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# Asymptotic behavior of the solution of a queueing system modeled by infinitely many partial differential equations with integral boundary conditions

Nurehemaiti Yiming[1] and Geni Gupur[1*]

*Correspondence:
genigupur@vip.163.com
[1]College of Mathematics and Systems Science, Xinjiang University, Urumqi, P.R. China

**Abstract**

By studying the spectrum on the imaginary axis of the underlying operator, which corresponds to the M/G/1 retrial queueing model with general retrial times described by infinitely many partial differential equations with integral boundary conditions, we prove that the time-dependent solution of the model strongly converges to its steady-state solution. Next, when the conditional completion rates for repeated attempts and service are constants, we describe the point spectrum of the underlying operator and verify that all points in an interval in the left real line including 0 are eigenvalues of the underlying operator. Lastly, by using these results and the spectral mapping theorem we prove that the $C_0$-semigroup generated by the underlying operator is not compact, but not eventually compact and even not quasi-compact, and it is impossible that the time-dependent solution exponentially converges to its steady-state solution. In other words, our result on convergence is optimal.

**MSC:** 47A10; 47D03

**Keywords:** M/G/1 retrial queueing model with general retrial times; $C_0$-semigroup; Time-dependent solution; Eigenvalue; Resolvent set

## 1 Introduction

Retrial queueing systems are widely used in teletraffic theory, computers networks, communication networks, and so on. So, retrial queueing systems received considerable attention over recent years; see, for instance, Artalejo et al. [1], Phung-Duc et al. [2], Aissani et al. [3], and Gomez-Corral [4]. This paper studies the M/G/1 retrial queueing system with general retrial times defined as follows: customers arrive according to a Poisson stream of rate $\lambda > 0$; upon arrival, the service of the arriving customer commences immediately; otherwise, the customer leaves the service area and enters a group of blocked customers called "orbit" in accordance with an FCFS (First Come, First Served) discipline; only the customer at the head of the orbit queue is allowed for access to the server; when a service is completed, the access from the orbit to the server is governed by an arbitrary law with common probability distribution function $\tilde{A}(x)$ ($\tilde{A}(0) = 0$), the density function $\tilde{a}(x)$, and the Laplace–Stieltjes transform $\tilde{\alpha}(\theta)$; the service times are independent with common probability distribution function $B(x)$ ($B(0) = 0$), the density function $b(x)$, the Laplace–Stieltjes transform $\tilde{\beta}(\theta)$, and the first moments $\tilde{\beta}_k = (-1)^k \tilde{\beta}^k(0)$, $k = 1, 2, 3$; interarrival times, retrial times, and service times are mutually independent. In 1999, Gomez-Corral [4] studied the M/G/1 retrial queueing system with general retrial times and obtained the following results: (1) a necessary and sufficient condition for the system to be stable by establishing the ergodicity of the embedded Markov chain at the departure points; (2) the steady-state distribution of the server state and the orbit length by using the supplementary variable technique; (3) the Laplace–Stieltjes transform of the waiting time distribution of a primary customer who arrives at the system at time $t$; (4) the Laplace–Stieltjes transform of the joint distribution of busy periods and idle times; (5) the Laplace–Stieltjes transform of the server state and the orbit length. To get the results mentioned, Gomez-Corral [4] firstly established a mathematical model of the M/G/1 retrial queueing system with general retrial times by using a supplementary variable technique and studied its steady-state solution under the following hypotheses:

$$p_0 = \lim_{t\to\infty} p_0(t), \qquad p_n(t) = \lim_{t\to\infty} p_n(x,t), \quad n \geq 1; \qquad Q_n(t) = \lim_{t\to\infty} Q_n(x,t), \quad n \geq 0,$$

which imply the following two hypotheses in view of partial differential equations:

**Hypothesis 1** *The equation system has a unique time-dependent solution.*

**Hypothesis 2** *The time-dependent solution converges to the steady-state solution.*

Hypothesis 2 does not hold for some queueing models; see, for instance, Zheng and Gupur [5], Kasim and Gupur [6], and Abla and Gupur [7]. On the other hand, three types of convergence are of interest in view of functional analysis: weak convergence, strong convergence, and uniform convergence, and Hypothesis 2 does not indicate which one holds for this queueing model. Hence, we need to study Hypotheses 1 and 2.

In 2005, by using the $C_0$-semigroup theory Gupur [8] has proved that the model has a unique positive time-dependent solution that satisfies the probability condition, that is, he showed that Hypothesis 1 holds under certain conditions. So far, no results on Hypothesis 2 have been found in the literature.

When $s(x)$ and $r(x)$ are constants, the M/G/1 retrial queueing model with general retrial times is called the M/M/1 retrial queueing model with special retrial times. In 2005, by studying the resolvent set of the adjoint operator of the operator corresponding to the M/M/1 retrial queueing model with special retrial times Zhang and Gupur [9] obtained that all points on the imaginary axis except 0 belong to the resolvent set of the operator. In 2006, Jiang and Gupur [10] proved that 0 is an eigenvalue of the operator with algebraic multiplicity one and 0 is an eigenvalue of its adjoint operator. By combining these results with Theorem 14 in Gupur et al. [11] (Theorem 1.96 in Gupur [12]) we deduce that the time-dependent solution of the M/M/1 retrial queueing model with special retrial times converges strongly to its steady-state solution. In 2009, Lv and Gupur [13] found that the operator has one eigenvalue on the left real line. After that, Ismayil and Gupur [14] and Gupur [15] proved that all points in an interval in the left real line belong to its point spectrum under different conditions. Until now, any other results on the M/M/1 retrial queueing model with special retrial times have not been found in the literature.

