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Study of a generalized logistic equation with nonlocal reaction term

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Abstract

In this paper, we consider the generalized logistic equation with nonlocal reaction term

$$-\Delta u = u\left(\lambda + b\int_{\Omega} u^r\,dx - f(u)\right) \quad \text{in } \Omega, \qquad u > 0 \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega.$$

Using the bifurcation and sub-supersolution method, we obtain the non-existence, existence, and uniqueness of positive solutions for different parameters on the nonlocal terms. Our works about the nonlocal elliptic problem improve the results in the previous literature.

MSC: 35R09; 45K05; 35J60; 35J25

Keywords: Logistic equation; Nonlocal term; Bifurcation method; Sub-supersolution method; Existence

1 Introduction

In this paper, we consider the nonlocal elliptic boundary value problem

$$\begin{cases} -\Delta u = u(\lambda + b\int_{\Omega} u^r\,dx - f(u)), & \text{in } \Omega, \\ u > 0, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

Here Ω is a bounded domain in $\mathbb{R}^N$, $N \geq 2$, with $C^{2,\beta}$ boundary $\partial\Omega$, λ, $b \in \mathbb{R}$, $r > 0$, $\beta \in (0,1)$, and $f(u)$ is a polynomial denoted by

$$f(u) = \sum_{i=1}^{n} a_i u^{k_i}, \quad a_i > 0, i = 1, 2, \dots, n \ (n \geq 1), \tag{1.2}$$

where

all k_i are integers with $1 = k_1 < k_2 < \cdots < k_n$.

This type of problem was studied initially by Delgado et al. in [8], where they proposed the equation

$$\begin{cases} \frac{\partial u}{\partial t} - \Delta u = u(\lambda + b\int_\Omega u^r\,dx - u), & \text{in } \Omega \times (0,+\infty), \\ u(x,t) = 0, & \text{in } \partial\Omega \times (0,+\infty), \\ u(x,0) = u_0, & \text{on } \Omega, \end{cases}$$

and the corresponding steady-state problem

$$\begin{cases} -\Delta u = u(\lambda + b\int_\Omega u^r\,dx - u), & \text{in } \Omega, \\ u > 0, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

Here $u(x,t)$ represents the density of a species in time $t > 0$ and at the point $x \in \Omega$, the habitat of the species that is surrounded by inhospitable areas, λ is the growth rate of species, term $-f(u)$ describes the limiting effect of crowding in the population. In this paper the authors proved the existence of an unbounded continuum of positive solutions of (1.3), presented some non-existence results, and discussed the local and global behavior of the continuum.

The introduction of nonlocal terms in the equation and in the boundary conditions models a number of processes in different fields such as mathematical physics, mechanics of deformable solids, mathematical biology, and many others (see [1, 2, 4, 5, 10–12, 16, 20]).

Obviously, problem (1.1) is a generalization of problem (1.3). In this paper, we present some results on the existence of an unbounded continuum of positive solutions of (1.1), the local and global behavior of the continuum, and prove the non-existence of positive solutions also.

The paper is organized as follows. In Sect. 2 we give some lemmas which show the relationship among the solution, sub-solution, and super-solution and the relationship between the solution and the nonlinear term f and prove the existence of an unbounded continuum of positive solutions of (1.1). Section 3 is devoted to proving the non-existence results and a priori bounds of positive solutions of (1.1). In Sect. 4 we presents some conditions for the existence of positive solutions for (1.1) and prove local and global behavior of the continuum of positive solutions of (1.1). Some ideas come from [13, 14].

Throughout our paper, we always suppose that (1.2) is true.

2 Bifurcation results

In order to discuss (1.1), we consider the following equation:

$$\begin{cases} -\Delta u = u(\lambda - f(u)), & \text{in } \Omega, \\ u > 0, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where $\lambda \in \mathbb{R}$.

Denote by φ_1 an eigenfunction corresponding to the principle eigenvalue λ_1 of

$$\begin{cases} -\Delta u = \lambda u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{2.2}$$

From [9] and [15], φ_1 belongs to $C^{2,\beta}(\overline{\Omega})$, $\varphi_1 > 0$ in Ω, and $\lambda_1 > 0$. Moreover, assume that $\|\varphi_1\|_\infty = 1$.

Lemma 2.1 (See [3]) *There exists a positive solution of* (2.1) *if and only if $\lambda > \lambda_1$. Moreover, if it exists, the solution is unique, and we denote it by θ_λ. Furthermore, the following inequalities hold*:

$$f^{-1}(\lambda - \lambda_1)\varphi_1 \le \theta_\lambda \le f^{-1}(\lambda). \tag{2.3}$$

Lemma 2.2 *Assume that u is the unique positive solution to* (2.1) *for $\lambda > \lambda_1$. Then*

$$u = (\lambda - \lambda_1)m_1^{-1}\varphi_1 + (\lambda - \lambda_1)^2 m_1^{-2} U_1 + O\big(|\lambda - \lambda_1|^3\big), \quad \text{as } \lambda \downarrow \lambda_1,$$

where

$$m_1 := a_1 \int_\Omega \varphi_1^3 \, dx \neq 0,$$

and we have denoted by β_1 the unique solution of the following linear problem in Ω under the homogeneous Dirichlet boundary condition:

$$(-\Delta - \lambda_1)\beta_1 = m_1\varphi_1 - a_1\varphi_1^2, \qquad \int_\Omega \beta_1\varphi_1 \, dx = 0,$$

and

$$\begin{aligned} m_2 &:= 2a_1 \int_\Omega \varphi_1^2 \beta_1 \, dx + a_2 \int_\Omega \varphi_1^4 \, dx - m_1 \int_\Omega \beta_1\varphi_1 \, dx, \quad \text{if } k_2 = 2, \\ m_2 &:= 2a_1 \int_\Omega \varphi_1^2 \beta_1 \, dx - m_1 \int_\Omega \beta_1\varphi_1 \, dx, \quad \text{if } k_2 > 2, \\ U_1 &:= \beta_1 - \frac{m_2}{m_1}\varphi_1. \end{aligned}$$

Lemma 4.3 in [9] proved the relationship between the solution u of problem (2.1) when $f(u) = u$ and the first eigenfunction φ_1 of problem (2.2). Lemma 2.2 obtains a similar result for the case $f(u) = \sum_{i=1}^n a_i u^{k_i}$. Since the proof is the same as that in [9], we omit it.

