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Existence of multiple positive solutions to integral boundary value systems with boundary multiparameters

Eunkyung Ko[1] and Eun Kyoung Lee[2*]

*Correspondence:
eklee@pusan.ac.kr
[2]Department of Mathematics
Education, Pusan National
University, Busan, South Korea
Full list of author information is
available at the end of the article

Abstract

We establish the existence, nonexistence, and multiplicity of positive solutions to semilinear elliptic systems with integral boundary conditions when positive multiparameters vary on the boundary. We prove the results by using sub- and supersolution argument and fixed point index theory.

MSC: 35J55; 35J25; 34B08; 34B10

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1 Introduction

In this paper, we study the existence of multiple positive radial solutions to the following nonlocal boundary value system with boundary parameters on an exterior domain:

$$
\begin{cases}
\Delta u + K_1(|x|)f_1(u(x),v(x)) = 0, & x \in \Omega_e, \\
\Delta v + K_2(|x|)f_2(u(x),v(x)) = 0, & x \in \Omega_e, \\
u(x) \to 0, \quad v(x) \to 0 & \text{as } |x| \to \infty, \\
u(x) - \int_{\Omega_e} l_1(|y|)v(y)\,dy = \lambda & \text{if } |x| = r_0, \\
v(x) - \int_{\Omega_e} l_2(|y|)u(y)\,dy = \mu & \text{if } |x| = r_0,
\end{cases}
\tag{1}
$$

where $\Omega_e = \{x \in (-\infty,\infty)^N : |x| \geq r_0 \text{ for } r_0 > 0, N \geq 3\}$, λ and μ are positive parameters, $K_i \in C((r_0,\infty),(0,\infty))$ is such that $\int_{r_0}^{\infty} r^\nu K_i(r)\,dr < \infty$ for some $\nu > 1$, $f_i \in C([0,\infty)^2,[0,\infty))$, and $l_i \in L^1((r_0,\infty))$ are nonnegative functions satisfying $0 < w_N r_0^{N-2} \int_{r_0}^{\infty} r l_i(r)\,dr < 1$ for $i = 1,2$, where w_N is the surface area of the unit sphere in $(-\infty,\infty)^N$.

Differential equations with integral boundary conditions arise in various areas of applied mathematics and physics like heat conduction, chemical engineering, underground water flow, and thermo-elasticity and plasma phenomena. For integral boundary value problems, we refer to [1, 5, 6, 8, 10–13], and the references therein.

Note that the change of variables $r = |x|$ and $t = (\frac{r}{r_0})^{2-N}$ transforms (1) into

$$
\begin{cases}
u''(t) + a_1(t)f_1(u(t), v(t)) = 0, & t \in (0, 1), \\
v''(t) + a_2(t)f_2(u(t), v(t)) = 0, & t \in (0, 1), \\
u(0) = 0 = v(0), \\
u(1) - \int_0^1 g_1(s)v(s)\, ds = \lambda, \\
v(1) - \int_0^1 g_2(s)u(s)\, ds = \mu,
\end{cases}
\tag{2}
$$

with

$$
a_i(t) = \left(\frac{1}{N-2}\right)^2 r_0^2 t^{\frac{-2(N-1)}{N-2}} K_i\left(r_0 t^{\frac{-1}{N-2}}\right),
$$

$$
g_i(t) = w_N \left(\frac{1}{N-2}\right) r_0^N t^{\frac{-2(N-1)}{N-2}} l_i\left(r_0 t^{\frac{-1}{N-2}}\right),
$$

where $a_i \in C((0,1), [0, \infty))$ are such that $\int_0^1 s^\alpha (1-s)^\beta a_i(s)\, ds < \infty$ for some $\alpha, \beta \in (0,1)$, and nonnegative functions $g_i \in L^1(0,1)$ are such that $0 < \int_0^1 s g_i(s)\, ds < 1$ for $i = 1, 2$. We further assume that there exists a closed interval $J \subset (0,1)$ of positive measure such that $a_i(t) > 0$ for all $t \in J$ and $i = 1, 2$. Note that the existence of positive solutions of system (2) guarantees the existence of positive radial solutions of (1). Hence, to investigate solutions of (1), we focus on system (2).

Such second-order ordinary differential systems with the coupled integral boundary conditions were considered in [2, 3], and [4]. Most of those results are about the existence of a solution obtained mainly by using the fixed point theorem, the mixed monotone method, and the monotone iterative method. In [4], the authors investigated the existence and uniqueness of positive solutions by using the a priori estimate method and the maximum principle. To the best of our knowledge, the multiplicity of solutions of differential systems with coupled integral boundary conditions has not been treated. In this paper, we study the existence, nonexistence, and multiplicity of positive solutions to a nonlocal boundary value system with integral boundary conditions when two positive multiparameters vary on the boundary. We establish our results by sub- and supersolution arguments and fixed point index theory.

In this paper, we assume the following hypotheses on f_i for $i = 1, 2$.

(H) $f_i : [0, \infty) \times [0, \infty) \to [0, \infty)$ is continuous and increasing in $[0, \infty) \times [0, \infty)$ for $i = 1, 2$.
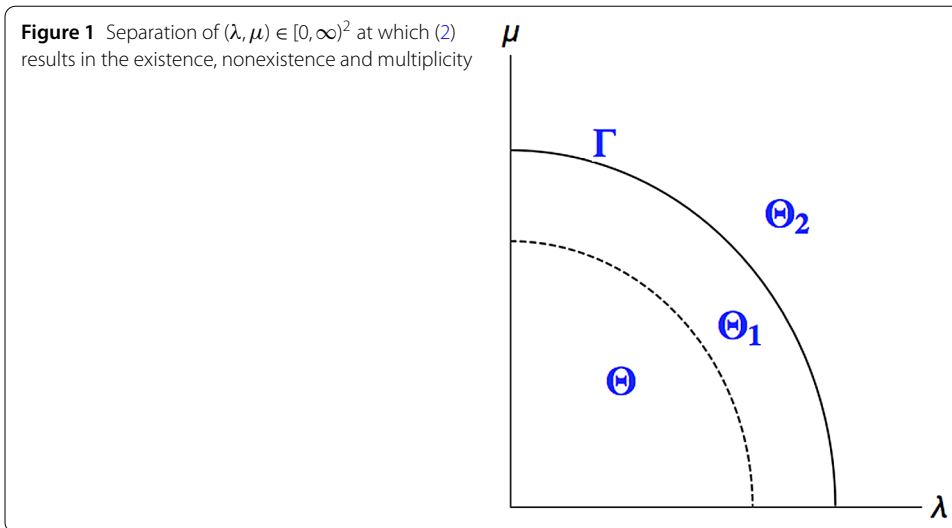
(H1) $f_{i,0} := \lim_{\|(u,v)\| \to 0} \frac{f_i(u,v)}{u+v} = 0$ for $i = 1, 2$.

(H2) $f_{i,\infty} := \lim_{\|(u,v)\| \to \infty} \frac{f_i(u,v)}{u+v} = \infty$ for $i = 1, 2$.

Here $\|(u, v)\| = \|u\|_\infty + \|v\|_\infty$. Now we state our main result.

