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Blow-up phenomena and lifespan for a quasi-linear pseudo-parabolic equation at arbitrary initial energy level

Gongwei Liu[1*] and Ruimin Zhao[1]

*Correspondence: gongweiliu@126.com
[1]College of Science, Henan University of Technology, Zhengzhou, China

Abstract

In this paper, we continue to study the initial boundary value problem of the quasi-linear pseudo-parabolic equation

$$u_t - \triangle u_t - \triangle u - \operatorname{div}(|\nabla u|^{2q}\nabla u) = u^p$$

which was studied by Peng et al. (Appl. Math. Lett. 56:17–22, 2016), where the blow-up phenomena and the lifespan for the initial energy $J(u_0) < 0$ were obtained. We establish the finite time blow-up of the solution for the initial data at arbitrary energy level and the lifespan of the blow-up solution. Furthermore, as a product, we obtain the blow-up rate and refine the lifespan when $J(u_0) < 0$.

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Keywords: Blow-up; Lifespan; Blow-up rate; Quasi-linear pseudo-parabolic equation

1 Introduction

In this paper, we investigate the initial boundary value problem of the following quasi-linear pseudo-parabolic equation:

$$\begin{cases} u_t - \triangle u_t - \triangle u - \operatorname{div}(|\nabla u|^{2q}\nabla u) = u^p, & (x,t) \in \Omega \times (0,T), \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T), \\ u(x,t) = u_0(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$ ($n \geq 3$) is a bounded domain with sufficiently smooth boundary $\partial\Omega$, $p > 1$ and $0 \leq 2q < p - 1$. $T \in (0, \infty]$ denotes the maximal existence time of the solution.

Problem (1.1) describes a variety of important physical and biological phenomena such as the aggregation of population [1], the unidirectional propagation of nonlinear, dispersive, long waves [2], and the nonstationary processes in semiconductors [3]. In the absence of the term $\operatorname{div}(|\nabla u|^{2q}\nabla u)$, Eq. (1.1) reduces to the following semilinear pseudo-parabolic equation:

$$u_t - \triangle u_t - \triangle u = u^p, \quad (x,t) \in \Omega \times (0,T). \tag{1.2}$$

There are many results for Eq. (1.2) such as the existence and uniqueness in [4], blow-up in [5–8], asymptotic behavior in [6, 9], and so on. Using the integral representation and the semigroup, Cao et al.[10] obtained the critical global existence exponent and the critical Fujita exponent for Eq. (1.2). Chen et al. [11] considered Eq. (1.2) with the logarithmic nonlinearity source term by the potential well methods.

Recently, Peng et al. [12] considered the blow-up phenomena on problem (1.1). By the way, Payne et al. [13] considered the blow-up phenomena of solutions on the initial boundary problem of the nonlinear parabolic equation

$$u_t - \operatorname{div}\bigl(\rho\bigl(|\nabla u|^2\bigr)\nabla u\bigr) = f(u).$$

In addition, Long et al. [14] investigated the blow-up phenomena for a nonlinear pseudo-parabolic equation with nonlocal source

$$u_t - \triangle u_t - \operatorname{div}\bigl(|\nabla u|^{2q}\nabla u\bigr) = u^p(x,t)\int_\Omega k(x,y)u^{p+1}(y,t)\,dy.$$

Finally, we mention some interesting works concerning quasi-linear or degenerate parabolic equations. For example, Winkert and Zacher [15] considered a generate class of quasi-linear parabolic problems and established global a priori bounds for the weak solutions of such problems; Fragnelli and Mugnai [16] established Carleman estimates for degenerate parabolic equations with interior degeneracy and non-smooth coefficients.

