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Solutions for a class of fractional Langevin equations with integral and anti-periodic boundary conditions

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Abstract

In this paper, we consider a class of fractional Langevin equations with integral and anti-periodic boundary conditions. By using some fixed point theorems and the Leray–Schauder degree theory, several new existence results of solutions are obtained.

MSC: 34A08; 34B15; 34B18

Keywords: Fractional Langevin equation; Mixed boundary condition; Leray–Schauder degree theory; Fixed point theorem

1 Introduction

In this paper, we consider the existence of solutions for the following fractional Langevin equation with integral and anti-periodic conditions:

$$\begin{cases} {}^cD_{0+}^{\beta}({}^cD_{0+}^{\alpha}+\lambda)x(t)=f(t,x(t)), \quad 0<t<1, \\ x(0)=0, \qquad x(1)=\mu\int_0^1 x(s)\,ds, \qquad {}^cD_{0+}^{\alpha}x(0)+{}^cD_{0+}^{\alpha}x(1)=0, \end{cases} \tag{1}$$

where $0<\alpha<1$, $1<\beta\leq 2$, $\lambda>0$, and $\mu>0$ are real numbers, ${}^cD_{0+}^{\alpha}x(t)$ and ${}^cD_{0+}^{\beta}x(t)$ are the Caputo fractional derivatives, and $f\in C([0,1]\times R,R)$.

Theory of fractional differential equations has important application in many areas. It has become a new research field in differential equations [1–3]. There are a lot of good research results on boundary value problems of fractional differential equations [4–24]. Recently fractional Langevin equations have been studied by some scholars (see, for example, [25–27]).

In [28], via fixed point theorems, Ahmad et al. discussed the existence of solutions for fractional Langevin equations with three-point nonlocal boundary value conditions.

In [29], Li et al. investigated the infinite-point boundary value problem of fractional Langevin equations. By means of the nonlinear alternative and Leray–Schauder degree theory, they got some existence results for the boundary value problem.

To our knowledge, for fractional Langevin equation, the boundary value problem with integral and anti-periodic boundary conditions is rarely studied, so the research of this paper is new.

The rest of this paper is organized as follows. In Sect. 2, we present some preliminaries and lemmas. In Sect. 3, we adopt some fixed point theorems and the Leray–Schauder degree theory to obtain the existence of solutions for boundary value problem (1).

2 Relevant lemmas

Some necessary definitions from fractional calculus theory can be found in [2, 3]. We omit them here.

Lemma 2.1 *Let* $h \in C(0,1) \cap L(0,1)$, $0 < \alpha < 1$, $1 < \beta \leq 2$, *then the following problem*

$$\begin{cases} {}^cD^{\beta}_{0^+}({}^cD^{\alpha}_{0^+} + \lambda)x(t) = h(t), \quad t \in (0,1) \\ x(0) = 0, \quad x(1) = \mu \int_0^1 x(s)\,ds, \quad {}^cD^{\alpha}_{0^+}x(0) + {}^cD^{\alpha}_{0^+}x(1) = 0 \end{cases} \tag{2}$$

has a solution $x(t)$ *satisfying*

$$\begin{aligned} x(t) = {} & \frac{1}{\Gamma(\alpha+\beta)}\int_0^t (t-s)^{\alpha+\beta-1}h(s)\,ds - \frac{t^{\alpha}(2t-\alpha-1)}{(1-\alpha)\Gamma(\alpha+\beta)}\int_0^1 (1-s)^{\alpha+\beta-1}h(s)\,ds \\ & + \frac{\lambda t^{\alpha}(2t-\alpha-1)}{(1-\alpha)\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}x(s)\,ds \\ & - \frac{t^{\alpha}(1-t)}{(1-\alpha)\Gamma(\alpha+1)\Gamma(\beta)}\int_0^1 (1-s)^{\beta-1}h(s)\,ds \\ & + \frac{\lambda\mu t^{\alpha}(1-t)}{(1-\alpha)\Gamma(\alpha+1)}\int_0^1 x(s)\,ds \\ & + \frac{\mu t^{\alpha}(2t-\alpha-1)}{1-\alpha}\int_0^1 x(s)\,ds - \frac{\lambda}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}x(s)\,ds. \end{aligned}$$

Proof From the relevant lemma in [23], it follows that

$$\begin{aligned} & \left({}^cD^{\alpha}_{0^+} + \lambda\right)x(t) = I^{\beta}_{0^+}h(t) + c_0 + c_1 t, \\ & {}^cD^{\alpha}_{0^+}x(t) = I^{\beta}_{0^+}h(t) + c_0 + c_1 t - \lambda x(t) = \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}h(s)\,ds + c_0 + c_1 t - \lambda x(t), \\ & x(t) = I^{\alpha+\beta}_{0^+}h(t) + I^{\alpha}_{0^+}c_0 + I^{\alpha}_{0^+}c_1 t - I^{\alpha}_{0^+}\lambda x(t) + c_2 \\ & \quad = \frac{1}{\Gamma(\alpha+\beta)}\int_0^t (t-s)^{\alpha+\beta-1}h(s)\,ds + \frac{c_0}{\Gamma(\alpha+1)}t^{\alpha} + \frac{c_1}{\Gamma(\alpha+2)}t^{\alpha+1} \\ & \qquad - \frac{\lambda}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}x(s)\,ds + c_2. \end{aligned}$$