Theorem 1.1 *There exists a continuous curve Γ separating $[0, \infty) \times [0, \infty) \setminus \{(0,0)\}$ into two disjoint sets Θ_1 and Θ_2, and there exists a subset $\Theta \subset \Theta_1$ such that (2) has at least two positive solutions for $(\lambda, \mu) \in \Theta$, at least one positive solution for $(\lambda, \mu) \in (\Theta_1 \setminus \Theta) \cup \Gamma$, and no positive solution when $(\lambda, \mu) \in \Theta_2$. (See Figure 1.)*

The structure of f_1 and f_2 near $(0, 0)$ and near ∞, that is, conditions (H1) and (H2), plays an important role in the construction of sub- and supersolutions when the parameters λ

Figure 1 Separation of $(\lambda, \mu) \in [0, \infty)^2$ at which (2) results in the existence, nonexistence and multiplicity

and μ vary. Hence system (2) in which two parameters λ and μ involved with only the boundary data is difficult for construction of sub- and supersolutions at the certain value of (λ, μ). To overcome this issue, we manipulate (2) in such a way that the two parameters λ and μ are related with f_1 and f_2 (see (4) in Sect. 2). Here we also emphasize that we consider the case where system (2) satisfies the condition $f_i(0, 0) = 0$ for $i = 1, 2$ and the boundary condition $u(0) = 0 = v(0)$ at the same time. This is a challenging case since these conditions do not allow us to construct a proper open set, which is very crucial for applying the fixed point index theory. Hence we perturb the boundary values of the manipulated system (4) so that the fixed point index argument works well (see (5) in Sect. 2).

The paper is organized as follows. In the next section, we present a system equivalent to (2) and introduce the sub- and supersolution theorem and fixed point index. Section 3 is devoted to the proof of the main result, Theorem 1.1. In Appendix, we prove the sub- and supersolution theorem for (2).

2 Preliminaries

Let $p_{\lambda,\mu}$ and $q_{\lambda,\mu}$ be the solutions, respectively, of the following problems:

$$\begin{cases} p''(t) = 0, & t \in (0, 1), \\ p(0) = 0, \\ p(1) - \int_0^1 g_1(s)q(s)\,ds = \lambda, \end{cases}$$

and

$$\begin{cases} q''(t) = 0, & t \in (0, 1), \\ q(0) = 0, \\ q(1) - \int_0^1 g_2(s)p(s)\,ds = \mu. \end{cases}$$

It is easy to see that

$$p_{\lambda,\mu}(t) = \frac{1}{1-\sigma_1\sigma_2}(\lambda + \sigma_1\mu)t, \qquad q_{\lambda,\mu}(t) = \frac{1}{1-\sigma_1\sigma_2}(\sigma_2\lambda + \mu)t, \quad t \in [0, 1],$$

where $\sigma_1 = \int_0^1 sg_1(s)\,ds$ and $\sigma_2 = \int_0^1 sg_2(s)\,ds$. We denote

$$p_{\lambda,\mu}(t) = (\rho\lambda + \rho_1\mu)t, \qquad q_{\lambda,\mu}(t) = (\rho_2\lambda + \rho\mu)t, \tag{3}$$

where $\rho = \frac{1}{1-\sigma_1\sigma_2}$, $\rho_1 = \frac{\sigma_1}{1-\sigma_1\sigma_2}$, and $\rho_2 = \frac{\sigma_2}{1-\sigma_1\sigma_2}$. Note that ρ, ρ_1, and ρ_2 are positive constants.

Now if (w_1, w_2) is a solution of the system

$$\begin{cases} w_1''(t) + a_1(t)f_1(w_1(t) + p_{\lambda,\mu}(t), w_2(t) + q_{\lambda,\mu}(t)) = 0, & t \in (0,1), \\ w_2''(t) + a_2(t)f_2(w_1(t) + p_{\lambda,\mu}(t), w_2(t) + q_{\lambda,\mu}(t)) = 0, & t \in (0,1), \\ w_1(0) = 0 = w_2(0), \\ w_1(1) - \int_0^1 g_1(s)w_2(s)\,ds = 0, \\ w_2(1) - \int_0^1 g_2(s)w_1(s)\,ds = 0, \end{cases} \tag{4}$$

then $(w_1 + p_{\lambda,\mu}, w_2 + q_{\lambda,\mu})$ is a solution of (2) since

$$\begin{cases} (w_1 + p_{\lambda,\mu})''(t) = w_1''(t) = -a_1(t)f_1(w_1 + p_{\lambda,\mu}, w_2 + q_{\lambda,\mu}), & t \in (0,1), \\ (w_1 + p_{\lambda,\mu})(0) = 0, \\ (w_1 + p_{\lambda,\mu})(1) = \int_0^1 g_1(s)(w_2(s) + q_{\lambda,\mu}(s))\,ds + \lambda, \end{cases}$$

and, similarly, $w_2 + q_{\lambda,\mu}$ satisfies (2). Hence, instead of studying system (2), we consider system (4) to get the existence and multiplicity of positive solutions of (2). However, for system (4) satisfying the conditions $f_i(0,0) = 0$ and $w_i(0) = 0$, $i = 1, 2$, there is an obstacle in using a fixed point index argument. To solve this issue, we employ the perturbed boundary condition.

For a while, let us consider the following problem with more general boundary conditions for $0 \leq \delta < \tilde{\delta}$ for sufficiently small $\tilde{\delta}$ ($\tilde{\delta}$ will be chosen in Lemma 3.3):

$$\begin{cases} w_1''(t) + a_1(t)f_1(w_1(t) + p_{\lambda,\mu}(t), w_2(t) + q_{\lambda,\mu}(t)) = 0, & t \in (0,1), \\ w_2''(t) + a_2(t)f_2(w_1(t) + p_{\lambda,\mu}(t), w_2(t) + q_{\lambda,\mu}(t)) = 0, & t \in (0,1), \\ w_1(0) = \delta = w_2(0), \\ w_1(1) - \int_0^1 g_1(s)w_2(s)\,ds = \delta, \\ w_2(1) - \int_0^1 g_2(s)w_1(s)\,ds = \delta. \end{cases} \tag{5}$$

We set up an operator equation for problem (5). We define

$$A_{\lambda,\mu,\delta}(w_1, w_2)(t) := \delta\big[1 + (\rho v_1 + \rho_1 v_2)t\big] + \int_0^1 H_2(t,s)a_1(s)f_1(w_1 + p_{\lambda,\mu}, w_2 + q_{\lambda,\mu})$$
$$+ K_1(t,s)a_2(s)f_2(w_1 + p_{\lambda,\mu}, w_2 + q_{\lambda,\mu})\,ds,$$

$$B_{\lambda,\mu,\delta}(w_1, w_2)(t) := \delta\big[1 + (\rho v_2 + \rho_2 v_1)t\big] + \int_0^1 K_2(t,s)a_1(s)f_1(w_1 + p_{\lambda,\mu}, w_2 + q_{\lambda,\mu})$$
$$+ H_1(t,s)a_2(s)f_2(w_1 + p_{\lambda,\mu}, w_2 + q_{\lambda,\mu})\,ds,$$

where

$$
\begin{cases}
H_1(t,s) = G(t,s) + \rho_2 t \int_0^1 G(\tau,s)g_1(\tau)\,d\tau, \\
H_2(t,s) = G(t,s) + \rho_1 t \int_0^1 G(\tau,s)g_2(\tau)\,d\tau,
\end{cases}
$$

$$
K_1(t,s) = \rho t \int_0^1 G(\tau,s)g_1(\tau)\,d\tau, \qquad K_2(t,s) = \rho t \int_0^1 G(\tau,s)g_2(\tau)\,d\tau,
$$

$$
G(t,s) =
\begin{cases}
s(1-t), & 0 \le s \le t \le 1, \\
t(1-s), & 0 \le t \le s \le 1,
\end{cases}
$$

and

$$
v_1 = \int_0^1 g_1(s)\,ds, \qquad v_2 = \int_0^1 g_2(s)\,ds.
$$

Then $(v_1(t), v_2(t)) = (A_{\lambda,\mu,\delta}(w_1,w_2)(t), B_{\lambda,\mu,\delta}(w_1,w_2)(t))$ is a solution of

$$
\begin{cases}
v_1''(t) + a_1(t)f_1(w_1(t) + p_{\lambda,\mu}(t), w_2(t) + q_{\lambda,\mu}(t)) = 0, & t \in (0,1), \\
v_2''(t) + a_2(t)f_2(w_1(t) + p_{\lambda,\mu}(t), w_2(t) + q_{\lambda,\mu}(t)) = 0, & t \in (0,1), \\
v_1(0) = \delta = v_2(0), \\
v_1(1) - \int_0^1 g_1(s)v_2(s)\,ds = \delta, \\
v_2(1) - \int_0^1 g_2(s)v_1(s)\,ds = \delta.
\end{cases}
\tag{6}
$$

