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A class of second-order nonlocal indefinite impulsive differential systems

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Abstract

We consider the second-order nonlocal impulsive differential system

$$\begin{cases} -x'' = a(t)xy + \omega(t)f(x), \quad 0 < t < 1, t \neq t_k, \\ -y'' = b(t)x, \quad 0 < t < 1, t \neq t_k, \\ \Delta x|_{t=t_k} = I_k(x(t_k)), \quad k = 1, 2, \ldots, n, \\ \Delta y|_{t=t_k} = J_k(y(t_k)), \quad k = 1, 2, \ldots, n, \\ x(0) = \int_0^1 h(t)x(t)\,dt, \qquad x'(1) = 0, \\ y(0) = \int_0^1 g(t)y(t)\,dt, \qquad y'(1) = 0, \end{cases}$$

where the weight functions $a(t)$, $b(t)$, and $\omega(t)$ change sign on $[0, 1]$, and $g(t) \not\equiv 0$ and $h(t) \not\equiv 0$ on $[0, 1]$. By constructing a cone $K_1 \times K_2$, which is the Cartesian product of two cones in space $PC[0, 1]$, and applying the well-known fixed point theorem of cone expansion and compression in $K_1 \times K_2$, we obtain conditions for the existence and multiplicity of positive solutions of a nonlocal indefinite impulsive differential system. An example is given to illustrate the main results.

Keywords: Indefinite weights; Nonlocal impulsive differential system; Positive solutions; Existence and multiplicity; Fixed point technology

1 Introduction

It is generally accepted that the theory and applications of differential equations with impulsive effects are an important area of investigation, since it is far richer than the corresponding theory of differential equations without impulsive effects. Various population models, biological system models, ecology models, biotechnology models, pharmacokinetics models, and optimal control models, which are characterized by the fact that per sudden changing of their state, can be expressed by impulsive differential equations. For an introduction of general theory of impulsive differential equations, we refer the reader to the references [1] and [2], whereas the applications of impulsive differential equations can be found in [3–5]. Some classical methods have been widely used to study impulsive differential equations: the theory of critical point theory and variational methods [6–8], fixed point theorems in cones [9–25], and bifurcation theory [26, 27]. In particular, we would like to mention some results of Lin and Jiang [28] and Feng and Xie [29]. Lin and Jiang [28] considered the following Dirichlet boundary value problem with impulse effects:

$$\begin{cases} -u''(t) = f(t, u(t)), \quad t \in J, t \neq t_k, \\ \triangle u'|_{t \neq t_k} = -I_k(u(t_k)), \quad k = 1, 2, \dots, m, \\ u(0) = u(1) = 0, \end{cases} \tag{1.1}$$

and by means of the fixed point index theory in cones the authors obtained some sufficient conditions for the existence of multiple positive solutions for problem (1.1).

Recently, using fixed point theorems in a cone, Feng and Xie [29] studied the existence of positive solutions for the following problem:

$$\begin{cases} -u''(t) = f(t, u(t)), \quad t \in J, t \neq t_k, \\ -\triangle u'|_{t \neq t_k} = I_k(u(t_k)), \quad k = 1, 2, \dots, n, \\ u(0) = \sum_{i=1}^{m-2} a_i u(\xi_i), \qquad u(1) = \sum_{i=1}^{m-2} b_i u(\xi_i). \end{cases} \tag{1.2}$$

In comparison with numerous results on the impulsive differential equations, it is less known about the impulsive differential systems, even for the nonlocal impulsive differential systems.

Moreover, we see that increasing attention has been paid to the study of boundary value problems with integral boundary conditions; for example, see Liu, Sun, Zhang, and Wu [30], Zhang, Feng, and Ge [31], Zhang and Ge [32], Hao et al. [33–35], Yan, Zuo, and Hao [36], Zhang et al. [37, 38], Sun, Liu, and Wu [39], Lin and Zhao [40], and Ahmad, Alsaedi, and Alghamdi [41]. This problem contains two-, three-, and multipoint boundary value problems as particular cases; for instance, see Karakostas and Tsamatos [42], Feng and Ge [43], Jiang, Liu, and Wu [44], Lan [45], Zhang et al. [46–50], Feng, Du, and Ge [51], Ahmad and Alsaedi [52], Mao and Zhao [53], Liu, Hao, and Wu [54], and the references therein. Specifically, Boucherif [55] exploited the fixed point theorem in cones to study the following problem:

$$\begin{cases} u''(t) = f(t, u(t)), \quad 0 < t < 1, \\ u(0) - cu'(0) = \int_0^1 g_0(t) u(t)\, dt, \\ u(1) - du'(1) = \int_0^1 g_1(t) u(t)\, dt. \end{cases} \tag{1.3}$$

The author obtained several excellent results on the existence of positive solutions to problem (1.3).

Feng, Ji, and Ge [56] began to study the following boundary value problem with integral boundary conditions in abstract spaces:

$$\begin{cases} u''(t) + f(t, u(t)) = \theta, \quad 0 < t < 1, \\ u(0) = \int_0^1 g(t) u(t)\, dt, \qquad u(1) = \theta. \end{cases} \tag{1.4}$$

Applying the fixed point theory in a cone for strict set contraction operators, the authors investigated the existence, nonexistence, and multiplicity of positive solutions for problem (1.4).

Recently, Kong [57] considered the existence and uniqueness of positive solutions for the second-order singular boundary value problem:

$$\begin{cases} u''(t) + \lambda f(u(t)) = 0, \quad t \in (0,1), \\ u(0) = \int_0^1 u(s)\,dA(s), \qquad u(1) = \int_0^1 u(s)\,dB(s). \end{cases} \tag{1.5}$$

The author examined the uniqueness of the solution and its dependence on the parameter λ for problem (1.5) by using the mixed monotone operator theory.

Simultaneously, an indefinite problem has attracted the attention of Ma and Han [58], López-Gómez and Tellini [59], Boscaggin and Zanolin [60], Feltrin and Zanolin [61], Boscaggin et al. [62, 63], Sovrano and Zanolin [64], Bravo and Torres [65], Wang and An [66], and Yao [67]. Ma and Han [58] considered the following boundary value problem:

$$\begin{cases} u'' + \lambda a(t) f(u) = 0, \quad 0 < t < 1, \\ u(0) = u(1) = 0, \end{cases} \tag{1.6}$$

where $a \in C[0,1]$ may change sign, and λ is a parameter. They proved the existence, multiplicity, and stability of positive solutions for problem (1.6) by applying bifurcation techniques.

Aapplying the shooting method, Sovrano and Zanolin [60] presented a multiplicity result for positive solutions for the Neumann problem

$$\begin{cases} u'' + a(t) f(u) = 0, \quad 0 < t < 1, \\ u(t) > 0, \quad t \in [0,T], \\ u'(0) = u'(T) = 0, \end{cases} \tag{1.7}$$

where the weight function $a \in C[0,1]$ has indefinite sign.

Recently, Wang and An [66] dealt with the existence and multiplicity of positive solutions for the second-order differential system

$$\begin{cases} -u'' = a(t)\varphi u + h(t) f(u), \quad 0 < t < 1, \\ -\varphi'' = b(t) u, \quad 0 < t < 1, \\ u(0) = u(1) = 0, \\ \varphi(0) = \varphi(1) = 0, \end{cases} \tag{1.8}$$

where $a(t)$, $b(t)$, and $g(t)$ are allowed to change sign on $[0,1]$. For the latest results of indefinite problems, please refer to Jiao and Zhang [68], Feltrin and Sovrano [69], and Zhang [70].

For all we know, in the literature there are no papers on multiple positive solutions for analogous indefinite impulsive differential systems with nonlocal boundary value conditions. More precisely, the study of $a(t)$, $b(t)$, and $\omega(t)$ changing sign on $[0,1]$ is still open for the second-order nonlocal impulsive differential system

$$\begin{cases} -x'' = a(t)xy + \omega(t)f(x), & 0 < t < 1, t \neq t_k, \\ -y'' = b(t)x, & 0 < t < 1, \\ \Delta x|_{t=t_k} = I_k(x(t_k)), & k = 1,2,\dots,n, \\ \Delta y|_{t=t_k} = J_k(y(t_k)), & k = 1,2,\dots,n, \\ x(0) = \int_0^1 h(t)x(t)\,dt, & x'(1) = 0, \\ y(0) = \int_0^1 g(t)y(t)\,dt, & y'(1) = 0, \end{cases} \tag{1.9}$$

where $a(t)$, $\omega(t)$, $b(t)$ change sign on $[0,1]$, t_k ($k = 1,2,\dots,n$; where n is a fixed positive integer) are fixed points such that $0 < t_1 < t_2 < \cdots < t_k < \cdots < t_n < 1$, $\Delta x|_{t=t_k}$ denotes the jump of $x(t)$ at $t = t_k$, that is, $\Delta x|_{t=t_k} = x(t_k^+) - x(t_k^-)$, where $x(t_k^+)$ and $x(t_k^-)$ represent the right- and left-hand limits of $x(t)$ at $t = t_k$, respectively; $\Delta y|_{t=t_k}$ has a similar meaning for $y(t)$. In addition, a, ω, b, f, I_k, and J_k ($k = 1,2,\dots,n$) satisfy (H_1) $a, \omega, b : [0,1] \to (-\infty,+\infty)$ are continuous, and there exists a constant $\xi \in (0,1)$ such that

$$\begin{cases} a(t), \omega(t), b(t) \geq 0, & \forall t \in [0,\xi], \\ a(t)), \omega(t), b(t) \leq 0, & \forall t \in [\xi,1]. \end{cases}$$

Moreover, $a(t)$, $\omega(t)$, $b(t)$ do not vanish identically on any subinterval of $[0,1]$.

(H_2) $f : [0,+\infty) \to [0,+\infty)$ is continuous.

(H_3) $I_k : [0,+\infty) \to [0,+\infty)$ is continuous.

(H_4) $J_k : [0,+\infty) \to [0,+\infty)$ is continuous.

(H_5) $h, g \in L^1[0,1]$ are nonnegative, and $\nu, \nu_1 \in [0,1)$, where

$$\nu = \int_0^1 g(s)\,ds, \qquad \nu_1 = \int_0^1 h(s)\,ds. \tag{1.10}$$

Let $J = [0,1]$ and $J' = J \backslash \{t_1, t_2, \dots, t_n\}$. The basic space used in this paper is $PC[0,1] = \{u | u : [0,1] \to R \text{ is continuous at } t \neq t_k, u(t_k^-) = u(t_k), \text{and } u(t_k^+) \text{ exists}, k = 1,2,\dots,n\}$. Then $PC[0,1]$ is a real Banach space with the norm

$$\|u\|_{PC_1} = \max_{t\in J} \bigl|u(t)\bigr|.$$

For convenience, consider $PC_1[0,1] = \{x : x \text{ is continuous at } t \neq t_k, x(t_k^-) = x(t_k), \text{and } x(t_k^+)$ exists, $k = 1,2,\dots,n\}$, which is a real Banach space with norm

$$\|x\|_{PC_1} = \max_{t\in J} \bigl|x(t)\bigr|,$$

and $PC_2[0,1] = \{y : y \text{ is continuous at } t \neq t_k, x(t_k^-) = x(t_k), \text{and } y(t_k^+) \text{ exists}, k = 1,2,\dots,n\}$, which is a real Banach space with norm

$$\|y\|_{PC_2} = \max_{t\in J} \bigl|y(t)\bigr|.$$

Clearly, $PC_1[0,1] \times PC_2[0,1]$ is also a real Banach space with norm

$$\|(x,y)\| = \max\{\|x\|_{PC_1}, \|y\|_{PC_2}\}.$$

By a positive solution of system (1.9) we mean a pair of functions (x,y) with $x \in C^2(J') \cap PC_1[0,1]$ and $y \in C^2(J') \cap PC_2[0,1]$ such that (x,y) satisfies system (1.9) and $x,y \geq 0$, $t \in J'$, $x,y \not\equiv 0$.

Remark 1.1 The technique to deal with the impulsive term is completely different from that of [6–27].

Remark 1.2 When we consider nonlocal differential systems with indefinite weights, another difficulty is to prove $T : K_1 \times K_2 \to K_1 \times K_2$; for details, see Lemma 2.3.

Remark 1.3 It is not difficult to see that Proposition 2.3 of [67] plays key roles in the proofs of main results of [66] and [67]. However, it is invalid for nonlocal problems; for details, see Corollary 4.1.

Remark 1.4 In comparison with other related indefinite problems [58–66], the main features of this paper are as follows.

(i) $I_k, J_k \not\equiv 0$ $(k = 1, 2, \ldots, n)$ are introduced.
(ii) Nonlocal boundary conditions are introduced.
(iii) $K_1 \times K_2$ is the Cartesian product of two cones in the space $PC[0,1]$.

We define $a^{\pm}(t)$, $\omega^{\pm}(t)$, and $b^{\pm}(t)$ as

$$a^+(t) = \max\{a(t), 0\}, \qquad a^-(t) = -\min\{a(t), 0\},$$
$$\omega^+(t) = \max\{\omega(t), 0\}, \qquad \omega^-(t) = -\min\{\omega(t), 0\},$$
$$b^+(t) = \max\{b(t), 0\}, \qquad b^-(t) = -\min\{b(t), 0\},$$

so that

$$a(t) = a^+(t) - a^-(t), \qquad \omega(t) = \omega^+(t) - \omega^-(t), \qquad b(t) = b^+(t) - b^-(t), \quad \forall t \in [0,1].$$

Inspired by the references mentioned, in this paper, we investigate the existence and multiplicity of positive solutions for system (1.9). By constructing a cone $K_1 \times K_2$, which is the Cartesian product of two cones in the space $PC[0,1]$, and using the well-known fixed point theorem of cone expansion and compression, we obtain conditions for the existence and multiplicity of positive solutions of system (1.9). We remark that this is probably the first time that the existence and multiplicity of positive solutions of impulsive differential systems with indefinite weight and integral boundary conditions have been studied.

The rest of this paper is organized as follows. In Sect. 2, we give some preliminary results. Section 3 is devoted to state and prove the main results. Finally, an example is given in Sect. 4.

