
&\qquad - 2K\eta_i\gamma^2|l|^{2\kappa+2} \\
&\quad \geq \frac{\gamma}{2|l|^{\kappa+1}}.
\end{aligned}$$

Therefore, the nonresonant condition is sufficient to keep the operators $\mathcal{L}_\omega$, $\tilde{\mathcal{L}}_\omega$, $\mathcal{J}_\omega$, $\tilde{\mathcal{J}}_\omega$, Λ and $1 + K\Lambda^{-1}$ invertible in a bigger space.

Lemma 4.1 *Let $\omega \in \mathcal{X}(\nu)$ and $\bar{\sigma} > \tilde{\sigma} \geq 0$. Then the "diagonal" operators $\mathcal{L}_\omega$, $\mathcal{J}_\omega$, $\tilde{\mathcal{L}}_\omega$ and $\tilde{\mathcal{J}}_\omega$ satisfy the following*:

(1) *For any $(u, w) \in X_\sigma \times Y_\sigma$,*

$$\mathcal{L}_\omega u = \mathcal{L}_\omega\left(\sum_{(l,j)\in\mathbb{Z}^2} u_{l,j}\varphi_j(x)e^{ilt}\right) = \lambda_{l,j}u,$$

$$\mathcal{J}_\omega w = \mathcal{J}_\omega\left(\sum_{(l,j)\in\mathbb{Z}^2} w_{l,j}\psi_j(x)e^{ilt}\right) = \mu_{l,j}w,$$

$$\tilde{\mathcal{L}}_\omega u = \tilde{\mathcal{L}}_\omega\left(\sum_{(l,j)\in\mathbb{Z}^2} u_{l,j}\varphi_j(x)e^{ilt}\right) = \tilde{\lambda}_{l,j}u,$$

$$\tilde{\mathcal{J}}_\omega w = \tilde{\mathcal{J}}_\omega\left(\sum_{(l,j)\in\mathbb{Z}^2} w_{l,j}\psi_j(x)e^{ilt}\right) = \tilde{\mu}_{l,j}w,$$

where

$$\begin{aligned} &\lambda_{l,j} = -\omega^2 m_b l^2 + Qj^2, \qquad \mu_{l,j} = -\omega^2 m_c l^2 + EIj^4, \\ &\tilde{\lambda}_{l,j} = -\omega^2 m_b l^2 + Qj^2 + \eta_i, \qquad \tilde{\mu}_{l,j} = -\omega^2 m_c l^2 + EIj^4 + \eta_i. \end{aligned} \tag{4.18}$$

Operators $\mathcal{L}_\omega$, $\tilde{\mathcal{L}}_\omega$, $\mathcal{J}_\omega$ and $\tilde{\mathcal{J}}_\omega$ are invertible and map the spaces $X_{\bar{\sigma},s}$ and $Y_{\bar{\sigma},s}$ onto space $X_{\bar{\sigma},s}$ and $Y_{\bar{\sigma},s}$, respectively, and

$$\mathcal{L}_\omega^{-1} u = \mathcal{L}_\omega^{-1}\left(\sum_{(l,j)\in\mathbb{Z}^2} u_{l,j}\varphi_j(x)e^{ilt}\right) = \lambda_{l,j}^{-1}u,$$

$$\mathcal{J}_\omega^{-1} w = \mathcal{J}_\omega^{-1}\left(\sum_{(l,j)\in\mathbb{Z}^2} w_{l,j}\psi_j(x)e^{ilt}\right) = \mu_{l,j}^{-1}w,$$

$$\tilde{\mathcal{L}}_\omega^{-1} u = \tilde{\mathcal{L}}_\omega^{-1}\left(\sum_{(l,j)\in\mathbb{Z}^2} u_{l,j}\varphi_j(x)e^{ilt}\right) = \tilde{\lambda}_{l,j}^{-1}u,$$

$$\tilde{\mathcal{J}}_\omega^{-1} w = \tilde{\mathcal{J}}_\omega^{-1}\left(\sum_{(l,j)\in\mathbb{Z}^2} w_{l,j}\psi_j(x)e^{ilt}\right) = \tilde{\mu}_{l,j}^{-1}w,$$

$$\Lambda^{-1}u = \left(\frac{1}{\mathcal{J}_\omega} + \frac{1}{\tilde{\mathcal{L}}_\omega} + \frac{\eta_i}{\mathcal{J}_\omega\tilde{\mathcal{L}}_\omega}\right)\left(\sum_{(l,j)\in\mathbb{Z}^2} u_{l,j}\varphi_j(x)e^{ilt}\right) = \left(\mu_{l,j}^{-1} + \tilde{\lambda}_{l,j}^{-1} + \eta_i\mu_{l,j}^{-1}\tilde{\lambda}_{l,j}^{-1}\right)u,$$

where $\lambda_{l,j}$, $\mu_{l,j}$, $\tilde{\lambda}_{l,j}$ and $\tilde{\mu}_{l,j}$ are defined in (4.18).

(2) *Set*

$$\Sigma_1(\varpi) := \sup_{(l,j)\in\mathbb{Z}^2/\{0\}}\left(\left|\omega^2 m_b l^2 - Qj^2\right|^{-1}e^{-\varpi|l|}\right),$$

$$\Sigma_2(\varpi) := \sup_{(l,j)\in\mathbb{Z}^2/\{0\}}\left(\left|\omega^2 m_c l^2 - EIj^4\right|^{-1}e^{-\varpi|l|}\right),$$

$$\Sigma_3(\varpi) := \sup_{(l,j)\in\mathbb{Z}^2/\{0\}}\left(\left|\omega^2 m_b l^2 - Qj^2 + \eta_i\right|^{-1}e^{-\varpi|l|}\right),$$

$$\Sigma_4(\varpi) := \sup_{(l,j)\in\mathbb{Z}^2/\{0\}}\left(\left|\omega^2 m_c l^2 - EIj^4 + \eta_i\right|^{-1}e^{-\varpi|l|}\right),$$

$$\Sigma_5(\varpi) := \sup_{(l,j)\in\mathbb{Z}^2/\{0\}}\left(\eta_i\left|\omega^2 m_c l^2 - EIj^4\right|^{-1}\left|\omega^2 m_b l^2 - Qj^2 + \eta_i\right|^{-1}e^{-2\varpi|l|}\right).$$

Then we have

$$\begin{aligned} &\left\|\mathcal{L}_\omega^{-1}u\right\|_{\tilde{\sigma}} \le \Sigma_1(\bar{\sigma}-\tilde{\sigma})\|u\|_{\bar{\sigma}}, \qquad \left\|\mathcal{J}_\omega^{-1}w\right\|_{\tilde{\sigma}} \le \Sigma_2(\bar{\sigma}-\tilde{\sigma})\|w\|_{\bar{\sigma}}, \\ &\left\|\tilde{\mathcal{L}}_\omega^{-1}u\right\|_{\tilde{\sigma}} \le \Sigma_3(\bar{\sigma}-\tilde{\sigma})\|u\|_{\bar{\sigma}}, \qquad \left\|\tilde{\mathcal{J}}_\omega^{-1}w\right\|_{\tilde{\sigma}} \le \Sigma_4(\bar{\sigma}-\tilde{\sigma})\|w\|_{\bar{\sigma}}, \\ &\left\|\Lambda^{-1}u\right\|_{\tilde{\sigma}} \le \left(\Sigma_2(\bar{\sigma}-\tilde{\sigma}) + \Sigma_3(\bar{\sigma}-\tilde{\sigma}) + \Sigma_5(\bar{\sigma}-\tilde{\sigma})\right)\|u\|_{\bar{\sigma}}, \end{aligned}$$

and

$$\Sigma_1(\varpi), \Sigma_2(\varpi), \Sigma_3(\varpi), \Sigma_4(\varpi) \leq \frac{C\gamma}{\varpi^\kappa}\left(\frac{\kappa}{e}\right)^\kappa, \qquad \Sigma_5(\varpi) \leq \frac{C^2\gamma^2}{\varpi^{2\kappa+2}}\left(\frac{2\kappa+2}{e}\right)^{2\kappa+2},$$

where C is a positive constant.

Proof The property of the operators $\mathcal{L}_\omega$, $\mathcal{J}_\omega$, $\tilde{\mathcal{L}}_\omega$ and $\tilde{\mathcal{J}}_\omega$ is obvious. Now we verify the property of the operators $\mathcal{L}_\omega^{-1}$, $\mathcal{J}_\omega^{-1}$, $\tilde{\mathcal{L}}_\omega^{-1}$ and $\tilde{\mathcal{J}}_\omega^{-1}$. We have

$$\begin{aligned}\left\|\mathcal{L}_\omega^{-1}u\right\|_{\tilde{\sigma}} &= \sum_{l\in\mathbb{Z}}\left(\left|\omega^2 m_b l^2 - Qj^2\right|^{-1} e^{-(\bar{\sigma}-\tilde{\sigma})|l|}\right)|u_{l,j}|\left|\varphi_j(x)\right|e^{|l|\bar{\sigma}} \\ &\leq \Sigma_1(\bar{\sigma}-\tilde{\sigma})\|u\|_{\bar{\sigma}}.\end{aligned}$$

Since $\omega \in \mathcal{D}_\gamma$, we have

$$\begin{aligned}\left|\omega^2 m_b l^2 - Qj^2\right|^{-1} &= |\omega\sqrt{m_b}l - \sqrt{Q}j|^{-1}|\omega\sqrt{m_b}l + \sqrt{Q}j|^{-1} \\ &\leq \frac{|l|^{\kappa+1}}{\gamma}\frac{\nu|l|}{\omega} = |l|^\kappa\gamma.\end{aligned}$$

Then, from $\sup_{x>0}(x^y e^{-x}) = (\frac{y}{e})^y$, $\forall y \geq 0$, we obtain

$$\Sigma_1(\varpi) \leq \frac{\gamma}{\varpi^\kappa}\left(\frac{\kappa}{e}\right)^\kappa.$$

In a same manner, we can get the property of the operators $\mathcal{J}_\omega^{-1}$, $\tilde{\mathcal{L}}_\omega^{-1}$, $\tilde{\mathcal{J}}_\omega^{-1}$ and Λ^{-1}. □

To solve (4.13), introduce the function spaces

$$W_3^{(i)} := \left\{a = \sum_{1\leq|l|\leq N_i} a_{l,j}\phi_j(x)e^{ilt}\right\},$$

where $\{\phi_j(x) = \sin(jx)\}$ is the complete orthonormal system of the eigenfunctions of the operator Λ.

Lemma 4.2 *Let $\omega \in \mathcal{X}(\nu)$. Then, for a constant $K > 0$, equation*

$$\Lambda a_i + K(a_i)^+ + E_{i-1} = 0 \tag{4.19}$$

has a unique solution $a_i \in W_3^{(i)}$. Especially, equation

$$\Lambda a_i + K(a_i)^+ = 0$$

has a unique solution $a_i = 0$. Furthermore,

$$\|a_i\|_\sigma \leq C\Sigma_5(\bar{\sigma}-\sigma)\|E_{i-1}\|_{\bar{\sigma}}, \tag{4.20}$$

where $i \in \mathbb{N}$, $\bar{\sigma} > \sigma$, E_{i-1} is periodic in time t and does not depend on a_i.

Proof For convenience, we denote $a = a_i$ and $E = E_{i-1}$, $i \in \mathbb{N}$. From the definition of operator Λ in (4.10), $a = \sum_{1\leq|l|\leq N_i} a_{l,j}\phi_j(x)e^{ilt}$ and $E = \sum_{1\leq|l|\leq N_i} E_{l,j}\phi_j(x)e^{ilt}$, Eq. (4.19) can be written as

$$\begin{aligned} &\sum_{1\leq|l|\leq N_i} \frac{\tilde{\lambda}_{l,j}\mu_{l,j}}{\tilde{\lambda}_{l,j}+\tilde{\mu}_{l,j}} a_{l,j}\phi_j(x)e^{ilt} + K\left(\sum_{1\leq|l|\leq N_i} a_{l,j}\phi_j(x)e^{ilt}\right)^+ \\ &\quad + \sum_{1\leq|l|\leq N_i} E_{l,j}\phi_j(x)e^{ilt} = 0. \end{aligned} \tag{4.21}$$

Denote the domain $\Omega^+ := \{(x,t) \mid a = \sum_{1\leq|l|\leq N_i} a_{l,j}\phi_j(x)e^{ilt} \geq 0\}$. Then, comparing the coefficients of the above equation in Ω^+, we get

$$\left(\frac{\tilde{\lambda}_{l,j}\mu_{l,j}}{\tilde{\lambda}_{l,j}+\tilde{\mu}_{l,j}} + K\right)a_{l,j} = -E_{l,j},$$

which implies that

$$a_{l,j} = \frac{-\tilde{\lambda}_{l,j}-\tilde{\mu}_{l,j}}{\tilde{\lambda}_{l,j}\mu_{l,j} + K\tilde{\lambda}_{l,j} + K\tilde{\mu}_{l,j}} E_{l,j}. \tag{4.22}$$

Denote the domain $\Omega^- := \{(x,t) \mid a = \sum_{1\leq|l|\leq N_i} a_{l,j}\phi_j(x)e^{ilt} \leq 0\}$. Then, from (4.21), it follows

$$\sum_{1\leq|l|\leq N_i} \frac{\tilde{\lambda}_{l,j}\mu_{l,j}}{\tilde{\lambda}_{l,j}+\tilde{\mu}_{l,j}} a_{l,j}\phi_j(x)e^{ilt} + \sum_{1\leq|l|\leq N_i} E_{l,j}\phi_j(x)e^{ilt} = 0.$$

Comparing the coefficients of the above equation, one derives

$$a_{l,j} = \frac{-\tilde{\lambda}_{l,j}-\tilde{\mu}_{l,j}}{\tilde{\lambda}_{l,j}\mu_{l,j}} E_{l,j}. \tag{4.23}$$

Note that $\phi_j(x) = \sin(jx)$, $j \in \mathbb{Z}$. So, we have

$$\begin{aligned} a\left(t, \frac{\pi}{j}\right) &= \sum_{1\leq|l|\leq N_i} \frac{-\tilde{\lambda}_{l,j}-\tilde{\mu}_{l,j}}{\tilde{\lambda}_{l,j}\mu_{l,j} + K\tilde{\lambda}_{l,j} + K\tilde{\mu}_{l,j}} E_{l,j}\phi_j\left(\frac{\pi}{j}\right)e^{ilt} \\ &= \sum_{1\leq|l|\leq N_i} \frac{-\tilde{\lambda}_{l,j}-\tilde{\mu}_{l,j}}{\tilde{\lambda}_{l,j}\mu_{l,j}} E_{l,j}\phi_j\left(\frac{\pi}{j}\right)e^{ilt} = 0. \end{aligned} \tag{4.24}$$

Combining (4.22)–(4.23) with (4.24), there exists a unique solution $a_i \in W_3^{(i)}$. Here, the solution a_i is in $\mathbb{C}((0,\pi),\mathbb{R})$. Especially, when $E_{i-1} = 0$, there exists a unique zero solution.