From Lemma 2.2, we obtain the following corollary directly.

Corollary 2.1 *Assume that u is the unique positive solution to* (2.1) *for $\lambda > \lambda_1$. There exist two positive constants $\delta > 0$ and $K > 0$ such that*

$$u \le K(\lambda - \lambda_1)\varphi_1, \quad \forall \lambda \in (\lambda_1, \lambda_1 + \delta]. \tag{2.4}$$

Lemma 2.3 *Assume that θ_λ is the unique positive solution to (2.1) for $\lambda > \lambda_1$.*

(1) *If $\underline{u} > 0$ is a strict sub-solution of (2.1), then $\underline{u} \le \theta_\lambda$.*

(2) *If $\overline{u} > 0$ is a strict super-solution of (2.1), then $\theta_\lambda \le \overline{u}$.*

Since $u^{-1}u(\lambda - f(u)) = \lambda - f(u)$ is decreasing, it is easy to get the proof from Lemma 2.3 in [19], and we omit it.

We consider the Banach space $X := C_0(\overline{\Omega})$, denote $B_\rho := \{u \in X : \|u\|_\infty < \rho\}$. Define

$$F(u) = u\left(\lambda + b\int_\Omega u_+^r\,dx - f(u)\right)$$

and the map

$$K_\lambda : X \to X, \qquad K_\lambda(u) = u - (-\Delta)^{-1}\big(F(u)\big),$$

where $u_+ = \max\{u(x), 0\}$ and $(-\Delta)^{-1}$ is the inverse of the operator $-\Delta$ under homogeneous Dirichlet boundary conditions. Agmon–Douglas–Nirenberg theorem, embedding theorem, and strong maximum theorem(see [17]) guarantee that $(-\Delta)^{-1}$ is positive and compact. It is clear that u is a nonnegative solution of (1.1) if and only if $K_\lambda u = 0$.

Now we give the main result of this section.

Theorem 2.1 *The value $\lambda = \lambda_1$ is the only bifurcation point from the trivial solution for (1.1). Moreover, there exists a continuum C_0 of nonnegative solutions of (1.1) unbounded in $\mathbb{R} \times X$ emanating from $(\lambda_1, 0)$. Furthermore,*

(i) *if $b \le 0$, the direction of bifurcation is supercritical.*

(ii) *Assume $b > 0$.*

  (a) *If $r < 1$, the direction of bifurcation is subcritical.*

  (b) *If $r > 1$, then the direction of bifurcation is supercritical.*

  (c) *Assume $r = 1$, and denote*

$$b_0 = \frac{a_1\int_\Omega \varphi_1^3\,dx}{\int_\Omega \varphi_1\,dx\int_\Omega \varphi_1^2\,dx}.$$

If $b > b_0$ (resp. $b < b_0$), the direction of bifurcation is subcritical (resp. supercritical).

Recall that we say that the direction of bifurcation is subcritical (resp. supercritical) if there exists a neighborhood V of $(\lambda_1, 0)$ such that for every solution $(\lambda, u) \in V$ satisfies $\lambda < \lambda_1$ (resp. $\lambda > \lambda_1$), see [8].

In order to prove this result, we use the Leray–Schauder degree of K_λ on B_ρ with respect to zero, denoted by $\deg(K_\lambda, B_\rho)$, and the index of the isolated zero of K_λ, denoted by $i(K_\lambda, u)$.

Lemma 2.4 *If $\lambda < \lambda_1$, then $i(K_\lambda, 0) = 1$.*

Lemma 2.5 *If $\lambda > \lambda_1$, then $i(K_\lambda, 0) = 0$.*

Since the proof is the same as that of Lemmas 2.3 and 2.4 in [8], we omit it.

Proof of Theorem 2.1 From Lemmas 2.4 and 2.5 and bifurcation theorem (see [18]), the same proof as that of Theorem 2.2 in [8] guarantees the existence of an unbounded continuum C_0 of positive solutions of (1.1). Moreover, conclusion (i) is true.

We only give the proof of (ii).

(a) Assume now that $b > 0$ and the existence of a sequence $(\lambda_n, u_n) \in C_0$ of positive solutions of (1.1) such that $\lambda_n \geq \lambda_1$ and $\|u_n\|_\infty \to 0$ as $n \to \infty$. By the property of f, there is $\delta_1 > 0$ such that

$$f(u) \leq (a_1 + 1)u, \quad \forall u \in [0, \delta_1]. \tag{2.5}$$

Take $M > 0$ such that

$$(a_1 + 1) - bM \int_\Omega \varphi_1 \, dx < 0. \tag{2.6}$$

Since $r < 1$, choose n large enough such that $u_n^r > Mu_n$, and then

$$-\Delta u_n > u_n\left(\lambda_n + bM \int_\Omega u_n \, dx - f(u_n)\right),$$

which implies that u_n is a strict super-solution of the following system:

$$\begin{cases} -\Delta u = u(\lambda_n + bM \int_\Omega u_n \, dx - f(u)), & x \in \Omega, \\ u|_{\partial\Omega} = 0. \end{cases} \tag{2.7}$$

Using Lemma 2.1, we get (2.7) has a unique positive solution θ_n and

$$\theta_n \geq f^{-1}\left(\lambda_n + bM \int_\Omega u_n \, dx - \lambda_1\right)\varphi_1.$$

Lemma 2.3 implies that

$$u_n \geq \theta_n \geq f^{-1}\left(\lambda_n + bM \int_\Omega u_n \, dx - \lambda_1\right)\varphi_1.$$

Integrating the above inequality yields that

$$\int_\Omega u_n \, dx \geq f^{-1}\left(\lambda_n + bM \int_\Omega u_n \, dx - \lambda_1\right) \int_\Omega \varphi_1 \, dx.$$

And then

$$f\left(\frac{\int_\Omega u_n \, dx}{\int_\Omega \varphi_1 \, dx}\right) \geq \lambda_n + bM \int_\Omega u_n \, dx - \lambda_1.$$

Using (2.5), one has

$$(a_1 + 1)\left(\int_\Omega \varphi_1 \, dx\right)^{-1} \int_\Omega u_n \, dx \geq f\left(\frac{\int_\Omega u_n \, dx}{\int_\Omega \varphi_1 \, dx}\right)$$

for n large enough. And then

$$(a_1+1)\left(\int_\Omega \varphi_1\,dx\right)^{-1}\int_\Omega u_n\,dx \geq \lambda_n + bM\int_\Omega u_n\,dx - \lambda_1,$$

which together with (2.6) implies

$$0 > \left((a_1+1)\left(\int_\Omega \varphi_1\,dx\right)^{-1} - bM\right)\int_\Omega u_n\,dx \geq \lambda_n - \lambda_1,$$

an absurdum.

