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Existence of nontrivial solutions for a class of nonlocal Kirchhoff type problems

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Abstract

With the aid of the three-critical-point theorem due to Brezis and Nirenberg (see Brezis and Nirenberg in Commun. Pure Appl. Math. 44:939–963, 1991), two existence results of at least two nontrivial solutions for a class of nonlocal Kirchhoff type problems are obtained.

MSC: 35D30; 35J50; 35J92

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1 Introduction and main results

Consider the existence of weak solutions for the following nonlocal Kirchhoff type problem:

$$\begin{cases} -(a+b\int_\Omega |\nabla u|^2\,dx)\Delta u = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where Ω is a smooth bounded domain in R^N ($N \geq 1$), $a > 0$, $b > 0$ are real numbers, and the nonlinearity $f \in C(\bar{\Omega} \times R, R)$.

Problem (1) is analogous to the stationary case of equations that arise in the study of string or membrane vibrations, that is,

$$u_{tt} - \left(a + b\int_\Omega |\nabla u|^2\,dx\right)\Delta u = f(x,u),$$

which was first proposed by Kirchhoff (see [3]) in 1883 to describe the transversal oscillations of a stretched string. Especially, in recent years, many solvability conditions with f (or F) near zero and infinity were considered to study the existence and multiplicity of weak solutions for problem (1) by using variational methods, for example, the nonlinearity f is asymptotically 3-linear at infinity (see [4, 6, 9]), the nonlinearity f is 3-suplinear at infinity (see [5, 7, 9]), and the nonlinearity f is 3-sublinear at infinity (see [9]). In this paper, motivated by [2, 7, 8], we prove the existence of at least two nontrivial solutions for problem (1) by using the variational method.

Let $H_0^1(\Omega)$ be the usual Hilbert space with the norm

$$\|u\| = \left(\int_\Omega |\nabla u|^2\,dx\right)^{\frac{1}{2}} \quad \text{for any } u \in H_0^1(\Omega).$$

From the Rellich embedding theorem, the embedding $H_0^1(\Omega) \hookrightarrow L^\theta(\Omega)$ is continuous for any $\theta \in [1, 2^*]$ and compact for any $\theta \in [1, 2^*)$, where $2^* = +\infty$ if $N = 1, 2$ and $2^* = \frac{2N}{N-2}$ if $N \geq 3$. Moreover, for any $\theta \in [1, 2^*)$, there is a constant $\tau_\theta > 0$ such that

$$\|u\|_{L^\theta} \leq \tau_\theta \|u\| \quad \text{for any } u \in H_0^1(\Omega), \tag{2}$$

where $\|\cdot\|_{L^\theta}$ denotes the norm of $L^\theta(\Omega)$. Let $m(x) \in C(\bar{\Omega})$ be positive on a subset of positive measure, the following eigenvalue problem

$$\begin{cases} -\Delta u = \lambda m(x) u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{3}$$

has a sequence of variational eigenvalues $\{\lambda_k(m)\}$ such that $\lambda_1(m) < \lambda_2(m) < \cdots < \lambda_k(m) \to \infty$ as $k \to \infty$. Let $M(x) \in C(\bar{\Omega})$ be positive on Ω. For the following nonlinear eigenvalue problem

$$\begin{cases} -\|u\|^2 \Delta u = \mu M(x) u^3 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4}$$

we define

$$\mu_1(M) = \inf\left\{\|u\|^4 : u \in H_0^1(\Omega), \int_\Omega M(x) u^4\,dx = 1\right\}.$$

Similar to Lemma 2.1 of [9], we can prove that $\mu_1(M)$ is the first eigenvalue of (4) and positive. Moreover, there is an eigenvalue Φ_1^M such that $\Phi_1^M > 0$ in Ω.