By the boundary value conditions $x(0) = 0$, $x(1) = \mu \int_0^1 x(s)\,ds$, and ${}^cD^{\alpha}_{0^+}x(0) + {}^cD^{\alpha}_{0^+}x(1) = 0$, we can get

$$\begin{aligned} & c_2 = 0, \\ & \frac{1}{\Gamma(\alpha+\beta)}\int_0^1 (1-s)^{\alpha+\beta-1}h(s)\,ds + \frac{c_0}{\Gamma(\alpha+1)} + \frac{c_1}{\Gamma(\alpha+2)} - \frac{\lambda}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}x(s)\,ds \\ & \quad = \mu\int_0^1 x(s)\,ds, \end{aligned}$$

and

$$c_0 + \frac{1}{\Gamma(\beta)} \int_0^1 (1-s)^{\beta-1} h(s)\,ds + c_0 + c_1 - \lambda\mu \int_0^1 x(s)\,ds = 0.$$

By direct computation, we have

$$\begin{aligned} c_0 = {}& \frac{\Gamma(\alpha+2)}{(1-\alpha)\Gamma(\alpha+\beta)} \int_0^1 (1-s)^{\alpha+\beta-1} h(s)\,ds - \frac{\lambda\alpha(1+\alpha)}{1-\alpha} \int_0^1 (1-s)^{\alpha-1} x(s)\,ds \\ &+ \frac{\lambda\mu}{1-\alpha} \int_0^1 x(s)\,ds - \frac{\mu\Gamma(\alpha+2)}{1-\alpha} \int_0^1 x(s)\,ds - \frac{1}{(1-\alpha)\Gamma(\beta)} \int_0^1 (1-s)^{\beta-1} h(s)\,ds, \end{aligned}$$

and

$$\begin{aligned} c_1 = {}& \frac{2\lambda\alpha(1+\alpha)}{1-\alpha} \int_0^1 (1-s)^{\alpha-1} x(s)\,ds + \frac{1+\alpha}{(1-\alpha)\Gamma(\beta)} \int_0^1 (1-s)^{\beta-1} h(s)\,ds \\ &- \frac{\lambda\mu(1+\alpha)}{1-\alpha} \int_0^1 x(s)\,ds \\ &+ \frac{2\mu\Gamma(\alpha+2)}{1-\alpha} \int_0^1 x(s)\,ds - \frac{2\Gamma(\alpha+2)}{(1-\alpha)\Gamma(\alpha+\beta)} \int_0^1 (1-s)^{\alpha+\beta-1} h(s)\,ds. \end{aligned}$$

Thus, the solution of (2) satisfies

$$\begin{aligned} x(t) = {}& \frac{1}{\Gamma(\alpha+\beta)} \int_0^t (t-s)^{\alpha+\beta-1} h(s)\,ds - \frac{t^\alpha(2t-\alpha-1)}{(1-\alpha)\Gamma(\alpha+\beta)} \int_0^1 (1-s)^{\alpha+\beta-1} h(s)\,ds \\ &+ \frac{\lambda t^\alpha(2t-\alpha-1)}{(1-\alpha)\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} x(s)\,ds \\ &- \frac{t^\alpha(1-t)}{(1-\alpha)\Gamma(\alpha+1)\Gamma(\beta)} \int_0^1 (1-s)^{\beta-1} h(s)\,ds \\ &+ \frac{\lambda\mu t^\alpha(1-t)}{(1-\alpha)\Gamma(\alpha+1)} \int_0^1 x(s)\,ds + \frac{\mu t^\alpha(2t-\alpha-1)}{1-\alpha} \int_0^1 x(s)\,ds \\ &- \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} x(s)\,ds. \end{aligned}$$