(b) Assume now that $b>0$, $r>1$ and the existence of a sequence $(\lambda_n, u_n) \in C_0$ of positive solutions of (1.1) such that $\lambda_n \leq \lambda_1$ and $\|u_n\|_\infty \to 0$ as $n\to\infty$. Without loss of generality, assume that $\|u_n\|_\infty \leq \delta$ defined in Corollary 2.1. Take $\varepsilon > 0$ such that

$$1 - bK\varepsilon|\Omega| > 0, \tag{2.8}$$

where K is defined in Corollary 2.1.

For n large we have $u_n^r < \varepsilon u_n$, and then

$$-\Delta u_n < u_n\left(\lambda_n + b\varepsilon\int_\Omega u_n\,dx - f(u_n)\right),$$

which implies that u_n is a strict sub-solution of the following problem:

$$\begin{cases} -\Delta v = v(\lambda_n + b\varepsilon\int_\Omega u_n\,dx - f(v)), & x\in\Omega,\\ v|_{\partial\Omega} = 0. \end{cases} \tag{2.9}$$

By Lemma 2.1, we get (2.9) has a unique positive solution θ_n. Moreover, from Corollary 2.1, we have

$$\theta_n \leq K\left(\lambda_n + b\varepsilon\int_\Omega u_n\,dx - \lambda_1\right)\varphi_1 \leq K\left(\lambda_n + b\varepsilon\int_\Omega u_n\,dx - \lambda_1\right)$$

for n large enough. By Lemma 2.3, we have

$$u_n \leq \theta_n \leq K\left(\lambda_n + b\varepsilon\int_\Omega u_n\,dx - \lambda_1\right)\varphi_1 \leq K\left(\lambda_n + b\varepsilon\int_\Omega u_n\,dx - \lambda_1\right).$$

Integrating the above inequality yields that

$$\int_\Omega u_n\,dx \leq K\left(\lambda_n + b\varepsilon\int_\Omega u_n\,dx - \lambda_1\right)|\Omega|,$$

which together with (2.8) implies that

$$0 < \left(1 - Kb\varepsilon|\Omega|\right)\int_\Omega u_n\,dx \leq K(\lambda_n - \lambda_1)|\Omega|,$$

an absurdum.

(c) Assume that $b>0$ and $r=1$. In this case, we apply the Crandall–Rabinowitz theorem (see [7]). Then there exist $\varepsilon > 0$ and two regular functions $\lambda(s)$, $u(s)$, $s\in(-\varepsilon,\varepsilon)$, such that in a neighborhood of $(\lambda_1, 0)$, the positive solutions are $u(s)$, $s \in (0, \varepsilon)$. We can write

$$u(s) = s\varphi_1 + s^2\varphi_2 + o(s^2)$$

and

$$\lambda(s) = \lambda_1 + s\lambda_2 + o(s),$$

where $\lambda_2 \in \mathbb{R}$, $\varphi_2 \in C^2(\overline{\Omega})$. It is evident that the sign of λ_2 determines the bifurcation direction. Substituting these expansions into (1.1) and identifying the terms of order one in s yield

$$-\Delta\varphi_2 - \lambda_1\varphi_2 = \lambda_2\varphi_1 - a_1\varphi_1^2 + b\varphi_1 \int_\Omega \varphi_1\,dx.$$

Multiplying by φ_1 and integrating in Ω, we conclude that

$$\lambda_2 = \frac{a_1 \int_\Omega \varphi_1^3\,dx - b\int_\Omega \varphi_1^2\,dx \int_\Omega \varphi_1\,dx}{\int_\Omega \varphi_1^2\,dx}.$$

This finishes the proof. □

3 A priori bounds and non-existence results of (1.1)

In this section, we obtain a priori bounds of the solutions for $b > 0$ as well as non-existence results of (1.1).

Theorem 3.1 *Assume that $b > 0$ and $r < 1$. Let u_λ be a positive solution of* (1.1) *such that $\lambda \in K \subset \mathbb{R}$ a compact set. Then there exists a constant $L_0 > 0$ such that*

$$\|u_\lambda\|_\infty \le L_0$$

for a constant independent of $\lambda \in K$. Moreover, there exists a constant L_1 such that, if

$$\lambda < L_1,$$

(1.1) *does not possess any positive solution.*

Proof Since K is compact, there is a positive constant $k_0 > 0$ such that

$$\lambda \le k_0, \quad \forall\lambda \in K.$$

Moreover, since u_λ is a positive solution of (1.1), we have, using Lemma 2.1 and Hölder's inequality, that

$$\begin{aligned} u_\lambda &\le f^{-1}\left(\lambda + b\int_\Omega u_\lambda^r\,dx\right) \\ &\le f^{-1}\left(\lambda + b|\Omega|^{1-r}\left(\int_\Omega u_\lambda\,dx\right)^r\right) \\ &\le f^{-1}\left(k_0 + b|\Omega|^{1-r}\left(\int_\Omega u_\lambda\,dx\right)^r\right), \quad x \in \Omega. \end{aligned} \tag{3.1}$$

Step 1. We show that there exists $c_1 > 0$ such that

$$\int_\Omega |u_\lambda|\,dx \le c_1. \tag{3.2}$$

In fact, suppose to the contrary that there exists $\{u_{\lambda_n}\}$ such that

$$\int_\Omega u_{\lambda_n}\,dx \to +\infty, \quad n \to +\infty.$$

Replacing λ in (3.1) by λ_n, one has

$$u_{\lambda_n} \le f^{-1}\left(k_0 + b\left(\int_\Omega u_{\lambda_n}\,dx\right)^r |\Omega|^{1-r}\right). \tag{3.3}$$

Integrating inequality (3.3) in Ω yields that

$$\int_\Omega u_{\lambda_n}\,dx \le f^{-1}\left(k_0 + b|\Omega|^{1-r}\left(\int_\Omega u_{\lambda_n}\,dx\right)^r\right)|\Omega|,$$

i.e.,

$$f\left(|\Omega|^{-1}\int_\Omega u_{\lambda_n}\,dx\right) \le \lambda_n + b|\Omega|^{1-r}\left(\int_\Omega u_{\lambda_n}\,dx\right)^r. \tag{3.4}$$

By the property of $f(u)$, one has

$$f(u) \ge a_1 u,$$

which together with (3.4) implies that

$$a_1|\Omega|^{-1}\int_\Omega u_{\lambda_n}\,dx \le k_0 + b|\Omega|^{1-r}\left(\int_\Omega |u_{\lambda_n}|\,dx\right)^r$$

for n large enough. This is a contradiction because $r < 1$.