Now we define

$$
T_{\lambda,\mu,\delta}(w_1,w_2)(t) := \big(A_{\lambda,\mu,\delta}(w_1,w_2)(t), B_{\lambda,\mu,\delta}(w_1,w_2)(t)\big).
$$

Let us denote $X := C([0,1], (-\infty,\infty)) \times C([0,1], (-\infty,\infty))$. Then $T_{\lambda,\mu,\delta} : X \to X$ is well defined on X, where X is the usual Banach space with the norm $\|(u,v)\| = \|u\|_\infty + \|v\|_\infty$. Notice that problem (5) is equivalent to

$$
(w_1,w_2) = T_{\lambda,\mu,\delta}(w_1,w_2) \quad \text{on } X.
\tag{7}
$$

Let $\mathcal{P} = \{(u,v) \in X : u(t) \ge 0, v(t) \ge 0 \text{ for all } t \in [0,1]\}$. Then $\mathcal{P}$ is a cone in X. It is clear that $T_{\lambda,\mu,\delta}(\mathcal{P}) \subset \mathcal{P}$ and $T_{\lambda,\mu,\delta}$ is completely continuous on X by a standard argument.

Now, we introduce a theorem on sub- and supersolutions to system (5). First, we state the following definition of subsolution and supersolution of system (5).

Definition 2.1 We say that (ψ_1, ψ_2) is a *subsolution* of problem (5) if $(\psi_1, \psi_2) \in C^2(0,1) \times C^2(0,1)$ satisfies

$$
\begin{cases}
\psi_1''(t) + a_1(t)f_1(\psi_1(t) + p_{\lambda,\mu}(t), \psi_2(t) + q_{\lambda,\mu}(t)) \ge 0, & t \in (0,1), \\
\psi_2''(t) + a_2(t)f_2(\psi_1(t) + p_{\lambda,\mu}(t), \psi_2(t) + q_{\lambda,\mu}(t)) \ge 0, & t \in (0,1), \\
\psi_1(0) \le \delta, \qquad \psi_2(0) \le \delta, \\
\psi_1(1) - \int_0^1 g_1(s)\psi_2(s)\,ds \le \delta, \\
\psi_2(1) - \int_0^1 g_2(s)\psi_1(s)\,ds \le \delta.
\end{cases}
$$

We also say that (ζ_1, ζ_2) is a *supersolution* of problem (5) if $(\zeta_1, \zeta_2) \in C^2(0,1) \times C^2(0,1)$ satisfies the reverse inequalities.

Theorem 2.1 *Assume that there exist a subsolution* (ψ_1, ψ_2) *and a supersolution* (ζ_1, ζ_2) *of problem* (5) *such that* $(\psi_1(t), \psi_2(t)) \le (\zeta_1(t), \zeta_2(t))$ *for all* $t \in [0,1]$. *Then* (5) *has at least one solution* (u, v) *such that*

$$\big(\psi_1(t), \psi_2(t)\big) \le \big(u(t), v(t)\big) \le \big(\zeta_1(t), \zeta_2(t)\big) \quad \text{for all } t \in [0,1].$$

Proof See Appendix. □

To show the existence of a second positive solution of (4), we need the following lemmas for the fixed point index argument in [7].

Lemma 2.1 *Let X be a Banach space, let P be cone in X, and let Ω be a bounded open set in X. Let $0 \in \Omega$, and let $T : P \cap \bar{\Omega} \to P$ be condensing. Suppose that $Tx \ne \nu x$ for all $x \in P \cap \partial\Omega$ and all $\nu \ge 1$. Then*

$$i(T, P \cap \Omega, P) = 1.$$

Lemma 2.2 *Let X be a Banach space, and let P be a cone in X. For $r > 0$, define $P_r = \{x \in P : \|x\| < r\}$. Assume that $T : \bar{P}_r \to P$ is a compact map such that $Tx \ne x$ for all $x \in \partial P_r$. If $\|x\| \le \|Tx\|$ for all $x \in \partial P_r$, then*

$$i(T, P_r, P) = 0.$$

3　Existence of multiple positive solutions

We recall that $J = [j_*, j^*] \subset (0,1)$ is a closed interval of positive measure such that $a_i(t) > 0$ for all $t \in J$ and $i = 1, 2$. Let $\gamma = \min\{j_*, 1 - j^*\} > 0$. Define the cone

$$\mathcal{K} = \Big\{ (w_1, w_2) \in \mathcal{P} : \min_J w_i(t) \ge \gamma \|w_i\|_\infty \text{ for } i = 1, 2 \Big\}.$$

Then we have the following:

Lemma 3.1 *For a given cone $\mathcal{P}$ in X, we have*

$$T_{\lambda,\mu,\delta}(\mathcal{P}) \subset \mathcal{K}. \tag{8}$$

Proof For any $(w_1, w_2) \in \mathcal{P}$, we first find

$$
\begin{aligned}
A_{\lambda,\mu,\delta}(w_1, w_2)(t) \le{}& \delta(1 + \rho\nu_1 + \rho_1\nu_2) \\
&+ \int_0^1 G(s,s)a_1(s)f_1(w_1 + p_{\lambda,\mu}, w_2 + q_{\lambda,\mu})\,ds \\
&+ \rho_1 \int_0^1 \left(\int_0^1 G(\tau,s)g_2(\tau)\,d\tau \right) a_1(s)f_1(w_1 + p_{\lambda,\mu}, w_2 + q_{\lambda,\mu})\,ds \\
&+ \rho \int_0^1 \left(\int_0^1 G(\tau,s)g_1(\tau)\,d\tau \right) a_2(s)f_2(w_1 + p_{\lambda,\mu}, w_2 + q_{\lambda,\mu})\,ds
\end{aligned}
$$

for all $t \in [0, 1]$. Thus we obtain

$$
\begin{aligned}
\left\|A_{\lambda,\mu,\delta}(w_1, w_2)\right\|_\infty &\leq \delta(1 + \rho v_1 + \rho_1 v_2) \\
&\quad + \int_0^1 G(s, s) a_1(s) f_1(w_1 + p_{\lambda,\mu}, w_2 + q_{\lambda,\mu}) \, ds + \rho_1 M_1 + \rho M_2,
\end{aligned}
\tag{9}
$$

where

$$
M_1 := \int_0^1 \left(\int_0^1 G(\tau, s) g_2(\tau) \, d\tau \right) a_1(s) f_1(w_1 + p_{\lambda,\mu}, w_2 + q_{\lambda,\mu}) \, ds
$$

and

$$
M_2 := \int_0^1 \left(\int_0^1 G(\tau, s) g_1(\tau) \, d\tau \right) a_2(s) f_2(w_1 + p_{\lambda,\mu}, w_2 + q_{\lambda,\mu}) \, ds.
$$