Throughout this paper, we use $\|\cdot\|_p = (\int_\Omega |\cdot|^p\,dx)^{\frac{1}{p}}$ and $\|\cdot\|_{W_0^{1,p}} = (\int_\Omega (|\cdot|^p + |\nabla\cdot|^p)\,dx)^{\frac{1}{p}}$ as the norms on the Banach spaces $L^p = L^p(\Omega)$ and $W_0^{1,p} = W_0^{1,p}(\Omega)$, respectively. As in [12], we define the energy functional and the Nehari functional of (1.1), respectively, by

$$J(u) := \frac{1}{2}\|\nabla u\|_2^2 + \frac{1}{2q+2}\|\nabla u\|_{2q+2}^{2q+2} - \frac{1}{p+1}\|u\|_{p+1}^{p+1}, \tag{1.3}$$

$$I(u) := \bigl(J'(u), u\bigr) = \|\nabla u\|_2^2 + \|\nabla u\|_{2q+2}^{2q+2} - \|u\|_{p+1}^{p+1}. \tag{1.4}$$

Let λ_1 be the first nontrivial eigenvalue of $-\triangle$ operator in Ω with homogeneous Dirichlet condition, then we have

$$\lambda_1\|u\|_2^2 \le \|\nabla u\|_2^2, \qquad \|\nabla u\|_2^2 \ge \frac{\lambda_1}{1+\lambda_1}\|u\|_{H_0^1}^2, \quad u \in H_0^1(\Omega). \tag{1.5}$$

In order to compare with our work, in this paper, we summarize the blow-up results obtained in [12] as follows.

(RES1) If $0 \le 2q < p-1$, $J(u_0) < 0$, and u is a nonnegative solution of (1.1), then u blows up at some finite time T, where T is bounded by

$$T \le T_1 := \frac{\|u_0\|_{H_0^1}^2}{(1-p^2)J(u_0)}. \tag{1.6}$$

From the above (RES1), we notice that (1) the blow-up rate is not given when $J(u_0) < 0$; (2) the blow-up phenomena and the lifespan are still unsolved when $J(u_0) \ge 0$.

Motivated by the above-mentioned facts, we investigate these two problems in this paper. Firstly, we state the local existence theorem of problem (1.1) by Faedo–Galerkin method (see Theorem 2.1 in [12]).

(RES2) For any $u_0 \in W_0^{1,2q+2}(\Omega)$, there exists $T > 0$ such that problem (1.1) has a unique local weak solution $u \in L^\infty(0,T;W_0^{1,2q+2}(\Omega))$ with $u_t \in L^2(0,T;H_0^1(\Omega))$ which satisfies

$$\langle u_t, v\rangle + \langle \nabla u_t, \nabla v\rangle + \langle \nabla u, \nabla v\rangle + \big\langle |\nabla u|^{2q}\nabla u, \nabla v\big\rangle = \big\langle u^p, v\big\rangle$$

for all $v \in W_0^{1,2q+2}(\Omega)$.

Our main result of this paper can be stated as the following theorem.

Theorem 1.1 *For all $0 \le 2q < p-1$, the nonnegative solution u of problem* (1.1) *blows up at finite time in H_0^1-norm provided that*

$$J(u_0) < \frac{(p-1)\lambda_1}{2(p+1)(1+\lambda_1)}\|u_0\|_{H_0^1}^2. \tag{1.7}$$

Furthermore, the lifespan T can be estimated by

$$T \le T_2 := \frac{8(p+1)(1+\lambda_1)\|u_0\|_{H_0^1}^2}{(p-1)^2[(p-1)\lambda_1\|u_0\|_{H_0^1}^2 - 2(p+1)(1+\lambda_1)J(u_0)]}. \tag{1.8}$$

Remark 1.1 For the case $J(u_0) < 0$, the initial data condition given in (1.7) is obviously satisfied. Noticing the values of T_1 and T_2 given in (1.6) and (1.8), we can refine the lifespan T as

$$T \le \min\{T_1, T_2\} = \begin{cases} T_2, & \text{if } -\frac{(p-1)^2\lambda_1\|u_0\|_{H_0^1}^2}{2(p+1)(3p+5)(1+\lambda_1)} \le J(u_0) < 0; \\ T_1, & \text{if } J(u_0) < -\frac{(p-1)^2\lambda_1\|u_0\|_{H_0^1}^2}{2(p+1)(3p+5)(1+\lambda_1)}. \end{cases}$$

2 Proof of Theorem 1.1

In this section, we prove Theorem 1.1 by using the following lemma (see [17]).