2 Preliminaries

In this section, we give some preliminary results for the convenience of later use and reference. It is clear that system (1.9) is equivalent to the following two boundary value problems:

$$\begin{cases} -y'' = b(t)x, & 0 < t < 1, t \neq t_k, \\ \Delta y|_{t=t_k} = J_k(y(t_k)), & k = 1, 2, \dots, n, \\ y(0) = \int_0^1 g(t)y(t)\,dt, & y'(1) = 0, \end{cases} \tag{2.1}$$

and

$$\begin{cases} -x'' = a(t)xy + \omega(t)f(x), & 0 < t < 1, t \neq t_k, \\ \Delta x|_{t=t_k} = I_k(x(t_k)), & k = 1, 2, \dots, n, \\ x(0) = \int_0^1 h(t)x(t)\,dt, & x'(1) = 0. \end{cases} \tag{2.2}$$

Lemma 2.1 *Assume that* (H_1), (H_2), (H_4), *and* (H_5) *hold. Then problem* (2.1) *has a unique solution* y, *which can be expressed in the form*

$$y(t) = \int_0^1 H(t,s)b(s)x(s)\,ds + \sum_{k=1}^n H'_s(t,t_k)J_k\big(y(t_k)\big), \tag{2.3}$$

where

$$H(t,s) = G(t,s) + \frac{1}{1-\nu}\int_0^1 G(\tau,s)g(\tau)\,d\tau, \tag{2.4}$$

$$H'_s(t,s) = G'_s(t,s) + \frac{1}{1-\nu}\int_0^1 G'_s(\tau,s)g(\tau)\,d\tau, \tag{2.5}$$

$$G(t,s) = \begin{cases} t, & 0 \le t \le s \le 1, \\ s, & 0 \le s \le t \le 1, \end{cases} \tag{2.6}$$

$$G'_s(t,s) = \begin{cases} 0, & 0 \le t \le s \le 1, \\ 1, & 0 \le s \le t \le 1. \end{cases} \tag{2.7}$$

Proof First, suppose that u is a solution of problem (2.1). It is easy to see by integration of problem (2.1) that

$$y'(t) - y'(0) = -\int_0^t b(s)x(s)\,ds. \tag{2.8}$$

Integrating again, we get

$$y(t) = y(0) + y'(0)t - \int_0^t (t-s)b(s)x(s)\,ds + \sum_{t_k<t} J_k\big(y(t_k)\big). \tag{2.9}$$

Letting $t = 1$ in (2.8), we find

$$y'(0) = \int_0^1 b(s)x(s)\,ds. \tag{2.10}$$

Substituting the boundary condition $y(0) = \int_0^1 g(t)y(t)\,dt$ and (2.10) into (2.9), we obtain

$$\begin{aligned} y(t) &= \int_0^1 g(s)y(s)\,ds + t\int_0^1 b(s)x(s)\,ds - \int_0^t (t-s)b(s)x(s)\,ds + \sum_{t_k<t} J_k\bigl(y(t_k)\bigr) \\ &= \int_0^1 g(s)y(s)\,ds + \int_0^1 G(t,s)b(s)x(s)\,ds + \sum_{k=1}^n G'_s(t,s)J_k\bigl(y(t_k)\bigr), \end{aligned}$$

where

$$\begin{aligned} \int_0^1 g(s)y(s)\,ds &= \int_0^1 g(s)\biggl[\int_0^1 g(\tau)y(\tau)\,d\tau + \int_0^1 G(s,\tau)b(\tau)x(\tau)\,d\tau \\ &\quad + \sum_{k=1}^n G'_\tau(s,\tau)J_k\bigl(y(t_k)\bigr)\biggr]\,ds \\ &= \int_0^1 g(s)\,ds\int_0^1 g(\tau)y(\tau)\,d\tau + \int_0^1 g(s)\biggl[\int_0^1 G(s,\tau)b(\tau)x(\tau)\,d\tau \\ &\quad + \sum_{k=1}^n G'_\tau(s,\tau)J_k\bigl(y(t_k)\bigr)\biggr]\,ds. \end{aligned}$$

Therefore we have

$$\int_0^1 g(s)y(s)\,ds = \frac{1}{1-\nu}\int_0^1 g(s)\biggl[\int_0^1 G(s,\tau)b(\tau)x(\tau)\,d\tau + \sum_{k=1}^n G'_\tau(s,\tau)J_k\bigl(y(t_k)\bigr)\biggr]\,ds$$

and

$$\begin{aligned} y(t) &= \frac{1}{1-\nu}\int_0^1 g(s)\biggl[\int_0^1 G(s,\tau)b(\tau)x(\tau)\,d\tau + \sum_{k=1}^n G'_\tau(s,\tau)J_k\bigl(y(t_k)\bigr)\biggr]\,ds \\ &\quad + \int_0^1 G(t,s)b(s)x(s)\,ds + \sum_{k=1}^n G'_s(t,s)J_k\bigl(y(t_k)\bigr) \\ &= \frac{1}{1-\nu}\int_0^1 \biggl[\int_0^1 G(\tau,s)g(\tau)\,d\tau\biggr]b(s)y(s)\,ds \\ &\quad + \frac{1}{1-\nu}\int_0^1 \biggl[\sum_{k=1}^n G'_s(\tau,s)g(\tau)J_k\bigl(y(t_k)\bigr)\biggr]\,d\tau \\ &\quad + \int_0^1 G(t,s)b(s)x(s)\,ds + \sum_{k=1}^n G'_s(t,s)J_k\bigl(y(t_k)\bigr). \end{aligned}$$

Let

$$\begin{aligned} H(t,s) &= G(t,s) + \frac{1}{1-\nu}\int_0^1 G(\tau,s)g(\tau)\,d\tau, \\ H'_s(t,s) &= G'_s(t,s) + \frac{1}{1-\nu}\int_0^1 G'_s(\tau,s)g(\tau)\,d\tau. \end{aligned}$$

Then

$$y(t) = \int_0^1 H(t,s)b(s)x(s)\,ds + \sum_{k=1}^{n} H_s'(t,t_k)J_k\big(y(t_k)\big).$$

The proof of sufficiency is complete.

Conversely, let u be a solution of (2.1). Direct differentiation of (2.4) and (2.5) implies, for $t \neq t_k$,

$$y'(t) = \int_0^1 b(s)x(s) - \int_0^t b(s)x(s).$$

Evidently,

$$\begin{aligned}
&-y'' = b(t)x, \\
&\Delta y|_{t=t_k} = J_k\big(y(t_k)\big), \quad k = 1,2,\dots,n, \\
&y(0) = \int_0^1 g(t)y(t)\,dt, \qquad y'(1) = 0.
\end{aligned}$$

The lemma is proved. □

Proposition 2.1 *Let $G(t,s)$, $G_s'(t,s)$, $H(t,s)$, and $H_s'(t,s)$ be given as in Lemma 2.1. If $\nu \in [0,1)$, then we get*

$$G(t,s) > 0, \qquad H(t,s) > 0, \quad \forall t,s \in (0,1), \tag{2.11}$$

$$G(t,t)G(s,s) \le G(t,s) \le G(s,s) = s \le 1, \quad \forall t,s \in J, \tag{2.12}$$

$$\rho G(t,t)G(s,s) \le H(t,s) \le H(s,s) = \gamma G(s,s) \le \gamma, \quad \forall t,s \in J, \tag{2.13}$$

$$G_s'(t,s) \le 1, \qquad 0 \le H_s'(t,s) \le \gamma, \quad \forall t,s \in J, \tag{2.14}$$

where

$$\gamma = \frac{1}{1-\nu}, \qquad \rho = 1 + \frac{\int_0^1 \tau g(\tau)\,d\tau}{1-\nu}. \tag{2.15}$$

Proof By the definition of $G(t,s)$ and $H(t,s)$, relations (2.11) and (2.12) are simple to prove.

Next, we consider (2.13). In fact, from (1.10) and (2.12) we get

$$\begin{aligned}
H(t,s) &\le G(s,s) + \frac{1}{1-\nu}\int_0^1 G(s,s)g(\tau)\,d\tau \\
&= G(s,s)\left(1 + \frac{1}{1-\nu}\int_0^1 g(\tau)\,d\tau\right) \\
&= \frac{1}{1-\nu}G(s,s) \\
&\le \gamma
\end{aligned}$$

and

$$\begin{aligned}
H(t,s) &\ge G(t,t)G(s,s) + \frac{1}{1-\nu}\int_0^1 G(s,s)G(\tau,\tau)g(\tau)\,d\tau \\
&= G(s,s)\left(G(t,t) + \frac{1}{1-\nu}\int_0^1 G(\tau,\tau)g(\tau)\,d\tau\right) \\
&\ge G(s,s)\left(G(t,t) + \frac{G(t,t)}{1-\nu}\int_0^1 G(\tau,\tau)g(\tau)\,d\tau\right) \\
&= G(s,s)G(t,t)\left(1 + \frac{G(t,t)}{1-\nu}\int_0^1 \tau g(\tau)\,d\tau\right) \\
&= \rho G(t,t)G(s,s).
\end{aligned}$$

This shows that (2.13) holds.

Similarly, by the the definition of $G'_s(t,s)$ and $H'_s(t,s)$, we can prove that (2.14) holds. □

Remark 2.1 From (2.5) we can prove that

$$H'_s(t,s) \ge \frac{1}{1-\nu}\int_\xi^1 g(\tau)\,d\tau, \quad \forall t,s \in [0,\xi].$$

Proof It follows from (2.5) and (2.7) that

$$\begin{aligned}
H'_s(t,s) &= \begin{cases} \frac{1}{1-\nu}\int_0^1 G'_s(\tau,s)g(\tau)\,d\tau, & 0 \le t \le s \le 1, \\ 1 + \frac{1}{1-\nu}\int_0^1 G'_s(\tau,s)g(\tau)\,d\tau, & 0 \le s \le t \le 1 \end{cases} \\
&= \begin{cases} \frac{1}{1-\nu}[\int_0^s G'_s(\tau,s)g(\tau)\,d\tau + \int_s^1 G'_s(\tau,s)g(\tau)\,d\tau], & 0 \le t \le s \le 1, \\ 1 + \frac{1}{1-\nu}[\int_0^s G'_s(\tau,s)g(\tau)\,d\tau + \int_s^1 G'_s(\tau,s)g(\tau)\,d\tau], & 0 \le s \le t \le 1 \end{cases} \\
&= \begin{cases} \frac{1}{1-\nu}\int_s^1 g(\tau)\,d\tau, & 0 \le t \le s \le 1, \\ 1 + \frac{1}{1-\nu}\int_s^1 g(\tau)\,d\tau, & 0 \le s \le t \le 1 \end{cases} \\
&\ge \frac{1}{1-\nu}\int_\xi^1 g(\tau)\,d\tau, \quad \forall t,s \in [0,\xi].
\end{aligned}$$
□

Lemma 2.2 *Assume that* (H_1)–(H_3) *and* (H_5) *hold. Then problem* (2.2) *has a unique solution* x *given by*

$$\begin{aligned}
x(t) = {} & \int_0^1 H_1(t,s)a(s)x(s)y(s)\,ds + \int_0^1 H_1(t,s)\omega(s)f\bigl(x(s)\bigr)\,ds \\
& + \sum_{k=1}^n H'_{1s}(t,t_k)I_k\bigl(x(t_k)\bigr),
\end{aligned} \tag{2.16}$$

where

$$H_1(t,s) = G(t,s) + \frac{1}{1-\nu_1}\int_0^1 G(\tau,s)h(\tau)\,d\tau, \tag{2.17}$$

$$H'_{1s}(t,s) = G'_s(t,s) + \frac{1}{1-\nu_1}\int_0^1 G'_s(\tau,s)h(\tau)\,d\tau. \tag{2.18}$$

Proof The proof of Lemma 2.2 is similar to that of Lemma 2.1. □

Proposition 2.2 *Let H_1 and H'_{1s} be given as in Lemma 2.1. If $\nu_1 \in [0,1)$, then we get*

$$\rho_1 G(t,t)G(s,s) \le H_1(t,s) \le H_1(s,s) = \gamma_1 G(s,s) \le \gamma_1, \quad \forall t,s \in J, \tag{2.19}$$

$$G'_s(t,s) \le 1, \qquad 0 \le H'_{1s}(t,s) \le \gamma_1, \quad \forall t,s \in J, \tag{2.20}$$

where

$$\gamma_1 = \frac{1}{1-\nu_1}, \qquad \rho_1 = 1 + \frac{\int_0^1 \tau h(\tau)\,d\tau}{1-\nu_1}. \tag{2.21}$$

Remark 2.2 From (2.18) we can prove that

$$H'_{1s}(t,s) \ge \frac{1}{1-\nu_1}\int_\xi^1 h(\tau)\,d\tau, \quad \forall t,s \in [0,\xi].$$

Remark 2.3 Let (x,y) be a solution of system (1.9). Then from Lemma 2.1 and Lemma 2.2 we have

$$\begin{aligned} x(t) &= \int_0^1\int_0^1 H_1(t,s)H(s,\tau)a(s)b(\tau)x(s)x(\tau)\,d\tau\,ds + \int_0^1 H_1(t,s)\omega(s)f\bigl(x(s)\bigr)\,ds \\ &\quad + \int_0^1 H_1(t,s)a(s)x(s)\left(\sum_{k=1}^n H'_\tau(s,t_k)J_k\bigl(y(t_k)\bigr)\right)ds + \sum_{k=1}^n H'_{1s}(t,t_k)I_k\bigl(x(t_k)\bigr), \\ y(t) &= \int_0^1 H(t,s)b(s)x(s)\,ds + \sum_{k=1}^n H'_s(t,t_k)J_k\bigl(y(t_k)\bigr), \end{aligned} \tag{2.22}$$

where

$$H'_\tau(s,\tau) = G'_\tau(s,\tau) + \frac{1}{1-\nu}\int_0^1 G'_\tau(\xi,\tau)g(\xi)\,d\xi.$$

To obtain the existence and multiplicity of a positive solution of system (1.9), we make the following hypotheses:

(H_6) There exists a constant σ_1 satisfying $0 < \sigma_1 < \xi$ such that

$$\sigma_1 \int_{\sigma_1}^\xi H(t,s)b^+(s)\,ds \ge \xi \int_\xi^1 H(t,s)b^-(s)\,ds;$$

(H_7) There exists a constant σ_2 satisfying $0 < \sigma_2 < \xi$ such that

$$\rho\sigma_2 \int_{\sigma_2}^\xi H_1(t,s)G(s,s)a^+(s)\,ds \ge \gamma\xi \int_\xi^1 H_1(t,s)a^-(s)\,ds;$$

(H_8) There exists a constant μ satisfying $0 < \mu \le 1$ such that

$$f(\omega) \ge \mu\varphi(\omega), \quad \omega \in [0,+\infty),$$

where $\varphi(\omega) = \max\{f(\rho) : 0 \le \rho \le \omega\}$;

(H_9) There exist constants $0 < \alpha < +\infty$ with $\alpha \neq 1$ and $k_1, k_2, l_1, l_2, m_1, m_2 > 0$ such that

$$\begin{aligned} &k_1 x^\alpha \leq f(x) \leq k_2 x^\alpha, \qquad l_1 x^\alpha \leq I_k(x) \leq l_2 x^\alpha, \\ &m_1 y^\alpha \leq J_k(y) \leq m_2 y^\alpha, \quad x, y \in [0, +\infty); \end{aligned}$$

(H_{10}) There exists $0 < \sigma_3 < \xi$ satisfying $\frac{\sigma_3}{2} < t_1 < \sigma_3$ such that

$$\sigma_3^\alpha \mu^2 k_1 \int_{\sigma_3}^{\xi} H_1(t,s)\omega^+(s)\,ds \geq k_2 \xi^\alpha \int_{\xi}^{1} H_1(t,s)\omega^-(s)\,ds.$$

Obviously, $\varphi : [0, +\infty) \to [0, +\infty)$ is nondecreasing. Moreover, if f is nondecreasing, then $f = \varphi$ and $\mu = 1$.