Let $m_0(x) \in C(\bar{\Omega})$ be positive on a subset of positive measure and $m_\infty(x) \in C(\bar{\Omega})$ be positive on Ω. Assume that

$$\lim_{|t|\to 0} \frac{2F(x,t)}{at^2} = m_0(x) \quad \text{uniformly in } x \in \Omega, \tag{5}$$

$$\lim_{|t|\to \infty} \frac{4F(x,t)}{bt^4} = m_\infty(x) \quad \text{uniformly in } x \in \Omega, \tag{6}$$

$$\lim_{|t|\to \infty} \bigl(f(x,t)t - 4F(x,t)\bigr) = +\infty \quad \text{uniformly in } x \in \Omega, \tag{7}$$

where $F(x,t) = \int_0^t f(x,s)\,ds$. We are ready to state our main results.

Theorem 1 *Let $N = 1, 2, 3$, and assume that the function F satisfies (5) with $\lambda_k(m_0) < 1 < \lambda_{k+1}(m_0)$ for some $k \geq 1$ and (6), and there exist $4 < p < 2^*$ and $c_0 > 0$ such that*

$$\bigl|f(x,t)\bigr| \leq c_0\bigl(1 + |t|^{p-1}\bigr) \quad \textit{for any } (x,t) \in \bar{\Omega} \times R, \tag{8}$$

then problem (1) *has at least two nontrivial solutions in each of the following cases*:

(i) $\mu_1(m_\infty) > 1$ *or*

(ii) $\mu_1(m_\infty) = 1$ *and* (7) *hold.*

Theorem 2 *Assume that the nonlinearity F satisfies* (5) *with $\lambda_k(m_0) < 1 < \lambda_{k+1}(m_0)$ for some $k \geq 1$ and the following condition*:

$$\lim_{|t|\to\infty} \frac{f(x,t)}{|t|^{p-1}} = 0 \quad \textit{uniformly in } x \in \Omega, \tag{9}$$

where $p = 4$ if $N = 1,2,3$ and $p = 2^*$ if $N \geq 4$, then problem (1) *has at least two nontrivial solutions.*

Remark If $N = 1,2,3$ and the nonlinearity f is 3-suplinear at infinity, Sun and Tang in [7] obtained a nontrivial solution for problem (1) by using the local linking theorem due to Li and Willem. In [8], when the nonlinearity F is some asymptotically 4-linear at infinity, Yang and Zhang proved the existence of at least two nontrivial solutions for problem (1) by means of the Morse theory and local linking. Since $p = 2^* \leq 4$ ($N \geq 4$), condition (9) implies that the nonlinearity f is 3-sublinear at infinity. Hence, our results are the complements for the ones of [7, 8].

2 Proof of the theorems

Define the functional $I : H_0^1(\Omega) \to R$ as follows:

$$I(u) = \frac{b}{4}\|u\|^4 + \frac{a}{2}\|u\|^2 - \int_\Omega F(x,u)\,dx. \tag{10}$$

From (8) (or (9)), by a standard argument, the functional $I \in C^1(H_0^1(\Omega), R)$, and a weak solution of problem (1) is a critical point of the functional I in $H_0^1(\Omega)$.

Recall that a sequence $\{u_n\} \subset H_0^1(\Omega)$ is called a $(PS)_c$ sequence for any $c \in R$ of the functional I on $H_0^1(\Omega)$ if $I(u_n) \to c$ and $I'(u_n) \to 0$ as $n \to \infty$. The functional I is called to satisfy the $(PS)_c$ condition if any $(PS)_c$ sequence has a convergent subsequence. We will prove our theorems by using the following three-critical-point theorem related to local linking due to Brezis and Nirenberg (see Theorem 4 in [1]).

Theorem A *Let X be a Banach space with a direct sum decomposition $X = X_1 \oplus X_2$ with $\dim X_1 < \infty$. Let I be a C^1 function on X with $I(0) = 0$ satisfying the (PS) condition, and assume that, for some $R > 0$,*

$$\begin{cases} I(u) \leq 0 & \textit{for } u \in X_1, \|u\| \leq R, \\ I(u) \geq 0 & \textit{for } u \in X_2, \|u\| \leq R. \end{cases}$$

Assume also that I is bounded below and $\inf_X I < 0$. Then I has at least two nonzero critical points.