Due to nonresonant condition and Remark 4.3, we have

$$\begin{aligned} \left|\frac{1}{\mu_{l,j}} + \frac{1}{\tilde{\lambda}_{l,j}} + \frac{\eta_i}{\tilde{\lambda}_{l,j}\mu_{l,j}}\right| &\leq \left|\frac{1}{\mu_{l,j}}\right| + \left|\frac{1}{\tilde{\lambda}_{l,j}}\right| + \eta_i\left|\frac{1}{\tilde{\lambda}_{l,j}\mu_{l,j}}\right| \\ &\leq |l|^{\kappa+1}\gamma\left(2+\eta_i|l|^{\kappa+1}\gamma\right). \end{aligned} \tag{4.25}$$

Furthermore, by (4.22)–(4.25), Remark 4.3 and $\sup_{x>0}(x^y e^{-x}) = (\frac{y}{e})^y$, one derives

$$
\begin{aligned}
\|a\|_\sigma &\le \max\left\{ \sum_{1\le|l|,|j|\le N_i} \left| \frac{\tilde{\lambda}_{l,j} + \tilde{\mu}_{l,j}}{\tilde{\lambda}_{l,j}\mu_{l,j} + K\tilde{\lambda}_{l,j} + K\tilde{\mu}_{l,j}} \right| |E_{l,j}| \left|\phi_j(x)\right| e^{|l|\sigma}, \right. \\
&\qquad \left. \sum_{1\le|l|,|j|\le N_i} \left| \frac{\tilde{\lambda}_{l,j} + \tilde{\mu}_{l,j}}{\tilde{\lambda}_{l,j}\mu_{l,j}} \right| |E_{l,j}| \left|\phi_j(x)\right| e^{|l|\sigma} \right\} \\
&\le \max\left\{ \sum_{1\le|l|,|j|\le N_i} \left| \frac{1}{\mu_{l,j}} + \frac{1}{\tilde{\lambda}_{l,j}} + \frac{\eta_i}{\tilde{\lambda}_{l,j}\mu_{l,j}} \right| \left| 1 + K\left(\frac{1}{\mu_{l,j}} + \frac{1}{\tilde{\lambda}_{l,j}} + \frac{\eta_i}{\tilde{\lambda}_{l,j}\mu_{l,j}} \right) \right| \right. \\
&\qquad \times |E_{l,j}| \left|\phi_j(x)\right| e^{|l|\sigma}, \\
&\qquad \left. \sum_{1\le|l|,|j|\le N_i} \left| \frac{1}{\mu_{l,j}} + \frac{1}{\tilde{\lambda}_{l,j}} + \frac{\eta_i}{\tilde{\lambda}_{l,j}\mu_{l,j}} \right| |E_{l,j}| \left|\phi_j(x)\right| e^{|l|\sigma} \right\} \\
&\le \max\left\{ \sum_{1\le|l|,|j|\le N_i} |l|^{2\kappa+2}\big(2 + \eta_i |l|^{\kappa+1}\gamma\big) |E_{l,j}| \left|\phi_j(x)\right| e^{|l|\sigma}, \right. \\
&\qquad \left. \gamma \sum_{1\le|l|,|j|\le N_i} |l|^{\kappa+1}\big(2 + \eta_i |l|^{\kappa+1}\gamma\big) |E_{l,j}| \left|\phi_j(x)\right| e^{|l|\sigma} \right\} \\
&\le \max\left\{ C\Sigma_5(\bar{\sigma}-\sigma)\|E\|_{\bar{\sigma}}, \gamma\Sigma_1(\bar{\sigma}-\sigma)\|E\|_{\bar{\sigma}} \right\} \\
&\le C\Sigma_5(\bar{\sigma}-\sigma)\|E\|_{\bar{\sigma}},
\end{aligned}
$$

where C is a constant, $\Sigma_1(\bar{\sigma}-\sigma)$ and $\Sigma_5(\bar{\sigma}-\sigma)$ are defined in Lemma 4.1. This completes the proof. □

Define

$$
\mathcal{M}(a, m_b g) = \Lambda a + K(a - f_2)^+ - m_b g - \epsilon \tilde{h}_2(x,t) = 0. \tag{4.26}
$$

Lemma 4.3 *Let $\omega \in \mathcal{X}(\nu)$. Then Eq. (4.26) possesses the first step approximation $a_1 \in W_3^{(1)}$ satisfying (4.20) for $i = 1$. For the range equation (4.1)–(4.2), we obtain the corresponding approximation solution*

$$
u_1 = \mathcal{L}_\omega^{-1}\left[K_0\big(a_0 + a_1 - f_2^{(0)}\big)^+ + (m_c g)_0 + (m_c g)_1 + \epsilon S_1 \tilde{h}_1\right] \in W_1^{(1)}, \tag{4.27}
$$

$$
w_1 = \mathcal{J}_\omega^{-1}\left[-K_0\big(a_0 + a_1 - f_2^{(0)}\big)^+ + (m_b g)_0 + (m_b g)_1 + \epsilon S_1 \tilde{h}_2\right] \in W_2^{(1)}, \tag{4.28}
$$

where

$$
\begin{aligned}
f_2^{(0)} &= \eta_0 \tilde{\mathcal{J}}_\omega A_3^{(0)} + \frac{(m_b g)_0 + (m_c g)_0}{\eta_0} + \epsilon \tilde{\mathcal{L}}_\omega^{-1}(\tilde{h}_1 + \tilde{h}_2), \\
A_1^{(0)} &= \tilde{\mathcal{J}}_\omega^{-1} u_0, \qquad A_2^{(0)} = \tilde{\mathcal{L}}_\omega^{-1} w_0, \qquad A_3^{(0)} = \tilde{\mathcal{L}}_\omega^{-1} A_1^{(0)} + \tilde{\mathcal{J}}_\omega^{-1} A_2^{(0)}, \\
E_0 &= \Lambda a_0 + K\big(a_0 - f_2^{(0)}\big)^+ - m_b g - \epsilon S_0 \tilde{h}_2(x,t).
\end{aligned}
$$

Proof Define

$$
\begin{aligned}
R_0 &= K\big(a_0 + a_1 - f_2^{(1)}\big)^+ - K\big(a_0 - f_2^{(0)}\big)^+ - K(a_1)^+ - (m_b g)_1 \\
&\quad + \epsilon(S_0 - S_1)\tilde{h}_2(x,t).
\end{aligned}
$$

Then we have

$$\begin{aligned}
&\mathcal{M}\big(a_0 + a_1, (m_b g)_0 + (m_b g)_1\big) \\
&\quad = \Lambda a_0 + \Lambda a_1 + K\big(a_0 + a_1 - f_2^{(1)}\big)^+ - (m_b g)_0 - (m_b g)_1 - \epsilon S_1 \tilde{h}_2(x,t) \\
&\quad = \Lambda a_0 + K\big(a_0 - f_2^{(0)}\big)^+ - (m_b g)_0 - \epsilon S_0 \tilde{h}_2(x,t) + \Lambda a_1 + K(a_1)^+ \\
&\qquad + K\big(a_0 + a_1 - f_2^{(1)}\big)^+ - K\big(a_0 - f_2^{(0)}\big)^+ - K(a_1)^+ - (m_b g)_1 + \epsilon(S_0 - S_1)\tilde{h}_2(x,t) \\
&\quad = E_0 + \Lambda a_1 + K(a_1)^+ + R_0.
\end{aligned}$$

On the basis of our approximation method, we need to solve the following equation:

$$\Lambda a_1 + K(a_1)^+ + E_0 = 0. \tag{4.29}$$

Lemma 4.2 shows that Eq. (4.29) has a unique solution $a_1 \in W_3^{(1)}$ which satisfies

$$\|a_1\|_\sigma \le C(\bar{\sigma} - \sigma)\|E_0\|_{\bar{\sigma}},$$

where $C(\bar{\sigma} - \sigma) = \Sigma_1(\bar{\sigma} - \sigma) + \Sigma_2(\bar{\sigma} - \sigma) + \Sigma_5(\bar{\sigma} - \sigma)$.

By (4.14)–(4.16), we obtain

$$\begin{aligned}
u_1 &= \mathcal{L}_\omega^{-1}\big[K\big(a_1 + a_0 - f_2^{(0)}\big)^+ + (m_c g)_0 + (m_c g)_1 + \epsilon S_1 \tilde{h}_1\big], \\
w_1 &= \mathcal{J}_\omega^{-1}\big[-K\big(a_1 + a_0 - f_2^{(0)}\big)^+ + (m_b g)_0 + (m_b g)_1 + \epsilon S_1 \tilde{h}_2\big].
\end{aligned}$$

This completes the proof. □

Using the same method in Lemma 4.3, the following result holds.

Lemma 4.4 *Let $\omega \in \mathcal{X}(\nu)$. Then (4.26) possesses the ith step approximation $a_i \in W_3^{(i)}$ satisfying (4.20). For the range equation (4.1)–(4.2), we obtain the corresponding approximation solution*

$$u_i = \mathcal{L}_\omega^{-1}\left[K\left(\sum_{k=0}^{i} a_k - f_2^{(i-1)}\right)^+ + \sum_{k=0}^{i}(m_c g)_k + \epsilon S_{i-1}\tilde{h}_1\right], \tag{4.30}$$

$$w_i = \mathcal{J}_\omega^{-1}\left[-K\left(\sum_{k=0}^{i} a_k - f_2^{(i-1)}\right)^+ + \sum_{k=0}^{i}(m_b g)_k + \epsilon S_{i-1}\tilde{h}_2\right], \tag{4.31}$$

where

$$f_2^{(i-1)} = \eta_{i-1}\tilde{\mathcal{J}}_\omega A_3^{(i-1)} + \frac{[(m_b g)_{i-1} + (m_c g)_{i-1}]}{\eta_{i-1}} + \epsilon \tilde{\mathcal{L}}_\omega^{-1} S_{i-1}(\tilde{h}_1 + \tilde{h}_2), \tag{4.32}$$

$$A_1^{(i-1)} = \tilde{\mathcal{J}}_\omega^{-1} u_{i-1}, \qquad A_2^{(i-1)} = \tilde{\mathcal{L}}_\omega^{-1} w_{i-1}, \qquad A_3^{(i-1)} = \tilde{\mathcal{L}}_\omega^{-1} A_1^{(i-1)} + \tilde{\mathcal{J}}_\omega^{-1} A_2^{(i-1)},$$

$$\begin{aligned}
R_{i-1} &= K\left(\sum_{k=0}^{i} a_k - f_2^{(i)}\right)^+ - K\left(\sum_{k=0}^{i-1} a_k - f_2^{(i-1)}\right)^+ - K(a_i)^+ + (m_b g)_i \\
&\quad + \epsilon(S_{i-1} - S_i)\tilde{h}_2,
\end{aligned} \tag{4.33}$$

$$E_i = \sum_{k=0}^{i} \Lambda a_k + K\left(\sum_{k=0}^{i} a_k - f_2^{(i)}\right)^+ - m_b g - \epsilon S_i \tilde{h}_2.$$

In order to prove the convergence of Nash–Moser algorithm, we need the following KAM estimates. For convenience, we choose the initial step $(u_0, w_0) = (0,0)$ and parameters $(m_b g)_0 = (m_c g)_0 = 0$. Then, by (4.11), it follows that $a_0 = 0$. Set

$$E_0 = K\big(-f_2^{(0)}\big)^+ - \epsilon S_0 \tilde{h}_2, \tag{4.34}$$

$$E_1 = R_0 = K\big(a_1 - f_2^{(1)}\big)^+ - K\big(-f_2^{(0)}\big)^+ - K(a_1)^+ - (m_b g)_1 + \epsilon(S_0 - S_1)\tilde{h}_2(x,t), \tag{4.35}$$

$$f_2^{(0)} = \epsilon \tilde{\mathcal{L}}_\omega^{-1} S_0(\tilde{h}_1 + \tilde{h}_2), \tag{4.36}$$

$$f_2^{(1)} = \eta_1 \tilde{\mathcal{J}}_\omega A_3^{(1)} + \frac{[(m_b g)_1 + (m_c g)_1]}{\eta_1} + \epsilon \tilde{\mathcal{L}}_\omega^{-1} S_1(\tilde{h}_1 + \tilde{h}_2), \tag{4.37}$$

$$A_1^{(1)} = \tilde{\mathcal{J}}_\omega^{-1} u_1, \qquad A_2^{(1)} = \tilde{\mathcal{L}}_\omega^{-1} w_1, \qquad A_3^{(1)} = \tilde{\mathcal{L}}_\omega^{-1} A_1^{(1)} + \tilde{\mathcal{J}}_\omega^{-1} A_2^{(1)}.$$