We denote

$$\begin{aligned} &PC_1^+[0,1] = \left\{ x \in PC_1[0,1] : \min_{0 \leq t \leq 1} x(t) \geq 0 \text{ and } x(0) = \int_0^1 h(t)x(t)\,dt, x'(1) = 0 \right\}, \\ &K_1 = \left\{ x \in PC_1^+[0,1] : x \text{ is concave on } [0,\xi], \text{ and convex on } [\xi, 1] \right\}, \\ &PC_2^+[0,1] = \left\{ y \in PC_2[0,1] : \min_{0 \leq t \leq 1} y(t) \geq 0 \text{ and } y(0) = \int_0^1 g(t)y(t)\,dt, y'(1) = 0 \right\}, \\ &K_2 = \left\{ y \in PC_2^+[0,1] : y \text{ is concave on } [0,\xi], \text{and convex on } [\xi, 1] \right\}. \end{aligned}$$

If $x \in K_1$, then it is easy to see that $\|x\|_{PC_1} = \max_{0 \leq t \leq \xi} |x(t)|$. Similarly, we have $\|y\|_{PC_2} = \max_{0 \leq t \leq \xi} |y(t)|$. Also, for a positive number r, we define

$$\Omega_r = \left\{ (x,y) \in K_1 \times K_2, \left\| (x,y) \right\| < r \right\},$$

and then we get

$$\partial \Omega_r = \left\{ (x,y) \in K_1 \times K_2, \left\| (x,y) \right\| = r \right\}.$$

For any $(x,y) \in K_1 \times K_2$, define the mappings $T_1 : K_1 \to PC_1[0,1]$, $T_2 : K_2 \to PC_2[0,1]$, and $T : K_1 \times K_2 \to PC_1[0,1] \times PC_2[0,1]$ as follows:

$$\begin{aligned} (T_1 x)(t) = {} & \int_0^1 \int_0^1 H_1(t,s)H(s,\tau)a(s)b(\tau)x(s)x(\tau)\,d\tau\,ds \\ & + \int_0^1 H_1(t,s)\omega(s)f\bigl(x(s)\bigr)\,ds \\ & + \int_0^1 H_1(t,s)a(s)x(s)\left(\sum_{k=1}^{n} H'_\tau(s,t_k)J_k\bigl(y(t_k)\bigr) \right) ds \\ & + \sum_{k=1}^{n} H'_{1s}(t,t_k)I_k\bigl(x(t_k)\bigr), \end{aligned} \tag{2.23}$$

$$\begin{aligned} &(T_2 y)(t) = \int_0^1 H(t,s)b(s)x(s)\,ds + \sum_{k=1}^{n} H'_s(t,t_k)J_k\bigl(y(t_k)\bigr), \\ &\bigl(T(x,y)\bigr)(t) = \bigl((T_1 x)(t), (T_2 y)(t)\bigr). \end{aligned}$$

Remark 2.4 It follows from Lemmas 2.1–2.2 and Remark 2.3 that (x, y) is a solution of system (1.9) if and only if (x, y) is a fixed point of operator T.

Lemma 2.3 *Assume that* (H_1)–(H_{10}) *hold. Then* $T(K_1 \times K_2) \subset K_1 \times K_2$, *and* $T : K_1 \times K_2 \to K_1 \times K_2$ *is completely continuous.*

Proof For any $(x, y) \in K_1 \times K_2$, we prove that $T(x, y) \in K_1 \times K_2$, that is, $T_1 x \in K_1$ and $T_2 y \in K_2$. In view of (2.20), we know that

$$(T_1 x)'(t) = \int_0^1 z(s)\,ds - \int_0^t z(s)\,ds,$$

where

$$z(s) = a(s)xy + \omega(s) f\bigl(x(s)\bigr),$$

and then we have $(T_1 x)'(1) = 0$. From (2.14) and (2.20) we get

$$\begin{aligned}
(T_1 x)(0) &= \int_0^1 \int_0^1 H_1(0,s)H(s,\tau)a(s)b(\tau)x(s)x(\tau)\,d\tau\,ds + \int_0^1 H_1(0,s)\omega(s)f\bigl(x(s)\bigr)\,ds \\
&\quad + \int_0^1 H_1(0,s)a(s)x(s)\left(\sum_{k=1}^n H'_\tau(s,t_k)J_k\bigl(y(t_k)\bigr)\right)ds + \sum_{k=1}^n H'_{1s}(0,t_k)I_k\bigl(x(t_k)\bigr) \\
&= \frac{1}{1-\nu_1}\int_0^1 a(s)x(s)\int_0^1 G(\tau,s)h(\tau)\,d\tau\,ds \int_0^1 H(s,\tau)b(\tau)x(\tau)\,d\tau \\
&\quad + \frac{1}{1-\nu_1}\int_0^1 a(s)x(s)\int_0^1 G(\tau,s)h(\tau)\,d\tau\left(\sum_{k=1}^n H'_\tau(s,t_k)J_k\bigl(y(t_k)\bigr)\right)ds \\
&\quad + \frac{1}{1-\nu_1}\int_0^1 \omega(s)f\bigl(x(s)\bigr)\int_0^1 G(\tau,s)h(\tau)\,d\tau\,ds \\
&\quad + \frac{1}{1-\nu_1}\sum_{k=1}^n \int_0^1 G'_s(\tau,s)h(\tau)\,d\tau\, I_k\bigl(x(t_k)\bigr)
\end{aligned}$$

and

$$\begin{aligned}
\int_0^1 h(t)(T_1 x)(t)\,dt &= \int_0^1 h(t)\biggl[\int_0^1\int_0^1 H_1(t,s)H(s,\tau)a(s)b(\tau)x(s)x(\tau)\,d\tau\,ds \\
&\quad + \int_0^1 H_1(t,s)a(s)x(s)\left(\sum_{k=1}^n H'_\tau(s,t_k)J_k\bigl(y(t_k)\bigr)\right)ds \\
&\quad + \int_0^1 H_1(t,s)\omega(s)f\bigl(x(s)\bigr)\,ds + \sum_{k=1}^n H'_{1s}(t,t_k)I_k\bigl(x(t_k)\bigr)\biggr]dt \\
&= \int_0^1 h(t)\,dt \int_0^1 H_1(t,s)a(s)x(s)\,ds \int_0^1 H(s,\tau)b(\tau)x(\tau)\,d\tau \\
&\quad + \int_0^1 h(t)\,dt \int_0^1 H_1(t,s)a(s)x(s)\left(\sum_{k=1}^n H'_\tau(s,t_k)J_k\bigl(y(t_k)\bigr)\right)ds
\end{aligned}$$

$$
\begin{aligned}
&+\int_0^1 h(t)\,dt\int_0^1 H_1(t,s)\omega(s)f\bigl(x(s)\bigr)\,ds\\
&+\sum_{k=1}^n\int_0^1 H'_{1s}(t,t_k)h(t)\,dtI_k\bigl(x(t_k)\bigr)\\
={}&\frac{1}{1-\nu_1}\int_0^1 a(s)x(s)\int_0^1 G(\tau,s)h(\tau)\,d\tau\,ds\int_0^1 H(s,\tau)b(\tau)x(\tau)\,d\tau\\
&+\frac{1}{1-\nu_1}\int_0^1 a(s)x(s)\int_0^1 G(\tau,s)h(\tau)\,d\tau\left(\sum_{k=1}^n H'_\tau(s,t_k)J_k\bigl(y(t_k)\bigr)\right)ds\\
&+\frac{1}{1-\nu_1}\int_0^1 \omega(s)f\bigl(x(s)\bigr)\int_0^1 G(\tau,s)h(\tau)\,d\tau\,ds\\
&+\frac{1}{1-\nu_1}\sum_{k=1}^n\int_0^1 G'_s(\tau,s)h(\tau)\,d\tau I_k\bigl(x(t_k)\bigr)\\
={}&(T_1x)(0).
\end{aligned}
$$

Similarly, we have $(T_2y)'(1)=0$, $(T_2y)(0)=\int_0^1 g(t)(T_2y)(t)\,dt$.

Define the function $q:[0,1]\to[0,1]$ as follows:

if $x(1)=0$, then

$$q(t)=\min\left\{\frac{t}{\xi},\frac{1-t}{1-\xi}\right\},\quad \forall t\in J;$$

if $x(1)>0$, then

$$q(t)=\min\left\{\frac{t}{\xi},1\right\},\quad \forall t\in J.$$

Since $\sigma_1<\xi$, $\max_{0\le t\le 1}q(t)=1$ and $\min_{\sigma_i\le t\le\xi}q(t)=\frac{\sigma_i}{\xi}$, $i=1,2,3$.

Let $x\in K_1$. Then x is concave on $[0,\xi]$ and convex on $[\xi,1]$. Noticing that $x(0)=\int_0^1 h(t)x(t)\,dt$ and $x'(1)=0$, we get

$$x(t)\ge x(\xi)q(t),\quad t\in[0,\xi];\qquad x(t)\le x(\xi)q(t),\quad t\in[\xi,1].$$

First of all, for any $x\in K_1$, we show that

$$\int_0^1 H(t,s)b(s)x(s)\,ds\ge\int_0^{\sigma_1}H(t,s)b^+(s)x(s)\,ds,\quad t\in J.$$

Indeed, for $x\in K_1$, we obtain

$$
\begin{aligned}
&\int_0^1 H(t,s)b(s)x(s)\,ds-\int_0^{\sigma_1}H(t,s)b^+(s)x(s)\,ds\\
&\quad=\int_{\sigma_1}^{\xi}H(t,s)b^+(s)x(s)\,ds-\int_{\xi}^1 H(t,s)b^-(s)x(s)\,ds\\
&\quad\ge\int_{\sigma_1}^{\xi}H(t,s)b^+(s)q(s)x(\xi)\,ds-\int_{\xi}^1 H(t,s)b^-(s)q(s)x(\xi)\,ds
\end{aligned}
$$

$$\geq x(\xi)\left[\min_{s\in[\sigma_1,\xi]} q(s)\int_{\sigma_1}^{\xi} H(t,s)b^+(s)\,ds - \max_{s\in[\xi,1]} q(s)\int_{\xi}^{1} H(t,s)b^-(s)\,ds\right]$$
$$= x(\xi)\left[\frac{\sigma_1}{\xi}\int_{\sigma_1}^{\xi} H(t,s)b^+(s)\,ds - \int_{\xi}^{1} H(t,s)b^-(s)\,ds\right].$$

Then by (H_6) we have

$$\int_0^1 H(t,s)b(s)x(s)\,ds \geq \int_0^{\sigma_1} H(t,s)b^+(s)x(s)\,ds.$$

Secondly, for any $x \in K_1$, we prove that

$$\int_0^1\int_0^1 H_1(t,s)H(s,\tau)a(s)b(\tau)x(s)x(\tau)\,d\tau\,ds$$
$$\geq \int_0^{\sigma_2}\int_0^1 H_1(t,s)H(s,\tau)a^+(s)b(\tau)x(s)x(\tau)\,d\tau\,ds.$$