In this paper, using Greiner's idea [16] and Gupur et al. [17], we prove that all points on the imaginary axis except 0 belong to the resolvent set of the underlying operator corresponding to the M/G/1 retrial queueing model with general retrial times, 0 is its eigenvalue with geometric multiplicity one,and 0 is an eigenvalue of its adjoint operator. Thus, using Theorem 14 in Gupur et al. [11] or Theorem 1.96 in Gupur [12], we deduce that the time-dependent solution of the M/G/1 retrial queueing model with general retrial times strongly converges to its steady-state solution. Hence, we answer that Hypothesis 2 holds in the sense of "strong convergence" under certain conditions. Next, we describe the point spectrum of the operator corresponding to the M/M/1 retrial queueing model with special retrial times and verify that an interval in the left real line that includes 0 belongs to the point spectrum of the operator. Moreover, we show that our result implies the main results obtained by Lv and Gupur [13], Ismayil and Gupur [14], and Gupur [15]. Lastly, by combining these results with the spectral mapping theorem we prove that the $C_0$-semigroup generated by the underlying operator is not compact, not eventually compact, and even not quasi-compact, and our result on the convergence of the time-dependent solution of the M/G/1 retrial queueing model with general retrial times is optimal, that is, it is impossible that the time-dependent solution exponentially converges to its steady-state solution, which means that Hypothesis 2 holds at most in the sense of strong convergence.

According to Gomez-Corral [4], the M/G/1 retrial queueing model with general retrial times can be described by the following system of equations:

$$\frac{dp_0(t)}{dt} = -\lambda p_0(t) + \int_0^\infty Q_0(x,t)s(x)\,dx, \tag{1.1}$$

$$\frac{\partial p_n(x,t)}{\partial t} + \frac{\partial p_n(x,t)}{\partial x} = -\bigl[\lambda + r(x)\bigr]p_n(x,t), \quad \forall n \geq 1, \tag{1.2}$$

$$\frac{\partial Q_0(x,t)}{\partial t} + \frac{\partial Q_0(x,t)}{\partial x} = -\bigl[\lambda + s(x)\bigr]Q_0(x,t), \tag{1.3}$$

$$\frac{\partial Q_n(x,t)}{\partial t} + \frac{\partial Q_n(x,t)}{\partial x} = -\bigl[\lambda + s(x)\bigr]Q_n(x,t) + \lambda Q_{n-1}(x,t), \quad \forall n \geq 1, \tag{1.4}$$

$$p_n(0,t) = \int_0^\infty Q_n(x,t)s(x)\,dx, \quad \forall n \geq 1, \tag{1.5}$$

$$Q_0(0,t) = \lambda p_0(t) + \int_0^\infty p_1(x,t)r(x)\,dx, \tag{1.6}$$

$$Q_n(0,t) = \lambda \int_0^\infty p_n(x,t)\,dx + \int_0^\infty p_{n+1}(x,t)r(x)\,dx, \quad \forall n \geq 1, \tag{1.7}$$

$$p_0(0) = u_0, \qquad p_n(x,0) = u_n(x), \quad \forall n \geq 1; \qquad Q_n(x,0) = v_n(x), \quad \forall n \geq 0, \tag{1.8}$$

where $(x,t) \in [0,\infty) \times [0,\infty)$, $u_0 \geq 0$, $u_n(x) \geq 0$ $(\forall n \geq 1)$, $v_n(x) \geq 0$ $(\forall n \geq 0)$, and $u_0 + \sum_{n=1}^\infty \int_0^\infty u_n(x)\,dx + \sum_{n=0}^\infty \int_0^\infty v_n(x)\,dx = 1$; $p_0(t)$ represents the probability that at time $t$ there is no customer in the system and the server is idle; $p_n(x,t)$ ( $n \geq 1$) represents the probability that at time $t$ the server is idle and there are $n$ customers in the system with elapsed retrial time $x$; $Q_n(x,t)$ $(n \geq 0)$ represents the probability that at time $t$ the server is busy and there are $n$ customers in the system with elapsed service time $x$ of the customer who is undergoing service; $\lambda$ is the arrival rate of customers; $r(x)$ is the conditional completion rate for repeated attempt at $x$ satisfying $r(x) \geq 0$ and $\int_0^\infty r(x)\,dx = \infty$; $s(x)$ is the conditional completion rate for service at $x$ satisfying $s(x) \geq 0$ and $\int_0^\infty s(x)\,dx = \infty$.

In this paper, we use the notations in Gupur [8] and choose the state space

$$X = \left\{ (p,Q) \middle| \begin{array}{l} p \in \mathbb{R} \times L^1[0,\infty) \times L^1[0,\infty) \times L^1[0,\infty) \times \cdots, \\ Q \in L^1[0,\infty) \times L^1[0,\infty) \times L^1[0,\infty) \times \cdots, \\ \|(p,Q)\| = |p_0| + \sum_{n=1}^{\infty} \|p_n\|_{L^1[0,\infty)} + \sum_{n=0}^{\infty} \|Q_n\|_{L^1[0,\infty)} < \infty \end{array} \right\}.$$

It is obvious that $X$ is a Banach space. In addition, $X$ is also a Banach lattice under the following order relation for almost all $x \in [0,\infty)$:

$$(p,Q) \le (y,z) \quad \Leftrightarrow \quad p_0 \le y_0, \qquad p_n(x) \le y_n(x), \quad n \ge 1; \qquad Q_n(x) \le z_n(x), \quad n \ge 0.$$

For convenience, we introduce

$$\begin{aligned} B_1 g(x) &= -\frac{dg(x)}{dx} - \bigl[\lambda + r(x)\bigr] g(x), \quad g \in W^{1,1}[0,\infty), \\ B_2 g(x) &= -\frac{dg(x)}{dx} - \bigl[\lambda + s(x)\bigr] g(x), \quad g \in W^{1,1}[0,\infty), \\ \phi f(x) &= \int_0^\infty s(x) f(x)\,dx, \quad f \in L^1[0,\infty), \\ \psi f(x) &= \int_0^\infty r(x) f(x)\,dx, \quad f \in L^1[0,\infty), \\ Hf(x) &= \int_0^\infty f(x)\,dx, \quad f \in L^1[0,\infty). \end{aligned}$$