Lemma 4.5 (KAM estimates) *Let $\omega \in \mathcal{X}(\nu)$. Then, for any $0 < \alpha < \sigma$, the following estimates hold*:

$$\|E_0\|_\sigma \le K\epsilon C(\alpha) N_0^\alpha \big(\|\tilde{h}_1\|_\sigma + \|\tilde{h}_2\|_\sigma\big) + \epsilon N_0^\alpha \|\tilde{h}_2\|_\sigma,$$

$$\|a_1\|_{\sigma - \frac{\alpha}{3}} \le \left(2C\left(\frac{\alpha}{3}\right) + C'\left(\frac{\alpha}{3}\right)\right)\|E_0\|_\sigma,$$

$$\|u_1\|_{\sigma - \frac{2\alpha}{3}} \le C(K)\left(C\left(\frac{\alpha}{3}\right)C'\left(\frac{\alpha}{3}\right) + C\left(\frac{2\alpha}{3}\right)\right)\|E_0\|_\sigma,$$

$$\|w_1\|_{\sigma - \frac{2\alpha}{3}} \le C(K)\left(C\left(\frac{\alpha}{3}\right)C'\left(\frac{\alpha}{3}\right) + C\left(\frac{2\alpha}{3}\right)\right)\|E_0\|_\sigma,$$

$$\begin{aligned}\|E_1\|_{\sigma - \frac{\alpha}{3}} &\le C(\eta_0, K, \alpha)\|E_0\|_\sigma^2 + \epsilon C\left(\frac{\alpha}{3}\right) N_0^{-\frac{\alpha}{3}}\big(\|\tilde{h}_1\|_\sigma + \|\tilde{h}_2\|_\sigma\big) \\ &\quad + \epsilon N_0^{-\frac{2\alpha}{3}} \big\|\tilde{h}_2(x,t)\big\|_\sigma + (m_b g)_1,\end{aligned}$$

where $C(K)$ and $C(\eta_0, K, \alpha)$ are constants, $C(\alpha)$ and $C'(\alpha)$ are defined in (4.38)–(4.39).

Proof Denote

$$C(\alpha) = \frac{\gamma}{\alpha^\kappa}\left(\frac{\kappa}{e}\right)^\kappa = C(\kappa, \gamma)\alpha^{-\kappa}, \tag{4.38}$$

$$C'(\alpha) = \frac{C^2\gamma^2}{\alpha^{2\kappa+2}}\left(\frac{2\kappa+2}{e}\right)^{2\kappa+2} \le C'(\kappa, \gamma)\alpha^{-2\kappa-2}. \tag{4.39}$$

From the definition of E_0 in (4.34) and $f_2^{(0)}$ in (4.36), it follows that

$$\begin{aligned}\big\|f_2^{(0)}\big\|_\sigma &= \big\|\epsilon \tilde{\mathcal{L}}_\omega^{-1} S_0(\tilde{h}_1 + \tilde{h}_2)\big\|_\sigma \le \epsilon \big\|\tilde{\mathcal{L}}_\omega^{-1} S_0(\tilde{h}_1 + \tilde{h}_2)\big\|_\sigma \\ &\le \epsilon C(\alpha) N_0^\alpha \big(\|\tilde{h}_1\|_\sigma + \|\tilde{h}_2\|_\sigma\big)\end{aligned} \tag{4.40}$$

and

$$\begin{aligned}\|E_0\|_\sigma &= \big\|K\big(-f_2^{(0)}\big)^+ - \epsilon S_0\tilde{h}_2\big\|_\sigma \\ &\le K\big\|f_2^{(0)}\big\|_\sigma + \epsilon\|S_0\tilde{h}_2\|_\sigma \\ &\le K\epsilon C(\alpha)N_0^\alpha\big(\|\tilde{h}_1\|_\sigma + \|\tilde{h}_2\|_\sigma\big) + \epsilon N_0^\alpha\|\tilde{h}_2\|_\sigma.\end{aligned}\tag{4.41}$$

By (4.20), for $i = 1$,

$$\|a_1\|_{\sigma-\frac{\alpha}{3}} \le C'\left(\frac{\alpha}{3}\right)\|E_0\|_\sigma.\tag{4.42}$$

Then, from the property of operator Λ in Lemma 4.1, (4.40) and (4.42), it follows that

$$\begin{aligned}\|u_1\|_{\sigma-\frac{2\alpha}{3}} &= \big\|\mathcal{L}_\omega^{-1}\big[K\big(a_1 - f_2^{(0)}\big)^+ + \epsilon S_0\tilde{h}_1\big]\big\|_{\sigma-\frac{2\alpha}{3}} \\ &\le \big\|K\mathcal{L}_\omega^{-1}a_1 + \mathcal{L}_\omega^{-1}\big[\big(-f_2^{(0)}\big)^+ + \epsilon S_0\tilde{h}_1\big]\big\|_{\sigma-\frac{2\alpha}{3}} \\ &\le KC\left(\frac{\alpha}{3}\right)\|a_1\|_{\sigma-\frac{\alpha}{3}} + C\left(\frac{2\alpha}{3}\right)\big\|K\big(-f_2^{(0)}\big)^+ + \epsilon S_0\tilde{h}_1\big\|_\sigma \\ &\le C(K)\left(C\left(\frac{\alpha}{3}\right)C'\left(\frac{\alpha}{3}\right) + C\left(\frac{2\alpha}{3}\right)\right)\|E_0\|_\sigma\end{aligned}\tag{4.43}$$

and

$$\begin{aligned}\|w_1\|_{\sigma-\frac{2\alpha}{3}} &= \big\|\mathcal{L}_\omega^{-1}\big[K\big(a_1 - f_2^{(0)}\big)^+ - \epsilon S_0\tilde{h}_1\big]\big\|_{\sigma-\frac{2\alpha}{3}} \\ &\le \big\|K\mathcal{L}_\omega^{-1}a_1 + \mathcal{L}_\omega^{-1}\big[\big(-f_2^{(0)}\big)^+ - \epsilon S_0\tilde{h}_1\big]\big\|_{\sigma-\frac{2\alpha}{3}} \\ &\le KC\left(\frac{\alpha}{3}\right)\|a_1\|_{\sigma-\frac{\alpha}{3}} + C\left(\frac{2\alpha}{3}\right)\big\|K\big(-f_2^{(0)}\big)^+ - \epsilon S_0\tilde{h}_1\big\|_\sigma \\ &\le C(K)\left(C\left(\frac{\alpha}{3}\right)C'\left(\frac{\alpha}{3}\right) + C\left(\frac{2\alpha}{3}\right)\right)\|E_0\|_\sigma,\end{aligned}\tag{4.44}$$

where $C(K)$ is a constant depending on K.

Denote

$$\tilde{f}_3^{(1)} = \tilde{L}^{-1}(\eta_1 u_1 - \eta_0 u_0), \qquad \tilde{f}_4^{(1)} = \tilde{L}^{-1}(\eta_1 w_1 - \eta_0 w_0).$$

Then

$$f_2^{(1)} - f_2^{(0)} = \tilde{f}_3^{(1)} + \tilde{f}_4^{(1)} + \epsilon\tilde{L}_\omega^{-1}(S_1 - S_0)(\tilde{h}_1 + \tilde{h}_2).\tag{4.45}$$

By (4.43)–(4.44), we derive

$$\begin{aligned}\big\|\tilde{f}_3^{(1)}\big\|_{\sigma-\alpha} &\le \big\|\tilde{L}^{-1}(\eta_1 u_1 - \eta_0 u_0)\big\|_{\sigma-\alpha} \le \eta_1 C\left(\frac{\alpha}{3}\right)\|u_1\|_{\sigma-\frac{2\alpha}{3}} \\ &\le \eta_1 C(K)C\left(\frac{\alpha}{3}\right)\left(C\left(\frac{\alpha}{3}\right)C'\left(\frac{\alpha}{3}\right) + C\left(\frac{2\alpha}{3}\right)\right)\|E_0\|_\sigma \\ &= C_1(\eta_1, K, \alpha)\|E_0\|_\sigma\end{aligned}\tag{4.46}$$

and

$$\begin{aligned}\|\tilde{f}_4^{(1)}\|_{\sigma-\alpha} &\leq \|\tilde{L}^{-1}(\eta_1 w_1 - \eta_1 w_0)\|_{\sigma-\alpha} \leq \eta_1 C\left(\frac{\alpha}{3}\right)\|w_1\|_{\sigma-\alpha} \\ &\leq \eta_1 C(K) C\left(\frac{\alpha}{3}\right)\left(C\left(\frac{\alpha}{3}\right)C'\left(\frac{\alpha}{3}\right) + C\left(\frac{2\alpha}{3}\right)\right)\|E_0\|_\sigma \\ &= C_2(\eta_1, K, \alpha)\|E_0\|_\sigma, \end{aligned} \tag{4.47}$$

where $C_1(\eta_1, K, \alpha)$ and $C_2(\eta_1, K, \alpha)$ are constants.

By the definition of E_1 in (4.35) and (4.45), we derive

$$\begin{aligned}\|E_1\|_{\sigma-\alpha} &= \big\|K\big(a_1 - f_2^{(1)}\big)^+ - K\big(-f_2^{(0)}\big)^+ - K(a_1)^+ - (m_b g)_1 \\ &\quad + \epsilon(S_0 - S_1)\tilde{h}_2(x,t)\big\|_{\sigma-\alpha} \\ &\leq \big\|K\big(a_1 - f_2^{(1)} + f_2^{(0)}\big)^+ - K(a_1)^+ + K\big(-f_2^{(0)}\big)^+ - K\big(-f_2^{(0)}\big)^+\big\|_{\sigma-\alpha} \\ &\quad + (m_b g)_1 + \epsilon\big\|(S_0 - S_1)\tilde{h}_2(x,t)\big\|_{\sigma-\frac{2\alpha}{3}} \\ &\leq \big\|K\big(\tilde{f}_3^{(1)} + \tilde{f}_4^{(1)}\big)^+\big\|_{\sigma-\alpha} + \epsilon\big\|\tilde{L}_\omega^{-1}(S_0 - S_1)(\tilde{h}_1 + \tilde{h}_2)\big\|_{\sigma-\alpha} \\ &\quad + (m_b g)_1 + \epsilon\big\|(S_0 - S_1)\tilde{h}_2(x,t)\big\|_{\sigma-\alpha} \\ &\leq K\big[\big\|\tilde{f}_3^{(1)}\big\|_{\sigma-\alpha} + \big\|\tilde{f}_4^{(1)}\big\|_{\sigma-\alpha}\big] + \epsilon C\left(\frac{\alpha}{3}\right)N_0^{-\frac{2\alpha}{3}}\big(\|\tilde{h}_1\|_\sigma + \|\tilde{h}_2\|_\sigma\big) \\ &\quad + \epsilon N_0^{-\alpha}\big\|\tilde{h}_2(x,t)\big\|_\sigma + (m_b g)_1, \end{aligned}$$

which, together with (4.42), (4.46) and (4.47), yields

$$\begin{aligned}\|E_1\|_{\sigma-\alpha} &\leq C(\eta_0, K, \alpha)\|E_0\|_\sigma^2 + \epsilon C\left(\frac{\alpha}{3}\right)N_0^{-\frac{\alpha}{3}}\big(\|\tilde{h}_1\|_\sigma + \|\tilde{h}_2\|_\sigma\big) \\ &\quad + \epsilon N_0^{-\frac{2\alpha}{3}}\big\|\tilde{h}_2(x,t)\big\|_\sigma + (m_b g)_1, \end{aligned}$$

where $C(\eta_0, K, \alpha)$ is a constant. This completes the proof. □

4.2 Convergence of Nash–Moser algorithm, and local uniqueness

In the following, we will give a sufficient condition on the convergence of Newton algorithm. For $i \in \mathbb{N}$ and $0 < \bar{\sigma} < \sigma < \tilde{\sigma}$, set

$$\sigma_i := \bar{\sigma} + \frac{\sigma - \bar{\sigma}}{2^i}, \tag{4.48}$$

$$\alpha_{i+1} := \sigma_i - \sigma_{i+1} = \frac{\sigma - \bar{\sigma}}{2^{i+1}}. \tag{4.49}$$

By (4.48), we have

$$\sigma_0 > \sigma_1 > \cdots > \sigma_i > \sigma_{i+1} > \cdots, \quad i \in \mathbb{N}.$$

For the convergence of the Nash–Moser algorithm, we need to choose

$$\big[(m_b g)_i + (m_c g)_i\big] = e^{-\frac{N_{i+1}}{2^{i-1}}}, \qquad \eta_i = \big[(m_b g)_i + (m_c g)_i\big] = e^{-\frac{N_{i+1}}{2^{i-1}}}, \quad i \in \mathbb{N}. \tag{4.50}$$

Furthermore, for convenience, choose

$$(m_b g)_i = (m_c g)_i = \frac{1}{2} e^{-\frac{N_{i+1}}{2^{i-1}}}, \quad i \in \mathbb{N}. \tag{4.51}$$

Remark 4.4 The choice of η_i in (4.50) depends on making

$$\eta_{i-1} \|a_i\|_{\sigma_{i-2}} \leq \eta_{i-1} e^{\frac{N_i}{2^{i-1}} + \frac{N_i}{2^{i-2}}} \|a_i\|_{\sigma_i} \leq \|a_i\|_{\sigma_i}$$

holds. Moreover, to make $\lim_{i\to\infty} \frac{[(m_b g)_i + (m_c g)_i]}{\eta_i} = C$ (C is a constant), the choice of $(m_b g)_i + (m_c g)_i$ in (4.50) is determined by η_i. For convenience, the values of $(m_b g)_i$ and $(m_c g)_i$ are chosen in (4.51). It is important to prove the convergence of the Nash–Moser algorithm.