Lemma 2.1 *Suppose that a nonnegative, twice-differentiable function $\theta(t)$ satisfies the inequality*

$$\theta''(t)\theta(t) - (1+r)\big(\theta'(t)\big)^2 \ge 0, \quad t > 0,$$

where $r > 0$ is some constant. If $\theta(0) > 0$ and $\theta'(0) > 0$, then there exists $0 < t_1 \le \frac{\theta(0)}{r\theta'(0)}$ such that $\theta(t) \to +\infty$ as $t \to t_1^-$.

Proof of Theorem 1.1 We give the proof in the following two steps.

Step 1: *Blow-up*. Let $u(t)$ be the solution of problem (1.1) with the initial data satisfying (1.7). We may assume $J(u(t)) \ge 0$; otherwise, there exists some $t_0 \ge 0$ such that $J(u(t_0)) < 0$, then $u(t)$ will blow up in finite time by (RES1), the proof of this step is complete. So, in the following, we give our proof by contradiction and assume that $u(t)$ exists globally and $J(u(t)) \ge 0$ for all $t \ge 0$.

Differentiating (1.3) and making use of (1.1) and (1.4), we have the following equalities:

$$\frac{d}{dt}J\big(u(t)\big) = -\|u_t\|_2^2 - \|\nabla u_t\|_2^2 = -\|u_t\|_{H_0^1}^2, \tag{2.1}$$

$$\begin{aligned}\frac{d}{dt}\left(\frac{1}{2}\|u(t)\|_{H_0^1}^2\right) &= -\|\nabla u\|_2^2 - \|\nabla u\|_{2q+2}^{2q+2} + \|u\|_{p+1}^{p+1} \\ &= -I\big(u(t)\big).\end{aligned} \tag{2.2}$$

Since

$$\int_0^t \|u_s(s)\|_{H_0^1}\,ds \ge \left\|\int_0^t u_s(s)\,ds\right\|_{H_0^1} = \|u(t)-u_0\|_{H_0^1} \ge \|u(t)\|_{H_0^1} - \|u_0\|_{H_0^1}, \quad t\ge 0,$$

by Hölder's inequality, (2.1), and $J(u_0) \ge J(u(t)) \ge 0$, we obtain that

$$\begin{aligned}\|u(t)\|_{H_0^1} &\le \|u_0\|_{H_0^1} + t^{\frac{1}{2}}\left[\int_0^t \|u_s(s)\|_{H_0^1}\,ds\right]^{\frac{1}{2}} \\ &= \|u_0\|_{H_0^1} + t^{\frac{1}{2}}\big[J(u_0) - J\big(u(t)\big)\big]^{\frac{1}{2}} \\ &\le \|u_0\|_{H_0^1} + t^{\frac{1}{2}}\big(J(u_0)\big)^{\frac{1}{2}}, \quad t\ge 0.\end{aligned} \tag{2.3}$$

Combining (1.5) and Hölder's inequality, we deduce that

$$\|\nabla u(t)\|_{2q+2}^{2q+2} \ge |\Omega|^{-q}\left(\frac{\lambda_1}{1+\lambda_1}\right)^{q+1}\|u(t)\|_{H_0^1}^{2q+2}.$$

On the other hand, by (1.3), (1.4), (2.2), and $0\le 2q < p-1$, we obtain

$$\begin{aligned}\frac{d}{dt}\left(\frac{1}{2}\|u(t)\|_{H_0^1}^2\right) &= \frac{p-1}{2}\|\nabla u(t)\|_2^2 + \frac{p-2q-1}{2q+2}\|\nabla u(t)\|_{2q+2}^{2q+2} - (p+1)J\big(u(t)\big) \\ &\ge \frac{(p-1)\lambda_1}{2(1+\lambda_1)}\|u(t)\|_{H_0^1}^2 + \frac{p-2q-1}{2q+2}\left(\frac{\lambda_1}{1+\lambda_1}\right)^{q+1}|\Omega|^{-q}\|u(t)\|_{H_0^1}^{2q+2} \\ &\quad - (p+1)J\big(u(t)\big) \\ &\ge \frac{(p-1)\lambda_1}{1+\lambda_1}\left[\frac{1}{2}\|u(t)\|_{H_0^1}^2 - \frac{(p+1)(1+\lambda_1)}{(p-1)\lambda_1}J\big(u(t)\big)\right].\end{aligned}$$