We define

$$A_m(p,Q)(x) = \left( \begin{pmatrix} -\lambda & 0 & 0 & 0 & \cdots \\ 0 & B_1 & 0 & 0 & \cdots \\ 0 & 0 & B_1 & 0 & \cdots \\ 0 & 0 & 0 & B_1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} p_0 \\ p_1(x) \\ p_2(x) \\ p_3(x) \\ \vdots \end{pmatrix} + \begin{pmatrix} \phi & 0 & \cdots \\ 0 & 0 & \cdots \\ 0 & 0 & \cdots \\ 0 & 0 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} Q_0(x) \\ Q_1(x) \\ Q_2(x) \\ Q_3(x) \\ \vdots \end{pmatrix}, \right.$$
$$\left. \begin{pmatrix} B_2 & 0 & 0 & 0 & \cdots \\ \lambda & B_2 & 0 & 0 & \cdots \\ 0 & \lambda & B_2 & 0 & \cdots \\ 0 & 0 & \lambda & B_2 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} Q_0(x) \\ Q_1(x) \\ Q_2(x) \\ Q_3(x) \\ \vdots \end{pmatrix} \right),$$

$$D(A_m) = \left\{ (p,Q) \in X \middle| \begin{array}{l} \frac{dp_n(x)}{dx} \in L^1[0,\infty)\ (n \ge 1),\ \frac{dQ_n(x)}{dx} \in L^1[0,\infty)\ (n \ge 0), \\ p_n(x) \text{ and } Q_n(x) \text{ are absolutely continuous functions} \\ \text{and } \sum_{n=1}^{\infty} \|\frac{dp_n}{dx}\|_{L^1[0,\infty)} < \infty, \sum_{n=0}^{\infty} \|\frac{dQ_n}{dx}\|_{L^1[0,\infty)} < \infty \end{array} \right\}.$$

We choose the boundary space

$$\partial X = l^1 \times l^1$$

and define the boundary operators

$$\Psi : D(A_m) \to \partial X, \qquad \Phi : D(A_m) \to \partial X,$$

$$\Psi(p,Q)(x)=\Psi\left(\begin{pmatrix}p_0\\p_1(x)\\p_2(x)\\p_3(x)\\\vdots\end{pmatrix},\begin{pmatrix}Q_0(x)\\Q_1(x)\\Q_2(x)\\Q_3(x)\\\vdots\end{pmatrix}\right)=\left(\begin{pmatrix}p_1(0)\\p_2(0)\\p_3(0)\\p_4(0)\\\vdots\end{pmatrix},\begin{pmatrix}Q_0(0)\\Q_1(0)\\Q_2(0)\\Q_3(0)\\\vdots\end{pmatrix}\right),$$

$$\Phi(p,Q)(x)=\left(\begin{pmatrix}0&\phi&0&0&\cdots\\0&0&\phi&0&\cdots\\0&0&0&\phi&\cdots\\0&0&0&0&\cdots\\\vdots&\vdots&\vdots&\vdots&\ddots\end{pmatrix}\begin{pmatrix}Q_0(x)\\Q_1(x)\\Q_2(x)\\Q_3(x)\\\vdots\end{pmatrix},\right.$$

$$\left.\begin{pmatrix}\lambda&\psi&0&0&\cdots\\0&\lambda H&\psi&0&\cdots\\0&0&\lambda H&\psi&\cdots\\0&0&0&\lambda H&\cdots\\\vdots&\vdots&\vdots&\vdots&\ddots\end{pmatrix}\begin{pmatrix}p_0\\p_1(x)\\p_2(x)\\p_3(x)\\\vdots\end{pmatrix}\right).$$

If we introduce the underlying operator $(A, D(A))$ by

$$A(p,Q)=A_m(p,Q),\qquad D(A)=\bigl\{(p,Q)\in D(A_m)\mid \Psi(p,Q)=\Phi(p,Q)\bigr\},$$

then Eqs. (1.1)–(1.8) can be written as an abstract Cauchy problem in the Banach space $X$, which is of the form given by Gupur [8]:

$$\begin{cases}\frac{d(p,Q)(t)}{dt}=A(p,Q)(t), & \forall t\in(0,\infty),\\ (p,Q)(0)=(p(0),Q(0)), \\ p(0)=(u_0,u_1,u_2,\ldots), & Q(0)=(v_0,v_1,v_2,\ldots).\end{cases}\tag{1.9}$$

Gupur [8] has proved the following result for system (1.9).

**Theorem 1.1** *If* $\sup_{x\in[0,\infty)} r(x)<\infty$, $\sup_{x\in[0,\infty)} s(x)<\infty$, *and* $(p(0),Q(0))\in D(A^2)$, *then* $(A, D(A))$ *generates a positive contraction* $C_0$*-semigroup* $T(t)$, *and system* (1.9) *has a unique positive time-dependent solution* $(p,Q)(x,t)=T(t)(p,Q)(0)$ *satisfying*

$$\bigl\|(p,Q)(\cdot,t)\bigr\|=p_0(t)+\sum_{n=1}^{\infty}\int_0^\infty p_n(x,t)\,dx+\sum_{n=0}^{\infty}\int_0^\infty Q_n(x,t)\,dx=1,\quad \forall t\in[0,\infty).$$

## 2 Asymptotic behavior of the time-dependent solution of system (1.9)

**Lemma 2.1** *If* $r(x)$ *and* $s(x)$ *satisfy*

$$\lambda\int_0^\infty e^{-\lambda x-\int_0^x r(\xi)\,d\xi}\,dx+\lambda\int_0^\infty xs(x)e^{-\int_0^x s(\xi)\,d\xi}\,dx<1,$$

*then* 0 *is an eigenvalue of* $A$ *with geometric multiplicity one*.