This completes the proof. □

3 Main results

Let $E = C[0,1]$. Obviously, the space E is a Banach space if it is endowed with the norm as follows:

$$\|x\| = \max_{t\in[0,1]} \bigl|x(t)\bigr|.$$

Define an operator $T : E \to E$ as

$$\begin{aligned} (Tx)(t) = {}& \frac{1}{\Gamma(\alpha+\beta)} \int_0^t (t-s)^{\alpha+\beta-1} f\bigl(s,x(s)\bigr)\,ds - \frac{t^\alpha(2t-\alpha-1)}{(1-\alpha)\Gamma(\alpha+\beta)} \\ &\times \int_0^1 (1-s)^{\alpha+\beta-1} f\bigl(s,x(s)\bigr)\,ds + \frac{\lambda t^\alpha(2t-\alpha-1)}{(1-\alpha)\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} x(s)\,ds \end{aligned}$$

$$
\begin{aligned}
&- \frac{t^\alpha(1-t)}{(1-\alpha)\Gamma(\alpha+1)\Gamma(\beta)} \int_0^1 (1-s)^{\beta-1} f\bigl(s,x(s)\bigr)\,ds + \frac{\lambda\mu t^\alpha(1-t)}{(1-\alpha)\Gamma(\alpha+1)} \int_0^1 x(s)\,ds \\
&+ \frac{\mu t^\alpha(2t-\alpha-1)}{1-\alpha} \int_0^1 x(s)\,ds - \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} x(s)\,ds, \quad t \in [0,1]. \qquad (3)
\end{aligned}
$$

It is easy to see that the solution for (1) is equivalent to the fixed point of T.

Lemma 3.1
(i) $\max_{t\in[0,1]} |t^\alpha(2t-\alpha-1)| = \max\{(\frac{\alpha}{2})^\alpha, 1-\alpha\}$;
(ii) $\max_{t\in[0,1]} t^\alpha(1-t) = \frac{\alpha^\alpha}{(1+\alpha)^{1+\alpha}}$.

Proof (i) Let $g(t) = t^\alpha(2t-\alpha-1)$, $0 \le t \le 1$, then $g'(t) = (\alpha+1)t^{\alpha-1}(2t-\alpha)$, $g(0) = 0$, $g(1) = 1-\alpha > 0$.

When $0 \le t < \frac{\alpha}{2}$, $g'(t) \le 0$; when $\frac{\alpha}{2} < t \le 1$, $g'(t) \ge 0$. In conclusion, $|g(t)|_{\max} = \max\{|g(\frac{\alpha}{2})|, g(1))\} = \max\{(\frac{\alpha}{2})^\alpha, 1-\alpha\}$.

(ii) Let $g(t) = t^\alpha(1-t)$, $0 \le t \le 1$, then $g'(t) = t^{\alpha-1}[\alpha-(\alpha+1)t]$.

When $0 \le t < \frac{\alpha}{1+\alpha}$, $g'(t) \ge 0$; when $\frac{\alpha}{1+\alpha} < t \le 1$, $g'(t) \le 0$. So $g(t)_{\max} = g(\frac{\alpha}{1+\alpha}) = \frac{\alpha^\alpha}{(1+\alpha)^{1+\alpha}}$.

The proof is completed. □

Let $\eta = \frac{1}{1-\alpha} \max\{(\frac{\alpha}{2})^\alpha, 1-\alpha\}$.

Lemma 3.2 *$T : E \to E$ is completely continuous.*

Proof Since the continuity of f, $T : E \to E$ is continuous. For any bounded set $D \subset E$, there exists $K > 0$ such that $\forall x \in D$, $\|x\| \le K$. There exists a constant $L_1 > 0$ such that $|f(t,x)| \le L_1$ for any $t \in [0,1]$ and $x \in [-K,K]$. Then $\forall x \in D$, it follows that

$$
\begin{aligned}
\bigl|(Tx)(t)\bigr| \le{}& \frac{1}{\Gamma(\alpha+\beta)} \int_0^t (t-s)^{\alpha+\beta-1} \bigl|f\bigl(s,x(s)\bigr)\bigr|\,ds \\
&+ \frac{\eta}{\Gamma(\alpha+\beta)} \int_0^1 (1-s)^{\alpha+\beta-1} \bigl|f\bigl(s,x(s)\bigr)\bigr|\,ds \\
&+ \frac{\lambda\eta}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} \bigl|x(s)\bigr|\,ds \\
&+ \frac{\alpha^\alpha}{(1+\alpha)^{1+\alpha}(1-\alpha)\Gamma(\alpha+1)\Gamma(\beta)} \int_0^1 (1-s)^{\beta-1} \bigl|f\bigl(s,x(s)\bigr)\bigr|\,ds \\
&+ \frac{\lambda\mu\alpha^\alpha}{(1+\alpha)^{1+\alpha}(1-\alpha)\Gamma(\alpha+1)} \int_0^1 \bigl|x(s)\bigr|\,ds + \mu\eta \int_0^1 \bigl|x(s)\bigr|\,ds \\
&+ \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \bigl|x(s)\bigr|\,ds \\
\le{}& \frac{L_1}{\Gamma(\alpha+\beta)} \int_0^t (t-s)^{\alpha+\beta-1}\,ds + \frac{L_1\eta}{\Gamma(\alpha+\beta)} \int_0^1 (1-s)^{\alpha+\beta-1}\,ds \\
&+ \frac{\eta\lambda\|x\|}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1}\,ds \\
&+ \frac{\alpha^\alpha L_1}{(1+\alpha)^{1+\alpha}(1-\alpha)\Gamma(\alpha+1)\Gamma(\beta)} \int_0^1 (1-s)^{\beta-1}\,ds \\
&+ \frac{\lambda\mu\alpha^\alpha\|x\|}{(1+\alpha)^{1+\alpha}(1-\alpha)\Gamma(\alpha+1)}
\end{aligned}
$$