Step 2. We show that there exists a constant $L_0 > 0$ such that

$$\|u_\lambda\|_\infty \le L_0, \quad \forall \lambda \in K.$$

Since Step 1 holds, (3.1) guarantees that

$$u_\lambda \le f^{-1}\left(\lambda + b\left(\int_\Omega u_\lambda\,dx\right)^r |\Omega|^{1-r}\right) \le f^{-1}\left(k_0 + bc_1^r|\Omega|^{1-r}\right).$$

Let $L_0 := f^{-1}(k_0 + bc_1^r|\Omega|^{1-r})$. We conclude

$$\|u_\lambda\|_\infty \le L_0, \quad \forall \lambda \in K.$$

Step 3. We show that there exists a constant L_1 such that, if

$$\lambda < L_1,$$

(1.1) does not possess any positive solution.

Now define a function

$$g(s) := f\big(|\Omega|^{-1}s\big) - b|\Omega|^{1-r}s^r, \quad s \in [0, +\infty).$$

From the property of f and $0 < r < 1$, one has

$$\lim_{s\to+\infty} \frac{f(|\Omega|^{-1}s)}{s^r} = +\infty,$$

which together with $g(0) = 0$ implies that there is $s_0 \geq 0$ such that

$$\min_{s\in[0,+\infty)} g(s) = g(s_0).$$

Let

$$L_1 := g(s_0).$$

Assume that u_λ is a positive solution to (1.1) for $\lambda \in \mathbb{R}$. From (3.4), we have

$$f\left(|\Omega|^{-1}\int_\Omega u_\lambda\,dx\right) - b\left(\int_\Omega u_\lambda\,dx\right)^r |\Omega|^{1-r} \leq \lambda,$$

which means that

$$\lambda \geq L_1.$$

Consequently, (1.1) has no positive solution if $\lambda < L_1$.

The proof is complete. □

Theorem 3.2 *Assume that $b > 0$, $k_n < r$, where k_n is the index of the last term of polynomial f. Let u_λ be a positive solution of (1.1) such that $\lambda \in K \subset \mathbb{R}$ a compact set. Then there exists a constant L_0 such that*

$$\|u_\lambda\|_\infty \leq L_0$$

for a constant independent of $\lambda \in K$. Moreover, there exists a constant L_1 such that

$$\lim_{b\to+\infty} L_1 = \lambda_1$$

and if

$$\lambda > L_1,$$

(1.1) *does not possess any positive solution.*

Proof Since K is compact, there is k_1 such that

$$k_1 \leq \lambda, \quad \forall\lambda \in K.$$

Moreover, using now the lower bound in Lemma 2.1, we get that

$$f^{-1}\left(k_1 + b\int_\Omega u_\lambda^r\,dx - \lambda_1\right)\varphi_1 \le f^{-1}\left(\lambda + b\int_\Omega u_\lambda^r\,dx - \lambda_1\right)\varphi_1$$
$$\le u_\lambda \le f^{-1}\left(\lambda + b\int_\Omega u_\lambda^r\,dx\right), \quad x \in \Omega. \tag{3.5}$$

Step 1. We show that there exists a constant $c_1 > 0$ such that

$$\int_\Omega u_\lambda^r\,dx \le c_1, \quad \forall \lambda \in K.$$

In fact, suppose to the contrary that there exists $\{\lambda_n\} \subseteq K$ such that

$$\int_\Omega u_{\lambda_n}^r\,dx \to +\infty, \quad \text{as } n \to +\infty.$$

By the property of $f(u)$, there is $M > 0$ such that

$$f(u) \le Mu^{k_n} \tag{3.6}$$

for u large enough. Integrating (3.5) in Ω yields that

$$f^{-1}\left(\lambda_n + b\int_\Omega u_{\lambda_n}^r\,dx - \lambda_1\right)\int_\Omega \varphi_1\,dx \le \int_\Omega u_{\lambda_n}\,dx,$$

that is, by Hölder's inequality and (3.6)

$$\begin{aligned}\lambda_n + b\int_\Omega u_{\lambda_n}^r\,dx - \lambda_1 &\le f\left(\frac{\int_\Omega u_{\lambda_n}\,dx}{\int_\Omega \varphi_1\,dx}\right)\\ &\le M\left(\frac{\int_\Omega u_{\lambda_n}\,dx}{\int_\Omega \varphi_1\,dx}\right)^{k_n}\\ &\le M\left(\int_\Omega \varphi_1\,dx\right)^{-k_n}\left(\int_\Omega u_{\lambda_n}^r\,dx\right)^{\frac{k_n}{r}}|\Omega|^{\frac{r-1}{r}}\end{aligned}$$

for n large enough.

This is a contradiction to $k_n < r$.

Step 2. We show that there exists a constant $L_0 > 0$ such that

$$\|u_\lambda\|_\infty \le L_0, \quad \forall \lambda \in K.$$

Since K is compact, there exists k_0 such that

$$\lambda \le k_0, \quad \forall \lambda \in K.$$

From (3.5) and Step 1, one has

$$u_\lambda \le f^{-1}\left(\lambda + b\int_\Omega u_\lambda^r\,dx\right) \le f^{-1}(k_0 + bc_1).$$

Let

$$L_0 := f^{-1}(k_0 + bc_1).$$

Then L_0 satisfies Step 2.

Step 3. We show that there exists a constant L_1 such that, if

$$\lambda > L_1,$$

(1.1) does not possess any positive solution.