*Proof* Consider the equation $A(p, Q) = 0$, which is equivalent to

$$\lambda p_0 = \int_0^\infty Q_0(x)s(x)\,dx, \tag{2.1}$$

$$\frac{dp_n(x)}{dx} = -\big[\lambda + r(x)\big]p_n(x), \quad \forall n \geq 1, \tag{2.2}$$

$$\frac{dQ_0(x)}{dx} = -\big[\lambda + s(x)\big]Q_0(x), \tag{2.3}$$

$$\frac{dQ_n(x)}{dx} = -\big[\lambda + s(x)\big]Q_n(x) + \lambda Q_{n-1}(x), \quad \forall n \geq 1, \tag{2.4}$$

$$p_n(0) = \int_0^\infty Q_n(x)s(x)\,dx, \quad \forall n \geq 1, \tag{2.5}$$

$$Q_0(0) = \lambda p_0 + \int_0^\infty p_1(x)r(x)\,dx, \tag{2.6}$$

$$Q_n(0) = \lambda \int_0^\infty p_n(x)\,dx + \int_0^\infty p_{n+1}(x)r(x)\,dx, \quad \forall n \geq 1. \tag{2.7}$$

Solving (2.2)∼(2.4), we have

$$p_n(x) = a_n e^{-\lambda x - \int_0^x r(\xi)\,d\xi}, \quad \forall n \geq 1, \tag{2.8}$$

$$Q_0(x) = b_0 e^{-\lambda x - \int_0^x s(\xi)\,d\xi}, \tag{2.9}$$

$$Q_n(x) = b_n e^{-\lambda x - \int_0^x s(\xi)\,d\xi} + \lambda e^{-\lambda x - \int_0^x s(\xi)\,d\xi} \int_0^x e^{\lambda \tau + \int_0^\tau s(\xi)\,d\xi} Q_{n-1}(\tau)\,d\tau, \quad \forall n \geq 1. \tag{2.10}$$

Using (2.9) and (2.10) repeatedly, we deduce

$$Q_n(x) = e^{-\lambda x - \int_0^x s(\xi)\,d\xi} \sum_{k=0}^n b_{n-k} \frac{(\lambda x)^k}{k!}, \quad \forall n \geq 0. \tag{2.11}$$

It is hard to determine concrete expressions of all $p_n(x)$ and $Q_n(x)$ and to prove that $(p, Q) \in D(A)$. We further use another method. We introduce the probability generating functions

$$p(x, z) = \sum_{n=1}^\infty p_n(x) z^n, \qquad Q(x, z) = \sum_{n=0}^\infty Q_n(x) z^n$$

for all complex variables $|z| < 1$. Theorem 1.1 ensures that $p(x, z)$ and $Q(x, z)$ are well-defined. Equation (2.2) gives

$$\begin{aligned} \frac{\partial \sum_{n=1}^\infty p_n(x) z^n}{\partial x} &= -\sum_{n=1}^\infty \big[\lambda + r(x)\big] p_n(x) z^n \\ &\Rightarrow \quad \frac{\partial p(x, z)}{\partial x} = -\big[\lambda + r(x)\big] p(x, z) \\ &\Rightarrow \quad p(x, z) = p(0, z) e^{-\lambda x - \int_0^x r(\xi)\,d\xi}, \end{aligned} \tag{2.12}$$

where the summation and differential are interchangeable because of the convergence of $\sum_{n=1}^\infty [\lambda + r(x)] p_n(x) z^n$ and the Lebesgue control theorem.

The convergence of $\sum_{n=0}^{\infty}[\lambda+s(x)]Q_n(x)z^n$ and $\sum_{n=1}^{\infty}Q_{n-1}(x)z^n$ allows us to change the order of summation and differential, so (2.3) and (2.4) imply

$$\begin{aligned}
&\frac{\partial\sum_{n=0}^{\infty}Q_n(x)z^n}{\partial x}=-\sum_{n=0}^{\infty}\bigl[\lambda+s(x)\bigr]Q_n(x)z^n+\lambda\sum_{n=1}^{\infty}Q_{n-1}(x)z^n\\
&\Rightarrow\quad \frac{\partial Q(x,z)}{\partial x}=-\bigl[\lambda+s(x)\bigr]Q(x,z)+\lambda zQ(x,z)=\bigl[\lambda(z-1)-s(x)\bigr]Q(x,z)\\
&\Rightarrow\quad Q(x,z)=Q(0,z)e^{\lambda(z-1)x-\int_0^x s(\xi)\,d\xi}.
\end{aligned}\tag{2.13}$$

Applying (2.5), (2.13), and (2.1) and noting the convergence of $\sum_{n=1}^{\infty}Q_n(x)z^n$ and $\int_0^{\infty}\sum_{n=1}^{\infty}Q_n(x)z^n\,dx$, we have

$$\begin{aligned}
p(0,z)&=\sum_{n=1}^{\infty}p_n(0)z^n\\
&=\sum_{n=1}^{\infty}\left(\int_0^{\infty}Q_n(x)s(x)\,dx\right)z^n\\
&=\int_0^{\infty}s(x)\sum_{n=1}^{\infty}Q_n(x)z^n\,dx\\
&=\int_0^{\infty}s(x)\sum_{n=0}^{\infty}Q_n(x)z^n\,dx-\int_0^{\infty}s(x)Q_0(x)\,dx\\
&=\int_0^{\infty}s(x)Q(x,z)\,dx-\lambda p_0\\
&=\int_0^{\infty}s(x)Q(0,z)e^{\lambda(z-1)x-\int_0^x s(\xi)\,d\xi}\,dx-\lambda p_0\\
&=Q(0,z)\int_0^{\infty}s(x)e^{\lambda(z-1)x-\int_0^x s(\xi)\,d\xi}\,dx-\lambda p_0.
\end{aligned}\tag{2.14}$$