We assume there exist sufficiently small K and ϵ such that

$$\Xi \leq 2^{-24\kappa} C^{-2}(\kappa, \gamma, \sigma, \bar{\sigma}) \rho^{-1}, \tag{4.52}$$

$$\theta \geq 2\theta_1 > 2 \ln \Xi^{-1}, \tag{4.53}$$

where Ξ is to be defined in (4.75) and depends on K and ϵ.

Lemma 4.6 (Convergence of Nash–Moser algorithm) *Let* $\omega \in \mathcal{X}(\nu)$. *Assume that* (4.52)–(4.53) *holds. Then there exist* $\sum_{i=0}^{\infty} a_i$ *and* $\sum_{i=0}^{\infty} (m_b g)_i$ *such that*

$$\mathcal{M}\left(\sum_{i=0}^{\infty} a_i, \sum_{i=0}^{\infty} (m_b g)_i\right) = 0.$$

Proof The target is to prove that the convergence of the error E_i of the ith iterative step, i.e.,

$$\lim_{i\to\infty} \|E_i\|_{\sigma_i} = 0.$$

Denote

$$\tilde{f}_3^{(i)} = \tilde{L}_\omega^{-1}(\eta_i u_i - \eta_{i-1} u_{i-1}), \qquad \tilde{f}_4^{(i)} = \tilde{L}_\omega^{-1}(\eta_i w_i - \eta_{i-1} w_{i-1}). \tag{4.54}$$

Then, by (4.54), we have

$$f_2^{(i)} - f_2^{(i-1)} = \tilde{f}_3^{(i)} + \tilde{f}_4^{(i)} + \epsilon \mathcal{L}_\omega^{-1}(S_{i-1} - S_{i-2})(\tilde{h}_1 + \tilde{h}_2). \tag{4.55}$$

From (4.33) and (4.55), it follows

$$\begin{aligned} \|E_i\|_{\sigma_i} &= \|R_{i-1}\|_{\sigma_i} \\ &= \left\| K\left(\sum_{k=0}^{i} a_k - f_2^{(i)}\right)^+ - K\left(\sum_{k=0}^{i-1} a_k - f_2^{(i-1)}\right)^+ - K(a_i)^+ \right. \\ &\quad \left. + \epsilon(S_{i-1} - S_i)\tilde{h}_2 + (m_b g)_i \right\|_{\sigma_i} \end{aligned}$$

$$
\begin{aligned}
&\leq \left\| K\big(a_i + f_2^{(i-1)} - f_2^{(i)}\big)^+ - K(a_i)^+ + K\left(\sum_{k=0}^{i-1} a_k - f_2^{(i-1)}\right)^+ - K\left(\sum_{k=0}^{i-1} a_k - f_2^{(i-1)}\right)^+ \right. \\
&\quad \left. + \epsilon(S_{i-1} - S_i)\tilde{h}_2 \right\|_{\sigma_i} + (m_b g)_i \\
&\leq \big\| K\big(\tilde{f}_3^{(i)} + \tilde{f}_4^{(i-1)}\big)^+ \big\|_{\sigma_i} + K\epsilon C(\alpha_i) e^{-(\frac{\theta}{2})^i(\sigma-\bar{\sigma})}\big(\|\tilde{h}_1\|_{\sigma_i} + \|\tilde{h}_2\|_{\sigma_i}\big) \\
&\quad + \epsilon \big\|(S_{i-1} - S_i)\tilde{h}_2\big\|_{\sigma_i} + (m_b g)_i \\
&\leq K\big(\big\|\tilde{f}_3^{(i)}\big\|_{\sigma_i} + \big\|\tilde{f}_4^{(i)}\big\|_{\sigma_i}\big) + K\epsilon C(\alpha_i) N_i^{-\alpha_i}\big(\|\tilde{h}_1\|_{\sigma_i} + \|\tilde{h}_2\|_{\sigma_i}\big) \\
&\quad + \epsilon N_i^{-\alpha_i}\|\tilde{h}_2\|_{\sigma_i} + e^{-\theta_1^i} \\
&\leq K\big(\big\|\tilde{f}_3^{(i)}\big\|_{\sigma_i} + \big\|\tilde{f}_4^{(i)}\big\|_{\sigma_i}\big) + K\epsilon C(\alpha_i) e^{-\theta_1^i(\sigma-\bar{\sigma})}\big(\|\tilde{h}_1\|_{\sigma_i} + \|\tilde{h}_2\|_{\sigma_i}\big) \\
&\quad + \epsilon e^{-\theta_1^i(\sigma-\bar{\sigma})}\|\tilde{h}_2\|_{\sigma_i} + e^{-\theta_1^i}.
\end{aligned} \tag{4.56}
$$

In what follows, we need to estimate $\|\tilde{f}_3^{(i)}\|_{\sigma_i}$ and $\|\tilde{f}_4^{(i)}\|_{\sigma_i}$. From the definition of $\tilde{f}_3^{(i)}$ in (4.54) and η_i in (4.50), we derive

$$
\begin{aligned}
\big\|\tilde{f}_3^{(i)}\big\|_{\sigma_i} &= \big\|\tilde{L}_\omega^{-1}(\eta_i u_i - \eta_{i-1}u_{i-1})\big\|_{\sigma_i} \\
&\leq \eta_{i-1}\left\| \tilde{L}_\omega^{-1} L_\omega^{-1} K\left(\left(\sum_{k=0}^{i} a_k - f_2^{(i-1)}\right)^+ - \left(\sum_{k=0}^{i-1} a_k - f_2^{(i-2)}\right)^+\right)\right\|_{\sigma_i} \\
&\quad + \epsilon\eta_{i-1}\big\|\mathcal{L}_\omega^{-1}(S_{i-1} - S_{i-2})\tilde{h}_1\big\|_{\sigma_i} \\
&\leq \eta_{i-1} C(\alpha_i) C(\alpha_{i-1})\big\|K\big[a_i + f_2^{(i-2)} - f_2^{(i-1)}\big]^+\big\|_{\sigma_{i-2}} \\
&\quad + \epsilon\eta_{i-1}\big\|\mathcal{L}_\omega^{-1}(S_{i-1} - S_{i-2})\tilde{h}_1\big\|_{\sigma_i} \\
&\leq \frac{\eta_{i-1}}{2} C(\alpha_i) C(\alpha_{i-1})\|a_i\|_{\sigma_{i-2}}^2 + \frac{\eta_{i-1}}{2} C(\alpha_i) C(\alpha_{i-1})\big\|\tilde{f}_3^{(i-1)}\big\|_{\sigma_{i-2}}^2 \\
&\quad + \frac{\eta_{i-1}}{2} C(\alpha_i) C(\alpha_{i-1})\big\|\tilde{f}_4^{(i-1)}\big\|_{\sigma_{i-2}}^2 \\
&\quad + \epsilon K\eta_{i-1} C(\alpha_i) C(\alpha_{i-1}) C(\alpha_{i-2}) N_{i-2}^{-\alpha_{i-2}}\big(\|\tilde{h}_1\|_{\sigma_{i-2}} + \|\tilde{h}_2\|_{\sigma_{i-2}}\big) \\
&\quad + \epsilon\eta_{i-1} C(\alpha_i) N_{i-1}^{-\alpha_{i-1}}\|h_1\|_{\sigma_i} + 2\eta_{i-1} C(\alpha_i) C(\alpha_{i-1}) K^2 \\
&\leq \eta_{i-1} e^{\frac{2N_i}{2^i}} e^{\frac{2N_{i-1}}{2^{i-1}}} C(\alpha_i) C(\alpha_{i-1})\|a_i\|_{\sigma_i}^2 + \eta_{i-1} e^{\frac{2N_{i-1}}{2^{i-1}}} C(\alpha_i) C(\alpha_{i-1})\big\|\tilde{f}_3^{(i-1)}\big\|_{\sigma_{i-1}}^2 \\
&\quad + \eta_{i-1} e^{\frac{2N_{i-1}}{2^{i-1}}} C(\alpha_i) C(\alpha_{i-1})\big\|\tilde{f}_4^{(i-1)}\big\|_{\sigma_{i-1}}^2 \\
&\quad + \epsilon K\eta_{i-1} C(\alpha_i) C(\alpha_{i-1}) C(\alpha_{i-2}) N_{i-2}^{-\alpha_{i-2}}\big(\|\tilde{h}_1\|_{\sigma_{i-2}} + \|\tilde{h}_2\|_{\sigma_{i-2}}\big) \\
&\quad + \epsilon\eta_{i-1} C(\alpha_i) N_{i-1}^{-\alpha_{i-1}}\|\tilde{h}_1\|_{\sigma_i} + 2\eta_{i-1} C(\alpha_i) C(\alpha_{i-1}) K^2 \\
&\leq \|a_i\|_{\sigma_i}^2 + C(\alpha_i) C(\alpha_{i-1})\big\|\tilde{f}_3^{(i-1)}\big\|_{\sigma_{i-1}}^2 \\
&\quad + C(\alpha_i) C(\alpha_{i-1})\big\|\tilde{f}_4^{(i-1)}\big\|_{\sigma_{i-1}}^2 \\
&\quad + \epsilon K\eta_{i-1} C(\alpha_i) C(\alpha_{i-1}) C(\alpha_{i-2}) N_{i-2}^{-\alpha_{i-2}}\big(\|\tilde{h}_1\|_{\sigma_{i-2}} + \|\tilde{h}_2\|_{\sigma_{i-2}}\big) \\
&\quad + \epsilon\eta_{i-1} C(\alpha_i) N_{i-1}^{-\alpha_{i-1}}\|\tilde{h}_1\|_{\sigma_i} + 2\eta_{i-1} C(\alpha_i) C(\alpha_{i-1}) K^2.
\end{aligned} \tag{4.57}
$$

In a similar manner, we get

$$\begin{aligned}
\left\|\tilde{f}_4^{(i)}\right\|_{\sigma_i} &= \left\|\tilde{L}_\omega^{-1}(\eta_i w_i - \eta_{i-1}w_{i-1})\right\|_{\sigma_i} \\
&\leq \|a_i\|_{\sigma_i}^2 + C(\alpha_i)C(\alpha_{i-1})\left\|\tilde{f}_3^{(i-1)}\right\|_{\sigma_{i-1}}^2 \\
&\quad + C(\alpha_i)C(\alpha_{i-1})\left\|\tilde{f}_4^{(i-1)}\right\|_{\sigma_{i-1}}^2 \\
&\quad + \epsilon K\eta_{i-1}C(\alpha_i)C(\alpha_{i-1})C(\alpha_{i-2})N_{i-2}^{-\alpha_{i-2}}\left(\|\tilde{h}_1\|_{\sigma_{i-2}} + \|\tilde{h}_2\|_{\sigma_{i-2}}\right) \\
&\quad + \epsilon\eta_{i-1}C(\alpha_i)N_{i-1}^{-\alpha_{i-1}}\|\tilde{h}_2\|_{\sigma_i} + 2\eta_{i-1}C(\alpha_i)C(\alpha_{i-1})K^2.
\end{aligned} \tag{4.58}$$

Hence, by (4.57)–(4.58), it follows that

$$\begin{aligned}
&\left\|\tilde{f}_3^{(i)}\right\|_{\sigma_i} + \left\|\tilde{f}_4^{(i)}\right\|_{\sigma_i} \\
&\quad\leq 2\|a_i\|_{\sigma_i}^2 + 2C(\alpha_i)C(\alpha_{i-1})\left(\left\|\tilde{f}_3^{(i-1)}\right\|_{\sigma_{i-1}}^2 + \left\|\tilde{f}_4^{(i-1)}\right\|_{\sigma_{i-1}}^2\right) \\
&\qquad + 2\epsilon K\eta_{i-1}C(\alpha_i)C(\alpha_{i-1})C(\alpha_{i-2})N_{i-2}^{-\alpha_{i-2}}\left(\|\tilde{h}_1\|_{\sigma_{i-2}} + \|\tilde{h}_2\|_{\sigma_{i-2}}\right) \\
&\qquad + \epsilon\eta_{i-1}C(\alpha_i)N_{i-1}^{-\alpha_{i-1}}\left(\|\tilde{h}_1\|_{\sigma_i} + \|\tilde{h}_2\|_{\sigma_i}\right) + 4\eta_{i-1}C(\alpha_i)C(\alpha_{i-1})K^2.
\end{aligned} \tag{4.59}$$

Furthermore, by using the Young inequality, we get

$$\begin{aligned}
\left\|\tilde{f}_3^{(i-1)}\right\|_{\sigma_{i-1}}^2 &\leq 6C^2(\alpha_{i-1})C^2(\alpha_{i-2})\|a_{i-1}\|_{\sigma_{i-1}}^{2^2} + 6C^2(\alpha_{i-1})C^2(\alpha_{i-2})\left\|\tilde{f}_3^{(i-2)}\right\|_{\sigma_{i-2}}^{2^2} \\
&\quad + 6C^2(\alpha_{i-1})C^2(\alpha_{i-2})\left\|\tilde{f}_4^{(i-2)}\right\|_{\sigma_{i-2}}^{2^2} \\
&\quad + 6\epsilon^2K^2\eta_{i-2}^2C^2(\alpha_{i-1})C^2(\alpha_{i-2})C^2(\alpha_{i-3})N_{i-3}^{-2\alpha_{i-3}}\left(\|\tilde{h}_1\|_{\sigma_{i-3}} + \|\tilde{h}_2\|_{\sigma_{i-3}}\right)^2 \\
&\quad + 6\epsilon^2\eta_{i-2}^2C^2(\alpha_{i-1})N_{i-2}^{-2\alpha_{i-2}}\|\tilde{h}_1\|_{\sigma_{i-1}}^2 \\
&\quad + 6\times 2^2\eta_{i-2}^2C^2(\alpha_{i-1})C^2(\alpha_{i-2})K^{2^2}.
\end{aligned} \tag{4.60}$$