For $t \in J$, since $\int_0^1 H(t,s)b(s)x(s)\,ds \geq 0$, we have

$$\int_0^1\int_0^1 H_1(t,s)H(s,\tau)a(s)b(\tau)x(s)x(\tau)\,d\tau\,ds$$
$$- \int_0^{\sigma_2}\int_0^1 H_1(t,s)H(s,\tau)a^+(s)b(\tau)x(s)x(\tau)\,d\tau\,ds$$
$$= \int_{\sigma_2}^{\xi}\int_0^1 H_1(t,s)H(s,\tau)a^+(s)b(\tau)x(s)x(\tau)\,d\tau\,ds$$
$$- \int_{\xi}^{1}\int_0^1 H_1(t,s)H(s,\tau)a^-(s)b(\tau)x(s)x(\tau)\,d\tau\,ds$$
$$\geq \int_{\sigma_2}^{\xi} H_1(t,s)a^+(s)q(s)x(\xi)\int_0^1 H(s,\tau)b(\tau)x(\tau)\,d\tau\,ds$$
$$- \int_{\xi}^{1} H_1(t,s)a^-(s)q(s)x(\xi)\int_0^1 H(s,\tau)b(\tau)x(\tau)\,d\tau\,ds$$
$$\geq \int_{\sigma_2}^{\xi} H_1(t,s)a^+(s)\min_{s\in[\sigma_2,\xi]} q(s)x(\xi)\int_0^1 H(s,\tau)b(\tau)x(\tau)\,d\tau\,ds$$
$$- \int_{\xi}^{1} H_1(t,s)a^-(s)\max_{s\in[\xi,1]} q(s)x(\xi)\int_0^1 H(s,\tau)b(\tau)x(\tau)\,d\tau\,ds$$
$$\geq x(\xi)\frac{\sigma_2}{\xi}\int_{\sigma_2}^{\xi} H_1(t,s)a^+(s)\int_0^1 H(s,\tau)b(\tau)x(\tau)\,d\tau\,ds$$
$$- x(\xi)\int_{\xi}^{1} H_1(t,s)a^-(s)\int_0^1 H(s,\tau)b(\tau)x(\tau)\,d\tau\,ds$$
$$\geq x(\xi)\frac{\sigma_2}{\xi}\int_{\sigma_2}^{\xi} H_1(t,s)a^+(s)\int_0^1 \rho G(s,s)G(\tau,\tau)b(\tau)x(\tau)\,d\tau\,ds$$
$$- x(\xi)\int_{\xi}^{1} H_1(t,s)a^-(s)\int_0^1 \gamma G(\tau,\tau)b(\tau)x(\tau)\,d\tau\,ds$$

$$
\begin{aligned}
&= x(\xi)\int_0^1 G(\tau,\tau)b(\tau)x(\tau)\,d\tau\left[\frac{\sigma_2}{\xi}\rho\int_{\sigma_2}^{\xi} H_1(t,s)G(s,s)a^+(s)\,ds\right.\\
&\quad\left.-\gamma\int_{\xi}^1 H_1(t,s)a^-(s)\,ds\right],
\end{aligned}
$$

and then it follows from (H_7) that

$$
\begin{aligned}
&\int_0^1\int_0^1 H_1(t,s)H(s,\tau)a(s)b(\tau)x(s)x(\tau)\,d\tau\,ds\\
&\quad\geq \int_0^{\sigma_2}\int_0^1 H_1(t,s)H(s,\tau)a^+(s)b(\tau)x(s)x(\tau)\,d\tau\,ds.
\end{aligned}
$$

Similarly, for any $y\in K_2$, since $\sum_{k=1}^n H'_\tau(s,t_k)I_k(y(t_k))\geq 0$, we get

$$
\begin{aligned}
&\int_0^1 H_1(t,s)a(s)x(s)\left(\sum_{k=1}^n H'_\tau(s,t_k)J_k\big(y(t_k)\big)\right)ds\\
&\quad\geq \int_0^{\sigma_2} H_1(t,s)a(s)x(s)\left(\sum_{k=1}^n H'_\tau(s,t_k)J_k\big(y(t_k)\big)\right)ds.
\end{aligned}
$$

Thirdly, for any $x\in K_1$, we prove that

$$
\int_0^1 H_1(t,s)\omega(s)f\big(x(s)\big)\,ds\geq\int_0^{\sigma_3} H_1(t,s)\omega^+(s)f\big(x(s)\big)\,ds,\quad t\in J.
$$

Since φ is nondecreasing, we also obtain

$$
\begin{aligned}
&\varphi\big(x(t)\big)\geq\varphi\big(q(t)x(\xi)\big),\quad t\in[0,\xi],\\
&\varphi\big(x(t)\big)\leq\varphi\big(q(t)x(\xi)\big),\quad t\in[\xi,1].
\end{aligned}
$$

Therefore, for any $x\in K_1$, it follows from (H_8)–(H_{10}) that

$$
\begin{aligned}
&\int_0^1 H_1(t,s)\omega(s)f\big(x(s)\big)\,ds-\int_0^{\sigma_3} H_1(t,s)\omega^+(s)f\big(x(s)\big)\,ds\\
&\quad=\int_{\sigma_3}^{\xi} H_1(t,s)\omega^+(s)f\big(x(s)\big)\,ds-\int_{\xi}^1 H_1(t,s)\omega^-(s)f\big(x(s)\big)\,ds\\
&\quad\geq\mu\int_{\sigma_3}^{\xi} H_1(t,s)\omega^+(s)\varphi\big(x(s)\big)\,ds-\int_{\xi}^1 H_1(t,s)\omega^-(s)\varphi\big(x(s)\big)\,ds\\
&\quad\geq\mu\int_{\sigma_3}^{\xi} H_1(t,s)\omega^+(s)\varphi\big(x(\xi)q(s)\big)\,ds-\int_{\xi}^1 H_1(t,s)\omega^-(s)\varphi\big(x(\xi)q(s)\big)\,ds\\
&\quad\geq\mu\int_{\sigma_3}^{\xi} H_1(t,s)\omega^+(s)f\big(x(\xi)q(s)\big)\,ds-\frac{1}{\mu}\int_{\xi}^1 H_1(t,s)\omega^-(s)f\big(x(\xi)q(s)\big)\,ds\\
&\quad\geq\mu\int_{\sigma_3}^{\xi} H_1(t,s)\omega^+(s)k_1x^\alpha(\xi)q^\alpha(s)\,ds-\frac{1}{\mu}\int_{\xi}^1 H_1(t,s)\omega^-(s)k_2x^\alpha(\xi)q^\alpha(s)\,ds\\
&\quad\geq x^\alpha(\xi)\left[\min_{\sigma_3\leq s\leq\xi}q^\alpha(s)\mu k_1\int_{\sigma_3}^{\xi} H_1(t,s)\omega^+(s)\,ds-\max_{\xi\leq s\leq 1}q^\alpha(s)\frac{1}{\mu}k_2\int_{\xi}^1 H_1(t,s)\omega^-(s)\,ds\right]
\end{aligned}
$$

$$
\begin{aligned}
&\geq x^{\alpha}(\xi)\left[\frac{\sigma_3^{\alpha}}{\xi^{\alpha}}\mu k_1\int_{\sigma_3}^{\xi}H_1(t,s)\omega^{+}(s)\,ds-\frac{1}{\mu}k_2\int_{\xi}^{1}H_1(t,s)\omega^{-}(s)\,ds\right]\\
&\geq 0,
\end{aligned}
$$

which shows that

$$\int_0^1 H_1(t,s)\omega(s)f\bigl(x(s)\bigr)\,ds\geq\int_0^{\sigma_3}H_1(t,s)\omega^{+}(s)f\bigl(x(s)\bigr)\,ds.$$

Thus, for $(x,y)\in K_1\times K_2$,

$$
\begin{aligned}
(T_1x)(t)={}&\int_0^1\int_0^1 H_1(t,s)H(s,\tau)a(s)b(\tau)x(s)x(\tau)\,d\tau\,ds+\int_0^1 H_1(t,s)\omega(s)f\bigl(x(t_k)\bigr)\,ds\\
&+\int_0^1 H_1(t,s)a(s)x(s)\left(\sum_{k=1}^{n}H'_{\tau}(s,t_k)J_k\bigl(y(t_k)\bigr)\right)ds+\sum_{k=1}^{n}H'_{1s}(t,t_k)I_k\bigl(x(t_k)\bigr)\\
\geq{}&\int_0^{\sigma_2}H_1(t,s)a^{+}(s)x(s)\int_0^1 H(s,\tau)b(\tau)x(\tau)\,d\tau\,ds\\
&+\int_0^{\sigma_3}H_1(t,s)\omega^{+}(s)f\bigl(x(t_k)\bigr)\,ds\\
&+\int_0^{\sigma_2}H_1(t,s)a^{+}(s)x(s)\left(\sum_{k=1}^{n}H'_{\tau}(s,t_k)J_k\bigl(y(t_k)\bigr)\right)ds+\sum_{k=1}^{n}H'_{1s}(t,t_k)I_k\bigl(x(t_k)\bigr)\\
\geq{}&0,\\
(T_2y)(t)={}&\int_0^1 H(t,s)b(s)x(s)\,ds+\sum_{k=1}^{n}H'_s(t,t_k)J_k\bigl(y(t_k)\bigr)\\
\geq{}&\int_0^{\sigma_1}H(t,s)b^{+}(s)x(s)\,ds+\sum_{k=1}^{n}H'_s(t,t_k)J_k\bigl(y(t_k)\bigr)\\
\geq{}&0.
\end{aligned}
$$

Moreover, by direct calculation we derive

$$
\begin{aligned}
&(T_1x)''(t)=-a^{+}(t)x(t)y(t)-\omega^{+}(t)f(x)\leq 0,\quad t\in[0,\xi],\\
&(T_2x)''(t)=a^{-}(t)x(t)y(t)+\omega^{-}(t)f(x)\geq 0,\quad t\in[\xi,1],\\
&(T_1y)''(t)=-b^{+}(t)x(t)\leq 0,\quad t\in[0,\xi],\\
&(T_2y)''(t)=b^{-}(t)x(t)\geq 0,\quad t\in[\xi,1],
\end{aligned}
$$

which shows that T_1x and T_2y are concave on $[0,\xi]$ and convex on $[\xi,1]$. It follows that $T_1x\in K_1$ and $T_2y\in K_2$. Thus $T(K_1\times K_2)\subset K_1\times K_2$.

Finally, by standard methods and the Arzelà–Ascoli theorem we can prove that T is completely continuous. □

Remark 2.5 In [66] and [67], it is not difficult to see that the function $q(t)$ plays an important role in the proof of completely continuous operator. If $x(0)=x(1)=0$, then we can define $q(t) = \min\{\frac{t}{\xi}, \frac{1-t}{1-\xi}\}$. However, if $x(0) = \int_0^1 h(t)x(t)\,dt$ and $x'(1) = 0$, then the definition of $q(t)$ is invalid. This shows that when $x(0) = \int_0^1 h(t)x(t)\,dt$ and $x'(1) = 0$, we require a special technique to give a fine definition of $q(t)$.

In fact, a fine definition of $q(t)$ is very difficult to give when $x(0) = \int_0^1 h(t)x(t)\,dt$ and $x'(1) = 0$. This is probably the main reason why there is almost no paper studying the existence of positive solutions for the class of second-order nonlocal differential systems with indefinite weights and even for second-order nonlocal impulsive differential systems with indefinite weights.

Remark 2.6 The idea of the proof of Lemma 2.3 comes from Theorem 3.1 of [67].

The following lemma is very crucial in our argument.

Lemma 2.4 (Theorem 2.3.4 of [71], Fixed point theorem of cone expansion and compression of norm type) *Let Ω_1 and Ω_2 be two bounded open sets in a real Banach space E such that $0 \in \Omega_1$ and $\bar{\Omega}_1 \subset \Omega_2$. Let an operator $T : K \cap (\bar{\Omega}_2 \backslash \Omega_1) \to K$ be completely continuous, where K is a cone in E. Suppose that one of the following two conditions is satisfied*:

(a) $\|Tx\| \le \|x\|, \forall x \in K \cap \partial\Omega_1$, *and* $\|Tx\| \ge \|x\|, \forall x \in K \cap \partial\Omega_2$,

and

(b) $\|Tx\| \ge \|x\|, \forall x \in K \cap \partial\Omega_1$, *and* $\|Tx\| \le \|x\|, \forall x \in K \cap \partial\Omega_2$,

is satisfied. Then T has at least one fixed point in $K \cap (\bar{\Omega}_2 \backslash \Omega_1)$.

3 Main results

In this part, applying Lemma 2.4, we obtain the following three existence theorems.

Theorem 3.1 *Assume that (H_1)–(H_{10}) hold. If $\alpha > 1$, then system (1.9) admits at least one positive solution.*

Proof On one hand, considering the case $\alpha > 1$, by (H_9) we get

$$\lim_{x\to 0} \frac{f(x)}{x} \le \lim_{x\to 0} \frac{k_2 x^\alpha}{x} = 0, \qquad \lim_{x\to 0} \frac{I_k(x)}{x} \le \lim_{x\to 0} \frac{l_2 x^\alpha}{x} = 0,$$
$$\lim_{y\to 0} \frac{J_k(y)}{y} \le \lim_{y\to 0} \frac{m_2 y^\alpha}{y} = 0.$$

Furthermore, there exist $r', r'' > 0$ such that

$$f(x) \le \varepsilon_1 x, \qquad I_k(x) \le \varepsilon_2 x, \quad k = 1, 2, \dots, n, 0 \le x \le r',$$
$$J_k(y) \le \varepsilon_3 y, \quad k = 1, 2, \dots, n, 0 \le y \le r'',$$

where ε_1, ε_2, ε_3 satisfy

$$4\gamma_1\varepsilon_1 \int_0^\xi \omega^+(s)\,ds < 1, \qquad \frac{4n\varepsilon_2}{1-\nu_1} < 1,$$
$$\varepsilon_3 < \min\left\{\frac{1}{4n\gamma r\int_0^\xi a^+(s)\,ds}, \frac{1}{n\gamma}\left(1 - \gamma\int_0^\xi b^+(s)\,ds\right)\right\}.$$