By combining (2.6) and (2.7) with (2.12) and (2.14) and noting the convergence of $\int_0^{\infty}\sum_{n=1}^{\infty}p_n(x)z^n\,dx$ and the Lebesgue control theorem we have

$$\begin{aligned}
Q(0,z)&=\sum_{n=0}^{\infty}Q_n(0)z^n\\
&=Q_0(0)+\sum_{n=1}^{\infty}Q_n(0)z^n\\
&=Q_0(0)+\lambda\int_0^{\infty}\sum_{n=1}^{\infty}p_n(x)z^n\,dx+\int_0^{\infty}\frac{1}{z}r(x)\sum_{n=0}^{\infty}p_{n+1}(x)z^{n+1}\,dx\\
&\quad-\int_0^{\infty}r(x)p_1(x)\,dx\\
&=\lambda p_0+\lambda\int_0^{\infty}p(x,z)\,dx+\int_0^{\infty}\frac{1}{z}r(x)p(x,z)\,dx\\
&=\lambda p_0+\int_0^{\infty}\left[\lambda+\frac{1}{z}r(x)\right]p(x,z)\,dx
\end{aligned}$$

$$
\begin{aligned}
&= \lambda p_0 + \int_0^\infty \left[\lambda + \frac{1}{z} r(x)\right] p(0,z) e^{-\lambda x - \int_0^x r(\xi)\,d\xi}\,dx \\
&= \lambda p_0 + p(0,z) \int_0^\infty \left[\lambda + \frac{1}{z} r(x)\right] e^{-\lambda x - \int_0^x r(\xi)\,d\xi}\,dx \\
&= \lambda p_0 + \left[Q(0,z) \int_0^\infty s(x) e^{\lambda(z-1)x - \int_0^x s(\xi)\,d\xi}\,dx - \lambda p_0\right] \\
&\quad \times \int_0^\infty \left[\lambda + \frac{1}{z} r(x)\right] e^{-\lambda x - \int_0^x r(\xi)\,d\xi}\,dx \\
&= \lambda p_0 - \lambda p_0 \int_0^\infty \left[\lambda + \frac{1}{z} r(x)\right] e^{-\lambda x - \int_0^x r(\xi)\,d\xi}\,dx \\
&\quad + Q(0,z) \int_0^\infty s(x) e^{\lambda(z-1)x - \int_0^x s(\xi)\,d\xi}\,dx \int_0^\infty \left[\lambda + \frac{1}{z} r(x)\right] e^{-\lambda x - \int_0^x r(\xi)\,d\xi}\,dx \\
&\Rightarrow
\end{aligned}
$$

$$
\begin{aligned}
Q(0,z) &= \frac{\lambda p_0 - \lambda p_0 \int_0^\infty [\lambda + \frac{1}{z} r(x)] e^{-\lambda x - \int_0^x r(\xi)\,d\xi}\,dx}{1 - \int_0^\infty s(x) e^{\lambda(z-1)x - \int_0^x s(\xi)\,d\xi}\,dx \int_0^\infty [\lambda + \frac{1}{z} r(x)] e^{-\lambda x - \int_0^x r(\xi)\,d\xi}\,dx} \\
&= \frac{z\lambda p_0 - \lambda p_0 \int_0^\infty [z\lambda + r(x)] e^{-\lambda x - \int_0^x r(\xi)\,d\xi}\,dx}{z - \int_0^\infty s(x) e^{\lambda(z-1)x - \int_0^x s(\xi)\,d\xi}\,dx \int_0^\infty [z\lambda + r(x)] e^{-\lambda x - \int_0^x r(\xi)\,d\xi}\,dx}.
\end{aligned}
\tag{2.15}
$$

From (2.14), (2.15), $\int_0^\infty s(x) e^{-\int_0^x s(\xi)\,d\xi}\,dx = 1$, $\int_0^\infty [\lambda + r(x)] e^{-\lambda x - \int_0^x r(\xi)\,d\xi}\,dx = 1$ (see (2.47), (2.48)), the Lebegue control theorem, and the l'Hospital rule it follows that

$$
\begin{aligned}
p(0,z) &= \frac{[z\lambda p_0 - \lambda p_0 \int_0^\infty [z\lambda + r(x)] e^{-\lambda x - \int_0^x r(\xi)\,d\xi}\,dx] \int_0^\infty s(x) e^{\lambda(z-1)x - \int_0^x s(\xi)\,d\xi}\,dx}{z - \int_0^\infty s(x) e^{\lambda(z-1)x - \int_0^x s(\xi)\,d\xi}\,dx \int_0^\infty [z\lambda + r(x)] e^{-\lambda x - \int_0^x r(\xi)\,d\xi}\,dx} \\
&\quad - \lambda p_0 \\
&= \frac{-z\lambda p_0 + z\lambda p_0 \int_0^\infty s(x) e^{\lambda(z-1)x - \int_0^x s(\xi)\,d\xi}\,dx}{z - \int_0^\infty s(x) e^{\lambda(z-1)x - \int_0^x s(\xi)\,d\xi}\,dx \int_0^\infty [z\lambda + r(x)] e^{-\lambda x - \int_0^x r(\xi)\,d\xi}\,dx} \\
&\Rightarrow
\end{aligned}
$$

$$
\lim_{z\to 1} p(0,z) = \frac{\lambda p_0 \int_0^\infty \lambda x s(x) e^{-\int_0^x s(\xi)\,d\xi}\,dx}{1 - \lambda \int_0^\infty e^{-\lambda x - \int_0^x r(\xi)\,d\xi}\,dx - \int_0^\infty \lambda x s(x) e^{-\int_0^x s(\xi)\,d\xi}\,dx}, \tag{2.16}
$$

$$
\lim_{z\to 1} Q(0,z) = \frac{\lambda p_0 - \lambda^2 p_0 \int_0^\infty e^{-\lambda x - \int_0^x r(\xi)\,d\xi}\,dx}{1 - \lambda \int_0^\infty e^{-\lambda x - \int_0^x r(\xi)\,d\xi}\,dx - \int_0^\infty \lambda x s(x) e^{-\int_0^x s(\xi)\,d\xi}\,dx}. \tag{2.17}
$$