Also,

$$\begin{aligned}
\left\|\tilde{f}_3^{(i-1)}\right\|_{\sigma_{i-1}}^2 &\leq 6C^2(\alpha_{i-1})C^2(\alpha_{i-2})\|a_{i-1}\|_{\sigma_{i-1}}^{2^2} + 6C^2(\alpha_{i-1})C^2(\alpha_{i-2})\left\|\tilde{f}_3^{(i-2)}\right\|_{\sigma_{i-2}}^{2^2} \\
&\quad + 6C^2(\alpha_{i-1})C^2(\alpha_{i-2})\left\|\tilde{f}_4^{(i-2)}\right\|_{\sigma_{i-2}}^{2^2} \\
&\quad + 6\epsilon^2K^2\eta_{i-2}^2C^2(\alpha_{i-1})C^2(\alpha_{i-2})C^2(\alpha_{i-3})N_{i-3}^{-2\alpha_{i-3}}\left(\|\tilde{h}_1\|_{\sigma_{i-3}} + \|\tilde{h}_2\|_{\sigma_{i-3}}\right)^2 \\
&\quad + 6\epsilon^2\eta_{i-2}^2C^2(\alpha_{i-1})N_{i-2}^{-2\alpha_{i-2}}\|\tilde{h}_2\|_{\sigma_{i-1}}^2 \\
&\quad + 6\times 2^2\eta_{i-2}^2C^2(\alpha_{i-1})C^2(\alpha_{i-2})K^{2^2}.
\end{aligned} \tag{4.61}$$

This, combining (4.60)–(4.61), shows that

$$\begin{aligned}
&\left\|\tilde{f}_3^{(i-1)}\right\|_{\sigma_{i-1}}^2 + \left\|\tilde{f}_4^{(i-1)}\right\|_{\sigma_{i-1}}^2 \\
&\quad\leq 6\times 2C^2(\alpha_{i-1})C^2(\alpha_{i-2})\|a_{i-1}\|_{\sigma_{i-1}}^{2^2} \\
&\qquad + 6\times 2C^2(\alpha_{i-1})C^2(\alpha_{i-2})\left\|\tilde{f}_3^{(i-2)}\right\|_{\sigma_{i-2}}^{2^2}
\end{aligned}$$

$$
\begin{aligned}
&+ 6 \times 2C^2(\alpha_{i-1})C^2(\alpha_{i-2})\big\|\tilde{f}_4^{(i-2)}\big\|_{\sigma_{i-2}}^{2^2} \\
&+ 6 \times 2\epsilon^2 K^2 C^2(\alpha_{i-1})C^2(\alpha_{i-2})C^2(\alpha_{i-3})N_{i-3}^{-2\alpha_{i-3}}\big(\|\tilde{h}_1\|_{\sigma_{i-3}} + \|\tilde{h}_2\|_{\sigma_{i-3}}\big)^2 \\
&+ 6\epsilon^2\eta_{i-2}^2 C^2(\alpha_{i-1})N_{i-2}^{-2\alpha_{i-2}}\big(\|\tilde{h}_1\|_{\sigma_{i-1}}^2 + \|\tilde{h}_2\|_{\sigma_{i-1}}^2\big) \\
&+ 6 \times 2^{2^2-1}\eta_{i-2}^2 C^2(\alpha_{i-1})C^2(\alpha_{i-2})K^{2^2}.
\end{aligned}
\tag{4.62}
$$

Then, applying the Young inequality to (4.60)–(4.61), we have

$$
\begin{aligned}
&\big\|\tilde{f}_3^{(i-2)}\big\|_{\sigma_{i-2}}^{2^2} + \big\|\tilde{f}_4^{(i-2)}\big\|_{\sigma_{i-2}}^{2^2} \\
&\quad\le 6^{2^2-1} \times 2^{2^2-1}C^{2^2}(\alpha_{i-2})C^{2^2}(\alpha_{i-3})\|a_{i-2}\|_{\sigma_{i-2}}^{2^3} \\
&\qquad+ 6^{2^2-1} \times 2^{2^2-1}C^{2^2}(\alpha_{i-2})C^{2^2}(\alpha_{i-3})\big\|\tilde{f}_3^{(i-3)}\big\|_{\sigma_{i-3}}^{2^3} \\
&\qquad+ 6^{2^2-1} \times 2^{2^2-1}C^{2^2}(\alpha_{i-2})C^{2^2}(\alpha_{i-3})\big\|\tilde{f}_4^{(i-3)}\big\|_{\sigma_{i-3}}^{2^3} \\
&\qquad+ 6^{2^2-1} \times 2^{2^2-1}\epsilon^{2^2}K^{2^2}C^{2^2}(\alpha_{i-2})C^{2^2}(\alpha_{i-3})C^{2^2}(\alpha_{i-4})N_{i-4}^{-2^2\alpha_{i-4}} \\
&\qquad\times \big(\|\tilde{h}_1\|_{\sigma_{i-4}} + \|\tilde{h}_2\|_{\sigma_{i-4}}\big)^{2^2} \\
&\qquad+ 6^{2^2-1}\epsilon^{2^2}\eta_{i-3}^{2^2}C^{2^2}(\alpha_{i-2})N_{i-3}^{-2^2\alpha_{i-3}}\big(\|\tilde{h}_1\|_{\sigma_{i-2}}^{2^2} + \|\tilde{h}_2\|_{\sigma_{i-2}}^{2^2}\big) \\
&\qquad+ 6^{2^2-1} \times 2^{2^3-1}\eta_{i-3}^{2^2}C^{2^2}(\alpha_{i-2})C^{2^2}(\alpha_{i-3})K^{2^3}, \\
&\ldots, \\
&\big\|\tilde{f}_3^{(2)}\big\|_{\sigma_2}^{2^{i-2}} + \big\|\tilde{f}_4^{(2)}\big\|_{\sigma_2}^{2^{i-2}} \\
&\quad\le 6^{2^{i-2}-1} \times 2^{2^{i-2}-1}C^{2^{i-2}}(\alpha_2)C^{2^{i-2}}(\alpha_1)\|a_2\|_{\sigma_2}^{2^{i-1}} \\
&\qquad+ 6^{2^{i-2}-1} \times 2^{2^{i-2}-1}C^{2^{i-2}}(\alpha_2)C^{2^{i-2}}(\alpha_1)\big\|\tilde{f}_3^{(1)}\big\|_{\sigma_1}^{2^{i-1}} \\
&\qquad+ 6^{2^{i-2}-1} \times 2^{2^{i-2}-1}C^{2^{i-2}}(\alpha_2)C^{2^{i-2}}(\alpha_1)\big\|\tilde{f}_4^{(1)}\big\|_{\sigma_1}^{2^{i-1}} \\
&\qquad+ 6^{2^{i-2}-1} \times 2^{2^{i-2}-1}\epsilon^{2^{i-2}}K^{2^{i-2}}C^{2^{i-2}}(\alpha_2)C^{2^{i-2}}(\alpha_1)C^{2^{i-2}}(\alpha_0)N_0^{-2^{i-2}\alpha_0} \\
&\qquad\times \big(\|\tilde{h}_1\|_{\sigma_0} + \|\tilde{h}_2\|_{\sigma_0}\big)^{2^{i-2}} \\
&\qquad+ 6^{2^{i-2}-1}\epsilon^{2^{i-2}}C^{2^{i-2}}(\alpha_2)N_0^{-2^{i-2}\alpha_0}\big(\|\tilde{h}_1\|_{\sigma_2}^{2^{i-2}} + \|\tilde{h}_2\|_{\sigma_2}^{2^{i-2}}\big) \\
&\qquad+ 6^{2^{i-2}-1} \times 2^{2^{i-1}-1}\eta_0^{2^{i-2}}C^{2^{i-2}}(\alpha_2)C^{2^{i-2}}(\alpha_1)K^{2^{i-1}}.
\end{aligned}
$$

Iterating the above estimates one by one, we obtain

$$
\big\|\tilde{f}_3^{(i-1)}\big\|_{\sigma_{i-1}}^2 + \big\|\tilde{f}_4^{(i-1)}\big\|_{\sigma_{i-1}}^2 \le R_1 + R_2 + R_3 + R_4 + R_5,
\tag{4.63}
$$

where

$$
\begin{aligned}
R_1 &= \sum_{k=2}^{i-1}\|a_{i-k+1}\|_{\sigma_{i-k+1}}^{2^k}\prod_{j=2}^{k}12^{2^{j-1}}C^{2^{j-1}}(\alpha_{i-j+1})C^{2^{j-1}}(\alpha_{i-j}), \\
R_2 &= \big(\big\|\tilde{f}_3^{(1)}\big\|_{\sigma_1}^{2^{i-1}} + \big\|\tilde{f}_4^{(1)}\big\|_{\sigma_1}^{2^{i-1}}\big)\prod_{k=2}^{i-1}12^{2^{k-1}}C^{2^{k-1}}(\alpha_{i-k+1})C^{2^{k-1}}(\alpha_{i-k}),
\end{aligned}
$$

$$
\begin{aligned}
R_3 &= \sum_{k=2}^{i-2} (K\epsilon)^{2^{k+1}} \big(C(\alpha_{i-k+1})C(\alpha_{i-k})C(\alpha_{i-k-1})\big)^{2^k} N_{i-k-1}^{-2^k\alpha_{i-k-1}} \\
&\quad \times \big(\|\tilde{h}_1\|_{\sigma_{i-k-1}} + \|\tilde{h}_2\|_{\sigma_{i-k-1}}\big)^{2^k} \prod_{j=2}^{k} 12^{2^{j-1}} C^{2^{j-1}}(\alpha_{i-j+1}) C^{2^{j-1}}(\alpha_{i-j}), \\
R_4 &= \sum_{k=2}^{i-2} \epsilon^{2^k} C^{2^k}(\alpha_{i-k}) N_{i-k}^{-2^{k-1}\alpha_{i-k}} \big(\|\tilde{h}_1\|_{\sigma_{i-k}}^{2^k} + \|\tilde{h}_2\|_{\sigma_{i-k}}^{2^k}\big) \\
&\quad \times \prod_{j=2}^{k} 6^{2^{j-1}} C^{2^{j-1}}(\alpha_{i-j+1}) C^{2^{j-1}}(\alpha_{i-j}), \\
R_5 &= \sum_{k=2}^{i-1} \eta_{i-k-2}^{2^k} K^{2^k} \prod_{j=2}^{k} 12^{2^j} C^{2^{j-1}}(\alpha_{i-j+1}) C^{2^{j-1}}(\alpha_{i-j}).
\end{aligned}
$$

By (4.48) and (4.49), there exist positive constants $C(\kappa,\gamma,\sigma,\bar{\sigma})$ and $C_1(\kappa,\gamma,\sigma,\bar{\sigma})$, depending on κ, σ, $\bar{\sigma}$ such that

$$
C(\alpha_i) \le C(\kappa,\gamma,\sigma,\bar{\sigma}) 2^{i\kappa}, \qquad C_1(\alpha_i) \le C_1(\kappa,\gamma,\sigma,\bar{\sigma}) 2^{2i\kappa}, \tag{4.64}
$$

where $C(\alpha)$ and $C'(\alpha)$ are defined in (4.38)–(4.39), $C_1(\alpha) = 2C(\alpha) + C'(\alpha)$.

On the other hand, from $N_n = e^{\theta^n}$, $\theta > 2\theta_1$ and $k \ge 1$, it follows that

$$
N_{i-k-1}^{-2^k\alpha_{i-k-1}} = e^{-(\frac{\theta}{2})^{i-1}} e^{-(\frac{4}{\theta})^k(\sigma-\bar{\sigma})} \le e^{-\theta_1^{i-1}}, \tag{4.65}
$$

$$
N_{i-k}^{-2^{k-1}\alpha_{i-k}} = e^{-(\frac{\theta}{2})^{i}} e^{-2(\frac{4}{\theta})^k(\sigma-\bar{\sigma})} \le e^{-\theta_1^{i-1}}. \tag{4.66}
$$

Note that $N_{i-k-1} = e^{\theta^{i-k-1}} > \theta^{i-k-1}$. We have

$$
\eta_{i-k-2}^{2^k} = e^{-N_{i-k-1}2^{2k-i+3}} \le e^{-\theta_1^{i-1}}. \tag{4.67}
$$