Proof of Theorem 1 (a) The functional I satisfies the local linking at zero with respect to $(V_k, V_k^\perp)$, where $V_k = \bigoplus_{i=1}^k \ker(-\Delta - \lambda_i(m_0))$ and $V_k^\perp = \bigoplus_{i=k+1}^{+\infty} \ker(-\Delta - \lambda_i(m_0))$ such that $H_0^1(\Omega) = V_k \oplus V_k^\perp$.

In fact, from (5), for any $\varepsilon > 0$, there is a positive constant L_0 such that

$$\left|2F(x,t) - am_0(x)t^2\right| \le a\varepsilon t^2 \quad \text{for any } x \in \Omega \text{ and } |t| \le L_0.$$

Combining the continuity of F, (8), and the above inequality, there is $M_0 = M_0(\varepsilon) > 0$ such that

$$F(x,t) \ge \frac{a}{2}m_0(x)t^2 - \frac{a\varepsilon}{2}t^2 - M_0|t|^p \quad \text{for any } (x,t) \in \Omega \times R, \quad \text{and} \tag{11}$$

$$F(x,t) \le \frac{a}{2}m_0(x)t^2 + \frac{a\varepsilon}{2}t^2 + M_0|t|^p \quad \text{for any } (x,t) \in \Omega \times R. \tag{12}$$

For any $u \in V_k$, from (2), (10), and (11), it follows that

$$\begin{aligned} I(u) &\le \frac{b}{4}\|u\|^4 + \frac{a}{2}\|u\|^2 - \frac{a}{2}\int_\Omega m_0(x)|u|^2\,dx + \frac{a\varepsilon}{2}\int_\Omega |u|^2\,dx + M_0\int_\Omega |u|^p\,dx \\ &\le \frac{a}{2}\left(1 - \frac{1}{\lambda_k(m_0)} + \varepsilon\tau_2^2\right)\|u\|^2 + \frac{b}{4}\|u\|^4 + M_0\tau_p^p\|u\|^p. \end{aligned} \tag{13}$$

On the other hand, for any $u \in V_k^\perp$, from (2), (10), and (12), we obtain

$$\begin{aligned} I(u) &\ge \frac{b}{4}\|u\|^4 + \frac{a}{2}\|u\|^2 - \frac{a}{2}\int_\Omega m_0(x)|u|^2\,dx - \frac{a\varepsilon}{2}\int_\Omega |u|^p\,dx - M_0\int_\Omega |u|^p\,dx \\ &\ge \frac{a}{2}\left(1 - \frac{1}{\lambda_{k+1}(m_0)} - \varepsilon\tau_2^2\right)\|u\|^2 + \frac{b}{4}\|u\|^4 - M_0\tau_p^p\|u\|^p. \end{aligned} \tag{14}$$

Noting that $\lambda_k(m_0) < 1 < \lambda_{k+1}(m_0)$ and $4 < p < 2^*$, (13) and (14), let $\varepsilon = \min\{(1 - \lambda_k(m_0))/\lambda_k(m_0), (\lambda_{k+1}(m_0) - 1)/\lambda_{k+1}(m_0)\}/2\tau_2^2$, there is a constant $r_0 > 0$ such that

$$I(u) < 0 \quad \text{for any } u \in V_k \text{ with } 0 < \|u\| \le r_0,$$
$$I(u) > 0 \quad \text{for any } u \in V_k^\perp \text{ with } 0 < \|u\| \le r_0.$$

(b) The functional I satisfies the (PS) condition. To the end, it suffices to say the functional I is coercive on $H_0^1(\Omega)$, i.e., $I(u) \to +\infty$ as $\|u\| \to \infty$.