Since $\frac{d}{dt}(J(u(t))) \le 0$, it follows from the above inequality that

$$\begin{aligned}&\frac{d}{dt}\left[\frac{1}{2}\|u(t)\|_{H_0^1}^2 - \frac{(p+1)(1+\lambda_1)}{(p-1)\lambda_1}J\big(u(t)\big)\right] \\ &\quad\ge \frac{(p-1)\lambda_1}{1+\lambda_1}\left[\frac{1}{2}\|u(t)\|_{H_0^1}^2 - \frac{(p+1)(1+\lambda_1)}{(p-1)\lambda_1}J\big(u(t)\big)\right].\end{aligned}$$

Let

$$H(t) = \frac{1}{2}\|u(t)\|_{H_0^1}^2 - \frac{(p+1)(1+\lambda_1)}{(p-1)\lambda_1}J\big(u(t)\big),$$

then

$$\frac{d}{dt}H(t) \geq \frac{(p-1)\lambda_1}{1+\lambda_1}H(t)$$

for all $t \geq 0$. By using Gronwall's inequality, we get

$$H(t) \geq e^{\frac{(p-1)\lambda_1}{1+\lambda_1}t}H(0).$$

Noticing that $H(0) > 0$ via (1.7) and the assumption $J(u(t)) \geq 0$ for $t \geq 0$, we deduce

$$\|u(t)\|_{H_0^1} \geq \sqrt{2H(0)}e^{\frac{(p-1)(\lambda_1}{2(1+\lambda_1)}t}, \quad t \geq 0,$$

which is a contradiction with (2.3) for t sufficiently large. Hence, $u(t)$ blows up at some finite time, i.e., $T < \infty$.

Step 2: *Lifespan*. We will find an upper bound for T. Firstly, we claim that

$$I(u(t)) = \|\nabla u(t)\|_2^2 + \|\nabla u(t)\|_{2q+2}^{2q+2} - \|u(t)\|_{p+1}^{p+1} < 0, \quad t \in [0, T). \tag{2.4}$$

Indeed, combining (1.3) and (1.4), after a simple calculation, we get

$$\begin{aligned} J(u(t)) = {} & \frac{p-1}{2(p+1)}\|\nabla u(t)\|_2^2 + \frac{p-2q-1}{2(q+1)(p+1)}\|\nabla u(t)\|_{2q+2}^{2q+2} \\ & + \frac{1}{p+1}I(u(t)), \quad t \in [0, T). \end{aligned} \tag{2.5}$$

It follows from (1.5), (1.7), and (2.5) that

$$\frac{(p-1)\lambda_1}{2(p+1)(1+\lambda_1)}\|u_0\|_{H_0^1}^2 > J(u_0) \geq \frac{p-1}{2(p+1)}\frac{\lambda_1}{1+\lambda_1}\|u_0\|_{H_0^1}^2 + \frac{1}{p+1}I(u_0),$$

where we also use $0 \leq 2q < p-1$, which implies $I(u_0) < 0$. Hence, if (2.4) does not hold, there must exist $t_0 \in (0, T)$ such that $I(u(t_0)) = 0$, $I(u(t)) < 0$ for $t \in [0, t_0)$. Then, by (2.2), we obtain that $\|u(t)\|_{H_0^1}^2$ is strictly increasing on $[0, t_0)$. Then it follows from (1.7) that

$$\begin{aligned} J(u_0) & < \frac{(p-1)\lambda_1}{2(p+1)(1+\lambda_1)}\|u_0\|_{H_0^1}^2 \\ & < \frac{(p-1)\lambda_1}{2(p+1)(1+\lambda_1)}\|u(t_0)\|_{H_0^1}^2. \end{aligned} \tag{2.6}$$