So, by (4.64)–(4.67), we estimate R_1, R_2, R_3 and R_4, having

$$
\begin{aligned}
R_1 &= \sum_{k=2}^{i-1} \|a_{i-k+1}\|_{\sigma_{i-k+1}}^{2^k} \prod_{j=2}^{k} 12^{2^{j-1}} C^{2^{j-1}}(\alpha_{i-j+1}) C^{2^{j-1}}(\alpha_{i-j}) \\
&\le \sum_{k=2}^{i-1} 12^{2^k} \times 2^{[2^{k+1}(i-k+2)-8(i-1)]\kappa} C^{2^{k+1}}(\kappa,\gamma,\sigma,\bar{\sigma}) \|a_{i-k+1}\|_{\sigma_{i-k+1}}^{2^k} \\
&\le \sum_{k=2}^{i-1} \big(12C^2(\kappa,\gamma,\sigma,\bar{\sigma})\big)^{2^k} \times 2^{[2^{k+1}(i-k+2)-8i+8]\kappa} \|a_{i-k+1}\|_{\sigma_{i-k+1}}^{2^k},
\end{aligned} \tag{4.68}
$$

$$
\begin{aligned}
R_2 &\le 12^{2^{i-1}} \times 2^{(2^{i+2}-4i)\kappa} C^{2^i}(\kappa,\gamma,\sigma,\bar{\sigma}) \big(\big\|\tilde{f}_3^{(1)}\big\|_{\sigma_1}^{2^{i-1}} + \big\|\tilde{f}_4^{(1)}\big\|_{\sigma_1}^{2^{i-1}}\big) \\
&\le \big[12 \times 2^{8\kappa} C^2(\kappa,\gamma,\sigma,\bar{\sigma}) \big(\big\|\tilde{f}_3^{(1)}\big\|_{\sigma_1} + \big\|\tilde{f}_4^{(1)}\big\|_{\sigma_1}\big)\big]^{2^{i-1}},
\end{aligned} \tag{4.69}
$$

$$
\begin{aligned}
R_3 &\le e^{-\theta_1^{i-1}} \sum_{k=2}^{i-2} 2^{[2^k(5i-5k+2)-8i+8]\kappa} \big(12K\epsilon C^5\big)^{2^k} \big(\|\tilde{h}_1\|_{\sigma_{i-k-2}} + \|\tilde{h}_2\|_{\sigma_{i-k-2}}\big)^{2^{k+1}} \\
&\le e^{-\theta_1^{i-1}} \sum_{k=2}^{i-2} 2^{[2^k(5i-5k+2)-8i+8]\kappa} \big[12K\epsilon C^5 \big(\|\tilde{h}_1\|_{\sigma_0} + \|\tilde{h}_2\|_{\sigma_0}\big)^2\big]^{2^k},
\end{aligned} \tag{4.70}
$$

$$
\begin{aligned}
R_4 &\le e^{-\theta_1^{i-1}} \sum_{k=2}^{i-2} 2^{[2^k(5i-5k+2)-8i+8]\kappa} \bigl(6\epsilon C^2(\kappa,\gamma,\sigma,\bar{\sigma})\bigr)^{2^k} \bigl(\|\tilde{h}_1\|_{\sigma_{i-k}}^{2^k} + \|\tilde{h}_2\|_{\sigma_{i-k}}^{2^k}\bigr) \\
&\le e^{-\theta_1^{i-1}} \sum_{k=2}^{i-2} 2^{[2^k(5i-5k+2)-8i+8]\kappa} \bigl[6\epsilon C^2(\kappa,\gamma,\sigma,\bar{\sigma})\bigl(\|\tilde{h}_1\|_{\sigma_0} + \|\tilde{h}_2\|_{\sigma_0}\bigr)\bigr]^{2^k},
\end{aligned} \tag{4.71}
$$

and

$$
R_5 \le e^{-\theta_1^{i-1}} \sum_{k=2}^{i-1} \bigl(12KC^2(\kappa,\gamma,\sigma,\bar{\sigma})\bigr)^{2^k} \times 2^{[2^{k+1}(i-k+2)-8i+8]\kappa}. \tag{4.72}
$$

Inserting (4.59), (4.63), (4.68)–(4.72) into (4.56), then, by (4.20), the relation between $\|E_i\|_{\sigma_i}$ and $\|E_{i-1}\|_{\sigma_{i-1}}$ is

$$
\begin{aligned}
\|E_i\|_{\sigma_i} &\le 2KC_1^2(\alpha_i)\|E_{i-1}\|_{\sigma_{i-1}}^2 + K\sum_{k=2}^{i-1} \rho^{2^k} 2^{[2^{k+2}(i-k+2)-4i]\kappa} \|E_{i-k}\|_{\sigma_{i-k}}^{2^k} \\
&\quad + \Upsilon_i^{(1)} + \Upsilon_i^{(2)} + \Upsilon_i^{(3)} + \Upsilon_i^{(4)} + \Upsilon_i^{(5)} \\
&\le 2KC_1^2(\alpha_i)\|E_{i-1}\|_{\sigma_{i-1}}^2 + K\sum_{k=1}^{i-1} \rho^{2^k} 2^{[2^{k+2}(i-k+2)-4i]\kappa} \|E_{i-k}\|_{\sigma_{i-k}}^{2^{k+1}} \\
&\quad + \Upsilon_i^{(1)} + \Upsilon_i^{(2)} + \Upsilon_i^{(3)} + \Upsilon_i^{(4)} + \Upsilon_i^{(5)},
\end{aligned} \tag{4.73}
$$

where

$$
\begin{aligned}
\rho &= 12C^2(\kappa,\gamma,\sigma,\bar{\sigma})C_1^2(\kappa,\gamma,\sigma,\bar{\sigma}), \\
\Upsilon_i^{(1)} &= C^2(\kappa,\gamma,\sigma,\bar{\sigma})\bigl[12\times 2^{8\kappa}C^2(\kappa,\gamma,\sigma,\bar{\sigma})\bigl(\bigl\|\tilde{f}_3^{(1)}\bigr\|_{\sigma_1} + \bigl\|\tilde{f}_4^{(1)}\bigr\|_{\sigma_1}\bigr)\bigr]^{2^{i-1}}, \\
\Upsilon_i^{(2)} &= e^{-\theta_1^{i-1}} C^2(\kappa,\gamma,\sigma,\tilde{\sigma}) \sum_{k=2}^{i-2} 2^{[2^k(5i-5k+2)-4i]\kappa} \bigl[12K\epsilon C^5\bigl(\|\tilde{h}_1\|_{\sigma_0} + \|\tilde{h}_2\|_{\sigma_0}\bigr)^2\bigr]^{2^k}, \\
\Upsilon_i^{(3)} &= e^{-\theta_1^{i-1}} \sum_{k=2}^{i-2} 2^{[2^k(5i-5k+2)-4i]\kappa} \bigl[6\epsilon C^2(\kappa,\gamma,\sigma,\bar{\sigma})\bigl(\|\tilde{h}_1\|_{\sigma_0} + \|\tilde{h}_2\|_{\sigma_0}\bigr)\bigr]^{2^k}, \\
\Upsilon_i^{(4)} &= e^{-\theta_1^{i}} C^2(\kappa,\gamma,\sigma,\tilde{\sigma}) \sum_{k=1}^{i-1} \bigl(12KC^2(\kappa,\gamma,\sigma,\bar{\sigma})\bigr)^{2^k} \times 2^{[2^{k+1}(i-k+2)-4i]\kappa}, \\
\Upsilon_i^{(5)} &= 3K\epsilon C(\alpha_i)e^{-\theta_1^i(\sigma-\bar{\sigma})}\bigl(\|\tilde{h}_1\|_{\sigma_{i-2}} + \|\tilde{h}_2\|_{\sigma_{i-2}}\bigr) + \epsilon e^{-\theta_1^i(\sigma-\bar{\sigma})}\|\tilde{h}_2\|_{\sigma_i} \\
&\quad + 4e^{-\theta_1^i}C(\alpha_i)C(\alpha_{i-1})K^2 + e^{-\theta_1^i}.
\end{aligned}
$$

By (4.53), it follows that

$$
\lim_{i\to+\infty} e^{-\theta_1^i} \sum_{k=1}^{i} 2^{(i-k)2^k} = 0.
$$

Hence, for small K and ϵ, by (4.46) and assumptions (4.52)–(4.53), it is easy to check that

$$
\lim_{i\to\infty} \Upsilon_i^{(j)} = 0, \quad j = 1,2,3,4,5.
$$

In what follows, we will prove $\lim_{i\to\infty} E_i = 0$ by induction.

When $i = 2$, by (4.73), we have

$$\begin{aligned} \|E_2\|_{\sigma_2} &\le 2KC_1^2(\alpha_2)\|E_1\|_{\sigma_1}^2 + \rho^2 2^{2^4\kappa}\|E_1\|_{\sigma_1}^{2^2} + \Upsilon_2^{(1)} + \Upsilon_2^{(2)} + \Upsilon_2^{(3)} + \Upsilon_2^{(4)} + \Upsilon_2^{(5)}, \\ &\le \Xi^2 C_1^2(\alpha_2)\bigl(2Kb^2 + \rho^2 2^{2^4\kappa} C_1^{-2}(\alpha_2)\Xi^2 b^{2^2} + \Phi_2\bigr) \\ &\le \Xi^2 C_1^2(\alpha_2)\Theta_2, \end{aligned} \tag{4.74}$$

where

$$\begin{aligned} &b\Xi = \|E_1\|_{\sigma_1}, \\ &\Xi^2\Phi_2 = \Upsilon_2^{(1)} + \Upsilon_2^{(2)} + \Upsilon_2^{(3)} + \Upsilon_2^{(4)} + \Upsilon_2^{(5)}, \\ &\Theta_2 = \bigl(2Kb^2 + \rho^2 2^{2^4\kappa} C_1^{-4}(\alpha_2)\Xi^2 b^{2^2} + \Phi_2\bigr). \end{aligned} \tag{4.75}$$

By assumptions (4.52)–(4.53), for a small $b > 0$, it is easy to check that

$$\Theta_2 < 1.$$

When $i = 3$, by (4.73), we derive

$$\begin{aligned} \|E_3\|_{\sigma_3} &\le 2KC_1^2(\alpha_3)\|E_2\|_{\sigma_2}^2 + \rho^2 2^{2^5\kappa}\|E_2\|_{\sigma_2}^{2^2} + \rho^{2^2} 2^{2^6\kappa}\|E_1\|_{\sigma_1}^{2^3} \\ &\quad + \Upsilon_3^{(1)} + \Upsilon_3^{(2)} + \Upsilon_3^{(3)} + \Upsilon_3^{(4)} + \Upsilon_3^{(5)}, \\ &\le 2KC_1^2(\alpha_3)C_1^{2^2}(\alpha_2)\Xi^{2^2}\Theta_2^2 + \rho^2 2^{2^5\kappa} C_1^{2^3}(\alpha_2)\Xi^{2^3}\Theta_2^2 \\ &\quad + \rho^{2^2} 2^{2^6\kappa} b^{2^3}\Xi^{2^3} + \Upsilon_3^{(1)} + \Upsilon_3^{(2)} + \Upsilon_3^{(3)} + \Upsilon_3^{(4)} + \Upsilon_3^{(5)} \\ &\le \Xi^{2^2} C_1^2(\alpha_3) C_1^{2^2}(\alpha_2)\Theta_3, \end{aligned}$$

where

$$\begin{aligned} \Xi^2\Phi_3 &= \Upsilon_3^{(1)} + \Upsilon_3^{(2)} + \Upsilon_3^{(3)} + \Upsilon_3^{(4)} + \Upsilon_3^{(5)}, \\ \Theta_3 &= 2K\Theta_2^2 + \rho^2 2^{2^5\kappa} C_1^{-2}(\alpha_3) C_1^{2^2}(\alpha_2)\Xi^{2^2}\Theta_2^{2^2} \\ &\quad + C_1^{-2}(\alpha_3) C_1^{-2^2}(\alpha_2)\rho^{2^2} 2^{2^6\kappa} b^{2^3}\Xi^{2^2} + C_1^{-2}(\alpha_3) C_1^{-2^2}(\alpha_2)\Phi_3. \end{aligned}$$

By assumptions (4.52)–(4.53), for a small $b > 0$, it is easy to check

$$\Theta_3 < 1.$$

For $2 \le k \le i-1$, assume that the following estimate holds:

$$\|E_k\|_{\sigma_k} \le \Xi^{2^{k-1}} C_1^2(\alpha_k) C_1^{2^2}(\alpha_{k-1}) \cdots C_1^{2^{k-1}}(\alpha_2)\Theta_k,$$

where

$$\Xi^{2^{k-1}}\Phi_k = \Upsilon_k^{(1)} + \Upsilon_k^{(2)} + \Upsilon_k^{(3)} + \Upsilon_k^{(4)} + \Upsilon_k^{(5)},$$

$$\begin{aligned}
\Theta_k &= 2K\Theta_{k-1}^2 + \rho^2 2^{(2^4-4)i\kappa} C_1^{-2}(\alpha_k) C_1^{2^2}(\alpha_{k-1}) \Xi^{2^{k-1}} \Theta_{k-1}^{2^2} \\
&\quad + \rho^{2^2} 2^{(2^5(i-1)-4i)\kappa} C_1^{-2}(\alpha_k) C_1^{2^2}(\alpha_{k-1}) \cdots C_1^{2^{i-1}}(\alpha_2) \Xi^{2^{k-1}} \Theta_{i-2}^{2^3} \\
&\quad + \cdots \\
&\quad + C_1^{-2}(\alpha_k) C_1^{-2^2}(\alpha_{k-1}) \cdots C_1^{-2^{k-1}}(\alpha_2) \rho^{2^{k-1}} 2^{(2^{k+3}(i-k+1)-4i)\kappa} \Xi^{2^{k-1}} b^{2^k} \\
&\quad + C_1^{-2}(\alpha_k) C_1^{-2^2}(\alpha_{k-1}) \cdots C_1^{-2^{k-1}}(\alpha_2) \Xi^{-2^{k-2}} K^{2^k} \\
&\quad + C_1^{-2}(\alpha_i) C_1^{-2^2}(\alpha_{i-1}) \cdots C_1^{-2^{k-1}}(\alpha_2) \Phi_k,
\end{aligned}$$