$$
\begin{aligned}
&+\eta\mu\|x\|+\frac{\lambda\|x\|}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\,ds \\
&=\frac{L_1 t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}+\frac{\eta L_1}{\eta\Gamma(\alpha+\beta+1)}+\frac{\lambda\|x\|}{\Gamma(\alpha+1)} \\
&\quad+\frac{\alpha^\alpha L_1}{(1+\alpha)^{1+\alpha}(1-\alpha)\Gamma(\alpha+1)\Gamma(\beta+1)} \\
&\quad+\frac{\lambda\mu\alpha^\alpha\|x\|}{(1+\alpha)^{1+\alpha}(1-\alpha)\Gamma(\alpha+1)}+\eta\mu\|x\|+\frac{\lambda\|x\|t^\alpha}{\Gamma(\alpha+1)} \\
&\le\frac{(1+\eta)L_1}{\Gamma(\alpha+\beta+1)}+\frac{(1+\eta)\lambda\|x\|}{\Gamma(\alpha+1)}+\frac{\alpha^\alpha L_1}{(1+\alpha)^{1+\alpha}(1-\alpha)\Gamma(\alpha+1)\Gamma(\beta+1)} \\
&\quad+\frac{\lambda\mu\alpha^\alpha\|x\|}{(1+\alpha)^{1+\alpha}(1-\alpha)\Gamma(\alpha+1)}+\eta\mu\|x\| \\
&\le\frac{(1+\eta)L_1}{\Gamma(\alpha+\beta+1)}+\frac{(1+\eta)\lambda K}{\Gamma(\alpha+1)}+\frac{\alpha^\alpha L_1}{(1+\alpha)^{1+\alpha}(1-\alpha)\Gamma(\alpha+1)\Gamma(\beta+1)} \\
&\quad+\frac{\lambda\mu\alpha^\alpha K}{(1+\alpha)^{1+\alpha}(1-\alpha)\Gamma(\alpha+1)}+\eta\mu K,
\end{aligned}
$$

which implies that TD is uniformly bounded.