Now define

$$g_1(s) := f\left(\frac{s^{\frac{1}{r}}}{(\int_\Omega \varphi_1^r\,dx)^{\frac{1}{r}}}\right) - bs + \lambda_1. \tag{3.7}$$

Since $k_n < r$, one has

$$\lim_{s\to+\infty} g_1(s) = -\infty,$$

which together with $g_1(0) = 0$ implies that there exists $s_1 \geq 0$ such that

$$\max_{s\in[0,+\infty)} g_1(s) = g_1(s_1).$$

Let

$$L_1 := g_1(s_1).$$

Assume that u_λ is a positive solution to (1.1) for $\lambda \in \mathbb{R}$. Integrating (3.5) in Ω yields that

$$\left(f^{-1}\left(\lambda + b\int_\Omega u_\lambda^r\,dx - \lambda_1\right)\right)^r \int_\Omega \varphi_1^r\,dx \leq \int_\Omega u_\lambda^r\,dx,$$

i.e.,

$$f^{-1}\left(\lambda + b\int_\Omega u_\lambda^r\,dx - \lambda_1\right) \leq \left(\frac{\int_\Omega u_\lambda^r\,dx}{\int_\Omega \varphi_1^r\,dx}\right)^{\frac{1}{r}}.$$

And then

$$\lambda + b\int_\Omega u_\lambda^r\,dx - \lambda_1 \leq f\left(\left(\frac{\int_\Omega u_\lambda^r\,dx}{\int_\Omega \varphi_1^r\,dx}\right)^{\frac{1}{r}}\right),$$

that is,

$$\lambda \leq -b\int_\Omega u_\lambda^r\,dx + f\left(\left(\frac{\int_\Omega u_\lambda^r\,dx}{\int_\Omega \varphi_1^r\,dx}\right)^{\frac{1}{r}}\right) + \lambda_1.$$

Hence

$$\lambda \le L_1.$$

Consequently, Step 3 is true.

Moreover, we consider

$$h(s) := As^q - \frac{b}{n}s + \frac{\lambda_1}{n}, \quad A > 0, 0 < q < 1.$$

It is easy to prove that $h(s)$ gets its maximum at $s = (\frac{b}{nAq})^{\frac{1}{1-q}}$ and

$$\max_{s\in[0,+\infty)} h(s) = A\left(\frac{b}{nAq}\right)^{\frac{q}{1-q}} - \frac{b}{n}\left(\frac{b}{nAq}\right)^{\frac{1}{1-q}} + \frac{\lambda_1}{n}.$$

Let

$$h_i(s) := a_i s^{\frac{k_i}{r}} - \frac{b}{n}s + \frac{\lambda_1}{n}, \quad i = 1, 2, \ldots.$$

Obviously, we have

$$\max_{s\in[0,+\infty)} h_i(s) = a_i\left(\frac{b}{nA\frac{k_i}{r}}\right)^{\frac{\frac{k_i}{r}}{1-\frac{k_i}{r}}} - \frac{b}{n}\left(\frac{b}{nA\frac{k_i}{r}}\right)^{\frac{1}{1-\frac{k_i}{r}}} + \frac{\lambda_1}{n}.$$

Since $0 < \frac{k_i}{r} < 1$, $i = 1, 2, \ldots, n$, one has that

$$\lim_{b\to+\infty} \max_{s\in[0,+\infty)} h_i(s) = \frac{\lambda_1}{n}, \quad i = 1, 2, 3, \ldots, n.$$

By the definition g_1 in (3.7), we have

$$g_1(s) = \sum_{i=1}^{n} h_i(s),$$

which implies that

$$\lim_{b\to+\infty} L_1 = \lim_{b\to+\infty} \max_{s\in[0,+\infty)} g_1(s) \le \lim_{b\to+\infty} \sum_{i=1}^{n} \max_{s\in[0,+\infty)} h_i(s) = \lambda_1.$$

On the other hand, since $L_1 > \lambda_1$, we have

$$\lim_{b\to+\infty} L_1 = \lambda_1.$$

The proof is complete. □

Theorem 3.3 *Assume that $b > 0$, $r = 1$. Let u_λ be a positive solution of* (1.1) *such that $\lambda \in K \subset \mathbb{R}$ a compact set.*

(1) *If* $k_n = 1 = r$ *and* $b < 1/|\Omega|$ *or* $b\int_\Omega \varphi_1\,dx > 1$, *then there exist a priori bounds of the solution of* (1.1). *Moreover, if* $b < 1/|\Omega|$ *and* $\lambda \le 0$ *or* $b\int_\Omega \varphi_1\,dx > 1$ *and* $\lambda \ge \lambda_1$, *then* (1.1) *does not possess a positive solution.*

(2) *If* $k_n > 1 = r$, *then there exists a constant* $L_0 > 0$ *such that*

$$\|u_\lambda\|_\infty \le L_0$$

for a constant independent of λ. *Moreover, there exists a constant* L_1 *such that, if*

$$\lambda < L_1,$$

(1.1) *does not possess any positive solution.*

Conclusion (i) is Proposition 3.1 in [8] and the proof of (ii) is similar to that in Theorem 3.1, and we omit it.

4 Existence and uniqueness results

In this section, first we introduce the method of sub-supersolution to some nonlocal elliptic problems.

Consider a continuous operator $B : L^\infty(\Omega) \to \mathbb{R}$ and $f : \Omega \times \mathbb{R}^2 \to \mathbb{R}$ a continuous function and the general problem

$$\begin{cases} -\Delta u = f(x,u,B(u)), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{4.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$, $N \ge 2$, with $C^{2,\beta}$ boundary $\partial\Omega$.

Definition 4.1 (See [6]) We say that the pair $(\underline{u}, \overline{u})$ with $\underline{u}, \overline{u} \in H^1(\Omega) \cap L^\infty(\Omega)$ is a pair of sub-supersolutions of (4.1) if

(a) $\underline{u} \le \overline{u}$ in Ω and $\underline{u} \le 0 \le \overline{u}$ on $\partial\Omega$;

(b) $-\Delta\underline{u} - f(x,\underline{u},B(u)) \le 0 \le -\Delta\overline{u} - f(x,\overline{u},B(u))$ in the weak sense for all $u \in [\underline{u}, \overline{u}]$.

Lemma 4.1 (See [6]) *Assume that there exists a pair of sub-supersolutions of* (4.1). *Then there exists a solution* $u \in H^1(\Omega) \cap L^\infty(\Omega)$ *of* (4.1) *such that* $u \in [\underline{u}, \overline{u}]$.