By using (2.12), (2.13), (2.16), (2.17), the condition of this lemma, Theorem 1.1, and the Lebesgue control theorem we derive

$$
\begin{aligned}
\sum_{n=1}^\infty p_n(x) &= \lim_{z\to 1} p(x,z) \\
&= \frac{\lambda p_0 e^{-\lambda x - \int_0^x r(\xi)\,d\xi} \int_0^\infty \lambda x s(x) e^{-\int_0^x s(\xi)\,d\xi}\,dx}{1 - \lambda \int_0^\infty e^{-\lambda x - \int_0^x r(\xi)\,d\xi}\,dx - \int_0^\infty \lambda x s(x) e^{-\int_0^x s(\xi)\,d\xi}\,dx} \\
&\Rightarrow
\end{aligned}
$$

$$\sum_{n=1}^{\infty}\int_0^{\infty} p_n(x)\,dx = \frac{\lambda p_0 \int_0^\infty e^{-\lambda x-\int_0^x r(\xi)\,d\xi}\,dx \int_0^\infty \lambda x s(x) e^{-\int_0^x s(\xi)\,d\xi}\,dx}{1-\lambda\int_0^\infty e^{-\lambda x-\int_0^x r(\xi)\,d\xi}\,dx - \int_0^\infty \lambda x s(x)e^{-\int_0^x s(\xi)\,d\xi}\,dx} < \infty, \tag{2.18}$$

$$\sum_{n=0}^{\infty} Q_n(x) = \lim_{z\to 1} Q(x,z) = \frac{\lambda p_0 e^{-\int_0^x s(\xi)\,d\xi}[1-\lambda\int_0^\infty e^{-\lambda x-\int_0^x r(\xi)\,d\xi}\,dx]}{1-\lambda\int_0^\infty e^{-\lambda x-\int_0^x r(\xi)\,d\xi}\,dx - \int_0^\infty \lambda x s(x)e^{-\int_0^x s(\xi)\,d\xi}\,dx}$$

$$\Rightarrow$$

$$\sum_{n=0}^{\infty}\int_0^{\infty} Q_n(x)\,dx = \frac{\lambda p_0 \int_0^\infty e^{-\int_0^x s(\xi)\,d\xi}\,dx[1-\lambda\int_0^\infty e^{-\lambda x-\int_0^x r(\xi)\,d\xi}\,dx]}{1-\lambda\int_0^\infty e^{-\lambda x-\int_0^x r(\xi)\,d\xi}\,dx - \int_0^\infty \lambda x s(x)e^{-\int_0^x s(\xi)\,d\xi}\,dx} < \infty. \tag{2.19}$$

Equations (2.18) and (2.19) show that 0 is an eigenvalue of $A$. Moreover, from (2.1), (2.5)–(2.8), and (2.11) we know that the eigenvectors corresponding to 0 span the following linear space:

$$\Upsilon := \left\{ \tilde{c}(p,Q) \;\middle|\; \begin{aligned} & p(x) = (p_0, p_1(x), p_2(x), \ldots), \qquad Q(x) = (Q_0(x), Q_1(x), Q_2(x), \ldots), \\ & p_n(x) = a_n e^{-\lambda x - \int_0^x r(\xi)\,d\xi}, \quad \forall n \ge 1, \\ & Q_n(x) = e^{-\lambda x-\int_0^x s(\xi)\,d\xi} \textstyle\sum_{k=0}^n b_{n-k}\frac{(\lambda x)^k}{k!}, \quad \forall n\ge 0, \\ & a_n = \textstyle\sum_{k=0}^n b_{n-k}\frac{\lambda^k}{k!}\int_0^\infty s(x)x^k e^{-\lambda x-\int_0^x s(\xi)\,d\xi}\,dx, \quad n\ge 1, \\ & b_n = \lambda \textstyle\sum_{k=0}^n b_{n-k}\frac{\lambda^k}{k!}\int_0^\infty s(x)x^k e^{-\lambda x-\int_0^x s(\xi)\,d\xi}\,dx \int_0^\infty e^{-\lambda x-\int_0^x r(\xi)\,d\xi}\,dx \\ & \qquad + \textstyle\sum_{k=0}^{n+1} b_{n+1-k}\frac{\lambda^k}{k!}\int_0^\infty s(x)x^k e^{-\lambda x-\int_0^x s(\xi)\,d\xi}\,dx \int_0^\infty e^{-\lambda x-\int_0^x r(\xi)\,d\xi}\,dx, \\ & n\ge 1, \quad b_0 = \frac{\lambda p_0}{\int_0^\infty s(x)e^{-\lambda x-\int_0^x s(\xi)\,d\xi}\,dx}, \quad \tilde{c}, p_0 \in \mathbb{R}. \end{aligned} \right\}$$

It is easy to see that $a_n$ and $b_n$ are decided by $p_0$, and $p_n(x)$ and $Q_n(x)$ are decided by $a_n$ and $b_n$, respectively. Therefore, $p_n(x)$ and $Q_n(x)$ are determined by $p_0$. Since $p_0 \in \mathbb{R}$ and $\tilde{c} \in \mathbb{R}$, $\Upsilon$ is a one-dimensional linear subspace of $X$, that is, the geometric multiplicity of 0 is one. □