Similarly, we have

$$
\begin{aligned}
&\left\|B_{\lambda,\mu,\delta}(w_1, w_2)\right\|_\infty \\
&\quad \leq \delta(1 + \rho v_2 + \rho_2 v_1) \\
&\qquad + \int_0^1 G(s, s) a_2(s) f_2(w_1 + p_{\lambda,\mu}, w_2 + q_{\lambda,\mu}) \, ds + \rho_2 M_2 + \rho M_1.
\end{aligned}
\tag{10}
$$

Then by (9) we find that, for all $t \in J$,

$$
\begin{aligned}
A_{\lambda,\mu,\delta}(w_1, w_2)(t) &= \delta\big[1 + (\rho v_1 + \rho_1 v_2)t\big] \\
&\quad + \int_0^1 G(t, s) a_1(s) f_1(w_1 + p_{\lambda,\mu}, w_2 + q_{\lambda,\mu}) \, ds + t\rho_1 M_1 + t\rho M_2 \\
&= \delta\big[1 + (\rho v_1 + \rho_1 v_2)t\big] + \int_0^t s(1 - t) a_1(s) f_1(w_1 + p_{\lambda,\mu}, w_2 + q_{\lambda,\mu}) \, ds \\
&\quad + \int_t^1 t(1 - s) a_1(s) f_1(w_1 + p_{\lambda,\mu}, w_2 + q_{\lambda,\mu}) \, ds + t\rho_1 M_1 + t\rho M_2 \\
&\geq \delta\big[1 + (\rho v_1 + \rho_1 v_2)j_*\big] + \left(1 - j^*\right) \int_0^t s a_1(s) f_1(w_1 + p_{\lambda,\mu}, w_2 + q_{\lambda,\mu}) \, ds \\
&\quad + j_* \int_t^1 (1 - s) a_1(s) f_1(w_1 + p_{\lambda,\mu}, w_2 + q_{\lambda,\mu}) \, ds + j_*(\rho_1 M_1 + \rho M_2) \\
&\geq \min\{j_*, 1 - j^*\} \bigg(\delta(1 + \rho v_1 + \rho_1 v_2) \\
&\quad + \int_0^t s(1 - s) a_1(s) f_1(w_1 + p_{\lambda,\mu}, w_2 + q_{\lambda,\mu}) \, ds \\
&\quad + \int_t^1 s(1 - s) a_1(s) f_1(w_1 + p_{\lambda,\mu}, w_2 + q_{\lambda,\mu}) \, ds + \rho_1 M_1 + \rho M_2 \bigg) \\
&= \min\{j_*, 1 - j^*\} \bigg(\delta(1 + \rho v_1 + \rho_1 v_2)
\end{aligned}
$$

$$+ \int_0^1 G(s,s) a_1(s) f_1(w_1 + p_{\lambda,\mu}, w_2 + q_{\lambda,\mu}) \, ds + \rho_1 M_1 + \rho M_2 \Big)$$

$$\geq \gamma \left\| A_{\lambda,\mu}(w_1, w_2) \right\|_\infty.$$

By the same argument, using (10), we also have that $B_{\lambda,\mu,\delta}(w_1, w_2)(t) \geq \gamma \|B_{\lambda,\mu}(w_1, w_2)\|_\infty$ for all $t \in J$. □

As a consequence of the lemma, note that if (w_1, w_2) is a solution of (5), then by (7) we have

$$\inf_{t \in J} w_i(t) \geq \gamma \|w_i\|_\infty \quad \text{for } i = 1, 2. \tag{11}$$

Lemma 3.2 *Assume* (H). *If* (5) *has a positive solution at* $(\bar{\lambda}, \bar{\mu})$, *then* (5) *also has a positive solution at* (λ, μ) *for all* $(\lambda, \mu) \leq (\bar{\lambda}, \bar{\mu})$.

Proof Let $(\bar{w}_1, \bar{w}_2)$ be a positive solution of (5) at $(\bar{\lambda}, \bar{\mu})$, and let $(\lambda, \mu) \in [0, \infty)^2 \setminus \{(0, 0)\}$ with $(\lambda, \mu) \leq (\bar{\lambda}, \bar{\mu})$. Then $(\bar{w}_1, \bar{w}_2)$ is a supersolution of (5) at (λ, μ) since $p_{\lambda,\mu}(t) \leq p_{\bar{\lambda},\bar{\mu}}(t)$ and $q_{\lambda,\mu}(t) \leq q_{\bar{\lambda},\bar{\mu}}(t)$ for $t \in (0, 1)$ and f_i is increasing for each $i = 1, 2$. Clearly, $(0, 0)$ is a subsolution of (5). Notice that $(0, 0)$ is not a solution of (5) since $f_i(p_{\lambda,\mu}(t), q_{\lambda,\mu}(t)) > 0$ for $t \in (0, 1)$ and $i = 1, 2$. Since $(\bar{w}_1, \bar{w}_2) \neq (0, 0)$ and $(\bar{w}_1, \bar{w}_2) \geq (0, 0)$, by Theorem 2.1, (5) has a positive solution at (λ, μ). □

Lemma 3.3 *Assume* (H) *and* (H1). *Then there exists* $(\tilde{\lambda}, \tilde{\mu}) > (0, 0)$ *such that* (5) *has a positive solution for all* $(\lambda, \mu) \leq (\tilde{\lambda}, \tilde{\mu})$.

Proof Let (ψ_1, ψ_2) be the unique solution of the following system:

$$\begin{cases} \psi_1''(t) + a_1(t) = 0 & t \in (0, 1), \\ \psi_2''(t) + a_2(t) = 0 & t \in (0, 1), \\ \psi_1(0) = 0 = \psi_2(0), \\ \psi_1(1) - \int_0^1 g_1(s) \psi_2(s) \, ds = 0, \\ \psi_2(1) - \int_0^1 g_2(s) \psi_1(s) \, ds = 0. \end{cases}$$

We recall that $p_{\lambda,\mu}(t) = (\rho\lambda + \rho_1\mu)t$ and $q_{\lambda,\mu}(t) = (\rho_2\lambda + \rho\mu)t$ and denote $\phi_1(t) = 1 + (\rho v_1 + \rho_1 v_2)t$ and $\phi_2(t) = 1 + (\rho v_2 + \rho_2 v_1)t$. Then let $\alpha = \|\psi_1\|_\infty + \|\psi_2\|_\infty + \rho + \rho_1 + \rho_2 + \rho v_2 + \rho_2 v_1 + \rho v_1 + \rho_1 v_2 + 2$. By (H1) there exist $\tilde{\lambda} \approx 0$, $\tilde{\mu} \approx 0$, and $\tilde{\delta} \approx 0$, sufficiently small, such that

$$f_1\big((\tilde{\lambda} + \tilde{\mu} + \tilde{\delta})\|\psi_1\|_\infty + p_{\tilde{\lambda},\tilde{\mu}}(1) + \tilde{\delta}\phi_1(1), (\tilde{\lambda} + \tilde{\mu} + \tilde{\delta})\|\psi_2\|_\infty + q_{\tilde{\lambda},\tilde{\mu}}(1) + \tilde{\delta}\phi_2(1)\big)$$

$$\leq \frac{1}{\alpha}\big((\tilde{\lambda} + \tilde{\mu} + \tilde{\delta})\|\psi_1\|_\infty + p_{\tilde{\lambda},\tilde{\mu}}(1) + \tilde{\delta}\phi_1(1)$$

$$+ (\tilde{\lambda} + \tilde{\mu} + \tilde{\delta})\|\psi_2\|_\infty + q_{\tilde{\lambda},\tilde{\mu}}(1) + \tilde{\delta}\phi_2(1)\big). \tag{12}$$