In addition, for $x\in D$, $0\le t_1<t_2\le 1$, we have

$$
\begin{aligned}
&\bigl|(Tx)(t_2)-(Tx)(t_1)\bigr| \\
&=\Bigl|\frac{1}{\Gamma(\alpha+\beta)}\int_0^{t_2}(t_2-s)^{\alpha+\beta-1}f\bigl(s,x(s)\bigr)\,ds-\frac{t_2^\alpha(2t_2-\alpha-1)}{(1-\alpha)\Gamma(\alpha+\beta)} \\
&\quad\times\int_0^1(1-s)^{\alpha+\beta-1}f\bigl(s,x(s)\bigr)\,ds+\frac{\lambda t_2^\alpha(2t_2-\alpha-1)}{(1-\alpha)\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}x(s)\,ds \\
&\quad-\frac{t_2^\alpha(1-t_2)}{(1-\alpha)\Gamma(\alpha+1)\Gamma(\beta)}\int_0^1(1-s)^{\beta-1}f\bigl(s,x(s)\bigr)\,ds+\frac{\lambda\mu t_2^\alpha(1-t_2)}{(1-\alpha)\Gamma(\alpha+1)}\int_0^1 x(s)\,ds \\
&\quad+\frac{\mu t_2^\alpha(2t_2-\alpha-1)}{1-\alpha}\int_0^1 x(s)\,ds-\frac{\lambda}{\Gamma(\alpha)}\int_0^{t_2}(t_2-s)^{\alpha-1}x(s)\,ds \\
&\quad-\frac{1}{\Gamma(\alpha+\beta)}\int_0^{t_1}(t_1-s)^{\alpha+\beta-1}f\bigl(s,x(s)\bigr)\,ds \\
&\quad+\frac{t_1^\alpha(2t_1-\alpha-1)}{(1-\alpha)\Gamma(\alpha+\beta)}\int_0^1(1-s)^{\alpha+\beta-1}f\bigl(s,x(s)\bigr)\,ds \\
&\quad-\frac{\lambda t_1^\alpha(2t_1-\alpha-1)}{(1-\alpha)\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}x(s)\,ds \\
&\quad+\frac{t_1^\alpha(1-t_1)}{(1-\alpha)\Gamma(\alpha+1)\Gamma(\beta)}\int_0^1(1-s)^{\beta-1}f\bigl(s,x(s)\bigr)\,ds \\
&\quad-\frac{\lambda\mu t_1^\alpha(1-t_1)}{(1-\alpha)\Gamma(\alpha+1)}\int_0^1 x(s)\,ds \\
&\quad-\frac{\mu t_1^\alpha(2t_1-\alpha-1)}{1-\alpha}\int_0^1 x(s)\,ds+\frac{\lambda}{\Gamma(\alpha)}\int_0^{t_1}(t_1-s)^{\alpha-1}x(s)\,ds \\
&=\biggl|\int_0^{t_1}\frac{f(s,x(s))}{\Gamma(\alpha+\beta)}\bigl[(t_2-s)^{\alpha+\beta-1}-(t_1-s)^{\alpha+\beta-1}\bigr]\,ds+\int_{t_1}^{t_2}\frac{f(s,x(s))}{\Gamma(\alpha+\beta)}(t_2-s)^{\alpha+\beta-1}\,ds
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{(\alpha+1)(t_2^\alpha - t_1^\alpha) + 2(t_1^{\alpha+1} - t_2^{\alpha+1})}{(1-\alpha)\Gamma(\alpha+\beta)} \int_0^1 (1-s)^{\alpha+\beta-1} f(s,x(s))\,ds \\
&+ \frac{\lambda(\alpha+1)(t_1^\alpha - t_2^\alpha) + 2\lambda(t_2^{\alpha+1} - t_1^{\alpha+1})}{(1-\alpha)\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} x(s)\,ds \\
&+ \frac{(t_2^{\alpha+1} - t_1^{\alpha+1}) + (t_1^\alpha - t_2^\alpha)}{(1-\alpha)\Gamma(\alpha+1)\Gamma(\beta)} \int_0^1 (1-s)^{\beta-1} f(s,x(s))\,ds \\
&+ \frac{\lambda\mu(t_1^{\alpha+1} - t_2^{\alpha+1}) + \lambda\mu(t_2^\alpha - t_1^\alpha)}{(1-\alpha)\Gamma(\alpha+1)} \int_0^1 x(s)\,ds \\
&+ \frac{2\mu(t_2^{\alpha+1} - t_1^{\alpha+1}) + \mu(\alpha+1)(t_1^\alpha - t_2^\alpha)}{1-\alpha} \int_0^1 x(s)\,ds \\
&+ \int_0^{t_1} \frac{\lambda x(s)}{\Gamma(\alpha)} \big[(t_1-s)^{\alpha-1} - (t_2-s)^{\alpha-1}\big]\,ds - \int_{t_1}^{t_2} \frac{\lambda x(s)}{\Gamma(\alpha)} (t_2-s)^{\alpha-1}\,ds \Bigg| \\
\le{}& \frac{L_1|t_2^{\alpha+\beta} - t_1^{\alpha+\beta}|}{\Gamma(\alpha+\beta+1)} + \frac{L_1|(\alpha+1)(t_2^\alpha - t_1^\alpha) + 2(t_1^{\alpha+1} - t_2^{\alpha+1})|}{(1-\alpha)\Gamma(\alpha+\beta+1)} \\
&+ \frac{|\lambda(\alpha+1)(t_1^\alpha - t_2^\alpha) + 2\lambda(t_2^{\alpha+1} - t_1^{\alpha+1})|\|x\|}{(1-\alpha)\Gamma(\alpha+1)} \\
&+ \frac{L_1|(t_2^{\alpha+1} - t_1^{\alpha+1}) + (t_1^\alpha - t_2^\alpha)|}{(1-\alpha)\Gamma(\alpha+1)\Gamma(\beta+1)} + \frac{|\lambda\mu(t_1^{\alpha+1} - t_2^{\alpha+1}) + \lambda\mu(t_2^\alpha - t_1^\alpha)|\|x\|}{(1-\alpha)\Gamma(\alpha+1)} \\
&+ \frac{|2\mu(t_2^{\alpha+1} - t_1^{\alpha+1}) + \mu(\alpha+1)(t_1^\alpha - t_2^\alpha)|\|x\|}{1-\alpha} + \frac{\lambda\|x\||t_1^\alpha - t_2^\alpha|}{\Gamma(\alpha+1)} \\
\le{}& \frac{L_1}{\Gamma(\alpha+\beta+1)}\big(t_2^{\alpha+\beta} - t_1^{\alpha+\beta}\big) + \bigg[\frac{2L_1}{(1-\alpha)\Gamma(\alpha+\beta+1)} + \frac{2\lambda K}{(1-\alpha)\Gamma(\alpha+1)} \\
&+ \frac{L_1}{(1-\alpha)\Gamma(\alpha+1)\Gamma(\beta+1)} + \frac{\lambda\mu K}{(1-\alpha)\Gamma(\alpha+1)} + \frac{2\mu K}{1-\alpha}\bigg]\big(t_2^{\alpha+1} - t_1^{\alpha+1}\big) \\
&+ \bigg[\frac{L_1(\alpha+1)}{(1-\alpha)\Gamma(\alpha+\beta+1)} + \frac{\lambda K(\alpha+1)}{(1-\alpha)\Gamma(\alpha+1)} + \frac{L_1}{(1-\alpha)\Gamma(\alpha+1)\Gamma(\beta+1)} \\
&+ \frac{\lambda\mu K}{(1-\alpha)\Gamma(\alpha+1)} + \frac{\mu K(\alpha+1)}{1-\alpha} + \frac{\lambda K}{\Gamma(\alpha+1)}\bigg]\big(t_2^\alpha - t_1^\alpha\big),
\end{aligned}
$$