If $\mu_1(m_\infty) > 1$, by (6), for any $\varepsilon > 0$, there is $L_1 > 0$ such that

$$\left|4F(x,t) - bm_\infty(x)t^4\right| \le b\varepsilon t^4 \quad \text{for any } x \in \Omega \text{ and } |t| \ge L_1.$$

Hence, from the continuity of F, there exists $M_1 = M_1(\varepsilon) > 0$ such that

$$F(x,t) \le \frac{b}{4}m_\infty(x)t^4 + \frac{b\varepsilon}{4}t^4 + M_1 \quad \text{for any } (x,t) \in \Omega \times R. \tag{15}$$

From (2), (10), and (15), we obtain

$$\begin{aligned} I(u) &\ge \frac{b}{4}\|u\|^4 + \frac{a}{2}\|u\|^2 - \frac{b}{4}\int_\Omega m_\infty(x)|u|^4\,dx - \frac{b\varepsilon}{4}\int_\Omega |u|^4\,dx - M_1|\Omega| \\ &\ge \frac{b}{4}\left(1 - \frac{1}{\mu_1(m_\infty)} - \varepsilon\tau_4^4\right)\|u\|^4 - M_1|\Omega|, \end{aligned}$$

where $|\Omega|$ denotes the Lebesgue measure of Ω. Hence, for $\varepsilon > 0$ small enough, it follows that the functional I is coercive on $H_0^1(\Omega)$.

If $\mu_1(m_\infty) = 1$ and (7) hold, let

$$H(x,t) = F(x,t) - \frac{b}{4} m_\infty(x) t^4.$$

By a simple computation, it follows that

$$H'(x,t)t - 4H(x,t) = f(x,t)t - 4F(x,t).$$

From (7), for any $M_2 > 0$, there is $L_2 > 0$ such that

$$H'(x,t)t - 4H(x,t) \geq M_2 \quad \text{for any } x \in \Omega \text{ and } |t| \geq L_2.$$

Hence, we have

$$\frac{d}{ds}\left(\frac{H(x,s)}{s^4}\right) = \frac{H'(x,s)s - 4H(x,s)}{s^5} \geq \frac{M_2}{s^5} \quad \text{for any } x \in \Omega \text{ and } |s| \geq L_2.$$

Integrating the above expression over the interval $[t,T] \subset [L_2,\infty)$, we obtain

$$\frac{H(x,t)}{t^4} \leq \frac{H(x,T)}{T^4} + \frac{M_2}{4}\left(\frac{1}{T^4} - \frac{1}{t^4}\right).$$

Noting that $\lim_{|T|\to\infty} H(x,T)/T^4 = 0$, let $T \to +\infty$, we obtain $H(x,t) \leq -M_2/4$ for $t \geq L_2$ and $x \in \Omega$. Similarly, $H(x,t) \leq -M_2/4$ for $t \leq -L_2$ and $x \in \Omega$. Hence, from the arbitrariness of $M_2(> 0)$, we have

$$\lim_{|t|\to\infty} H(x,t) = -\infty \quad \text{uniformly in } x \in \Omega.$$

Moreover, from the continuity of F, there is a positive constant M_3 such that

$$H(x,t) < M_3 \quad \text{for any } (x,t) \in \Omega \times R. \tag{16}$$

If the functional I is not coercive on $H_0^1(\Omega)$, there are a sequence $\{u_n\} \subset H_0^1(\Omega)$ and a positive constant M_4 such that $\|u_n\| \to \infty$ as $n \to \infty$ and $I(u_n) \leq M_4$. By the definition of $\mu_1(m_\infty)$ and $\mu_1(m_\infty) = 1$, we have that $\int_\Omega m_\infty(x)|u_n|^4\,dx \leq \|u_n\|^4$. Hence, from (16), it follows that

$$\begin{aligned} M_4 \geq I(u_n) &= \frac{b}{4}\|u_n\|^4 + \frac{a}{2}\|u_n\|^2 - \frac{b}{4}\int_\Omega m_\infty(x)|u_n|^4\,dx - \int_\Omega H(x,u_n)\,dx \\ &\geq \frac{b}{4}\|u_n\|^4 + \frac{a}{2}\|u_n\|^2 - \frac{b}{4}\int_\Omega m_\infty(x)|u_n|^4\,dx - M_3|\Omega| \\ &\geq \frac{a}{2}\|u_n\|^2 - M_3|\Omega| \\ &\to +\infty \quad \text{as } n \to \infty, \end{aligned}$$

which is a contradiction, and the conclusion is proved.