and

$$\Theta_k < 1.$$

By (4.73), we get

$$\begin{aligned}
\|E_i\|_{\sigma_i} &\le C_1^2(\alpha_i) \|E_{i-1}\|_{\sigma_{i-1}}^2 + \sum_{k=1}^{i-1} \rho^{2^k} 2^{[2^{k+2}(i-k+2)-4i]\kappa} \|E_{i-k}\|_{\sigma_{i-k}}^{2^{k+1}} \\
&\quad + \Upsilon_i^{(1)} + \Upsilon_i^{(2)} + \Upsilon_i^{(3)} + \Upsilon_i^{(4)} \\
&\le C_1^2(\alpha_i) C_1^{2^2}(\alpha_{i-1}) \cdots C_1^{2^{i-1}}(\alpha_2) \Xi^{2^{i-1}} \Theta_{i-1}^2 \\
&\quad + \rho^2 2^{(2^4-4)i\kappa} C_1^{2^3}(\alpha_{i-1}) \cdots C_1^{2^i}(\alpha_2) \Xi^{2^i} \Theta_{i-1}^{2^2} \\
&\quad + \rho^{2^2} 2^{(2^5(i-1)-4i)\kappa} C_1^{2^4}(\alpha_{i-2}) \cdots C_1^{2^i}(\alpha_2) \Xi^{2^i} \Theta_{i-2}^{2^3} \\
&\quad + \cdots + \rho^{2^{i-1}} 2^{(2^{i+3}-4i)\kappa} b^{2^i} \Xi^{2^i} + \Upsilon_i^{(1)} + \Upsilon_i^{(2)} + \Upsilon_i^{(3)} + \Upsilon_i^{(4)} + \Upsilon_i^{(5)} \\
&\le C_1^2(\alpha_i) C_1^{2^2}(\alpha_{i-1}) \cdots C_1^{2^{i-1}}(\alpha_2) \Xi^{2^{i-1}} \Theta_i,
\end{aligned} \tag{4.76}$$

where

$$\begin{aligned}
&\Xi^{2^{i-1}} \Phi_i = \Upsilon_i^{(1)} + \Upsilon_i^{(2)} + \Upsilon_i^{(3)} + \Upsilon_i^{(4)} + \Upsilon_i^{(5)}, \\
&\Theta_i = 2K\Theta_{i-1}^2 + \rho^2 2^{(2^4-4)i\kappa} C_1^{-2}(\alpha_i) C_1^{2^2}(\alpha_{i-1}) \Xi^{2^{i-1}} \Theta_{i-1}^{2^2} \\
&\quad + \rho^{2^2} 2^{(2^5(i-1)-4i)\kappa} C_1^{-2}(\alpha_i) C_1^{2^2}(\alpha_{i-1}) \cdots C_1^{2^{i-1}}(\alpha_2) \Xi^{2^{i-1}} \Theta_{i-2}^{2^3} \\
&\quad + \cdots \\
&\quad + C_1^{-2}(\alpha_i) C_1^{-2^2}(\alpha_{i-1}) \cdots C_1^{-2^{i-1}}(\alpha_2) \rho^{2^{i-1}} 2^{(2^{i+3}-4i)\kappa} \Xi^{2^{i-1}} b^{2^i} \\
&\quad + C_1^{-2}(\alpha_i) C_1^{-2^2}(\alpha_{i-1}) \cdots C_1^{-2^{i-1}}(\alpha_2) \Phi_i.
\end{aligned}$$

Note that

$$C_1^2(\alpha_i) C_1^{2^2}(\alpha_{i-1}) \cdots C_1^{2^{i-1}}(\alpha_2) \le \left(2^{8\kappa} C_1^2(\kappa, \gamma, \sigma, \bar{\sigma})\right)^{2^{i-1}}.$$

Hence, by (4.76), it shows that

$$\|E_i\|_{\sigma_i} \le \left(2^{8\kappa} C_1^2(\kappa, \gamma, \sigma, \bar{\sigma}) \Xi\right)^{2^{i-1}} \Theta_i \tag{4.77}$$

and

$$\Theta_i \le 2K\Theta_{i-1}^2 + \rho^2 C_1^2(\kappa,\gamma,\sigma,\bar{\sigma})\Xi^{2^{i-1}}\Theta_{i-1}^{2^2} + \rho^{2^2}\bigl(2^{8\kappa}C_1(\kappa,\gamma,\sigma,\bar{\sigma})\Xi\bigr)^{2^{i-1}}\Theta_{i-2}^{2^3} + \cdots + \bigl(2^{24\kappa}C_1^2(\kappa,\gamma,\sigma,\bar{\sigma})\Xi\rho b^2\bigr)^{2^{i-1}} + \bigl(4\times 2^{8\kappa}C_1^2(\kappa,\gamma,\sigma,\bar{\sigma})\bigr)^{-2^{i-1}}\Phi_i.$$

By assumptions (4.52)–(4.53), for small $b > 0$, it is also easy to check

$$\Theta_i < \sum_{i=1}^{\infty} 2^{-i} = 1.$$

Hence, we have

$$0 \le \lim_{i\to\infty}\|E_i\|_{\sigma_i} \le \lim_{i\to\infty}\bigl(2^{8\kappa}C_1^2(\kappa,\gamma,\sigma,\bar{\sigma})\Xi\bigr)^{2^{i-1}}\Theta_i \to 0,$$

which implies that

$$\lim_{i\to\infty}\|E_i\|_{\sigma_i} = 0.$$

This means that Eq. (4.26) has a solution $a_\infty = \sum_{k=1}^{\infty} a_k$. This completes the proof. □

Lemma 4.7 (Existence of solution) *Let $\omega \in \mathcal{X}(\nu)$. Assume (4.52)–(4.53) hold. Then system (4.1)–(4.2) has a solution $(u_\infty, w_\infty) \in X_{\bar{\sigma}} \times Y_{\bar{\sigma}}$ to be defined in (4.82)–(4.83).*

Proof This result is to prove the existence of solution for system (4.1)–(4.2), i.e., $\lim_{j\to\infty}\|u_j\|_{\sigma_j}$ and $\lim_{j\to\infty}\|w_j\|_{\sigma_j}$ exist. Since the methods for the two are the same, we only prove the former.

From the definition of u_i in (4.30), for $j \ge i$, we have

$$\begin{aligned}
&\|u_i - u_{i-1}\|_{\sigma_{j+1}} \\
&\quad\le \|u_i - u_{i-1}\|_{\sigma_{i+1}} \\
&\quad\le \Biggl\| L_\omega^{-1} K\Biggl(\Biggl(\sum_{k=0}^{i} a_k - f_2^{(i-1)}\Biggr)^{+} - \Biggl(\sum_{k=0}^{i-1} a_k - f_2^{(i-2)}\Biggr)^{+}\Biggr)\Biggr\|_{\sigma_{i+1}} \\
&\qquad + \epsilon \bigl\|\mathcal{L}_\omega^{-1}(S_{i-1} - S_{i-2})\tilde{h}_1\bigr\|_{\sigma_{i+1}} \\
&\quad\le C(\alpha_{i+1})\Biggl\| K\bigl(a_i + f_2^{(i-2)} - f_2^{(i-1)}\bigr)^{+} + \Biggl(\sum_{k=0}^{i-1} a_k - f_2^{(i-2)}\Biggr)^{+} - \Biggl(\sum_{k=0}^{i-1} a_k - f_2^{(i-2)}\Biggr)^{+}\Biggr\|_{\sigma_i} \\
&\qquad + \epsilon \bigl\|\mathcal{L}_\omega^{-1}(S_{i-1} - S_{i-2})\tilde{h}_1\bigr\|_{\sigma_{i+1}} \\
&\quad= C(\alpha_{i+1})\bigl\|K\bigl(a_i + f_2^{(i-2)} - f_2^{(i-1)}\bigr)^{+}\bigr\|_{\sigma_i} + \epsilon\bigl\|\mathcal{L}_\omega^{-1}(S_{i-1} - S_{i-2})\tilde{h}_1\bigr\|_{\sigma_{i+1}} \\
&\quad\le C(\alpha_{i+1})\bigl(\|a_i\|_{\sigma_i} + \bigl\|\tilde{f}_3^{(i-1)}\bigr\|_{\sigma_{i-1}} + \bigl\|\tilde{f}_4^{(i-1)}\bigr\|_{\sigma_{i-1}}\bigr) + K\epsilon C(\alpha_i)C(\alpha_{i-1})N_{i-1}^{-\alpha_{i-1}} \\
&\qquad \times \bigl(\|\tilde{h}_1\|_{\sigma_{i-1}} + \|\tilde{h}_2\|_{\sigma_{i-1}}\bigr) + \epsilon C(\alpha_i)N_{i-1}^{-\alpha_{i-1}}\|h_1\|_{\sigma_i}.
\end{aligned} \tag{4.78}$$

In what follows, we estimate the term $\|\tilde{f}_3^{(i-1)}\|_{\sigma_{i-1}}$ and $\|\tilde{f}_4^{(i-1)}\|_{\sigma_{i-1}}$. By (4.59), (4.63), (4.68)–(4.71), it follows that

$$\begin{aligned}
&\big\|\tilde{f}_3^{(i-1)}\big\|_{\sigma_{i-1}} + \big\|\tilde{f}_4^{(i-1)}\big\|_{\sigma_{i-1}} \\
&\quad\le 2\|a_{i-1}\|_{\sigma_{i-1}}^2 + 2C(\kappa,\gamma,\sigma,\bar{\sigma})\sum_{k=2}^{i-2}\rho^{2^k}2^{[2^{k+1}(i-k+1)-4i+4]}\|a_{i-k}\|_{\sigma_{i-k}}^{2^k} \\
&\qquad + \Upsilon_{i-1}^{(1)} + \Upsilon_{i-1}^{(2)} + \Upsilon_{i-1}^{(3)} + \Upsilon_{i-1}^{(4)} + R_6,
\end{aligned} \tag{4.79}$$

where

$$\begin{aligned}
\Upsilon_{i-1}^{(1)} &= C^2(\kappa,\gamma,\sigma,\bar{\sigma})\big[12\times 2^{8\kappa}C^2(\kappa,\gamma,\sigma,\bar{\sigma})\big(\big\|\tilde{f}_3^{(1)}\big\|_{\sigma_1} + \big\|\tilde{f}_4^{(1)}\big\|_{\sigma_1}\big)\big]^{2^{i-2}}, \\
\Upsilon_{i-1}^{(2)} &= e^{-\theta_1^{i-2}}C^2(\kappa,\gamma,\sigma,\tilde{\sigma})\sum_{k=2}^{i-3}2^{[2^k(5i-5k-3)-4i+4]\kappa}\big[12K\epsilon C^5\big(\|\tilde{h}_1\|_{\sigma_0} + \|\tilde{h}_2\|_{\sigma_0}\big)^2\big]^{2^k}, \\
\Upsilon_{i-1}^{(3)} &= e^{-\theta_1^{i-2}}\sum_{k=2}^{i-3}2^{[2^k(5i-5k-3)-4i+4]\kappa}\big[6\epsilon C^2(\kappa,\gamma,\sigma,\bar{\sigma})\big(\|\tilde{h}_1\|_{\sigma_0} + \|\tilde{h}_2\|_{\sigma_0}\big)\big]^{2^k}, \\
\Upsilon_{i-1}^{(4)} &= e^{-\theta_1^{i-1}}C^2(\kappa,\gamma,\sigma,\tilde{\sigma})\sum_{k=1}^{i-2}\big(12KC^2(\kappa,\gamma,\sigma,\bar{\sigma})\big)^{2^k}\times 2^{[2^{k+1}(i-k+1)-4i+4]\kappa}, \\
R_6 &= 2\epsilon K\eta_{i-2}C(\alpha_{i-1})C(\alpha_{i-2})C(\alpha_{i-3})N_{i-3}^{-\alpha_{i-3}}\big(\|\tilde{h}_1\|_{\sigma_{i-3}} + \|\tilde{h}_2\|_{\sigma_{i-3}}\big) \\
&\quad + \epsilon\eta_{i-2}C(\alpha_{i-1})N_{i-2}^{-\alpha_{i-2}}\big(\|\tilde{h}_1\|_{\sigma_{i-1}} + \|\tilde{h}_2\|_{\sigma_{i-1}}\big) + 4\eta_{i-2}C(\alpha_{i-1})C(\alpha_{i-2})K^2.
\end{aligned}$$

Combining (4.78) with (4.79), we get

$$\begin{aligned}
&\|u_i - u_{i-1}\|_{\sigma_{j+1}} \\
&\quad\le C(\alpha_{i+1})C_1(\alpha_i)\|E_{i-1}\|_{\sigma_{i-1}} + 2C(\alpha_{i+1})C(\alpha_{i-1})\|E_{i-2}\|_{\sigma_{i-2}}^2 \\
&\qquad + 2C^2(\kappa,\gamma,\sigma,\bar{\sigma})\sum_{k=2}^{i-2}\big(C(\kappa,\gamma,\sigma,\bar{\sigma})\rho\big)^{2^k}2^{2^{k+2}(i-k+1)-2i}\|E_{i-k-1}\|_{\sigma_{i-k-1}}^{2^k} \\
&\qquad + C(\alpha_{i+1})\big(\Upsilon_{i-1}^{(1)} + \Upsilon_{i-1}^{(2)} + \Upsilon_{i-1}^{(3)} + \Upsilon_{i-1}^{(4)} + R_6\big) + R_7,
\end{aligned} \tag{4.80}$$

where

$$\begin{aligned}
\rho &= 12C^2(\kappa,\gamma,\sigma,\bar{\sigma})C_1^2(\kappa,\gamma,\sigma,\bar{\sigma}), \\
R_7 &= K\epsilon C^2(\kappa,\gamma,\sigma,\bar{\sigma})2^{(2i-1)\kappa}e^{-\theta_1^{i-1}(\sigma-\bar{\sigma})}(\|\tilde{h}_1\|_{\sigma_{i-1}} + \|\tilde{h}_2\|_{\sigma_{i-1}}) \\
&\quad + \epsilon C(\kappa,\gamma,\sigma,\bar{\sigma})2^{(i)\kappa}e^{-\theta_1^{i-1}(\sigma-\bar{\sigma})}\|h_1\|_{\sigma_i}.
\end{aligned}$$