Now we define $Z_1(t) = (\tilde{\lambda} + \tilde{\mu} + \tilde{\delta})\psi_1 + \tilde{\delta}\phi_1(t)$ and $Z_2(t) = (\tilde{\lambda} + \tilde{\mu} + \tilde{\delta})\psi_2 + \tilde{\delta}\phi_2(t)$. Then, from (H) and (12) we have $Z_1(0) = \tilde{\delta}$,

$$Z_1''(t) + a_1(t) f_1\big(Z_1(t) + p_{\tilde{\lambda},\tilde{\mu}}(t), Z_2(t) + q_{\tilde{\lambda},\tilde{\mu}}(t)\big)$$

$$= -(\tilde{\lambda} + \tilde{\mu} + \tilde{\delta}) a_1(t) + a_1(t) f_1\big(Z_1(t) + p_{\tilde{\lambda},\tilde{\mu}}(t), Z_2(t) + q_{\tilde{\lambda},\tilde{\mu}}(t)\big)$$

$$\leq a_1(t)\big\{f_1\big((\tilde{\lambda}+\tilde{\mu}+\tilde{\delta})\|\psi_1\|_\infty + p_{\tilde{\lambda},\tilde{\mu}}(1) + \tilde{\delta}\phi_1(1),$$

$$(\tilde{\lambda}+\tilde{\mu}+\tilde{\delta})\|\psi_2\|_\infty + q_{\tilde{\lambda},\tilde{\mu}}(1) + \tilde{\delta}\phi_2(1)\big) - (\tilde{\lambda}+\tilde{\mu}+\tilde{\delta})\big\}$$

$$\leq a_1(t)\bigg[\frac{1}{\alpha}\big((\tilde{\lambda}+\tilde{\mu}+\tilde{\delta})\|\psi_1\|_\infty + p_{\tilde{\lambda},\tilde{\mu}}(1) + \tilde{\delta}\phi_1(1)$$

$$+ (\tilde{\lambda}+\tilde{\mu}+\tilde{\delta})\|\psi_2\|_\infty + q_{\tilde{\lambda},\tilde{\mu}}(1) + \tilde{\delta}\phi_2(1)\big) - (\tilde{\lambda}+\tilde{\mu}+\tilde{\delta})\bigg]$$

$$\leq 0,$$

and

$$Z_1(1) - \int_0^1 g_1(s)Z_2(s)\,ds = \tilde{\delta}\phi_1(1) - \tilde{\delta}\int_0^1 g_1(s)\phi_2(s)\,ds = \tilde{\delta} \geq \delta.$$

Similarly, we can show that Z_2 satisfies the same inequalities. This shows that (Z_1, Z_2) is a supersolution of (5) at $(\tilde{\lambda}, \tilde{\mu})$. On the other hand, $(0,0)$ is a strict subsolution such that $(0,0) \leq (Z_1, Z_2)$ in $[0,1]$. Hence, by Theorem 2.1, (5) has a positive solution at $(\tilde{\lambda}, \tilde{\mu})$, and then Lemma 3.2 completes the proof. □

Lemma 3.4 *Assume* (H2). *There exists $M > 0$ such that $\|(w_1, w_2)\| \leq M$ for all possible solutions (w_1, w_2) of* (5).

Proof As (w_1, w_2) is a solution of (5), we find

$$\|w_1\|_\infty \geq w_1(j_*)$$

$$= \delta\big[1 + (\rho v_1 + \rho_1 v_2)j_*\big]$$

$$+ \int_0^1 \bigg(G(j_*, s) + \rho_1 j_* \int_0^1 G(\tau, s)g_2(\tau)\,d\tau\bigg)a_1(s)f_1(w_1 + p_{\lambda,\mu}, w_2 + q_{\lambda,\mu})$$

$$+ \bigg(\rho j_* \int_0^1 G(\tau, s)g_1(\tau)\,d\tau\bigg)a_2(s)f_2(w_1 + p_{\lambda,\mu}, w_2 + q_{\lambda,\mu})\,ds$$

$$\geq \int_{j_*}^{j^*} \bigg(\rho j_* \int_{j_*}^{j^*} G(\tau, s)g_1(\tau)\,d\tau\bigg)a_2(s)f_2(w_1 + p_{\lambda,\mu}, w_2 + q_{\lambda,\mu})\,ds$$

$$\geq \rho j_* \int_{j_*}^{j^*} g_1(s)\bigg(\int_{j_*}^s \tau(1-s)a_2(\tau)f_2(w_1 + p_{\lambda,\mu}, w_2 + q_{\lambda,\mu})\,d\tau$$

$$+ \int_s^{j^*} s(1-\tau)a_2(\tau)f_2(w_1 + p_{\lambda,\mu}, w_2 + q_{\lambda,\mu})\,d\tau\bigg)\,ds$$

$$\geq \rho j_* \int_{j_*}^{j^*} g_1(s)j_*(1-j^*)\bigg(\int_{j_*}^{j^*} a_2(\tau)f_2(w_1 + p_{\lambda,\mu}, w_2 + q_{\lambda,\mu})\,d\tau\bigg)\,ds$$

$$\geq \rho j_*^2(1-j^*)\bigg(\int_{j_*}^{j^*} g_1(s)\,ds\bigg)\bigg(\int_{j_*}^{j^*} a_2(s)\,ds\bigg)f_2\big(\gamma\|w_1\|_\infty, \gamma\|w_2\|_\infty\big).$$

Hence we obtain

$$\|w_1\|_\infty + \|w_2\|_\infty \geq \gamma^2 \rho j_* \bigg(\int_{j_*}^{j^*} g_1(s)\,ds\bigg)\bigg(\int_{j_*}^{j^*} a_2(s)\,ds\bigg)f_2\big(\gamma\|w_1\|_\infty, \gamma\|w_2\|_\infty\big),$$

which implies that

$$\left(\gamma^3 \rho j_*\right)^{-1} \left(\int_{j_*}^{j^*} g_1(s)\,ds\right)^{-1} \left(\int_{j_*}^{j^*} a_2(s)\,ds\right)^{-1} \geq \frac{f_2(\gamma\|w_1\|_\infty, \gamma\|w_2\|_\infty)}{\gamma\|w_1\|_\infty + \gamma\|w_2\|_\infty}. \tag{13}$$

If $\|w_1\|_\infty \to \infty$, then (13) contradict to (H2). Similarly, we can show that $\|w_2\|_\infty$ is bounded. $\qquad\square$

Define $\mathcal{S} = \{(\lambda, \mu) \in [0, \infty)^2 \setminus \{(0,0)\} : (5) \text{ has a positive solution at } (\lambda, \mu)\}$. Then $\mathcal{S} \neq \emptyset$ by Lemma 3.3.