(c) From (b), we have that the functional I is bounded from below. From the fact that $I(u) < 0$ for any $u \in V_k$ with $0 < \|u\| \leq r_0$, we have $\inf_{u\in H_0^1(\Omega)} I(u) < 0$. Moreover, $I(0) = 0$. Therefore, Theorem 1 is proved by Theorem A. □

Proof of Theorem 2 First of all, from (a) of the proof of Theorem 1, we have that the functional I satisfies the local linking at zero with respect to $(V_k, V_k^{\perp})$. And then, we know from (9) that $f(x,t)$ is 3-sublinear at infinity, which implies that the functional I is coercive on $H_0^1(\Omega)$ by a standard argument. We obtain that the functional I is bounded from below and satisfies the (PS) condition for $N = 1, 2, 3$. In the following, we only prove that the functional I also satisfies the (PS) condition for $p = 2^*$ $(N \geq 4)$, where $f(x,t)$ is not only 3-sublinear at infinity, but also is asymptotically critical growth at infinity.

In fact, let $\{u_n\}$ be a (PS) sequence of I, that is,

$$I(u_n) \to c, \qquad I'(u_n) \to 0 \quad \text{as } n \to \infty. \tag{17}$$

Noting that the functional I is coercive on $H_0^1(\Omega)$, we obtain that $\{u_n\}$ is bounded in $H_0^1(\Omega)$. Going if necessary to a subsequence, we can assume $u_n \rightharpoonup u$ in $H_0^1(\Omega)$, and by the Rellich theorem, $u_n \to u$ in $L^r(\Omega)$ $(1 \leq r < 2^*)$. From (17) and the boundedness of $\{u_n\}$, we have

$$\langle I'(u_n), u_n - u\rangle = \left(a + b\|u_n\|^2\right) \int_\Omega \nabla u_n(\nabla u_n - \nabla u)\,dx + \int_\Omega f(x,u_n)(u_n - u)\,dx \to 0 \tag{18}$$

as $n \to \infty$. From (9), for any $\varepsilon > 0$, there is $M_5 > 0$ such that

$$\left|f(x,t)\right| \leq \varepsilon|t|^{p-1} + M_5 \quad \text{for any } (x,t) \in \Omega \times R.$$

Hence, from Hölder's inequality, (2), the boundedness of $\{u_n\}$, and the arbitrariness of ε, we have

$$\begin{aligned}
\left|\int_\Omega f(x,u_n)(u_n - u)\,dx\right| &\leq \int_\Omega \left(\varepsilon|u_n|^{p-1} + M_5\right)|u_n - u|\,dx \\
&\leq \varepsilon \int_\Omega \left(|u_n|^p + |u_n|^{p-1}|u|\right)dx + M_5\|u_n - u\|_{L^1} \\
&\leq \varepsilon\|u_n\|_{L^p}^{p-1}\left(\|u_n\|_{L^p} + \|u\|_{L^p}\right) + M_5\|u_n - u\|_{L^1} \\
&\to 0 \quad \text{as } n \to \infty.
\end{aligned}$$

Combining with (18), we have

$$\int_\Omega \nabla u_n(\nabla u_n - \nabla u)\,dx \to 0 \quad \text{as } n \to \infty.$$

Since $u_n \rightharpoonup u$ weakly in $H_0^1(\Omega)$, we have

$$\int_\Omega \nabla u(\nabla u_n - \nabla u)\,dx \to 0 \quad \text{as } n \to \infty.$$

Then $u_n \to u$ strongly in $H_0^1(\Omega)$ as $n \to \infty$.

At last, similar to (c) of the proof of Theorem 1, Theorem 2 is proved. □

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