According to Theorem 14 in Gupur, Li, and Zhu [11] or Theorem 1.96 in Gupur [12], we know that to obtain the asymptotic behavior of the time-dependent solution of system (1.9), we need to know the spectrum of $A$ on the imaginary axis. By comparing to Zhang and Gupur [9] we find that the main difficult point is the boundary conditions and that there are infinitely many equations. In 1987, Greiner [16] put forward an idea to study the spectrum of $A$ by perturbing boundary conditions when he studied a population equation which was described by a partial differential equation with an integral boundary condition. Using Greiner's idea, Haji and Radl [18] obtained the resolvent set of the operator corresponding to the $M/M^B/1$ queueing model where all parameters are constants and gave a result described by the Dirichlet operator. In the following, by applying the result, we deduce the resolvent set of $A$ on the imaginary axis. To do this, define $(A_0, D(A_0))$ as

$$A_0(p,Q) = A_m(p,Q) \quad \text{and} \quad D(A_0) = \big\{(p,Q)\in D(A_m) \mid \Psi(p,Q) = 0\big\}$$

and discuss the inverse of $A_0$. For any given $(y,z)\in X$, consider the equation $(\gamma I - A_0)(p,Q) = (y,z)$, that is,

$$(\gamma+\lambda)p_0 = y_0 + \int_0^\infty Q_0(x)s(x)\,dx, \tag{2.20}$$

$$\frac{dp_n(x)}{dx} = -\big[\gamma+\lambda+r(x)\big]p_n(x) + y_n(x), \quad \forall n\ge 1, \tag{2.21}$$

$$\frac{dQ_0(x)}{dx} = -\big[\gamma + \lambda + s(x)\big]Q_0(x) + z_0(x), \tag{2.22}$$

$$\frac{dQ_n(x)}{dx} = -\big[\gamma + \lambda + s(x)\big]Q_n(x) + \lambda Q_{n-1}(x) + z_n(x), \quad \forall n \geq 1, \tag{2.23}$$

$$p_n(0) = 0, \quad \forall n \geq 1, \qquad Q_n(0) = 0, \quad \forall n \geq 0. \tag{2.24}$$

By solving (2.20)–(2.24) we have

$$p_n(x) = e^{-(\gamma+\lambda)x-\int_0^x r(\xi)\,d\xi} \int_0^x y_n(\tau)e^{(\gamma+\lambda)\tau+\int_0^\tau r(\xi)\,d\xi}\,d\tau, \quad \forall n \geq 1, \tag{2.25}$$

$$Q_0(x) = e^{-(\gamma+\lambda)x-\int_0^x s(\xi)\,d\xi} \int_0^x z_0(\tau)e^{(\gamma+\lambda)\tau+\int_0^\tau s(\xi)\,d\xi}\,d\tau, \tag{2.26}$$

$$\begin{aligned} Q_n(x) &= e^{-(\gamma+\lambda)x-\int_0^x s(\xi)\,d\xi} \int_0^x z_n(\tau)e^{(\gamma+\lambda)\tau+\int_0^\tau s(\xi)\,d\xi}\,d\tau \\ &\quad + \lambda e^{-(\gamma+\lambda)x-\int_0^x s(\xi)\,d\xi} \int_0^x Q_{n-1}(\tau)e^{(\gamma+\lambda)\tau+\int_0^\tau s(\xi)\,d\xi}\,d\tau, \quad \forall n \geq 1, \end{aligned} \tag{2.27}$$

$$\begin{aligned} p_0 &= \frac{y_0}{\gamma+\lambda} + \frac{1}{\gamma+\lambda}\int_0^\infty Q_0(x)s(x)\,dx \\ &= \frac{y_0}{\gamma+\lambda} + \frac{1}{\gamma+\lambda}\int_0^\infty s(x)e^{-(\gamma+\lambda)x-\int_0^x s(\xi)\,d\xi} \int_0^x z_0(\tau)e^{(\gamma+\lambda)\tau+\int_0^\tau s(\xi)\,d\xi}\,d\tau\,dx. \end{aligned} \tag{2.28}$$

If we set

$$E_r f(x) = e^{-(\gamma+\lambda)x-\int_0^x r(\xi)\,d\xi} \int_0^x f(\tau)e^{(\gamma+\lambda)\tau+\int_0^\tau r(\xi)\,d\xi}\,d\tau, \quad \forall f \in L^1[0,\infty),$$

$$E_s f(x) = e^{-(\gamma+\lambda)x-\int_0^x s(\xi)\,d\xi} \int_0^x f(\tau)e^{(\gamma+\lambda)\tau+\int_0^\tau s(\xi)\,d\xi}\,d\tau, \quad \forall f \in L^1[0,\infty),$$

then Eqs. (2.25)–(2.28) give, provided that the resolvent of $A_0$ exists,

$$\begin{aligned} (\gamma I - A_0)^{-1}(y,z) = \Biggl( & \begin{pmatrix} \frac{1}{\gamma+\lambda} & 0 & 0 & \cdots \\ 0 & E_r & 0 & \cdots \\ 0 & 0 & E_r & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} y_0 \\ y_1(x) \\ y_2(x) \\ \vdots \end{pmatrix} \\ & + \begin{pmatrix} \frac{1}{\gamma+\lambda}\phi E_s & 0 & \cdots \\ 0 & 0 & \cdots \\ 0 & 0 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} z_0(x) \\ z_1(x) \\ z_2(x) \\ \vdots \end{pmatrix}, \\ & \begin{pmatrix} E_s & 0 & 0 & \cdots \\ \lambda E_s^2 & E_s & 0 & \cdots \\ \lambda^2 E_s^3 & \lambda E_s^2 & E_s & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} z_0(x) \\ z_1(x) \\ z_2(x) \\ \vdots \end{pmatrix} \Biggr). \end{aligned}$$

From this, together with the definition of the resolvent set, we obtain the following result.