Let

$$\begin{aligned} A &= \gamma_1 \int_0^{\xi} \int_0^{\xi} H(s,\tau) a^+(s) b^+(\tau)\,d\tau\,ds, \\ A' &= \gamma n \int_0^{\xi} a^+(s)\,ds, \end{aligned} \tag{3.1}$$

and choose $r = \min\{(4A)^{-1}, (4\varepsilon_3 A')^{-1}, r', r''\}$. Then for any $(x,y) \in (K_1 \times K_2) \cap \partial\Omega_r$, we have $\|(x,y)\| = r$, and by (2.20) we get

$$\begin{aligned} (T_1x)(t) &= \int_0^1 \int_0^1 H_1(t,s)H(s,\tau)a(s)b(\tau)x(s)x(\tau)\,d\tau\,ds + \int_0^1 H_1(t,s)\omega(s)f\bigl(x(s)\bigr)\,ds \\ &\quad + \int_0^1 H_1(t,s)a(s)x(s)\left(\sum_{k=1}^n H'_\tau(s,t_k)J_k\bigl(y(t_k)\bigr)\right)ds + \sum_{k=1}^n H'_{1s}(t,t_k)I_k\bigl(x(t_k)\bigr) \\ &= \int_0^{\xi} \int_0^1 H_1(t,s)H(s,\tau)a^+(s)b(\tau)x(s)x(\tau)\,d\tau\,ds \\ &\quad - \int_{\xi}^1 \int_0^1 H_1(t,s)H(s,\tau)a^-(s)b(\tau)x(s)x(\tau)\,d\tau\,ds \\ &\quad + \int_0^{\xi} H_1(t,s)\omega^+(s)f\bigl(x(s)\bigr)\,ds - \int_{\xi}^1 H_1(t,s)\omega^-(s)f\bigl(x(s)\bigr)\,ds \\ &\quad + \int_0^{\xi} H_1(t,s)a^+(s)x(s)\left(\sum_{k=1}^n H'_\tau(s,t_k)J_k\bigl(y(t_k)\bigr)\right)ds \\ &\quad - \int_{\xi}^1 H_1(t,s)a^-(s)x(s)\left(\sum_{k=1}^n H'_\tau(s,t_k)J_k\bigl(y(t_k)\bigr)\right)ds + \sum_{k=1}^n H'_{1s}(t,t_k)I_k\bigl(x(t_k)\bigr) \\ &\le \int_0^{\xi} \int_0^1 H_1(t,s)H(s,\tau)a^+(s)b(\tau)x(s)x(\tau)\,d\tau\,ds + \int_0^{\xi} H_1(t,s)\omega^+(s)f\bigl(x(s)\bigr)\,ds \\ &\quad + \int_0^{\xi} H_1(t,s)a^+(s)x(s)\left(\sum_{k=1}^n H'_\tau(s,t_k)J_k\bigl(y(t_k)\bigr)\right)ds + \sum_{k=1}^n H'_{1s}(t,t_k)I_k\bigl(x(t_k)\bigr) \\ &= \int_0^{\xi} \int_0^{\xi} H_1(t,s)H(s,\tau)a^+(s)b^+(\tau)x(s)x(\tau)\,d\tau\,ds \\ &\quad - \int_0^{\xi} \int_0^{\xi} H_1(t,s)H(s,\tau)a^+(s)b^-(\tau)x(s)x(\tau)\,d\tau\,ds + \int_0^{\xi} H_1(t,s)\omega^+(s)f\bigl(x(s)\bigr)\,ds \\ &\quad + \int_0^{\xi} H_1(t,s)a^+(s)x(s)\left(\sum_{k=1}^n H'_\tau(s,t_k)J_k\bigl(y(t_k)\bigr)\right)ds + \sum_{k=1}^n H'_{1s}(t,t_k)I_k\bigl(x(t_k)\bigr) \\ &\le \int_0^{\xi} \int_0^{\xi} H_1(t,s)H(s,\tau)a^+(s)b^+(\tau)x(s)x(\tau)\,d\tau\,ds + \int_0^{\xi} H_1(t,s)\omega^+(s)f\bigl(x(s)\bigr)\,ds \\ &\quad + \int_0^{\xi} H_1(t,s)a^+(s)x(s)\left(\sum_{k=1}^n H'_\tau(s,t_k)J_k\bigl(y(t_k)\bigr)\right)ds + \sum_{k=1}^n H'_{1s}(t,t_k)I_k\bigl(x(t_k)\bigr) \\ &\le \gamma_1 \int_0^{\xi} \int_0^{\xi} H(s,\tau)a^+(s)b^+(\tau)x(s)x(\tau)\,d\tau\,ds + \gamma_1 \int_0^{\xi} \omega^+(s)\varepsilon_1 x\,ds \end{aligned}$$

$$
\begin{aligned}
&\quad + \gamma_1 \frac{1}{1-\nu} \sum_{k=1}^{n} \varepsilon_3 y \int_0^{\xi} a^+(s)x(s)\,ds + \frac{1}{1-\nu_1} \sum_{k=1}^{n} \varepsilon_2 x \\
&\le A\|x\|_{PC_1}^2 + \gamma_1\varepsilon_1 \int_0^{\xi} \omega^+(s)\,ds\|x\|_{PC_1} + \gamma_1 \frac{1}{1-\nu} n\varepsilon_3 \int_0^{\xi} a^+(s)\,ds\|x\|_{PC_1}\|y\|_{PC_2} \\
&\quad + \frac{1}{1-\nu_1} n\varepsilon_2\|x\|_{PC_1} \\
&\le Ar^2 + \gamma_1\varepsilon_1 r \int_0^{\xi} \omega^+(s)\,ds + A'\varepsilon_3 r^2 + \frac{1}{1-\nu_1} n\varepsilon_2 r \\
&< \frac{1}{4}r + \frac{1}{4}r + \frac{1}{4}r + \frac{1}{4}r \\
&= r,
\end{aligned}
\tag{3.2}
$$

$$
\begin{aligned}
(T_2 y)(t) &= \int_0^1 H(t,s)b(s)x(s)\,ds + \sum_{k=1}^{n} H_s'(t,t_k)J_k\bigl(y(t_k)\bigr) \\
&\le \int_0^{\xi} H(t,s)b^+(s)x(s)\,ds + \sum_{k=1}^{n} H_s'(t,t_k)J_k\bigl(y(t_k)\bigr) \\
&\le \gamma \int_0^{\xi} b^+(s)x(s)\,ds + \frac{1}{1-\nu} \sum_{k=1}^{n} J_k\bigl(y(t_k)\bigr) \\
&\le \gamma \int_0^{\xi} b^+(s)\,ds\|x\|_{PC_1} + \frac{1}{1-\nu} n\varepsilon_3\|y\|_{PC_2} \\
&\le \gamma \int_0^{\xi} b^+(s)\,dsr + \gamma n\varepsilon_3 r \\
&< r.
\end{aligned}
\tag{3.3}
$$

Consequently,

$$
\bigl\|T(x,y)\bigr\| < \bigl\|(x,y)\bigr\|, \quad \forall (x,y) \in (K_1 \times K_2) \cap \partial\Omega_r. \tag{3.4}
$$

On the other hand, since $\alpha > 1$, it follows from (H_9) that

$$
\begin{aligned}
&\lim_{x\to\infty} \frac{f(x)}{x} \ge \lim_{x\to\infty} \frac{k_1 x^{\alpha}}{x} = \infty, \\
&\lim_{x\to\infty} \frac{I_k(x)}{x} \ge \lim_{x\to\infty} \frac{l_1 x^{\alpha}}{x} = \infty, \\
&\lim_{y\to\infty} \frac{J_k(y)}{y} \ge \lim_{y\to\infty} \frac{m_1 y^{\alpha}}{y} = \infty,
\end{aligned}
$$

which shows that there exist $R', R'' > 0$ such that

$$
\begin{aligned}
&f(x) \ge \varepsilon_4 x, \qquad I_k(x) \ge \varepsilon_5 x, \quad k = 1,2,\ldots,n, x \ge R', \\
&J_k(y) \ge \varepsilon_6 y, \quad k = 1,2,\ldots,n, y \ge R'',
\end{aligned}
$$

where ε_4, ε_5, ε_6 satisfy

$$3\rho_1\frac{\sigma_2}{2}\varepsilon_4\min_{\frac{\sigma_3}{2}\leq t\leq\sigma_3}\delta(t)\int_{\frac{\sigma_3}{2}}^{\sigma_3}G_1(s,s)\omega^+(s)\,ds>1,$$
$$3\frac{1}{1-\nu_1}\varepsilon_5\min_{\frac{\sigma_3}{2}\leq t\leq\sigma_3}\delta(t)\int_{\xi}^{1}h(\tau)\,d\tau>1,$$
$$\frac{1}{1-\nu}\int_{\xi}^{1}g(\tau)\,d\tau\,\varepsilon_6\min_{\frac{\sigma_3}{2}\leq t\leq\sigma_3}\delta(t)>1,$$

where

$$\delta(t)=\min\left\{\frac{t}{\xi},\frac{\xi-t}{\xi}\right\},\quad t\in[0,\xi]. \tag{3.5}$$

If $(x,y)\in K_1\times K_2$, then x, y are two nonnegative concave functions on $[0,\xi]$. So we get

$$\begin{aligned}&x(t)\geq\delta(t)\|x\|_{PC_1},\quad t\in[0,\xi],\\&y(t)\geq\delta(t)\|y\|_{PC_2},\quad t\in[0,\xi].\end{aligned} \tag{3.6}$$

It follows that $\min_{\frac{\sigma_i}{2}\leq t\leq\sigma_i}x(t)\geq\theta_i\|x\|_{PC_1}$, $\min_{\frac{\sigma_i}{2}\leq t\leq\sigma_i}y(t)\geq\theta_i\|y\|_{PC_2}$, $i=1,2,3$, where

$$\theta_i=\min_{\frac{\sigma_i}{2}\leq t\leq\sigma_i}\delta(t)=\min\left\{\frac{\sigma_i}{2\xi},1-\frac{\sigma_i}{\xi}\right\}>0. \tag{3.7}$$

Let

$$B=\rho_1\frac{\sigma_2}{2}\theta_1\theta_2\int_{\frac{\sigma_2}{2}}^{\sigma_2}\int_{\frac{\sigma_1}{2}}^{\sigma_1}G_1(s,s)H(s,\tau)a^+(s)b^+(\tau)\,d\tau\,ds>0, \tag{3.8}$$

$R_1>\max\{(3B)^{-1},\frac{R'}{\theta_3},r\}$, $R_2>\max\{\frac{R''}{\theta_3},r\}$, and $R=\max\{R_1,R_2\}$. Then for any $(x,y)\in(K_1\times K_2)\cap\partial\Omega_R$, we have

$$R=\|(x,y)\|=\max\{\|x\|_{PC_1},\|y\|_{PC_2}\}=\max\{R_1,R_2\},$$
$$\min_{\frac{\sigma_3}{2}\leq t\leq\sigma_3}x(t)\geq\min_{\frac{\sigma_3}{2}\leq t\leq\sigma_3}\delta(t)\|x\|_{PC_1}\geq\theta_3R_1>R',$$
$$\min_{\frac{\sigma_3}{2}\leq t\leq\sigma_3}y(t)\geq\min_{\frac{\sigma_3}{2}\leq t\leq\sigma_3}\delta(t)\|y\|_{PC_2}\geq\theta_3R_2>R'',$$

and

$$\begin{aligned}&\|T_1x\|_{PC_1}\\&\quad=\max_{t\in J}\Biggl\{\int_0^1\int_0^1H_1(t,s)H(s,\tau)a(s)b(\tau)x(s)x(\tau)\,d\tau\,ds+\int_0^1H_1(t,s)\omega(s)f\bigl(x(s)\bigr)\,ds\\&\qquad+\int_0^1H_1(t,s)a(s)x(s)\left(\sum_{k=1}^{n}H'_\tau(s,t_k)J_k\bigl(y(t_k)\bigr)\right)ds+\sum_{k=1}^{n}H'_{1s}(t,t_k)I_k\bigl(x(t_k)\bigr)\Biggr\}\end{aligned}$$

$$
\begin{aligned}
&\geq \max_{t\in J}\left\{\int_0^{\sigma_2} H_1(t,s)a^+(s)x(s)\int_0^1 H(s,\tau)b(\tau)x(\tau)\,d\tau\,ds+\int_0^{\sigma_3} H_1(t,s)\omega^+(s)f\big(x(s)\big)\,ds\right.\\
&\quad\left.+\int_0^{\sigma_2} H_1(t,s)a^+(s)x(s)\left(\sum_{k=1}^{n} H'_\tau(s,t_k)J_k\big(y(t_k)\big)\right)ds+\sum_{k=1}^{n} H'_{1s}(t,t_k)I_k\big(x(t_k)\big)\right\}\\
&\geq \min_{\frac{\sigma_2}{2}\leq t\leq\sigma_2}\int_{\frac{\sigma_2}{2}}^{\sigma_2} H_1(t,s)a^+(s)x(s)\int_{\frac{\sigma_1}{2}}^{\sigma_1} H(s,\tau)b^+(\tau)x(\tau)\,d\tau\,ds\\
&\quad+\min_{\frac{\sigma_2}{2}\leq t\leq\sigma_2}\int_{\frac{\sigma_2}{2}}^{\sigma_2} H_1(t,s)a(s)x(s)\left(\sum_{\frac{\sigma_3}{2}<t_k<\sigma_3} H'_\tau(s,t_k)J_k\big(y(t_k)\big)\right)ds\\
&\quad+\min_{\frac{\sigma_2}{2}\leq t\leq\sigma_2}\int_{\frac{\sigma_3}{2}}^{\sigma_3} H_1(t,s)\omega^+(s)f\big(x(s)\big)\,ds+\min_{\sigma_3\leq t\leq\xi}\sum_{\frac{t}{2}<t_k<t} H'_{1s}(t,t_k)I_k\big(x(t_k)\big)\\
&\geq \rho_1\frac{\sigma_2}{2}\int_{\frac{\sigma_2}{2}}^{\sigma_2} G_1(s,s)a^+(s)\delta(s)\|x\|_{PC_1}\int_{\frac{\sigma_1}{2}}^{\sigma_1} H(s,\tau)b^+(\tau)\delta(\tau)\|x\|_{PC_1}\,d\tau\,ds\\
&\quad+\rho_1\frac{\sigma_2}{2}\int_{\frac{\sigma_3}{2}}^{\sigma_3} G_1(s,s)\omega^+(s)\varepsilon_3 x(s)\,ds+\frac{1}{1-\nu_1}\int_\xi^1 h(\tau)\,d\tau\sum_{\frac{\sigma_3}{2}<t_k<\sigma_3}\varepsilon_4 x(t_k)\\
&\geq \rho_1\frac{\sigma_2}{2}\int_{\frac{\sigma_2}{2}}^{\sigma_2} G_1(s,s)a^+(s)\delta(s)\|x\|_{PC_1}\int_{\frac{\sigma_1}{2}}^{\sigma_1} H(s,\tau)b^+(\tau)\delta(\tau)\|x\|_{PC_1}\,d\tau\,ds\\
&\quad+\rho_1\frac{\sigma_2}{2}\int_{\frac{\sigma_3}{2}}^{\sigma_3} G_1(s,s)\omega^+(s)\varepsilon_3 x(s)\,ds+\frac{1}{1-\nu_1}\int_\xi^1 h(\tau)\,d\tau\,\varepsilon_4\sum_{\frac{\sigma_3}{2}<t_1<\sigma_3} x(t_1)\\
&\geq B\|x\|_{PC_1}^2+\rho_1\frac{\sigma_2}{2}\varepsilon_3\min_{\frac{\sigma_3}{2}\leq t\leq\sigma_3}\delta(t)\int_{\frac{\sigma_3}{2}}^{\sigma_3} G_1(s,s)\omega^+(s)\,ds\|x\|_{PC_1}\\
&\quad+\frac{1}{1-\nu_1}\int_\xi^1 h(\tau)\,d\tau\,\varepsilon_4\min_{\frac{\sigma_3}{2}\leq t\leq\sigma_3}\delta(t)\|x\|_{PC_1}\\
&\geq B\|x\|_{PC_1}^2+\rho_1\frac{\sigma_2}{2}\varepsilon_3\min_{\frac{\sigma_3}{2}\leq t\leq\sigma_3}\delta(t)\int_{\frac{\sigma_3}{2}}^{\sigma_3} G_1(s,s)\omega^+(s)\,ds\|x\|_{PC_1}\\
&\quad+\frac{1}{1-\nu_1}\int_\xi^1 h(\tau)\,d\tau\,\varepsilon_4\min_{\frac{\sigma_3}{2}\leq t\leq\sigma_3}\delta(t)\|x\|_{PC_1}\\
&>\frac{1}{3}\|x\|_{PC_1}+\frac{1}{3}\|x\|_{PC_1}+\frac{1}{3}\|x\|_{PC_1}\\
&=\|x\|_{PC_1},
\end{aligned}
\tag{3.9}
$$