On the other hand, combining (2.1) and (2.5), we get

$$\begin{aligned} J(u_0) \geq J(u(t_0)) & = \frac{p-1}{2(p+1)}\|\nabla u(t_0)\|_2^2 + \frac{p-2q-1}{2(q+1)(p+1)}\|\nabla u(t_0)\|_{2q+2}^{2q+2} + \frac{1}{p+1}I(u(t_0)) \\ & \geq \frac{(p-1)\lambda_1}{2(p+1)(1+\lambda_1)}\|u(t_0)\|_{H_0^1}^2, \end{aligned}$$

which is a contradiction with (2.6). Hence, $I(u(t)) < 0$ and $\|u(t)\|_{H_0^1}^2$ is strictly increasing on $[0, T)$.

We define the functional

$$F(t) = \int_0^t \|u(s)\|_{H_0^1}^2\,ds + (T-t)\|u_0\|_{H_0^1}^2 + \beta(t+\gamma)^2, \quad t \in [0,T),$$

with two positive constants β, γ to be chosen later. Since $\|u(t)\|_{H_0^1}^2$ is strictly increasing, we get

$$\begin{aligned} F'(t) &= \|u(t)\|_{H_0^1}^2 - \|u_0\|_{H_0^1}^2 + 2\beta(t+\gamma) \\ &= \int_0^t \frac{d}{ds}\|u(s)\|_{H_0^1}^2\,ds + 2\beta(t+\gamma) \geq 2\beta(t+\gamma) > 0 \end{aligned} \tag{2.7}$$

and

$$\begin{aligned} F''(t) &= \frac{d}{dt}\|u(t)\|_{H_0^1}^2 + 2\beta \\ &= (p-1)\|\nabla u(t)\|_2^2 + \frac{p-2q-1}{q+1}\|\nabla u(t)\|_{2q+2}^{2q+2} - 2(p+1)J\big(u(t)\big) + 2\beta \\ &\geq \frac{(p-1)\lambda_1}{1+\lambda_1}\|u(t)\|_{H_0^1}^2 + 2(p+1)\int_0^t \|u_s\|_{H_0^1}^2\,ds - 2(p+1)J(u_0). \end{aligned} \tag{2.8}$$

Noticing that

$$F(0) = T\|u_0\|_{H_0^1}^2 + \beta\gamma^2 > 0$$

and

$$F'(0) = 2\beta\gamma > 0,$$

by using Young's inequality, Hölder's inequality, and the element algebraic inequality

$$ab + cd \leq \sqrt{a^2+c^2}\sqrt{b^2+d^2},$$

we can deduce

$$\begin{aligned} \xi(t) :=& \left(\int_0^t \|u(s)\|_{H_0^1}^2\,ds + \beta(t+\gamma)^2\right)\left(\int_0^t \|u_s\|_{H_0^1}^2\,ds + \beta\right) \\ &- \left(\int_0^t \frac{1}{2}\frac{d}{ds}\|u(s)\|_{H_0^1}^2\,ds + \beta(t+\gamma)\right)^2 \\ \geq& \ 0. \end{aligned}$$

Hence, it follows from the above inequality and (2.7) that

$$\begin{aligned} -\big(F'(t)\big)^2 &= -4\left[\frac{1}{2}\int_0^t \frac{d}{ds}\|u(s)\|_{H_0^1}^2\,ds + 2\beta(t+\gamma)\right]^2 \\ &= 4\left(\xi(t) - \big(F(t) - (T-t)\|u_0\|_{H_0^1}^2\big)\left(\int_0^2 \frac{d}{ds}\|u(s)\|_{H_0^1}^2\,ds + \beta\right)\right) \\ &\geq -4F(t)\left(\int_0^t \frac{d}{ds}\|u(s)\|_{H_0^1}^2\,ds + \beta\right). \end{aligned}$$

By the above equality, (2.8), and the fact that $\|u(t)\|_{H_0^1}^2$ is strictly increasing, we have

$$F(t)F''(t) - \frac{p+1}{2}\big(F'(t)\big)^2 \ge F(t)\left[F''(t) - 2(p+1)\left(\int_0^t \frac{d}{ds}\big\|u(s)\big\|_{H_0^1}^2\,ds + \beta\right)\right]$$
$$\ge 2(p+1)F(t)\left[\frac{(p-1)\lambda_1}{2(p+1)(1+\lambda_1)}\|u_0\|_{H_0^1}^2 - J(u_0) - \beta\right].$$