Now we give the main theorems.

Theorem 4.1 *Assume that* $b < 0$. *Then* (1.1) *has a positive solution if and only if* $\lambda > \lambda_1$. *Moreover, if there exists the unique positive solution, denoted by* $u_{\lambda,b}$, *then*

$$\lim_{b\to-\infty} \|u_{\lambda,b}\|_\infty = 0.$$

Proof By Theorem 2.1 we know the existence of an unbounded continuum C_0 of positive solutions bifurcating from the trivial solution at $\lambda = \lambda_1$. Assume that $(\lambda, u_\lambda) \in C_0$. Now we show that $\lambda > \lambda_1$.

In fact, if $\lambda \le \lambda_1$, then

$$-\Delta u_\lambda \le u_\lambda\bigl(\lambda - f(u_\lambda)\bigr),$$

which implies that u_λ is a sub-solution (2.1).

Choose a constant K large enough such that

$$\lambda < f(K) \text{ and } K > \max_{x\in\overline{\Omega}} u_\lambda.$$

Obviously, (u_λ, K) is a pair of sub-supersolutions to (2.1). Then (2.1) has a positive solution for $\lambda \le \lambda_1$. This is a contradiction to Lemma 2.1.

We know that positive solutions do not exist for $\lambda \le \lambda_1$, hence we conclude that if $(\lambda, u) \in C_0$, we have $\lambda > \lambda_1$.

Moreover, if $(\lambda, u) \in C_0$, since $b < 0$, we have

$$-\Delta u_\lambda < u_\lambda\bigl(\lambda - f(u_\lambda)\bigr),$$

and Lemma 2.3 implies that $u_\lambda \le \theta_\lambda$, where θ_λ is a solution to (2.1). Lemma 2.1 guarantees that (2.1) has a unique positive solution θ_λ for all $\lambda > \lambda_1$, which together with the unboundedness of C_0 implies that (1.1) has at least one positive solution $u_{\lambda,b}$ for all $\lambda > \lambda_1$.

We show now the uniqueness.

Assume that there exist two positive solutions $u \neq v$ for $b < 0$. If $\int_\Omega u^r\,dx = \int_\Omega v^r\,dx$, u and v satisfy

$$\begin{cases} -\Delta u = u(\lambda + k - f(u)), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

where $k = b\int_\Omega u^r\,dx = b\int_\Omega v^r\,dx < 0$. This is a contradiction to Lemma 2.1.

So, assume that for instance

$$\int_\Omega u^r\,dx < \int_\Omega v^r\,dx,$$

then

$$-\Delta u = u\left(\lambda + b\int_\Omega u^r\,dx - f(u)\right) > u\left(\lambda + b\int_\Omega v^r\,dx - f(u)\right),$$

and then by Lemma 2.3 we get $u > v$, an absurdum.

On the other hand, we have that

$$u_{\lambda,b} \le f^{-1}\left(\lambda + b\int_\Omega u_{\lambda,b}^r\,dx\right)$$

and then $f(u_{\lambda,b}) < \lambda$. So, as $b \to -\infty$, we get

$$\int_\Omega u_{\lambda,b}^r\,dx \to 0.$$

Moreover, Lemma 2.1 implies

$$f^{-1}\left(\lambda + b\int_\Omega u_{\lambda,b}^r\,dx - \lambda_1\right)\varphi_1 \le u_{\lambda,b}$$

and

$$\lambda + b\int_\Omega u_{\lambda,b}^r\,dx - \lambda_1 > 0,$$

we conclude that

$$b\int_\Omega u_{\lambda,b}^r\,dx \to \lambda_1 - \lambda.$$

This implies that $\|u_{\lambda,b}\|_\infty \to 0$. □

Theorem 4.2 *Assume that* $b > 0$, $0 < r < 1$. *Then there exists* $\lambda_* < \lambda_1$ *such that* (1.1) *possesses at least a positive solution if and only if* $\lambda \ge \lambda_*$. *Moreover,*

$$\lim_{b\to 0^+}\lambda_*(b) = \lambda_1 \quad \textit{and} \quad \lim_{b\to +\infty}\lambda_*(b) = -\infty.$$

Proof Define

$$\lambda_* = \inf\bigl\{\lambda \in \mathbb{R} : (1.1) \text{ possesses at least a positive solution}\bigr\}.$$

We know by Theorems 2.1 and 3.1 that $-\infty < \lambda_* < \lambda_1$.

Step 1. We show that (1.1) has at least one positive solution for all $\lambda > \lambda_*$.

Take $\lambda > \lambda_*$, then there exists $\mu \in [\lambda_*, \lambda)$ such that (1.1) possesses at least a positive solution, denoted by u_μ. Choose K large enough such that

$$f(K) - bK^r|\Omega| > \lambda \quad \text{and} \quad K > \max_{x\in\overline{\Omega}} u_\mu(x). \tag{4.2}$$

Let $(\underline{u}, \overline{u}) = (u_\mu, K)$. Since u_μ is a positive solution of (1.1) and (4.2) is true, we have

(a) $\underline{u} = u_\mu < K = \overline{u}$ in Ω and $\underline{u} = u_\mu = 0 < K = \overline{u}$ on $\partial\Omega$;

(b)

$$\begin{aligned}
&-\Delta\underline{u} - \underline{u}\left(\lambda + b\int_\Omega u^r\,dx - f(\underline{u})\right)\\
&\quad = u_\mu\left(\mu + b\int_\Omega u_\mu^r\,dx - f(u_\mu)\right) - u_\mu\left(\lambda + b\int_\Omega u^r\,dx - f(u_\mu)\right)\\
&\quad = bu_\mu\int_\Omega \left(u_\mu^r - u^r\right)dx + (\mu - \lambda)u_\lambda\\
&\quad \le 0, \quad x \in \Omega, \forall u \in [\underline{u}, \overline{u}]
\end{aligned}$$

and

$$\begin{aligned}-\Delta\overline{u}-\overline{u}\left(\lambda+b\int_\Omega u^r\,dx-f(\overline{u})\right)&=-0-K\left(\lambda+b\int_\Omega u^r\,dx-f(K)\right)\\&=K\left(-\lambda-b\int_\Omega u^r\,dx+f(K)\right)\\&\geq K\bigl(-\lambda-bK|\Omega|^r+f(K)\bigr)\\&>0,\quad x\in\Omega,\forall u\in[\overline{u},\overline{u}],\end{aligned}$$

which implies that $(\underline{u},\overline{u})$ is a pair of sub-supersolutions to (1.1). Theorem 4.2 guarantees that (1.1) has at least one positive solution for all $\lambda>\lambda_*$.