$$
\begin{aligned}
\|T_2y\|_{PC_2}&=\max_{t\in J}\left\{\int_0^1 H(t,s)b(s)x(s)\,ds+\sum_{k=1}^{n} H'_s(t,t_k)J_k\big(y(t_k)\big)\right\}\\
&\geq \max_{t\in J}\left\{\int_0^{\sigma_2} H(t,s)b^+(s)x(s)\,ds+\sum_{k=1}^{n} H'_s(t,t_k)J_k\big(y(t_k)\big)\right\}\\
&\geq \min_{\frac{\sigma_2}{2}\leq t\leq\sigma_2}\int_0^{\sigma_2} H(t,s)b^+(s)x(s)\,ds+\min_{\sigma_3\leq t\leq\xi}\sum_{\frac{t}{2}<t_k<t}\sum_{k=1}^{n} H'_s(t,t_k)J_k\big(y(t_k)\big)\\
&\geq \frac{1}{1-\nu}\int_\xi^1 g(\tau)\,d\tau\,\varepsilon_6\sum_{\frac{\sigma_3}{2}<t_1<\sigma_3} y(t_1)
\end{aligned}
$$

$$\geq \frac{1}{1-\nu}\int_{\xi}^{1} g(\tau)\,d\tau\,\varepsilon_6 \min_{\frac{\sigma_3}{2}\leq t\leq \sigma_3} \delta(t)\|y\|_{PC_2}$$
$$= \|y\|_{PC_2}. \tag{3.10}$$

Consequently,

$$\|T(x,y)\| > \|(x,y)\|, \quad \forall (x,y)\in (K_1\times K_2)\cap \partial\Omega_R. \tag{3.11}$$

Therefore, applying Lemma 2.4 to (3.4) and (3.11), we can show that T has at least one fixed point

$$(x,y)\in (K_1\times K_2)\cap (\bar{\Omega}_R \setminus \Omega_r).$$

The proof of Theorem 3.1 is completed. □

The following theorem deals with the multiplicity of system (1.9). For convenience, we introduce the following notations:

$$D = 3\gamma_1\int_0^{\xi}\omega^+(s)\,ds, \qquad \Lambda = 3n\gamma_1, \qquad \Gamma = \frac{1-\nu}{n}\left(1-\gamma\int_0^{\xi} b^+(s)\,ds\right),$$
$$D^* = \rho_1\sigma_2\int_{\frac{\sigma_3}{2}}^{\sigma_3} G_1(s,s)\omega^+(s)\,ds, \qquad \Lambda^* = 2\gamma_1\int_{\xi}^{1} h(\tau)\,d\tau,$$
$$\Gamma^* = \left(\frac{1}{1-\nu}n\int_{\xi}^{1} g(\tau)\,d\tau\right)^{-1}.$$

Theorem 3.2 *Assume that* (H_1)–(H_{10}) *hold. Suppose that* $0<\alpha<1$ *and there exist constants* d *and* r *satisfying* $0<d<\min\{(4A)^{-1},(4A'\Gamma)^{-1},r\}$ *such that*

$$\begin{aligned}
&\min_{(x,y)\in(K_1\times K_2)\cap\partial\Omega_d} f(x) < D^{-1}d,\\
&\min_{(x,y)\in(K_1\times K_2)\cap\partial\Omega_d} I_k(x) < \Lambda^{-1}d,\\
&\min_{(x,y)\in(K_1\times K_2)\cap\partial\Omega_d} J_k(y) < \Gamma d,
\end{aligned} \tag{3.12}$$

where A *and* A' *are defined in* (3.1). *Then system* (1.9) *admits at least two positive solutions.*

Proof If $0<\alpha<1$, then by (H_9) we know that

(i) $\lim_{x\to 0}\frac{f(x)}{x}\geq \lim_{x\to 0}\frac{k_1x^{\alpha}}{x}=\infty$, $\lim_{x\to 0}\frac{I_k(x)}{x}\geq \lim_{x\to 0}\frac{l_1x^{\alpha}}{x}=\infty$, $\lim_{y\to 0}\frac{J_k(y)}{y}\geq \lim_{y\to 0}\frac{m_1y^{\alpha}}{y}=\infty$;

(ii) $\lim_{x\to\infty}\frac{f(x)}{x}\leq \lim_{x\to\infty}\frac{k_2x^{\alpha}}{x}=0$, $\lim_{x\to\infty}\frac{I_k(x)}{x}\leq \lim_{x\to\infty}\frac{l_2x^{\alpha}}{x}=0$, $\lim_{y\to\infty}\frac{J_k(y)}{y}\leq \lim_{y\to\infty}\frac{m_2y^{\alpha}}{y}=0$.

From (i) it follows that there exists a sufficiently small positive constant r such that

$$f(x)\geq \varepsilon_7 x, \qquad I_k(x)\geq \varepsilon_8 x, \quad k=1,2,\ldots,n, 0\leq x\leq r,$$
$$J_k(y)\geq \varepsilon_9 y, \quad k=1,2,\ldots,n, 0\leq y\leq r,$$

where ε_7, ε_8, ε_9 satisfy

$$\begin{aligned}
&\rho_1\sigma_2\varepsilon_7 \min_{\frac{\sigma_3}{2}\leq t\leq\sigma_3} \delta(t)\int_{\frac{\sigma_3}{2}}^{\sigma_3} G_1(s,s)\omega^+(s)\,ds > 1,\\
&2\varepsilon_8\frac{1}{1-\nu_1}\int_\xi^1 h(\tau)\,d\tau \min_{\frac{\sigma_3}{2}\leq t\leq\sigma_3} \delta(t) > 1,\\
&\frac{1}{1-\nu}\int_\xi^1 g(\tau)\,d\tau\,\varepsilon_9 \min_{\frac{\sigma_3}{2}\leq t\leq\sigma_3} \delta(t) > 1,
\end{aligned}$$

with $\delta(t)$ defined in (3.5).

Therefore, for any $(x,y)\in(K_1\times K_2)\cap\partial\Omega_r$, we get from (3.6) that

$$\begin{aligned}
\|T_1x\|_{PC_1} &= \max_{t\in J}\left\{\int_0^1\int_0^1 H_1(t,s)H(s,\tau)a(s)b(\tau)x(s)x(\tau)\,d\tau\,ds\right.\\
&\quad+\int_0^1 H_1(t,s)\omega(s)f\bigl(x(s)\bigr)\,ds\\
&\quad\left.+\int_0^1 H_1(t,s)a(s)x(s)\left(\sum_{k=1}^n H'_\tau(s,t_k)J_k\bigl(y(t_k)\bigr)\right)ds+\sum_{k=1}^n H'_{1s}(t,t_k)I_k\bigl(x(t_k)\bigr)\right\}\\
&\geq \max_{t\in J}\left\{\int_0^{\sigma_2} H_1(t,s)a^+(s)x(s)\int_0^1 H(s,\tau)b(\tau)x(\tau)\,d\tau\,ds\right.\\
&\quad+\int_0^{\sigma_3} H_1(t,s)\omega^+(s)f\bigl(x(s)\bigr)\,ds\\
&\quad\left.+\int_0^{\sigma_2} H_1(t,s)a(s)x(s)\left(\sum_{k=1}^n H'_\tau(s,t_k)J_k\bigl(y(t_k)\bigr)\right)ds+\sum_{k=1}^n H'_{1s}(t,t_k)I_k\bigl(x(t_k)\bigr)\right\}\\
&\geq \min_{\frac{\sigma_2}{2}\leq t\leq\sigma_2}\int_{\frac{\sigma_2}{2}}^{\sigma_2} H_1(t,s)a^+(s)x(s)\int_{\frac{\sigma_1}{2}}^{\sigma_1} H(s,\tau)b^+(\tau)x(\tau)\,d\tau\,ds\\
&\quad+\min_{\frac{\sigma_2}{2}\leq t\leq\sigma_2}\int_{\frac{\sigma_3}{2}}^{\sigma_3} H_1(t,s)\omega^+(s)f\bigl(x(s)\bigr)\,ds+\min_{0\leq t\leq\xi}\sum_{0<t_k<\xi} H'_{1s}(t,t_k)I_k\bigl(x(t_k)\bigr)\\
&\geq \rho_1\frac{\sigma_2}{2}\int_{\frac{\sigma_3}{2}}^{\sigma_3} G_1(s,s)\omega^+(s)f\bigl(x(s)\bigr)\,ds+\frac{1}{1-\nu_1}\int_\xi^1 h(\tau)\,d\tau\sum_{0<t_k<\xi} I_k\bigl(x(t_k)\bigr)\\
&\geq \rho_1\frac{\sigma_2}{2}\int_{\frac{\sigma_3}{2}}^{\sigma_3} G_1(s,s)\omega^+(s)\varepsilon_7x(s)\,ds+\frac{1}{1-\nu_1}\int_\xi^1 h(\tau)\,d\tau\sum_{\frac{\sigma_3}{2}<t_1<\sigma_3}\varepsilon_8x(t_1)\\
&\geq \rho_1\frac{\sigma_2}{2}\varepsilon_7\min_{\frac{\sigma_3}{2}\leq t\leq\sigma_3}\delta(t)\int_{\frac{\sigma_3}{2}}^{\sigma_3} G_1(s,s)\omega^+(s)\,ds\|x\|_{PC_1}\\
&\quad+\varepsilon_8\frac{1}{1-\nu_1}\int_\xi^1 h(\tau)\,d\tau\min_{\frac{\sigma_3}{2}\leq t\leq\sigma_3}\delta(t)\|x\|_{PC_1}\\
&> \frac{1}{2}\|x\|_{PC_1}+\frac{1}{2}\|x\|_{PC_1}\\
&= \|x\|_{PC_1},
\end{aligned}\tag{3.13}$$

$$
\begin{aligned}
\|T_2y\|_{PC_2} &= \max_{t\in J}\left\{\int_0^1 H(t,s)b(s)x(s)\,ds + \sum_{k=1}^{n} H'_s(t,t_k)J_k\bigl(y(t_k)\bigr)\right\} \\
&\geq \max_{t\in J}\left\{\int_0^{\sigma_2} H(t,s)b^+(s)x(s)\,ds + \sum_{k=1}^{n} H'_s(t,t_k)J_k\bigl(y(t_k)\bigr)\right\} \\
&\geq \min_{\frac{\sigma_2}{2}\leq t\leq \sigma_2}\int_0^{\sigma_2} H(t,s)b^+(s)x(s)\,ds + \min_{\sigma_3\leq t\leq \xi}\sum_{\frac{t}{2}<t_k<t}\sum_{k=1}^{n} H'_s(t,t_k)J_k\bigl(y(t_k)\bigr) \\
&\geq \frac{1}{1-\nu}\int_\xi^1 g(\tau)\,d\tau\,\varepsilon_9 \sum_{\frac{\sigma_3}{2}<t_1<\sigma_3} y(t_1) \\
&\geq \frac{1}{1-\nu}\int_\xi^1 g(\tau)\,d\tau\,\varepsilon_9 \min_{\frac{\sigma_3}{2}\leq t\leq \sigma_3}\delta(t)\|y\|_{PC_2} \\
&= \|y\|_{PC_2}.
\end{aligned} \tag{3.14}
$$

Consequently,

$$
\bigl\|T(x,y)\bigr\| > \bigl\|(x,y)\bigr\|, \quad (x,y)\in (K_1\times K_2)\cap \partial\Omega_r. \tag{3.15}
$$

Next, let us turn to (ii), which shows that there exist $R', R'' > r$ such that

$$
\begin{aligned}
&f(x)\leq \varepsilon_1 x, \qquad I_k(x)\leq \varepsilon_2 x, \quad x\geq R', \\
&J_k(y)\leq \varepsilon_3 y, \quad y\geq R',
\end{aligned}
$$

where ε_1, ε_2, ε_3 satisfy

$$
\begin{aligned}
&5\gamma_1\varepsilon_1\int_0^\xi \omega^+(s)\,ds < 1, \qquad \frac{5n\varepsilon_2}{1-\nu_1} < 1, \\
&\varepsilon_3 < \max\left\{\frac{1}{6A'R}, \frac{1}{\gamma n} - \frac{1}{n}\int_0^\xi b^+(s)\,ds\right\}.
\end{aligned}
$$