From (1.7), we can choose β sufficiently small such that

$$0 < \beta \le \beta_0 := \frac{(p-1)\lambda_1}{2(p+1)(1+\lambda_1)}\|u_0\|_{H_0^1}^2 - J(u_0). \tag{2.9}$$

Then the conditions of Lemma 2.1 are satisfied with $r = \frac{p-1}{2}$, so we have

$$T \le \frac{2F(0)}{(p-1)F'(0)} = \frac{\|u_0\|_{H_0^1}^2}{(p-1)\beta\gamma}T + \frac{\gamma}{p-1}. \tag{2.10}$$

Fixing arbitrary β satisfying (2.9), then let γ be sufficiently large such that

$$\frac{\|u_0\|_{H_0^1}^2}{(p-1)\beta} < \gamma < +\infty,$$

then it follows from (2.10) that

$$T \le \frac{\beta\gamma^2}{(p-1)\beta\gamma - \|u_0\|_{H_0^1}^2}. \tag{2.11}$$

Define a function $T_\beta(\gamma)$ by

$$T_\beta(\gamma) = \frac{\beta\gamma^2}{(p-1)\beta\gamma - \|u_0\|_{H_0^1}^2}, \quad \gamma \in \left(\frac{\|u_0\|_{H_0^1}^2}{(p-1)\beta}, +\infty\right).$$

It is easy to prove that the function $T_\beta(\gamma)$ has a unique minimum at

$$\gamma_\beta := \frac{2\|u_0\|_{H_0^1}^2}{(p-1)\beta} \in \left(\frac{\|u_0\|_{H_0^1}^2}{(p-1)\beta}, +\infty\right).$$

Then it follows from (2.11) that

$$T \le \inf_{\gamma\in(\frac{\|u_0\|_{H_0^1}^2}{(p-1)\beta}, +\infty)} T_\beta(\gamma) = T_\beta(\gamma_\beta) = \frac{4\|u_0\|_{H_0^1}^2}{(p-1)^2\beta}$$

for any β satisfying (2.9). Finally, we obtain

$$T \le \inf_{\beta\in(0,\beta_0]} \frac{4\|u_0\|_{H_0^1}^2}{(p-1)^2\beta} = \frac{4\|u_0\|_{H_0^1}^2}{(p-1)^2\beta_0} = \frac{8(p+1)(1+\lambda_1)\|u_0\|_{H_0^1}^2}{(p-1)^2[(p-1)\lambda_1\|u_0\|_{H_0^1}^2 - 2(p+1)(1+\lambda_1)J(u_0)]}.$$

This completes the proof of Theorem 1.1. □

Corollary 2.1 *For all $0 \le 2q < p-1$ and any $M > 0$, there exists initial $u_{0M} \in W_0^{1,2q+2}(\Omega)$ such that the weak solution for corresponding problem (1.1) will blow up in finite time.*

Proof Let $M > 0$, and Ω_1 and Ω_2 be two arbitrary disjoint open subdomains of Ω. We assume that $v \in W_0^{1,2q+2}(\Omega_1) \subset W_0^{1,2q+2}(\Omega) \subset H_0^1(\Omega)$ is an arbitrary nonzero function, then we can take $\alpha_1 > 0$ sufficiently large such that

$$\begin{aligned}\|\alpha_1 v\|_{H_0^1}^2 &= \alpha_1^2 \int_\Omega |v^2|\,dx + \alpha_1^2 \int_\Omega |\nabla v|^2\,dx = \alpha_1^2 \int_{\Omega_1} |v^2|\,dx + \alpha_1^2 \int_{\Omega_1} |\nabla v|^2\,dx \\ &> \frac{2(p+1)(1+\lambda_1)}{(p-1)\lambda_1} M.\end{aligned}$$

We claim that there exist $w \in W_0^{1,2q+2}(\Omega_2) \subset W_0^{1,2q+2}(\Omega)$ and $\alpha > \alpha_1$ such that $J(w) = M - J(\alpha v)$.