Lemma 3.5 *Assume* (H) *and* (H2). *Then* $(\mathcal{S}, \leq)$ *is bounded above.*

Proof We claim that there exist $\bar{\lambda}, \bar{\mu} > 0$ such that (5) has no solution for $\lambda > \bar{\lambda}$ or $\mu > \bar{\mu}$. Suppose on the contrary that there exists a sequence (λ_n, μ_n) such that either $\lambda_n \to \infty$ or $\mu_n \to \infty$ and (5) has a positive solution (w_{1n}, w_{2n}) at $\lambda = \lambda_n$ and $\mu = \mu_n$. Without loss of generality, we assume that $\lambda_n \to \infty$. First, we observe from (H2) that there exists R_{f_2} sufficiently large such that

$$f_2(u, v) \geq u + v \quad \text{for all } u + v \geq R_{f_2}. \tag{14}$$

Now we choose λ_n sufficiently large such that

$$\gamma\left(\|w_{1n}\|_\infty + \|w_{2n}\|_\infty\right) + j_*(\rho + \rho_2)\lambda_n + j_*(\rho_1 + \rho)\mu_n \geq R_{f_2}.$$

Then, for such λ_n, from (14) it follows that

$$f_2\left(\gamma\|w_{1n}\|_\infty + (\rho\lambda_n + \rho_1\mu_n)j_*, \gamma\|w_{2n}\|_\infty + (\rho_2\lambda_n + \rho\mu_n)j_*\right)$$
$$\geq \gamma\left(\|w_{1n}\|_\infty + \|w_{2n}\|_\infty\right) + j_*(\rho + \rho_2)\lambda_n + j_*(\rho_1 + \rho)\mu_n. \tag{15}$$

Since w_{1n} is a solution of (5), using (3), (11), and (15), we have

$$\begin{aligned}
\|w_{1n}\|_\infty &\geq w_{1n}(j_*) \\
&= \delta\left[1 + (\rho\nu_1 + \rho_1\nu_2)j_*\right] \\
&\quad + \int_0^1 H_2(j_*, s)a_1(s)f_1(w_{1n} + p_{\lambda_n, \mu_n}, w_{2n} + q_{\lambda_n, \mu_n}) \\
&\quad + K_1(j_*, s)a_2(s)f_2(w_{1n} + p_{\lambda_n, \mu_n}, w_{2n} + q_{\lambda_n, \mu_n})\,ds \\
&\geq \int_{j_*}^{j^*} K_1(j_*, s)a_2(s)f_2\left(\gamma\|w_{1n}\|_\infty + p_{\lambda_n, \mu_n}(j_*), \gamma\|w_{2n}\|_\infty + q_{\lambda_n, \mu_n}(j_*)\right)ds \\
&\geq \left(\gamma\left(\|w_{1n}\|_\infty + \|w_{2n}\|_\infty\right) + j_*(\rho + \rho_2)\lambda_n \right. \\
&\quad \left. + j_*(\rho_1 + \rho)\mu_n\right) \int_{j_*}^{j^*} \left(\rho j_* \int_0^1 G(\tau, s)g_1(\tau)\,d\tau\right)a_2(s)\,ds,
\end{aligned}$$

which implies that $\|w_{1n}\|_\infty \to \infty$, a contradiction to Lemma 3.4. Hence, we conclude that there exist $\bar{\lambda}, \bar{\mu} > 0$ such that (5) has no solution for $\lambda > \bar{\lambda}$ or $\mu > \bar{\mu}$. $\qquad\square$

The following Lemmas 3.6, 3.7, and 3.8 can be proved by the same ideas as in Lemma 3.5, Lemma 3.6, and Theorem 3.1 in [9], respectively.

Lemma 3.6 *Assume* (H) *and* (H2). *Then every chain in* $\mathcal{S}$ *has a unique supremum in* $\mathcal{S}$.

Lemma 3.7 *Assume* (H) *and* (H2). *Then there exists* $\mu^* \in [\tilde{\mu}, \bar{\mu}]$ *such that* (5) *has a positive solution at* $(0, \mu)$ *for all* $0 < \mu \leq \mu^*$ *and no solution at* $(0, \mu)$ *for all* $\mu > \mu^*$. *Similarly, there exists* $\lambda^* \in [\tilde{\lambda}, \bar{\lambda}]$ *such that* (5) *has a positive solution at* $(\lambda, 0)$ *for all* $0 < \lambda \leq \lambda^*$ *and no solution at* $(\lambda, 0)$ *for all* $\lambda > \lambda^*$.

Lemma 3.8 *Assume* (H1) *and* (H2). *Then there exists a continuous curve* Γ_δ *splitting* $[0, \infty)^2 \setminus \{(0, 0)\}$ *into two disjoint subsets* $\Theta_{\delta,1}$ *and* $\Theta_{\delta,2}$ *such that* (5) *has at least one positive solution if* $(\lambda, \mu) \in \Theta_1 \cup \Gamma_\delta$ *and no positive solution if* $(\lambda, \mu) \in \Theta_{\delta,2}$.

Notice that by the sub- and supersolution theorem we can show that if $0 < \bar{\delta} < \delta$, then $\Theta_{\delta,1} \cup \Gamma_\delta \subset \Theta_{\bar{\delta},1} \cup \Gamma_{\bar{\delta}}$. Let $\Theta = \bigcup_{\delta > 0}(\Theta_{\delta,1} \cup \Gamma_\delta)$. Then $\Theta \subset \Theta_{0,1}$. Now we prove our main theorem, Theorem 1.1, by taking $\Theta_1 = \Theta_{0,1}$, $\Theta_2 = \Theta_{0,2}$, and $\Gamma = \Gamma_0$ in Theorem 3.1.

Theorem 3.1 *Assume* (H), (H1), *and* (H2). *Then there exists a continuous curve* Γ_0 *splitting* $[0, \infty)^2 \setminus \{(0, 0)\}$ *into two disjoint subsets* $\Theta_{0,1}$, $\Theta_{0,2}$ *and a subset* $\Theta \subset \Theta_{0,1}$ *such that* (4) *has at least two positive solutions for* $(\lambda, \mu) \in \Theta$, *at least one positive solution for* $(\lambda, \mu) \in (\Theta_{0,1} \setminus \Theta) \cup \Gamma_0$, *and no positive solution for* $(\lambda, \mu) \in \Theta_{0,2}$.

Proof We prove that (4) has a positive second solution for $(\lambda, \mu) \in \Theta$. If $(\lambda, \mu) \in \Theta$, then there exists $\delta > 0$ such that $(\lambda, \mu) \in (\Theta_{\delta,1} \cup \Gamma_\delta)$. Now we let $(w_{\delta,1}, w_{\delta,2})$ be a positive solution of (5) at (λ, μ) and define $\Omega = \{(w_1, w_2) \in X : -\epsilon < w_1(t) < w_{\delta,1}(t), -\epsilon < w_2(t) < w_{\delta,2}(t), t \in [0, 1]\}$. Then Ω is a bounded open set in X such that $0 \in \Omega$. Here we denote

$$\left(A_{\lambda,\mu}(w_1, w_2)(t), B_{\lambda,\mu}(w_1, w_2)(t)\right) := \left(A_{\lambda,\mu,0}(w_1, w_2)(t), B_{\lambda,\mu,0}(w_1, w_2)(t)\right)$$

and

$$T_{\lambda,\mu}(w_1, w_2)(t) := T_{\lambda,\mu,0}(w_1, w_2)(t).$$

Then $T_{\lambda,\mu} : \mathcal{K} \cap \bar{\Omega} \to \mathcal{K}$ is condensing. Let $(w_1, w_2) \in \mathcal{K} \cap \partial\Omega$. Then there exists $t_0 \in [0, 1]$ such that either $w_1(t_0) = w_{\delta,1}(t_0)$ or $w_2(t_0) = w_{\delta,2}(t_0)$. Suppose that $w_1(t_0) = w_{\delta,1}(t_0)$. Then by (H)

$$\begin{aligned}
A_{\lambda,\mu}(w_1, w_2)(t_0) &= \int_0^1 H_2(t_0, s) a_1(s) f_1(w_1 + p_{\lambda,\mu}, w_2 + q_{\lambda,\mu}) \\
&\quad + K_1(t_0, s) a_2(s) f_2(w_1 + p_{\lambda,\mu}, w_2 + q_{\lambda,\mu}) \, ds \\
&< \delta \left[1 + (\rho v_1 + \rho_1 v_2) t_0\right] \\
&\quad + \int_0^1 H_2(t_0, s) a_1(s) f_1(w_{\delta,1} + p_{\lambda,\mu}, w_{\delta,2} + q_{\lambda,\mu}) \\
&\quad + K_1(t_0, s) a_2(s) f_2(w_{\delta,1} + p_{\lambda,\mu}, w_{\delta,2} + q_{\lambda,\mu}) \, ds
\end{aligned}$$