which implies that TD is equicontinuous. Thus, by the *Arzel à–Ascoli* theorem, $T : E \to E$ is completely continuous.

The proof is completed. □

Theorem 3.1 *Suppose that f satisfies the Lipschitz condition*

$$
\big|f(t,x) - f(t,y)\big| \le L|x-y|, \quad \forall t \in [0,1], x, y \in R,
$$

and

$$
\begin{aligned}
A ={}& \frac{(1+\eta)L}{\Gamma(\alpha+\beta+1)} + \frac{(1+\eta)\lambda}{\Gamma(\alpha+1)} + \frac{\alpha^\alpha L}{(1+\alpha)^{1+\alpha}(1-\alpha)\Gamma(\alpha+1)\Gamma(\beta+1)} \\
&+ \frac{\lambda\mu\alpha^\alpha}{(1+\alpha)^{1+\alpha}(1-\alpha)\Gamma(\alpha+1)} + \eta\mu < 1.
\end{aligned}
$$

Then (1) *has a unique solution.*

Proof Define $Q = \max_{t\in[0,1]} |f(t,0)|$ and select $r \geq \frac{\frac{2Q}{\Gamma(\alpha+\beta+1)}+\frac{\alpha^\alpha Q}{(1+\alpha)^{1+\alpha}(1-\alpha)\Gamma(\alpha+1)\Gamma(\beta+1)}}{1-A}$, define a closed ball as $B_r = \{x \in E : \|x\| \leq r\}$, then for $x \in B_r$, we have

$$\begin{aligned}
\bigl|(Tx)(t)\bigr| \leq{}& \frac{1}{\Gamma(\alpha+\beta)}\int_0^t (t-s)^{\alpha+\beta-1}\bigl|f\bigl(s,x(s)\bigr)\bigr|\,ds \\
&+ \frac{\eta}{\Gamma(\alpha+\beta)}\int_0^1 (1-s)^{\alpha+\beta-1}\bigl|f\bigl(s,x(s)\bigr)\bigr|\,ds \\
&+ \frac{\lambda\eta}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}\bigl|x(s)\bigr|\,ds \\
&+ \frac{\alpha^\alpha}{(1+\alpha)^{1+\alpha}(1-\alpha)\Gamma(\alpha+1)\Gamma(\beta)}\int_0^1 (1-s)^{\beta-1}\bigl|f\bigl(s,x(s)\bigr)\bigr|\,ds \\
&+ \frac{\lambda\mu\alpha^\alpha}{(1+\alpha)^{1+\alpha}(1-\alpha)\Gamma(\alpha+1)}\int_0^1 \bigl|x(s)\bigr|\,ds + \mu\eta\int_0^1 \bigl|x(s)\bigr|\,ds \\
&+ \frac{\lambda}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\bigl|x(s)\bigr|\,ds \\
\leq{}& \frac{1}{\Gamma(\alpha+\beta)}\int_0^t (t-s)^{\alpha+\beta-1}\bigl(\bigl|f\bigl(s,x(s)\bigr)-f(s,0)\bigr| + \bigl|f(s,0)\bigr|\bigr)\,ds \\
&+ \frac{\eta}{\Gamma(\alpha+\beta)}\int_0^1 (1-s)^{\alpha+\beta-1}\bigl(\bigl|f\bigl(s,x(s)\bigr)-f(s,0)\bigr| + \bigl|f(s,0)\bigr|\bigr)\,ds \\
&+ \frac{\lambda\eta}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}\bigl|x(s)\bigr|\,ds + \frac{\alpha^\alpha}{(1+\alpha)^{1+\alpha}(1-\alpha)\Gamma(\alpha+1)\Gamma(\beta)} \\
&\times \int_0^1 (1-s)^{\beta-1}\bigl(\bigl|f\bigl(s,x(s)\bigr)-f(s,0)\bigr| + \bigl|f(s,0)\bigr|\bigr)\,ds \\
&+ \frac{\lambda\mu\alpha^\alpha}{(1+\alpha)^{1+\alpha}(1-\alpha)\Gamma(\alpha+1)}\int_0^1 \bigl|x(s)\bigr|\,ds + \mu\eta\int_0^1 \bigl|x(s)\bigr|\,ds \\
&+ \frac{\lambda}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\bigl|x(s)\bigr|\,ds \\
\leq{}& \frac{(Lr+Q)t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + \frac{\eta(Lr+Q)}{\Gamma(\alpha+\beta+1)} + \frac{\lambda r\eta}{\Gamma(\alpha+1)} \\
&+ \frac{\alpha^\alpha(Lr+Q)}{(1+\alpha)^{1+\alpha}(1-\alpha)\Gamma(\alpha+1)\Gamma(\beta+1)} \\
&+ \frac{\lambda\mu\alpha^\alpha r}{(1+\alpha)^{1+\alpha}(1-\alpha)\Gamma(\alpha+1)} + \eta\mu r + \frac{\lambda r t^\alpha}{\Gamma(\alpha+1)} \\
\leq{}& \frac{(1+\eta)(Lr+Q)}{\Gamma(\alpha+\beta+1)} + \frac{(1+\eta)\lambda r}{\Gamma(\alpha+1)} + \frac{\alpha^\alpha(Lr+Q)}{(1+\alpha)^{1+\alpha}(1-\alpha)\Gamma(\alpha+1)\Gamma(\beta+1)} \\
&+ \frac{\lambda\mu\alpha^\alpha r}{(1+\alpha)^{1+\alpha}(1-\alpha)\Gamma(\alpha+1)} + \eta\mu r \leq r,
\end{aligned}$$