In fact, we choose a function $w_k \in C_0^1(\Omega_2)$ such that $\|\nabla w_k\|_2 \ge k$ and $\|w_k\|_\infty \le c_0$. Hence,

$$\begin{aligned}&\frac{1}{2}\int_{\Omega_2} |\nabla w_k|^2\,dx + \frac{1}{2q+2}\int_{\Omega_2} |\nabla w_k|^{2q+2}\,dx - \frac{1}{p+1}\int_{\Omega_2} |w_k|^{p+1}\,dx \\ &\quad\ge \frac{1}{2}\int_{\Omega_2} |\nabla w_k|^2\,dx + \frac{1}{2q+2}|\Omega_2|^{-q}\left(\int_{\Omega_2} |\nabla w_k|^2\,dx\right)^{q+1} - \frac{1}{p+1}c_0^{p+1}|\Omega_2|.\end{aligned}$$

On the other hand, since $0 \le 2q < p-1$, it holds that

$$\begin{aligned}M - J(\alpha v) &= M - \frac{\alpha^2}{2}\int_{\Omega_1} |\nabla v|^2\,dx - \frac{\alpha^{2q+2}}{2q+2}\int_{\Omega_1} |\nabla|^{2q+2}\,dx \\ &\quad + \frac{\alpha^{p+1}}{p+1}\int_{\Omega_1} |v|^{p+1}\,dx \to +\infty, \quad \text{as } \alpha \to +\infty.\end{aligned}$$

Hence, there exist $k > 0$ and $\alpha > \alpha_1$ both sufficiently large such that

$$M - J(\alpha v) = \frac{1}{2}\int_{\Omega_2} |\nabla w_k|^2\,dx + \frac{1}{2q+2}\int_{\Omega_2} |\nabla w_k|^{2q+2}\,dx - \frac{1}{p+1}\int_{\Omega_2} |w_k|^{p+1}\,dx.$$

Then we choose $w = w_k$ and denote $u_{0M} := \alpha v + w$. Hence, we have

$$\begin{aligned}\|u_{0M}\|_{H_0^1}^2 &= \int_\Omega \left|u_{0M}^2\right|dx + \int_\Omega |\nabla u_{0M}|^2\,dx \ge \alpha^2 \int_{\Omega_1} \left|v^2\right|dx + \alpha^2 \int_{\Omega_1} |\nabla v|^2\,dx \\ &> \frac{2(p+1)(1+\lambda_1)}{(p-1)\lambda_1} M\end{aligned}$$

and

$$M = J(\alpha v) + J(w) = J(u_{0M}) < \frac{(p-1)\lambda_1}{2(p+1)(1+\lambda_1)}\|u_{0M}\|_{H_0^1}^2.$$

The proof is complete. □

Remark 2.1 In this remark, we establish the blow-up rate for $J(u_0) < 0$. We define the functionals $\varphi(t) = \|u(t)\|_{H_0^1}^2$ and $\psi(t) = -2(p+1)J(u(t))$ as these in [12]. It was shown in (4.8) of

[12] that

$$\frac{\varphi'(t)}{[\varphi(t)]^{\frac{p+1}{2}}} \geq \frac{\psi(0)}{[\varphi(0)]^{\frac{p+1}{2}}}.$$

Now, we integrate the inequality from t to T, noticing $\lim_{t\to T^-} \varphi(t) = +\infty$ (by (RES1)), we obtain

$$\varphi(t) \leq \left[\frac{(p-1)\psi(0)}{2[\varphi(0)]^{\frac{p+1}{2}}}\right]^{\frac{2}{1-p}}.$$

Then it follows from the definitions of $\varphi(t)$ and $\psi(t)$ that

$$\|u(t)\|_{H_0^1} \leq \left[\frac{(1-p^2)J(u_0)}{\|u_0\|_{H_0^1}^{p+1}}\right]^{\frac{1}{1-p}} (T-t)^{-\frac{1}{p-1}}.$$

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Authors' contributions
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