Step 2. We show that, for $\lambda=\lambda_*$, (1.1) has a positive solution.

By the definition of λ_*, there exists $\{\lambda_n\}$ such that $\lambda_n\geq\lambda_*$ and $\lambda_n\to\lambda_*$. Thanks to the bounds of Theorem 3.1, we have that $u_n\to u_*\geq 0$, u_* is a solution for $\lambda=\lambda_*$. Since $\lambda_*<\lambda_1$ and λ_1 is the unique bifurcation point from the trivial solution, we conclude that $u_*>0$.

Step 3. We show that

$$\lim_{b\to 0^+}\lambda_*(b)=\lambda_1\quad\text{and}\quad\lim_{b\to+\infty}\lambda_*(b)=-\infty.$$

Since u is bounded and

$$\lambda+b\int_\Omega u^r\,dx>\lambda_1,$$

and then taking $b\to 0$, we have that $\lambda\geq\lambda_1$, that is,

$$\lim_{b\to 0^+}\lambda_*(b)=\lambda_1.$$

Now we prove

$$\lim_{b\to+\infty}\lambda_*(b)=-\infty.$$

It suffices to show that, for any $\lambda<\lambda_1$, there exists $b>0$ big enough such that (1.1) possesses at least one positive solution.

In fact, for any $\lambda<\lambda_1$, there exists $b>0$ large enough such that for the function

$$(\lambda_1-\lambda)-b\int_\Omega\left(\frac{1}{2}\varphi_1\right)^r dx+f\left(\frac{1}{2}\varphi_1\right)<0,\quad\forall x\in\Omega.\tag{4.3}$$

For above b, take $K>1+|\lambda|+\frac{1}{2}\|\varphi_1\|_\infty$ large enough such that

$$f(K)>bK^r|\Omega|.\tag{4.4}$$

Let $\underline{u}=\frac{1}{2}\varphi_1(x)$ and $\overline{u}=K$. From (4.3) and (4.4), we have

(a) $\underline{u}=\frac{1}{2}\varphi_1<K=\overline{u}$ in Ω and $\underline{u}=\frac{1}{2}\varphi_1(x)=0<K=\overline{u}$ on $\partial\Omega$;

(b)

$$\begin{aligned}
-\Delta\underline{u}-\underline{u}\left(\lambda+b\int_\Omega u^r\,dx-f(\underline{u})\right) &= \frac{1}{2}\lambda_1\varphi_1-\frac{1}{2}\varphi_1\left(\lambda+b\int_\Omega u^r\,dx-f\left(\frac{1}{2}\varphi_1\right)\right)\\
&= \frac{1}{2}\varphi_1\left((\lambda_1-\lambda)-b\int_\Omega u^r\,dx+f\left(\frac{1}{2}\varphi_1\right)\right)\\
&< 0,\quad x\in\Omega,\forall u\in[\underline{u},\overline{u}]
\end{aligned}$$

and

$$\begin{aligned}
-\Delta\overline{u}-\overline{u}\left(\lambda+b\int_\Omega u^r\,dx-f(\overline{u})\right) &= -0-K\left(\lambda+b\int_\Omega u^r\,dx-f(K)\right)\\
&= K\left(-\lambda-b\int_\Omega u^r\,dx+f(K)\right)\\
&\geq K\left(-\lambda-bK^r|\Omega|+f(K)\right)\\
&> 0,\quad x\in\Omega,\forall u\in[\overline{u},\overline{u}],
\end{aligned}$$

which implies that $(\underline{u},\overline{u})$ is a pair of sub-supersolutions to (1.1). Theorem 4.2 guarantees that (1.1) has at least one positive solution for all $\lambda<\lambda_1$.

The proof is complete. □

Theorem 4.3 *Assume that $b>0$ and $k_n<r$. There exists $\lambda^*>\lambda_1$ such that* (1.1) *possesses at least a positive solution if and only if $\lambda\leq\lambda^*$. Moreover,*

$$\lim_{b\to0^+}\lambda^*(b)=+\infty\quad \textit{and}\quad \lim_{b\to+\infty}\lambda^*(b)=\lambda_1.$$

Proof Assume that $b>0$ and $k_n<r$. Define now

$$\lambda^*=\sup\bigl\{\lambda\in\mathbb{R}:(1.1)\text{ possesses at least a positive solution}\bigr\}.$$

We know by Theorems 2.1 and 3.2 that $\lambda_1<\lambda^*<+\infty$.

Step 1. We prove now that there exists a positive solution for all $\lambda\in(-\infty,\lambda^*)$.

Indeed, take $\lambda\in[\lambda_1,\lambda^*)$, then there exists $\mu\in(\lambda_1,\lambda^*]$ such that (1.1) possesses at least a positive solution, denoted by u_μ. Choose ε small enough such that $\varepsilon\varphi_1<u_\mu$ for all $x\in\Omega$ such that

$$f(\varepsilon\varphi_1)-b\int_\Omega(\varepsilon\varphi_1)^r\,dx\leq\lambda-\lambda_1. \tag{4.5}$$

Let $\underline{u}=\varepsilon\varphi_1$ and $\overline{u}=u_\mu$. Since u_μ is a positive solution of (1.1), from (4.5), one has

(a) $\underline{u}=\varepsilon\varphi_1<u_\mu$ in Ω and $\underline{u}=0=\overline{u}$ on $\partial\Omega$;