$$= w_{\delta,1}(t_0)$$

$$= w_1(t_0) \leq \nu w_1(t_0)$$

for all $\nu \geq 1$. Thus $T_{\lambda,\mu}(w_1, w_2) \neq \nu(w_1, w_2)$ for all $(w_1, w_2) \in \mathcal{K} \cap \partial\Omega$ and $\nu \geq 1$. By Lemma 2.1 we conclude

$$i(T_{\lambda,\mu}, \mathcal{K} \cap \Omega, \mathcal{K}) = 1.$$

Next, we denote $\kappa = \min_{J \times J} K_1(t, s)$. By (H2) there exists $R_{f_2} > 0$ sufficiently large such that

$$f_2(u, v) \geq \eta(u + v) \quad \text{for } u + v \geq R_{f_2},$$

where $\eta > 0$ can be chosen such that $\eta \kappa \gamma \int_{j_*}^{j^*} a_2(s)\,ds > 1$. Let $R = \max\{M + 1, \frac{1}{\gamma}R_{f_2}\}$, where M is defined in Lemma 3.4. Let $\mathcal{K}_R = \{(w_1, w_2) \in \mathcal{K} : \|(w_1, w_2)\| < R\}$. Then, by Lemma 3.4, $(w_1, w_2) \neq T_{\lambda,\mu}(w_1, w_2)$ for $(w_1, w_2) \in \partial\mathcal{K}_R$. Moreover, if $(w_1, w_2) \in \partial\mathcal{K}_R$, then

$$\min_J \big(w_1(t) + w_2(t)\big) \geq \gamma \big(\|w_1\|_\infty + \|w_2\|_\infty\big) \geq R_{f_2},$$

which implies that $f_2(w_1(t), w_2(t)) \geq \eta(w_1(t) + w_2(t))$ for all $t \in J$. Finally, we find that, for $(w_1, w_2) \in \partial\mathcal{K}_R$,

$$
\begin{aligned}
A_{\lambda,\mu}(w_1, w_2)(j_*) &= \int_0^1 H_2(j_*, s) a_1(s) f_1(w_1 + p_{\lambda,\mu}, w_2, q_{\lambda,\mu}) \\
&\quad + K_1(j_*, s) a_2(s) f_2(w_1 + p_{\lambda,\mu}, w_2 + q_{\lambda,\mu})\,ds \\
&\geq \kappa \int_{j_*}^{j^*} a_2(s) f_2(w_1 + p_{\lambda,\mu}, w_2 + q_{\lambda,\mu})\,ds \\
&\geq \kappa \eta \int_{j_*}^{j^*} a_2(s)\big(w_1(s) + w_2(s)\big)\,ds \\
&\geq \kappa \eta \gamma \, \|(w_1, w_2)\| \int_{j_*}^{j^*} a_2(s)\,ds \\
&> \big\|(w_1, w_2)\big\|.
\end{aligned}
$$

Therefore $\|T_{\lambda,\mu}(w_1, w_2)\| \geq \|A_{\lambda,\mu}(w_1, w_2)\|_\infty \geq A_{\lambda,\mu}(w_1, w_2)(j_*) > \|(w_1, w_2)\|$, and by Lemma 2.2 we find

$$i(T_{\lambda,\mu}, \mathcal{K}_R, \mathcal{K}) = 0.$$

By the additivity of the fixed point index we obtain

$$0 = i(T_{\lambda,\mu}, \mathcal{K}_R, \mathcal{K}) = i(T_{\lambda,\mu}, \mathcal{K} \cap \Omega, \mathcal{K}) + i(T_{\lambda,\mu}, \mathcal{K}_R \setminus \overline{\mathcal{K} \cap \Omega}, \mathcal{K}).$$

Since $i(T_{\lambda,\mu}, \mathcal{K} \cap \Omega, \mathcal{K}) = 1$, we find $i(T_{\lambda,\mu}, \mathcal{K}_R \setminus \overline{\mathcal{K} \cap \Omega}, \mathcal{K}) = -1$, which implies that $T_{\lambda,\mu}$ has a fixed point on $\mathcal{K} \cap \Omega$ and on $\mathcal{K}_R \setminus \overline{\mathcal{K} \cap \Omega}$. Hence the proof is completed. □

Now, we give a simple example for the main results.

Example 3.2 Consider $N = 3$ and $r_0 = 1$ for system (1). Then $\Omega_e = \{x \in (-\infty, \infty)^3 : |x| \geq 1\}$. Let $K_i(r) = r^{-\alpha_i}$ for $r \in (1, \infty)$, where $\alpha_i > 2$ for $i = 1, 2$. Then there exists ν such that $1 < \nu < \min\{\alpha_1, \alpha_2\} - 1$, and hence we get that $\nu - \alpha_i < -1$ and $\int_1^\infty r^\nu K_i(r)\, dr = \int_1^\infty r^{\nu - \alpha_i}\, dr < \infty$ for $i = 1, 2$. Let $l_i(r) = r^{-\beta_i}$ for $r \in (1, \infty)$, where $\beta_i > 4\pi + 2$ ($i = 1, 2$). It is easy to see that $w_3 \int_1^\infty r l_i(r)\, dr = 4\pi \int_1^\infty r^{1-\beta_i}\, dr = \frac{4\pi}{\beta_i - 2} < 1$ as $w_3 = 4\pi$. Now, if $f_1(u, v) = u^2 + v^3$ and $f_2(u, v) = e^{u+v} - (u + v)$, then (H), (H1), and (H2) hold for f_1 and f_2. Thus the conclusion of Theorem 1.1 is valid. We note that, for the corresponding radial transformed problem (2) of (1), the conclusion of Theorem 3.1 is also valid.

Appendix: sub- and supersolution theorem

In this subsection, we prove Theorem 2.1. We consider the system with more general boundary conditions:

$$\begin{cases} u''(t) + a_1(t) f_1(u(t), v(t)) = 0, & t \in (0, 1), \\ v''(t) + a_2(t) f_2(u(t), v(t)) = 0, & t \in (0, 1), \\ u(0) = a, v(0) = b, \\ u(1) - \int_0^1 g_1(s) v(s)\, ds = \lambda, \\ v(1) - \int_0^1 g_2(s) u(s)\, ds = \mu, \end{cases} \tag{16}$$

where λ, μ, a, and b are constants.

Definition 4.1 We say that $(\psi_1, \psi_2) \in C^2(0, 1) \times C^2(0, 1)$ is a *subsolution* of problem (16) if

$$\begin{cases} \psi_1''(t) + a_1(t) f_1(\psi_1(t), \psi_2(t)) \geq 0, & t \in (0, 1), \\ \psi_2''(t) + a_2(t) f_2(\psi_1(t), \psi_2(t)) \geq 0, & t \in (0, 1), \\ \psi_1(0) \leq a, \psi_2(0) \leq b, \\ \psi_1(1) - \int_0^1 g_1(s) \psi_2(s)\, ds \leq \lambda, \\ \psi_2(1) - \int_0^1 g_2(s) \psi_1(s)\, ds \leq \mu. \end{cases}$$

We also say that $(\zeta_1, \zeta_2) \in C^2(0, 1) \times C^2(0, 1)$ is an *supersolution* of problem (16) if the reverse inequalities are satisfied.

A theorem for sub- and supersolutions to problem (16) is as follows.