which implies that $\|Tx\| \leq r$, that is, $T(B_r) \subset B_r$. In what follows, for $x, y \in E$, for each $t \in [0,1]$, we can get that

$$\begin{aligned}
&\bigl|(Tx)(t) - (Ty)(t)\bigr| \\
&\quad\leq \frac{1}{\Gamma(\alpha+\beta)}\int_0^t (t-s)^{\alpha+\beta-1}\bigl|f\bigl(s,x(s)\bigr) - f\bigl(s,y(s)\bigr)\bigr|\,ds
\end{aligned}$$

$$
\begin{aligned}
&+\frac{\eta}{\Gamma(\alpha+\beta)}\int_0^1(1-s)^{\alpha+\beta-1}\bigl|f(s,x(s))-f(s,y(s))\bigr|\,ds\\
&+\frac{\lambda\eta}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}\bigl|x(s)-y(s)\bigr|\,ds\\
&+\frac{\alpha^\alpha}{(1+\alpha)^{1+\alpha}(1-\alpha)\Gamma(\alpha+1)\Gamma(\beta)}\int_0^1(1-s)^{\beta-1}\bigl|f(s,x(s))-f(s,y(s))\bigr|\,ds\\
&+\frac{\lambda\mu\alpha^\alpha}{(1+\alpha)^{1+\alpha}(1-\alpha)\Gamma(\alpha+1)}\int_0^1\bigl|x(s)-y(s)\bigr|\,ds+\mu\eta\int_0^1\bigl|x(s)-y(s)\bigr|\,ds\\
&+\frac{\lambda}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\bigl|x(s)-y(s)\bigr|\,ds\\
\le{}&\frac{(1+\eta)L\|x-y\|}{\Gamma(\alpha+\beta+1)}+\frac{(1+\eta)\lambda\|x-y\|}{\Gamma(\alpha+1)}+\frac{\alpha^\alpha L\|x-y\|}{(1+\alpha)^{1+\alpha}(1-\alpha)\Gamma(\alpha+1)\Gamma(\beta+1)}\\
&+\frac{\lambda\mu\alpha^\alpha\|x-y\|}{(1+\alpha)^{1+\alpha}(1-\alpha)\Gamma(\alpha+1)}+\eta\mu\|x-y\|=A\|x-y\|,
\end{aligned}
$$

which implies T is a contraction. Thus, by Banach fixed point theorem [30], T has a unique fixed point, that is, (1) has a unique solution.

The proof is completed. □

Theorem 3.2 *Suppose there exist $M>0$ and $c\ge 0$ with $\frac{(1+\eta)c}{\Gamma(\alpha+\beta+1)}+\frac{(1+\eta)\lambda}{\Gamma(\alpha+1)}+\frac{\alpha^\alpha c}{(1+\alpha)^{1+\alpha}(1-\alpha)\Gamma(\alpha+1)\Gamma(\beta+1)}+\frac{\lambda\mu\alpha^\alpha}{(1+\alpha)^{1+\alpha}(1-\alpha)\Gamma(\alpha+1)}+\eta\mu<1$ such that $|f(t,x)|\le c|x|+M$ for $t\in[0,1]$ and $x\in R$, then (1) has at least one solution.*

Proof First we analyze the a priori bound of solutions of the equation $x=\sigma Tx$ for some $\sigma\in[0,1]$.