(b)

$$\begin{aligned}
-\Delta\underline{u}-\underline{u}\left(\lambda+b\int_\Omega u^r\,dx-f(\underline{u})\right) &= \varepsilon\lambda_1\varphi_1-\varepsilon\varphi_1\left(\lambda+b\int_\Omega u^r\,dx-f(\varepsilon\varphi_1)\right)\\
&= \varphi_1\varepsilon\left((\lambda_1-\lambda)-b\int_\Omega u^r\,dx+f(\varepsilon\varphi_1)\right)\\
&< 0,\quad x\in\Omega,\forall u\in[\underline{u},\overline{u}]
\end{aligned}$$

and

$$
\begin{aligned}
&-\Delta\overline{u}-\overline{u}\left(\lambda+b\int_{\Omega}u^{r}\,dx-f(\overline{u})\right)\\
&\quad=u_{\mu}\left(\mu+b\int_{\Omega}u_{\mu}^{r}\,dx-f(u_{\mu})\right)-u_{\mu}\left(\lambda+b\int_{\Omega}u^{r}\,dx-f(u_{\mu})\right)\\
&\quad=u_{\mu}\left(\mu-\lambda+b\int_{\Omega}\left(u_{\mu}^{r}-u^{r}\right)dx\right)\\
&\quad>0,\quad x\in\Omega,\forall u\in[\overline{u},\overline{u}],
\end{aligned}
$$

which implies that $(\underline{u},\overline{u})$ is a pair of sub-supersolutions to (1.1). Theorem 4.2 guarantees that (1.1) has at least one positive solution for all $\lambda\in[\lambda_1,\lambda_*)$.

Now, Theorem 2.1 implies C_0 is supercritical, which implies that there exists $(\lambda,u)\in C_0$ with $\lambda_0>\lambda_1$. For any $\lambda<\lambda_1$, let $K=[\lambda,\lambda_0]$. Theorem 3.2 guarantees that $\|u\|_\infty\le L_0$ for all $\lambda\in K$, which together with the unboundedness of C_0 implies that there is u such that $(\lambda,u)\in C_0$.

Step 2. We show that, for $\lambda=\lambda^*$, (1.1) has a positive solution.

Taking a sequence of positive solutions (λ_n,u_n) of (1.1) such that $\lambda_n\le\lambda^*$ and $\lambda_n\to\lambda^*$. Thanks to the bounds of Theorem 3.2, we have that $u_n\to u^*\ge 0$, u^* is a solution for $\lambda=\lambda^\star$. Since $\lambda^*>\lambda_1$ and λ_1 is the unique bifurcation point from the trivial solution, we conclude that $u^*>0$.

Step 3. We show that

$$\lim_{b\to0^+}\lambda^*(b)=+\infty\quad\text{and}\quad\lim_{b\to+\infty}\lambda^*(b)=\lambda_1.$$

Observe that since $\lambda_1<\lambda^*\le L_1$ defined in Theorem 3.2 and $\lim_{b\to\infty}L_1=\lambda_1$, we concluded that

$$\lim_{b\to\infty}\lambda^*(b)=\lambda_1.$$

Now we prove

$$\lim_{b\to0^+}\lambda^*(b)=+\infty.$$

It suffices to show that, for any $\lambda>\lambda_1$, there exists $b>0$ small enough such that (1.1) possesses at least a positive solution. For $\lambda>\lambda_1$, take $\tilde{\Omega}\supset\Omega$ and consider $\tilde{\varphi}_1$ and $\tilde{\lambda_1}$ the positive eigenfunction and eigenvalue associated with $\tilde{\Omega}$. Choose K large enough such that

$$f\bigl(K\tilde{\varphi}_1(x)\bigr)-\lambda>0,\quad\forall x\in\overline{\Omega}.$$

Choose $b>0$ small enough such that

$$f\bigl(K\tilde{\varphi}_1(x)\bigr)-\lambda-b\int_{\Omega}(K\tilde{\varphi}_1)^r\,dx>0,\quad\forall x\in\overline{\Omega}.\tag{4.6}$$

Choose $\varepsilon > 0$ small enough such that $\varepsilon\varphi_1 < K\tilde{\varphi}_1$ and

$$\lambda - \lambda_1 + b\int_\Omega (\varepsilon\varphi_1)^r\,dx - f(\varepsilon\varphi_1) > 0, \quad \forall x \in \Omega. \tag{4.7}$$

Let $\underline{u}(x) = \varepsilon\varphi_1(x)$ and $\overline{u}(x) = K\tilde{\varphi}_1(x)$ for $x \in \overline{\Omega}$. From (4.6) and (4.7), we have

(a) $\underline{u} = \varepsilon\varphi_1 < K\tilde{\varphi}_1 = \overline{u}(x)$ in Ω and $\underline{u} = 0 < K\tilde{\varphi}_1 = \overline{u}$ on $\partial\Omega$;

(b)

$$\begin{aligned} -\Delta\underline{u} - \underline{u}\left(\lambda + b\int_\Omega u^r\,dx - f(\underline{u})\right) &= \varepsilon\lambda_1\varphi_1 - \varepsilon\varphi_1\left(\lambda + b\int_\Omega u^r\,dx - f(\varepsilon\varphi_1)\right) \\ &= \varepsilon\varphi_1\left((\lambda_1 - \lambda) - b\int_\Omega u^r\,dx + f(\varepsilon\varphi_1)\right) \\ &< 0, \quad x \in \Omega, \forall u \in [\underline{u}, \overline{u}] \end{aligned}$$

and

$$\begin{aligned} -\Delta\overline{u} - \overline{u}\left(\lambda + b\int_\Omega u^r\,dx - f(\overline{u})\right) &= K\tilde{\lambda}\tilde{\varphi}_1 - K\tilde{\varphi}_1\left(\lambda + b\int_\Omega u^r\,dx - f(K\tilde{\varphi}_1)\right) \\ &= K\tilde{\varphi}_1\left(\tilde{\lambda} - \lambda - b\int_\Omega u^r\,dx + f(K\tilde{\varphi}_1)\right) \\ &> 0, \quad x \in \Omega, \forall u \in [\overline{u}, \overline{u}], \end{aligned}$$

which implies that $(\underline{u}, \overline{u})$ is a pair of sub-supersolutions to (1.1). Theorem 4.2 guarantees that (1.1) has at least one positive solution for $\lambda > \lambda_1$.

The proof is complete. □

Theorem 4.4 *Assume that $b > 0$ and $r = 1 < k_n$. There exists $\lambda_* < \lambda_1$ such that (1.1) possesses at least a positive solution if and only if $\lambda \geq \lambda_*$. Moreover,*

$$\lim_{b\to 0^+} \lambda_*(b) = \lambda_1 \quad \textit{and} \quad \lim_{b\to+\infty} \lambda_*(b) = -\infty.$$

The proof of Theorem 4.4 is similar to that of Theorem 4.3, we omit it.

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