By (4.77), for $\forall 1\le k\le i$, it follows that

$$\|E_{i-k}\|_{\sigma_{i-k}} \le \big(2^{8\kappa}C_1^2(\kappa,\gamma,\sigma,\bar{\sigma})\Xi\big)^{2^{i-k-1}}\Theta_{i-k}, \tag{4.81}$$

where

$$\Theta_i \le 2K\Theta_{i-1}^2 + \rho^2 C_1^2(\kappa,\gamma,\sigma,\bar{\sigma})\Xi^{2^{i-1}}\Theta_{i-1}^{2^2} + \rho^{2^2}\bigl(2^{8\kappa}C_1(\kappa,\gamma,\sigma,\bar{\sigma})\Xi\bigr)^{2^{i-1}}\Theta_{i-2}^{2^3}$$
$$+ \cdots + \bigl(2^{24\kappa}C_1^2(\kappa,\gamma,\sigma,\bar{\sigma})\Xi\rho b^2\bigr)^{2^{i-1}} + \bigl(4\times 2^{8\kappa}C_1^2(\kappa,\gamma,\sigma,\bar{\sigma})\bigr)^{-2^{i-1}}\Phi_i$$
$$< 1.$$

Inserting (4.81) into (4.80), we derive

$$\begin{aligned}\|u_i - u_{i-1}\|_{\sigma_{j+1}} \le{}& C(\alpha_{i+1})C_1(\alpha_i)\bigl(2^{8\kappa}C_1^2(\kappa,\gamma,\sigma,\bar{\sigma})\Xi\bigr)^{2^{i-2}}\Theta_{i-1}\\ &+ 2C(\alpha_{i+1})C(\alpha_{i-1})\bigl(2^{8\kappa}C_1^2(\kappa,\gamma,\sigma,\bar{\sigma})\Xi\bigr)^{2^{i-2}}\Theta_{i-2}^2\\ &+ 2C^2(\kappa,\gamma,\sigma,\bar{\sigma})\bigl(2^{8\kappa}C_1^2(\kappa,\gamma,\sigma,\bar{\sigma})\Xi\bigr)^{2^{i-1}}\\ &\times\sum_{k=2}^{i-2}\bigl(C(\kappa,\gamma,\sigma,\bar{\sigma})\rho\bigr)^{2^k}2^{2^{k+2}(i-k+1)-2i}\Theta_{i-2}^{2^k}\\ &+ C(\alpha_{i+1})\bigl(\Upsilon_{i-1}^{(1)} + \Upsilon_{i-1}^{(2)} + \Upsilon_{i-1}^{(3)} + \Upsilon_{i-1}^{(4)} + R_6\bigr) + R_7.\end{aligned}$$

By (4.52)–(4.53), the above estimate implies that $i \to \infty$ as $j \to \infty$. Then

$$\|u_i - u_{i-1}\|_{\sigma_j} \to 0.$$

Therefore, by the relation $\|u_j\|_{\sigma_j} = \|u_0 + \sum_{i=1}^{j}(u_i - u_{i-1})\|_{\sigma_j} \le \sum_{i=1}^{j}\|u_i - u_{i-1}\|_{\sigma_i}$, we find that $\lim_{j\to\infty}\|u_j\|_{\sigma_j}$ exists.

Thus, the solution of system (4.1)–(4.2) is

$$\tilde{u}_\infty = \mathcal{L}_\omega^{-1}\Biggl[K\Biggl(\sum_{k=1}^{\infty}a_k - f_2^{(\infty)}\Biggr)^+ + \sum_{k=1}^{\infty}(m_b g)_k + \epsilon\tilde{h}_1\Biggr], \tag{4.82}$$

$$\tilde{w}_\infty = \mathcal{J}_\omega^{-1}\Biggl[-K\Biggl(\sum_{k=1}^{\infty}a_k - f_2^{(\infty)}\Biggr)^+ + \sum_{k=1}^{\infty}(m_c g)_k + \epsilon\tilde{h}_2\Biggr], \tag{4.83}$$

where

$$f_2^{(\infty)} = \frac{K}{2} + \epsilon\tilde{L}_\omega(\tilde{h}_1 + \tilde{h}_2). \tag{4.84}$$

□

Lemma 4.8 (Local uniqueness of solution) *System* (4.1)–(4.2) *has a unique solution* $(u_\infty, w_\infty) \in X_{\bar{\sigma}} \times Y_{\bar{\sigma}}$, *which is obtained in* (4.82)–(4.83).

Proof Suppose that there exist two solutions (u,w) and (u',w') of system (4.1)–(4.2). Then we denote by $\bar{a}$ and $\tilde{a}$ the corresponding solutions of Eq. (4.26), respectively. Let $(h^{(1)},h^{(2)}) = (u-u', w-w')$ and $h^{(3)} = \bar{a} - \tilde{a}$. Then, by (4.26), we have

$$\begin{aligned}&\mathcal{M}\bigl(h^{(3)},(m_b g)-(m_b g)'\bigr)\\ &\quad= \Lambda h^{(3)} + K\bigl(h^{(3)}\bigr)^+ + K(\bar{a}-f_2)^+ - K(\tilde{a}-f_2)^+ - K\bigl(h^{(3)}\bigr)^+ - (m_b g) + (m_b g)'\\ &\quad= \Psi_1\bigl(h^{(3)}\bigr) + \Psi_2\bigl(h^{(3)}\bigr) = 0,\end{aligned} \tag{4.85}$$

where f_2 is defined in (4.84), and

$$\Psi_1\big(h^{(3)}\big) = \Lambda h^{(3)} + K\big(h^{(3)}\big)^+,$$
$$\Psi_2\big(h^{(3)}\big) = K(\bar{a} - f_2)^+ - K(\tilde{a} - f_2)^+ - K\big(h^{(3)}\big)^+ - (m_b g) + (m_b g)'.$$

Take $(m_b g) = (m_b g)'$. Then, by the definition of $\Psi_2(h^{(3)})$, we derive

$$\begin{aligned}\big\|\Psi_2\big(h^{(3)}\big)\big\|_0 &= \big\|K(\bar{a} - f_2)^+ - K(\tilde{a} - f_2)^+ - K\big(h^{(3)}\big)^+ - (m_b g) + (m_b g)'\big\|_0 \\ &\le \big\|K\big(h^{(3)}\big)^+ + K(\tilde{a} - f_2)^+ - K(\tilde{a} - f_2)^+ - K\big(h^{(3)}\big)^+\big\|_0 \\ &\le 0,\end{aligned}$$

which means $\Psi_2(h^{(3)}) = 0$. Then, from (4.85), we get $\Psi_1(h^{(3)}) = \Lambda h^{(3)} + K(h^{(3)})^+ = 0$. Therefore, by Lemma 4.2, $\bar{a} = \tilde{a}$. Then, from the definition (u, w) and (u', w') in (4.82)–(4.83), we obtain $u = u'$ and $w = w'$. This completes the proof. □

Appendix

The following result shows the regular dependence on the parameter ω.

Lemma A.1 *Let $\omega \in \mathcal{X}(\nu)$. Assume (4.52)–(4.53) hold. Then all the maps*

$$r_i^{(1)} := u_i - u_{i-1} : \mathcal{D}_\gamma \to W_{1\sigma}^{(i)},$$
$$r_i^{(2)} := w_i - w_{i-1} : \mathcal{D}_\gamma \to W_{2\sigma}^{(i)},$$

are differentiable w.r.t. ω.

Proof Based on the classical implicit function theorem, Lemma 4.6 and Lemma 4.8, it is easy to prove this result. So we omit it. □

Lemma A.2 (Measure estimate) *There exists a constant $C(Q, EI)$ such that*

$$\frac{|\mathcal{X}(\nu) \cap (\omega_1, \omega_2)|}{\omega_2 - \omega_1} \ge 1 - C(Q, EI)\gamma, \quad \forall \nu \in [\nu_1, \nu_2].$$

Furthermore,

$$\frac{|\mathcal{D}_\gamma \cap \mathcal{Y}|}{|\mathcal{Y}|} \ge 1 - C(Q, EI)\gamma,$$

where $\forall \mathcal{Y} = (\nu_1, \nu_2) \times (\omega_1, \omega_2) \subset \{(\nu, \omega) \in [\nu', \nu''] \times [\gamma, \infty) : \frac{\nu}{\omega} < C\gamma^2\}$.

Proof Define

$$\Omega_{l,j} = \bigcup_{k=0}^{3} \bigcup_{l,j \ge 1} \bar{\Omega}_{l,j}^{(k)},$$

where

$$\bar{\Omega}_{l,j}^{(0)} = \left\{\omega : \left|\omega\sqrt{m_b+m_c}l - \sqrt{EIj^4+Qj^2}\right| \le \frac{\gamma}{|l|^{\kappa+1}}\right\},$$
$$\bar{\Omega}_{l,j}^{(1)} = \left\{\omega : \left|\omega\sqrt{m_c}l - \sqrt{Q}j^2\right| \le \frac{\gamma}{|l|^{\kappa+1}}\right\},$$
$$\bar{\Omega}_{l,j}^{(2)} = \left\{\omega : |\omega\sqrt{m_b}l - \sqrt{Q}j| \le \frac{\gamma}{|l|^{\kappa+1}}\right\},$$
$$\bar{\Omega}_{l,j}^{(3)} = \left\{\omega : \left|1 + K\left(\frac{1}{2\omega\sqrt{m_b}l(\omega\sqrt{m_b}l - \sqrt{Q}j)} + \frac{1}{2\omega\sqrt{m_c}l(\omega\sqrt{m_c}l - \sqrt{EI}j^2)}\right)\right| \le \frac{\gamma}{|l|^{\kappa+1}}\right\}.$$

We now prove the set $\Omega_{l,j}^{(k)}$ being a small measure. Because the method of the measure estimate of sets $\Omega_{l,j}^{(k)}$, $k = 0,1,2,3$ are the same, we only estimate $\bigcup_{l,j\ge 1}\bar{\Omega}_{l,j}^{(0)}$. It is obvious that

$$\partial_\omega\left(\omega\sqrt{m_b+m_c}l - \sqrt{EIj^4+Qj^2}\right) \ge \sqrt{m_b+m_c}l, \quad \forall l \ge 1. \tag{A.1}$$

If the set $\bar{\Omega}_{l,j}^{(0)}$ is nonempty, then we must have

$$j \ge \sqrt{\sqrt{EI^{-1}\left[\left(\omega_1\sqrt{m_b+m_c}l - \frac{\gamma}{|l|^{\kappa+1}}\right)^2 + \frac{Q^2}{4EI}\right]} - \frac{Q}{2EI}} := A$$

and

$$j \le \sqrt{\sqrt{EI^{-1}\left[\left(\omega_2\sqrt{m_b+m_c}l + \frac{\gamma}{|l|^{\kappa+1}}\right)^2 + \frac{Q^2}{4EI}\right]} - \frac{Q}{2EI}} := B.$$

By the above inequality, it follows that

$$\begin{aligned}
\sharp\{j\} &\le B - A \\
&\le \frac{\sqrt{EI^{-1}[(\omega_2\sqrt{m_b+m_c}l + \frac{\gamma}{|l|^{\kappa+1}})^2 + \frac{Q^2}{4EI}]} - \sqrt{EI^{-1}[(\omega_1\sqrt{m_b+m_c}l - \frac{\gamma}{|l|^{\kappa+1}})^2 + \frac{Q^2}{4EI}]}}{A+B} \\
&\le \frac{EI^{-1}[(\omega_2-\omega_1)\sqrt{m_b+m_c}l + \frac{2\gamma}{|l|^{\kappa+1}}][(\omega_2+\omega_1)\sqrt{m_b+m_c}l]}{(A+B)(D+F)},
\end{aligned} \tag{A.2}$$

where

$$D := \sqrt{EI^{-1}\left[\left(\omega_2\sqrt{m_b+m_c}l + \frac{\gamma}{|l|^{\kappa+1}}\right)^2 + \frac{Q^2}{4EI}\right]},$$
$$F := \sqrt{EI^{-1}\left[\left(\omega_1\sqrt{m_b+m_c}l - \frac{\gamma}{|l|^{\kappa+1}}\right)^2 + \frac{Q^2}{4EI}\right]}.$$

Note that $\omega\sqrt{m_b+m_c} \geq \omega\sqrt{m_c} \geq \frac{1}{C\gamma^2} > 1$. Then, for suitably big $\frac{Q}{2EI}$, we have

$$A + B \geq B \geq EI^{-\frac{1}{4}} \tag{A.3}$$

and

$$D + F \geq EI^{-\frac{1}{2}}(\omega_2 + \omega_1)\sqrt{m_b+m_c}\,l. \tag{A.4}$$

Hence, by (A.2)–(A.4), for sufficiently small γ, we get

$$\begin{aligned}\sharp\{j\} &\leq EI^{-\frac{1}{4}}\left[(\omega_2-\omega_1)\sqrt{m_b+m_c}\,l + \frac{2\gamma}{|l|^{\kappa+1}}\right]\\ &\leq 2EI^{-\frac{1}{4}}(\omega_2-\omega_1)\sqrt{m_b+m_c}\,l.\end{aligned} \tag{A.5}$$

Combining (A.1)–(A.5), we obtain

$$\begin{aligned}\left|\bigcup_{l,j\geq 1}\bar{\Omega}_{l,j}^{(0)}\right| &\leq 2EI^{-\frac{1}{4}}(\omega_2-\omega_1)\sqrt{m_b+m_c}\sum_{l=1}^{\infty}\left|\bar{\Omega}_{l,j}^{(0)}\right|\\ &\leq 2EI^{-\frac{1}{4}}(\omega_2-\omega_1)\gamma\sum_{l=1}^{\infty}\frac{1}{|l|^{\kappa+1}}\\ &\leq 2C'EI^{-\frac{1}{4}}(\omega_2-\omega_1)\gamma,\end{aligned} \tag{A.6}$$

where C' denotes a constant. This completes the proof. □

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