Theorem 4.1 *Assume that $f_1(t, s)$ and $f_2(t, s)$ is quasimonotone increasing with respect to s and t, respectively (i.e., $f_1(t, s_1) \leq f_1(t, s_2)$ for $s_1 \leq s_2$ and $f_2(t_1, s) \leq f_2(t_2, s)$ for $t_1 \leq t_2$) and there exist a subsolution (ψ_1, ψ_2) and a supersolution (ζ_1, ζ_2) of problem (16) such that $(\psi_1(t), \psi_2(t)) \leq (\zeta_1(t), \zeta_2(t))$ for all $t \in [0, 1]$. Then problem (16) has at least one solution (u, v) such that*

$$\big(\psi_1(t), \psi_2(t)\big) \leq \big(u(t), v(t)\big) \leq \big(\zeta_1(t), \zeta_2(t)\big) \quad \text{for all } t \in [0, 1].$$

Proof Define $\gamma_i : [0,1] \times (-\infty, \infty) \to [0, \infty)$ for $i = 1, 2$ by

$$
\gamma_i(t,s) := \begin{cases} \zeta_i(t), & s \geq \zeta_i(t), \\ s, & \psi_i(t) \leq s \leq \zeta_i(t), \\ \psi_i(t), & s \leq \psi_i(t), \end{cases}
$$

and consider the modified problem

$$
\begin{cases} u''(t) + a_1(t)f_1(\gamma_1(t, u(t)), \gamma_2(t, v(t))) = 0, & t \in (0, 1), \\ v''(t) + a_2(t)f_2(\gamma_1(t, u(t)), \gamma_2(t, v(t))) = 0, & t \in (0, 1), \\ u(0) = a, \qquad v(0) = b, \\ u(1) - \int_0^1 g_1(s)\gamma_2(s, v(s))\, ds = \lambda, \\ v(1) - \int_0^1 g_2(s)\gamma_1(s, u(s))\, ds = \mu. \end{cases} \tag{17}
$$

For given $(w, z) \in C[0,1] \times C[0,1]$, define $g_{w,z} : (-\infty, \infty)^2 \to (-\infty, \infty)^2$ by

$$
g_{w,z}(x, y) := \left(\int_0^1 g_1(s)\gamma_2\big(s, z(s) + ys + b\big)\, ds + \lambda - a, \right.
$$
$$
\left. \int_0^1 g_2(s)\gamma_1\big(s, w(s) + xs + a\big)\, ds + \mu - b \right)
$$

for $(x, y) \in (-\infty, \infty)^2$. Since $|\gamma_i(s, x) - \gamma_i(s, y)| \leq |x - y|$ for any $x, y \in (-\infty, \infty)$ and $s \in [0, 1]$, $g_{w,z}$ is a contraction mapping on $(-\infty, \infty)^2$, and thus there exists a unique fixed point $(A_{w,z}, B_{w,z}) \in (-\infty, \infty)^2$ of $g_{w,z}$, which satisfies

$$
(A_{w,z}, B_{w,z}) = \left(\int_0^1 g_1(s)\gamma_2\big(s, z(s) + B_{w,z}s + b\big)\, ds + \lambda - a, \right.
$$
$$
\left. \int_0^1 g_2(s)\gamma_1\big(s, w(s) + A_{w,z}s + a\big)\, ds + \mu - b \right).
$$

If (w, z) is a solution of

$$
\begin{cases} w''(t) + a_1(t)f_1(\gamma_1(t, w(t) + A_{w,z}t + a), \gamma_2(t, z(t) + B_{w,z}t + b)) = 0, \\ z''(t) + a_2(t)f_2(\gamma_1(t, w(t) + A_{w,z}t + a), \gamma_2(t, z(t) + B_{w,z}t + b)) = 0, \\ w(0) = w(1) = z(0) = z(1) = 0, \end{cases} \tag{18}
$$

then $(u(t), v(t)) := (w(t) + A_{w,z}t + a, z(t) + B_{w,z}t + b)$ is a solution of (17).

Now, we define $T : \mathcal{P} \to \mathcal{P}$ by

$$
T(w, z)(t) := \big(T_1(w, z)(t), T_2(w, z)(t)\big),
$$

where

$$
T_1(w, z)(t) = \int_0^1 G(t, s)a_1(s)f_1\big(\gamma_1(s, w(s) + A_{w,z}s + a), \gamma_2(s, z(s) + B_{w,z}s + b)\big)\, ds,
$$

$$T_2(w,z)(t) = \int_0^1 G(t,s)a_2(s)f_2\big(\gamma_1\big(s,w(s)+A_{w,z}s+a\big),\gamma_2\big(s,z(s)+B_{w,z}s+b\big)\big)\,ds.$$

Then (w,z) is a fixed point of T in $\mathcal{P}$ if and only if (w,z) is a nonnegative solution of (18). Since T_γ is completely continuous on $\mathcal{P}$ and $T(\mathcal{P})$ is bounded in $C[0,1] \times C[0,1]$, T has a fixed point (w,z), and consequently (17) has a nonnegative solution (u,v). Now if we prove that $(\psi_1(t),\psi_2(t)) \le (u(t),v(t)) \le (\zeta_1(t),\zeta_2(t))$ for $t \in [0,1]$, then, by the definition of γ_i, (16) has a solution (u,v) such that $(\psi_1(t),\psi_2(t)) \le (u(t),v(t)) \le (\zeta_1(t),\zeta_2(t))$ for all $t \in [0,1]$, and the proof is done.

To show that $u(t) \le \zeta_1(t)$, we set $X(t) := u(t) - \zeta_1(t)$. Note that

$$X(0) = u(0) - \zeta_1(0) \le 0,$$

$$X(1) = u(1) - \zeta_1(1) \le \int_0^1 g_1(s)\big[\gamma_2\big(s,v(s)\big) - \zeta_2(s)\big]\,ds \le 0.$$

Now let us assume on the contrary that there is $t_0 \in (0,1)$ such that $X(t_0) = u(t_0) - \zeta_1(t_0) > 0$. It is clear that there exists $\sigma \in (0,1)$ such that $X(\sigma) = \max_{t \in [0,1]} X(t) > 0$. Then $X'(\sigma) = 0$, and there is $a \in (\sigma,1)$ such that $X'(t) < 0$ and $X(t) > 0$ for $t \in (\sigma,a]$, which means that

$$u'(\sigma) = \zeta_1'(\sigma), \qquad u'(t) < \zeta_1'(t), \quad \text{and} \quad u(t) > \zeta_1(t) \quad \text{for } t \in (\sigma,a]. \tag{19}$$

By the quasimonotonicity of $f_1(t,s)$ with respect to s and the definition of γ_i for $i = 1,2$, we obtain

$$-u''(t) = a_1(t)f_1\big(\gamma_1\big(t,u(t)\big),\gamma_2\big(t,v(t)\big)\big)$$
$$\le a_1(t)f_1\big(\zeta_1(t),\zeta_2(t)\big) \le -\zeta_1''(t) \quad \text{for } t \in (\sigma,a].$$

Integrating this from σ to $t \in (\sigma,a]$, we find

$$u'(t) \ge \zeta_1'(t), \quad \text{for } t \in (\sigma,a],$$

which contradicts to (19) Thus $u(t) \le \zeta_1(t)$ for $t \in [0,1]$. In a similar manner, we can prove that $\psi_1(t) \le u(t)$, $v(t) \le \zeta_2(t)$ and $\psi_2(t) \le v(t)$ for $t \in [0,1]$, and hence the proof is complete. □

Availability of data and materials
Not applicable.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Both authors have contributed equally to the paper. Both authors read and approved the final manuscript.

Author details
^{1}Major in Mathematics, College of Natural Science, Keimyung University, Daegu, South Korea. ^{2}Department of Mathematics Education, Pusan National University, Busan, South Korea.

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