If $x=\sigma Tx$ for some $\sigma[0,1]$, $x\in E$, then we get

$$
\begin{aligned}
\forall t\in[0,1],\quad \bigl|x(t)\bigr|={}&\bigl|\sigma Tx(t)\bigr|\le\bigl|Tx(t)\bigr|\le\frac{1}{\Gamma(\alpha+\beta)}\int_0^t(t-s)^{\alpha+\beta-1}\bigl|f(s,x(s))\bigr|\,ds\\
&+\frac{\eta}{\Gamma(\alpha+\beta)}\int_0^1(1-s)^{\alpha+\beta-1}\bigl|f(s,x(s))\bigr|\,ds\\
&+\frac{\lambda\eta}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}\bigl|x(s)\bigr|\,ds\\
&+\frac{\alpha^\alpha}{(1+\alpha)^{1+\alpha}(1-\alpha)\Gamma(\alpha+1)\Gamma(\beta)}\int_0^1(1-s)^{\beta-1}\bigl|f(s,x(s))\bigr|\,ds\\
&+\frac{\lambda\mu\alpha^\alpha}{(1+\alpha)^{1+\alpha}(1-\alpha)\Gamma(\alpha+1)}\int_0^1\bigl|x(s)\bigr|\,ds+\mu\eta\int_0^1\bigl|x(s)\bigr|\,ds\\
&+\frac{\lambda}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\bigl|x(s)\bigr|\,ds\\
\le{}&\frac{(1+\eta)2(c\|x\|+M)}{\Gamma(\alpha+\beta+1)}+\frac{(1+\eta)\lambda\|x\|}{\Gamma(\alpha+1)}\\
&+\frac{\alpha^\alpha(c\|x\|+M)}{(1+\alpha)^{1+\alpha}(1-\alpha)\Gamma(\alpha+1)\Gamma(\beta+1)}\\
&+\frac{\lambda\mu\alpha^\alpha\|x\|}{(1+\alpha)^{1+\alpha}(1-\alpha)\Gamma(\alpha+1)}+\eta\mu\|x\|.
\end{aligned}
$$

So

$$\|x\| \le \frac{(1+\eta)(c\|x\|+M)}{\Gamma(\alpha+\beta+1)} + \frac{(1+\eta)\lambda\|x\|}{\Gamma(\alpha+1)} + \frac{\alpha^\alpha(c\|x\|+M)}{(1+\alpha)^{1+\alpha}(1-\alpha)\Gamma(\alpha+1)\Gamma(\beta+1)} + \frac{\lambda\mu\alpha^\alpha\|x\|}{(1+\alpha)^{1+\alpha}(1-\alpha)\Gamma(\alpha+1)} + \eta\mu\|x\|.$$

This implies

$$\|x\| \le \frac{\frac{2M}{\Gamma(\alpha+\beta+1)} + \frac{\alpha^\alpha M}{(1+\alpha)^{1+\alpha}(1-\alpha)\Gamma(\alpha+1)\Gamma(\beta+1)}}{1-(\frac{(1+\eta)c}{\Gamma(\alpha+\beta+1)} + \frac{(1+\eta)\lambda}{\Gamma(\alpha+1)} + \frac{\alpha^\alpha c}{(1+\alpha)^{1+\alpha}(1-\alpha)\Gamma(\alpha+1)\Gamma(\beta+1)} + \frac{\lambda\mu\alpha^\alpha}{(1+\alpha)^{1+\alpha}(1-\alpha)\Gamma(\alpha+1)} + \eta\mu)} := B.$$

Let $\gamma = B + 1$. Set $P_\gamma = \{x \in E : \| x \| < \gamma\}$, then $\forall x \in \partial P_\gamma$, $\| x \| = \gamma > B$. Now, we consider $T : \overline{P_\gamma} \to E$. By the above analysis, it follows that $x \neq \sigma Tx$ for $\forall x \in \partial P_\gamma, \forall\sigma \in [0,1]$.

Define operators $H_\sigma : E \to E$ ($\forall\sigma \in [0,1]$), as $H_\sigma(x) = x - \sigma Tx$. It is easy to see that

$$\forall\sigma \in [0,1], \forall x \in \partial P_\gamma, \quad H_\sigma(x) = x - \sigma Tx \neq 0.$$

According to Lemma 3.2, T is completely continuous. This yields that $\forall\sigma \in [0,1]$, H_σ is a completely continuous field.

Hence by the homotopy invariance of Leray–Schauder degree, we know that

$$\deg(H_\sigma, P_\gamma, 0) = \deg(H_1, P_\gamma, 0) = \deg(H_0, P_\gamma, 0) = \deg(I, P_\gamma, 0) = 1 \neq 0.$$

By the nonzero property of Leray–Schauder degree, the equation $H_1(x) = x - Tx = 0$ has at least one solution in P_γ, that is, problem (1) has at least one solution. The proof is completed. □

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