Let

$$
\begin{aligned}
&\eta_1 = \max_{x\in[0,R']}\bigl\{f(x)\bigr\}, \qquad \eta_2 = \max_{x\in[0,R']}\bigl\{I_k(x)\bigr\}, \\
&\eta_3 = \max_{y\in[0,R'']}\bigl\{J_k(y)\bigr\}, \quad k=1,2,\ldots,n.
\end{aligned}
$$

Then

$$
f(x)\leq \varepsilon_1 x + \eta_1, \qquad I_k(x)\leq \varepsilon_2 x + \eta_2, \qquad J_k(y)\leq \varepsilon_3 y + \eta_3, \quad \forall x,y\geq 0. \tag{3.16}
$$

Let

$$
M = \gamma_1\eta_1\int_0^\xi \omega^+(s)\,ds, \qquad M^* = \frac{1}{1-\nu_1}n\eta_2.
$$

Choosing $\max\{6M, 6M^*, r\} < R < (6A)^{-1}$, for any $(x,y) \in (K_1 \times K_2) \cap \partial\Omega_R$, similarly to the proof of (3.2) and (3.3), we get

$$\begin{aligned}
(T_1x)(t) &\le \int_0^\xi \int_0^\xi H_1(t,s)H(s,\tau)a^+(s)b^+(\tau)x(s)x(\tau)\,d\tau\,ds + \int_0^\xi H_1(t,s)\omega^+(s)f\bigl(x(s)\bigr)\,ds \\
&\quad + \int_0^\xi H_1(t,s)a^+(s)x(s)\Biggl(\sum_{k=1}^n H'_\tau(s,t_k)J_k\bigl(y(t_k)\bigr)\Biggr)\,ds + \sum_{k=1}^n H'_{1s}(t,t_k)I_k\bigl(x(t_k)\bigr) \\
&\le \gamma_1 \int_0^\xi \int_0^\xi H(s,\tau)a^+(s)b^+(\tau)\|x\|^2_{PC_1}\,d\tau\,ds + \gamma_1 \int_0^\xi \omega^+(s)\bigl(\varepsilon_1\|x\|_{PC_1} + \eta_1\bigr)\,ds \\
&\quad + \gamma_1 \frac{1}{1-\nu} n \int_0^\xi a^+(s)\,ds\|x\|_{PC_1}\bigl(\varepsilon_3\|y\|_{PC_2} + \eta_3\bigr) + \frac{1}{1-\nu_1}\sum_{k=1}^n \bigl(\varepsilon_2\|x\|_{PC_1} + \eta_2\bigr) \\
&\le AR^2 + \gamma_1\varepsilon_1 \int_0^\xi \omega^+(s)\,dsR + M + \varepsilon_3 A'R^2 + \Biggl(\gamma_1 \frac{1}{1-\nu} n\eta_3 \int_0^\xi a^+(s)\,ds \\
&\quad + \frac{1}{1-\nu_1} n\varepsilon_2\Biggr)R + M^* \\
&< \frac{1}{6}R + \frac{1}{6}R + \frac{1}{6}R + \frac{1}{6}R + \frac{1}{6}R + \frac{1}{6}R \\
&= R,
\end{aligned} \tag{3.17}$$

$$\begin{aligned}
(T_2y)(t) &= \int_0^1 H(t,s)b(s)x(s)\,ds + \sum_{k=1}^n H'_s(t,t_k)J_k\bigl(y(t_k)\bigr) \\
&\le \int_0^\xi H(t,s)b^+(s)x(s)\,ds + \sum_{k=1}^n H'_s(t,t_k)J_k\bigl(y(t_k)\bigr) \\
&\le \gamma \int_0^\xi b^+(s)x(s)\,ds + \frac{1}{1-\nu}\sum_{k=1}^n J_k\bigl(y(t_k)\bigr) \\
&\le \gamma \int_0^\xi b^+(s)\,ds\|x\|_{PC_1} + \frac{1}{1-\nu} n\varepsilon_3\|y\|_{PC_2} \\
&\le \gamma \int_0^\xi b^+(s)\,dsR + \gamma n\varepsilon_3 R \\
&= R,
\end{aligned} \tag{3.18}$$

which shows that

$$\bigl\|T(x,y)\bigr\| < \bigl\|(x,y)\bigr\|, \quad \forall (x,y) \in (K_1 \times K_2) \cap \partial\Omega_R. \tag{3.19}$$

Finally, since $0 < d < \min\{(4A)^{-1}, (4A'\Gamma)^{-1}, r\}$, for $(x,y) \in (K_1 \times K_2) \cap \partial\Omega_d$, it follows from (3.12) that

$$\begin{aligned}
(T_1x)(t) &\le \int_0^\xi \int_0^\xi H_1(t,s)H(s,\tau)a^+(s)b^+(\tau)x(s)x(\tau)\,d\tau\,ds + \int_0^\xi H_1(t,s)\omega^+(s)f\bigl(x(s)\bigr)\,ds \\
&\quad + \int_0^\xi H_1(t,s)a^+(s)x(s)\Biggl(\sum_{k=1}^n H'_\tau(s,t_k)J_k\bigl(y(t_k)\bigr)\Biggr)\,ds + \sum_{k=1}^n H'_{1s}(t,t_k)I_k\bigl(x(t_k)\bigr)
\end{aligned}$$

$$
\begin{aligned}
&\le \gamma_1 \int_0^\xi \int_0^\xi H(s,\tau)a^+(s)b^+(\tau)\|x\|_{PC_1}^2 \, d\tau \, ds + \gamma_1 \int_0^\xi \omega^+(s) f\bigl(x(s)\bigr)\, ds \\
&\quad + \gamma_1 \frac{1}{1-\nu} n \int_0^\xi a^+(s) J_k\bigl(y(t_k)\bigr)\, ds \|x\|_{PC_1} + \frac{1}{1-\nu_1} \sum_{k=1}^n I_k\bigl(x(t_k)\bigr) \\
&\le Ad^2 + \gamma_1 \int_0^\xi \omega^+(s)\, ds (D)^{-1} d + A'\Gamma d^2 + \frac{1}{1-\nu_1} n(\Lambda)^{-1} d \\
&< \frac{1}{4}d + \frac{1}{4}d + \frac{1}{4}d + \frac{1}{4}d \\
&= d,
\end{aligned}
\tag{3.20}
$$

$$
\begin{aligned}
(T_2 y)(t) &= \int_0^1 H(t,s)b(s)x(s)\, ds + \sum_{k=1}^n H_s'(t,t_k) J_k\bigl(y(t_k)\bigr) \\
&\le \int_0^\xi H(t,s)b^+(s)x(s)\, ds + \sum_{k=1}^n H_s'(t,t_k) J_k\bigl(y(t_k)\bigr) \\
&\le \gamma \int_0^\xi b^+(s)x(s)\, ds + \frac{1}{1-\nu} \sum_{k=1}^n J_k\bigl(y(t_k)\bigr) \\
&\le \gamma \int_0^\xi b^+(s)\, ds d + \frac{1}{1-\nu} n\Gamma d \\
&= d,
\end{aligned}
\tag{3.21}
$$

which shows that

$$
\bigl\| T(x,y) \bigr\| < \bigl\| (x,y) \bigr\|, \quad \forall (x,y) \in (K_1 \times K_2) \cap \partial \Omega_d. \tag{3.22}
$$

Therefore, applying Lemma 2.4 to (3.15), (3.19), and (3.22), we can show that T has at least two fixed points

$$
(x_1,y_1) \in (K_1 \times K_2) \cap (\bar{\Omega}_R \setminus \Omega_r), \qquad (x_2,y_2) \in (K_1 \times K_2) \cap (\bar{\Omega}_r \setminus \Omega_d).
$$

The proof of Theorem 3.2 is completed. □

Corollary 3.1 *Assume that* (H_1)–(H_{10}) *hold. If* $0 < \alpha < 1$, *then system* (1.9) *admits at least one positive solution.*

Proof It follows from the proof of Theorem 3.2 that Corollary 3.1 holds. □

Corollary 3.2 *Assume that* (H_1)–(H_{10}) *hold. Suppose that* $\alpha > 1$ *and there exist two constants* d_1 *and* r *satisfying* $0 < d_1 < r = \min\{(4A)^{-1}, (4\varepsilon_3 A')^{-1}, r', r''\}$ *such that*

$$
\begin{aligned}
&\min_{(x,y)\in(K_1\times K_2)\cap\partial\Omega_d} f(x) > \bigl(D^*\bigr)^{-1} d_1, \\
&\min_{(x,y)\in(K_1\times K_2)\cap\partial\Omega_d} I_k(x) > \bigl(\Lambda^*\bigr)^{-1} d_1, \\
&\min_{(x,y)\in(K_1\times K_2)\cap\partial\Omega_d} J_k(y) > \Gamma^* d_1,
\end{aligned}
$$

where A, A', ε_3, r' and r'' are defined in Theorem 3.1. Then system (1.9) admits at least two positive solutions.

Proof Similarly to the proof of Theorem 3.1, we can obtain that (3.4) and (3.11) hold. Then, similarly to the proof (3.22), we get

$$\|T(x,y)\| > \|(x,y)\|, \quad \forall (x,y) \in (K_1 \times K_2) \cap \partial\Omega_{d_1}. \tag{3.23}$$

This finishes the proof of Corollary 3.2. □

Finally, in the case $0 < \alpha < 1$, we consider the existence of three positive solutions for system (1.9).

Theorem 3.3 *Assume that (H_1)–(H_{10}) hold and there exist four positive numbers η, η_1, η_2, and γ such that one of the following conditions is satisfied:*

(H_{11}) $0 < \alpha < 1$, $0 < \eta = \max\{\eta_1, \eta_2\} < \min\{r, (4A)^{-1}, (4A'\Gamma)^{-1}\} \leq r \leq \max\{6M, 6M^*, r\} < R < (6A)^{-1} < \gamma$, *and*

$$\begin{aligned} & f(x) < (D)^{-1}\eta, \qquad I_k(x) < (\Lambda)^{-1}\eta, \\ & J_k(y) < \Gamma\eta, \quad \forall x \in [\theta_3\eta_1, \eta_1], y \in [\theta_3\eta_2, \eta_2], \\ & f(x) > \left(D^*\right)^{-1}\gamma, \qquad I_k(x) > \left(\Lambda^*\right)^{-1}\gamma, \\ & J_k(y) > \Gamma^*\gamma, \quad \forall x, y \in [0, \gamma], \end{aligned}$$

where r, R, A, M, M^*, D, D^*, γ, γ^*, θ_3, Λ, and Λ^* are defined in Theorems 3.1 and 3.2, respectively. Then system (1.9) admits at least three positive solutions.

Proof Since $0 < \alpha < 1$, from the proof of Theorem 3.2 we know that

$$\|T(x,y)\| > \|(x,y)\|, \quad \forall (x,y) \in (K_1 \times K_2) \cap \partial\Omega_r, \tag{3.24}$$

$$\|T(x,y)\| < \|(x,y)\|, \quad \forall (x,y) \in (K_1 \times K_2) \cap \partial\Omega_R. \tag{3.25}$$

By the first part of (H_{11}), for any $(x,y) \in (K_1 \times K_2) \cap \partial\Omega_\eta$, we obtain

$$\begin{aligned} & \|(x,y)\| = \max\left\{\|x\|_{PC_1}, \|y\|_{PC_2}\right\} = \eta = \max\{\eta_1, \eta_2\}, \\ & \theta_3\eta_1 \leq \theta_3\|x\|_{PC_1} \leq \min_{\frac{\sigma_3}{2} \leq t \leq \sigma_3} x(t) \leq x(t) \leq \eta_1, \\ & \theta_3\eta_2 \leq \theta_3\|y\|_{PC_2} \leq \min_{\frac{\sigma_3}{2} \leq t \leq \sigma_3} y(t) \leq x(t) \leq \eta_2, \end{aligned}$$

and similarly to the proof of (3.22), we get

$$\|T(x,y)\| < \|(x,y)\|, \quad \forall (x,y) \in (K_1 \times K_2) \cap \partial\Omega_\eta. \tag{3.26}$$

Considering the second part of (H_{11}), for any $(x,y)\in(K_1\times K_2)\cap\partial\Omega_\gamma$, we have

$$\begin{aligned}
\|T_1x\|_{PC_1} &= \max_{t\in J}\Biggl\{\int_0^1\int_0^1 H_1(t,s)H(s,\tau)a(s)b(\tau)x(s)x(\tau)\,d\tau\,ds \\
&\quad+\int_0^1 H_1(t,s)\omega(s)f\bigl(x(s)\bigr)\,ds \\
&\quad+\int_0^1 H_1(t,s)a(s)x(s)\Biggl(\sum_{k=1}^n H'_\tau(s,t_k)J_k\bigl(y(t_k)\bigr)\Biggr)\,ds+\sum_{k=1}^n H'_{1s}(t,t_k)I_k\bigl(x(t_k)\bigr)\Biggr\} \\
&\ge \max_{t\in J}\Biggl\{\int_0^{\sigma_2} H_1(t,s)a^+(s)x(s)\int_0^1 H(s,\tau)b(\tau)x(\tau)\,d\tau\,ds \\
&\quad+\int_0^{\sigma_3} H_1(t,s)\omega^+(s)f\bigl(x(s)\bigr)\,ds \\
&\quad+\int_0^{\sigma_2} H_1(t,s)a^+(s)x(s)\Biggl(\sum_{k=1}^n H'_\tau(s,t_k)J_k\bigl(y(t_k)\bigr)\Biggr)\,ds+\sum_{k=1}^n H'_{1s}(t,t_k)I_k\bigl(x(t_k)\bigr)\Biggr\} \\
&\ge \min_{\frac{\sigma_2}{2}\le t\le\sigma_2}\int_{\frac{\sigma_2}{2}}^{\sigma_2} H_1(t,s)a^+(s)x(s)\int_{\frac{\sigma_1}{2}}^{\sigma_1} H(s,\tau)b^+(\tau)x(\tau)\,d\tau\,ds \\
&\quad+\min_{\frac{\sigma_2}{2}\le t\le\sigma_2}\int_{\frac{\sigma_2}{2}}^{\sigma_2} H_1(t,s)a(s)x(s)\Biggl(\sum_{\frac{\sigma_3}{2}<t_k<\sigma_3} H'_\tau(s,t_k)J_k\bigl(y(t_k)\bigr)\Biggr)\,ds \\
&\quad+\min_{\frac{\sigma_2}{2}\le t\le\sigma_2}\int_{\frac{\sigma_3}{2}}^{\sigma_3} H_1(t,s)\omega^+(s)f\bigl(x(s)\bigr)\,ds+\min_{0\le t\le\xi}\sum_{0<t_k<\xi} H'_{1s}(t,t_k)I_k\bigl(x(t_k)\bigr) \\
&\ge \min_{\frac{\sigma_2}{2}\le t\le\sigma_2}\int_{\frac{\sigma_3}{2}}^{\sigma_3} H_1(t,s)\omega^+(s)f\bigl(x(s)\bigr)\,ds+\min_{0\le t\le\xi}\sum_{0<t_k<\xi} H'_{1s}(t,t_k)I_k\bigl(x(t_k)\bigr) \\
&\ge \rho_1\frac{\sigma_2}{2}\int_{\frac{\sigma_3}{2}}^{\sigma_3} G_1(s,s)\omega^+(s)f(x)\,ds+\frac{1}{1-\nu_1}\int_\xi^1 h(\tau)\,d\tau\sum_{\frac{\sigma_3}{2}<t_1<\sigma_3} I_k(x) \\
&\ge \rho_1\frac{\sigma_2}{2}\int_{\frac{\sigma_3}{2}}^{\sigma_3} G_1(s,s)\omega^+(s)\,ds\bigl(D^*\bigr)^{-1}\gamma+\frac{1}{1-\nu_1}\int_\xi^1 h(\tau)\,d\tau\bigl(\Lambda^*\bigr)^{-1}\gamma \\
&> \frac{1}{2}\gamma+\frac{1}{2}\gamma=\gamma,
\end{aligned}\tag{3.27}$$

$$\begin{aligned}
\|T_2y\|_{PC_2} &= \max_{t\in J}\Biggl\{\int_0^1 H(t,s)b(s)x(s)\,ds+\sum_{k=1}^n H'_s(t,t_k)J_k\bigl(y(t_k)\bigr)\Biggr\} \\
&\ge \max_{t\in J}\Biggl\{\int_0^{\sigma_2} H(t,s)b^+(s)x(s)\,ds+\sum_{k=1}^n H'_s(t,t_k)J_k\bigl(y(t_k)\bigr)\Biggr\} \\
&\ge \min_{t\in[0,\xi]}\int_0^{\sigma_2} H(t,s)b^+(s)x(s)\,ds+\min_{t\in[0,\xi]}\sum_{k=1}^n H'_s(t,t_k)J_k\bigl(y(t_k)\bigr) \\
&\ge \min_{t\in[0,\xi]}\sum_{k=1}^n H'_s(t,t_k)J_k\bigl(y(t_k)\bigr) \\
&\ge n\frac{1}{1-\nu}\int_\xi^1 g(\tau)\,d\tau\,\Gamma^*\gamma=\gamma.
\end{aligned}\tag{3.28}$$

This shows that

$$\|T(x,y)\| > \|(x,y)\|, \quad \forall (x,y) \in (K_1 \times K_2) \cap \partial\Omega_\gamma. \tag{3.29}$$

Therefore, applying Lemma 2.4 to (3.24), (3.25), (3.26), and (3.29) respectively, we can show that T has at least three fixed points (x_i, y_i) $(i = 1, 2, 3)$ satisfying

$$\begin{aligned}
&(x_1, y_1) \in (K_1 \times K_2) \cap (\bar{\Omega}_\gamma \setminus \Omega_R),\\
&(x_2, y_2) \in (K_1 \times K_2) \cap (\bar{\Omega}_R \setminus \Omega_r),\\
&(x_3, y_3) \in (K_1 \times K_2) \cap (\bar{\Omega}_r \setminus \Omega_\eta).
\end{aligned}$$

This gives the proof of Theorem 3.3. □

4 An example

Example 4.1 Let $n = 1$ and $t_1 = \frac{1}{10}$. Consider the following system:

$$\begin{cases}
-x'' = a(t)xy + \omega(t)x^3, \quad 0 < t < 1, t \neq \frac{1}{10},\\
-y'' = b(t)x, \quad 0 < t < 1, t \neq \frac{1}{10},\\
\Delta x|_{t=\frac{1}{10}} = I_1(x(\frac{1}{10})),\\
\Delta y|_{t=\frac{1}{10}} = J_1(y(\frac{1}{10})),\\
x(0) = \int_0^1 x(t)\,dt, \qquad x'(1) = 0,\\
y(0) = \int_0^1 ty(t)\,dt, \qquad y'(1) = 0,
\end{cases} \tag{4.1}$$

where $I_1(x) = \frac{x^3}{2}$, $J_1(y) = \frac{y^3}{4}$, $h(t) \equiv 1$, $g(t) = t$, and

$$b(t) = \begin{cases} 48(\frac{1}{3} - t), & t \in [0, \frac{1}{3}],\\ -\frac{1}{8}(t - \frac{1}{3}), & t \in [\frac{1}{3}, 1], \end{cases}$$

$$a(t) = \begin{cases} \frac{1728}{5}(\frac{1}{3} - t), & t \in [0, \frac{1}{3}],\\ -\frac{1}{8}(t - \frac{1}{3}), & t \in [\frac{1}{3}, 1], \end{cases}$$

$$g(t) = \begin{cases} 192(\frac{1}{3} - t), & t \in [0, \frac{1}{3}],\\ -\frac{1}{8}(t - \frac{1}{3}), & t \in [\frac{1}{3}, 1]. \end{cases}$$

Conclusion 4.1 System (4.1) admits at least one positive solution.

For convenience, we give a corollary of Proposition 2.3 in [67].

Corollary 4.1 *Consider the following system*:

$$\begin{cases}
-x'' = k(t)x^\alpha, \quad 0 < t < 1,\\
x(0) = \int_0^1 h(t)x(t)\,dt, \qquad x'(1) = 0,
\end{cases} \tag{4.2}$$

$$\begin{cases}
-y'' = k(t)y^\alpha, \quad 0 < t < 1,\\
y(0) = \int_0^1 g(t)y(t)\,dt, \qquad y'(1) = 0,
\end{cases} \tag{4.3}$$

where $\alpha > 0$, *and* $k(t)$ *satisfies the changing-sign condition*

$$\begin{cases} k(t) \geq 0, & t \in [0, \xi], \\ k(t) \leq 0, & t \in [\xi, 1], \end{cases}$$

and

$$c_1 x^\alpha \leq f(x) = x^\alpha \leq c_2 x^\alpha, \qquad c_1 y^\alpha \leq f(y) = y^\alpha \leq c_2 y^\alpha, \quad c_1, c_2 > 0.$$

If there exists $0 < \sigma < \xi$ *such that*

$$c_1 \frac{\xi - \sigma}{1 - \xi} \sigma^{\alpha+1} \mu^2 k^+ \left(\xi - \frac{\xi - \sigma}{1 - \xi} \eta \right) \geq c_2 \xi^\alpha k^-(\xi + \eta), \quad \eta \in [0, 1 - \xi], \tag{4.4}$$

then the following inequalities hold:

$$\sigma^\alpha \mu^2 \int_\sigma^\xi H(t,s) k^+(s)\, ds \geq \frac{c_2}{c_1} \xi^\alpha \int_\xi^1 H(t,s) k^-(s)\, ds, \tag{4.5}$$

$$\sigma^\alpha \mu^2 \int_\sigma^\xi H_1(t,s) k^+(s)\, ds \geq \frac{c_2}{c_1} \xi^\alpha \int_\xi^1 H_1(t,s) k^-(s)\, ds. \tag{4.6}$$

Proof Similarly to the proof of Proposition 2.3 in [67], we can prove that

$$G\left(t, \xi - \frac{\xi - \sigma}{1 - \xi} \eta\right) \geq \sigma G(t, \xi + \eta), \quad \eta \in [0, 1 - \xi].$$

Hence it follows from (2.4) that

$$\begin{aligned} H\left(t, \xi - \frac{\xi - \sigma}{1 - \xi} \eta\right) &= G\left(t, \xi - \frac{\xi - \sigma}{1 - \xi} \eta\right) + \frac{1}{1 - \nu} \int_0^1 G\left(\tau, \xi - \frac{\xi - \sigma}{1 - \xi} \eta\right) g(\tau)\, d\tau \\ &\geq \sigma G(t, \xi + \eta) + \frac{\sigma}{1 - \nu'} \int_0^1 G(\tau, \xi + \eta) g(\tau)\, d\tau \\ &= \sigma \left[G(t, \xi + \eta) + \frac{1}{1 - \nu} \int_0^1 G(\tau, \xi + \eta) g(\tau)\, d\tau \right] \\ &= \sigma H(t, \xi + \eta), \quad \eta \in [0, 1 - \xi]. \end{aligned}$$

Next, letting $s = \xi - \frac{\xi - \sigma}{1 - \xi} \eta$, $\eta \in [0, 1 - \xi]$, we get

$$\int_\sigma^\xi H(t,s) k^+(s)\, ds = \frac{\xi - \sigma}{1 - \xi} \int_0^{1-\xi} H\left(t, \xi - \frac{\xi - \sigma}{1 - \xi} \eta\right) k^+\left(\xi - \frac{\xi - \sigma}{1 - \xi} \eta\right) d\eta;$$

letting $s = \xi + \eta$, $\eta \in [0, 1 - \xi]$, we have

$$\int_\xi^1 H(t,s) k^-(s)\, ds = \int_0^{1-\xi} H(t, \xi + \eta) k^-(\xi + \eta)\, d\eta.$$

Now, from assumption (4.4), for all $(t, \eta) \in [0,1] \times [0, 1 - \xi]$, we have

$$c_1 \frac{\xi - \sigma}{1 - \xi} \sigma^\alpha \mu^2 H\left(t, \xi - \frac{\xi - \sigma}{1 - \xi} \eta\right) k^+\left(\xi - \frac{\xi - \sigma}{1 - \xi} \eta\right) \geq c_2 \xi^\alpha H(t, \xi + \eta) k^-(\xi + \eta). \tag{4.7}$$

Finally, by integrating in η both sides of (4.7) from 0 to $1-\xi$ it follows that inequality (4.5) holds. □

Similarly, we can show that inequality (4.6) holds.

Proof of Example 4.1 From the definitions of $a(t)$, $b(t)$, and $g(t)$ we know that $\xi = \frac{1}{3}$.

Step 1. We show that (H_6) holds. For fixed $c_1 = c_2 = 1$, $\sigma_1 = \frac{1}{6}$, $\mu = 1$, and $\alpha = 1$, (4.4) is equivalent to the inequality

$$\frac{1}{48}b^+\left(\frac{1}{3}-\frac{1}{4}\eta\right) \geq b^-\left(\frac{1}{3}+\eta\right), \quad \eta \in \left[0, \frac{2}{3}\right].$$

Letting $\frac{1}{3}-\frac{1}{4}\tau = \varrho$, this inequality is equivalent to

$$\frac{1}{48}b^+(\varrho) \geq b^-\left(\frac{5}{3}-4\varrho\right), \quad \varrho \in \left[\frac{1}{4}, \frac{1}{3}\right].$$

By the definition of $b(t)$ the last inequality holds obviously. It is clear that by (4.5) (H_6) is reasonable.

Step 2. We show (H_7) holds. Similarly to Step 1, letting $c_1 = 1$, $c_2 = \frac{36}{5}$, $\sigma_2 = \frac{1}{6}$, $\mu = 1$, and $\alpha = 1$, by (4.6) we get

$$\frac{1}{6}\int_{\frac{1}{6}}^{\frac{1}{3}} H_1(t,s)a^+(s)\,ds \geq \frac{12}{5}\int_{\frac{1}{3}}^{1} H_1(t,s)a^-(s)\,ds. \tag{4.8}$$

It is easy to see by calculating that

$$\nu = \int_0^1 g(s)\,ds = \int_0^1 s\,ds = \frac{1}{2},$$
$$\gamma = \frac{1}{1-\nu} = 2, \qquad \rho = 1 + \frac{\int_0^1 \tau g(\tau)\,d\tau}{1-\nu} = \frac{5}{3}.$$

Furthermore, from inequality (4.8) it follows that

$$\begin{aligned}
&\frac{1}{6}\cdot\frac{5}{3}\int_{\frac{1}{6}}^{\frac{1}{3}} H_1(t,s)\frac{1}{6}a^+(s)\,ds \geq 2\cdot\frac{1}{3}\int_{\frac{1}{3}}^{1} H_1(t,s)a^-(s)\,ds \\
&\quad\Leftrightarrow\quad \frac{1}{6}\cdot\frac{5}{3}\int_{\frac{1}{6}}^{\frac{1}{3}} H_1(t,s)\min_{s\in[\frac{1}{6},\frac{1}{3}]} G(s,s)a^+(s)\,ds \geq 2\cdot\frac{1}{3}\int_{\frac{1}{3}}^{1} H_1(t,s)a^-(s)\,ds \\
&\quad\Rightarrow\quad \frac{1}{6}\cdot\frac{5}{3}\int_{\frac{1}{6}}^{\frac{1}{3}} H_1(t,s)G(s,s)a^+(s)\,ds \geq 2\cdot\frac{1}{3}\int_{\frac{1}{3}}^{1} H_1(t,s)a^-(s)\,ds.
\end{aligned}$$

So, (H_7) holds.

Step 3. Similarly to Step 1, letting $c_1 = c_2 = 1$, $\sigma_3 = \frac{1}{6}$, $\mu = 1$, and $\alpha = 3$, we get that (H_{10}) holds.

Hence it follows from Theorem 3.1 that system (4.1) admits at least one positive solution for $\alpha > 1$. □

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