**Lemma 2.2** *Let $r(x)$, $s(x) : [0,\infty) \to [0,\infty)$ be measurable, $0 \le \inf_{x\in[0,\infty)} r(x) \le \sup_{x\in[0,\infty)} r(x) < \infty$, and $0 < \inf_{x\in[0,\infty)} s(x) \le \sup_{x\in[0,\infty)} s(x) < \infty$. Then*

$$\left\{ \gamma \in \mathbb{C} \left| \begin{array}{l} \Re\gamma + \lambda > 0, \\ \Re\gamma + \inf_{x\in[0,\infty)} s(x) > 0 \end{array} \right. \right\} \subset \rho(A_0).$$

*Proof.* For any $f \in L^1[0,\infty)$, using integration by parts, as in Gupur and Ehmet [17], we estimate

$$\|E_r\| \le \frac{1}{\Re\gamma + \lambda + \inf_{x\in[0,\infty]} r(x)}, \tag{2.29}$$

$$\|E_s\| \le \frac{1}{\Re\gamma + \lambda + \inf_{x\in[0,\infty)} s(x)}. \tag{2.30}$$

Since an absolutely convergent number series converges to the original limit if the original orders of its terms are changed, (2.29), (2.30), $\|\phi\| \le \sup_{x\in[0,\infty)} s(x)$, and $\|\psi\| \le \sup_{x\in[0,\infty)} r(x)$ imply, for $(y,z) \in X$,

$$\begin{aligned}
&\big\|(\gamma I - A_0)^{-1}(y,z)\big\| \\
&\quad= \left| \frac{y_0}{\gamma+\lambda} + \frac{1}{\gamma+\lambda}\phi E_s z_0 \right| \\
&\qquad+ \|E_r y_1\|_{L^1[0,\infty)} + \|E_r y_2\|_{L^1[0,\infty)} + \|E_r y_3\|_{L^1[0,\infty)} + \cdots \\
&\qquad+ \|E_s z_0\|_{L^1[0,\infty)} + \big\|\lambda E_s^2 z_0 + E_s z_1\big\|_{L^1[0,\infty)} + \big\|\lambda^2 E_s^3 z_0 + \lambda E_s^2 z_1 + E_s z_2\big\|_{L^1[0,\infty)} \\
&\qquad+ \big\|\lambda^3 E_s^4 z_0 + \lambda^2 E_s^3 z_1 + \lambda E_s^2 z_2 + E_s z_3\big\|_{L^1[0,\infty)} + \cdots \\
&\quad\le \frac{|y_0|}{|\gamma+\lambda|} + \frac{1}{|\gamma+\lambda|}\|\phi E_s z_0\|_{L^1[0,\infty)} + \sum_{n=1}^{\infty} \|E_r y_n\|_{L^1[0,\infty)} \\
&\qquad+ \|E_s z_0\|_{L^1[0,\infty)} + \big\|\lambda E_s^2 z_0\big\|_{L^1[0,\infty)} + \|E_s z_1\|_{L^1[0,\infty)} + \big\|\lambda^2 E_s^3 z_0\big\|_{L^1[0,\infty)} \\
&\qquad+ \big\|\lambda E_s^2 z_1\big\|_{L^1[0,\infty)} + \|E_s z_2\|_{L^1[0,\infty)} + \big\|\lambda^3 E_s^4 z_0\big\|_{L^1[0,\infty)} + \big\|\lambda^2 E_s^3 z_1\big\|_{L^1[0,\infty)} \\
&\qquad+ \big\|\lambda E_s^2 z_2\big\|_{L^1[0,\infty)} + \|E_s z_3\|_{L^1[0,\infty)} + \cdots \\
&\quad\le \frac{|y_0|}{|\gamma+\lambda|} + \frac{1}{|\gamma+\lambda|}\|\phi\|\|E_s\|\|z_0\|_{L^1[0,\infty)} + \sum_{n=1}^{\infty} \|E_r\|\|y_n\|_{L^1[0,\infty)} \\
&\qquad+ \|E_s\|\|z_0\|_{L^1[0,\infty)} + \lambda\|E_s\|^2\|z_0\|_{L^1[0,\infty)} + \|E_s\|\|z_1\|_{L^1[0,\infty)} \\
&\qquad+ \lambda^2\|E_s\|^3\|z_0\|_{L^1[0,\infty)} + \lambda\|E_s\|^2\|z_1\|_{L^1[0,\infty)} + \|E_s\|\|z_2\|_{L^1[0,\infty)} \\
&\qquad+ \lambda^3\|E_s\|^4\|z_0\|_{L^1[0,\infty)} + \lambda^2\|E_s\|^3\|z_1\|_{L^1[0,\infty)} + \lambda\|E_s\|^2\|z_2\|_{L^1[0,\infty)} \\
&\qquad+ \|E_s\|\|z_3\|_{L^1[0,\infty)} + \cdots \\
&\quad= \frac{|y_0|}{|\gamma+\lambda|} + \frac{1}{|\gamma+\lambda|}\|\phi\|\|E_s\|\|z_0\|_{L^1[0,\infty)} + \sum_{n=1}^{\infty} \|E_r\|\|y_n\|_{L^1[0,\infty)} \\
&\qquad+ \sum_{n=0}^{\infty} \lambda^n\|E_s\|^{n+1} \sum_{n=0}^{\infty} \|z_n\|_{L^1[0,\infty)}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{|y_0|}{|\gamma+\lambda|} + \frac{1}{|\gamma+\lambda|}\|\phi\|\|E_s\|\|z_0\|_{L^1[0,\infty)} + \|E_r\|\sum_{n=1}^{\infty}\|y_n\|_{L^1[0,\infty)} \\
&\quad + \|E_s\|\sum_{n=0}^{\infty}\lambda^n\|E_s\|^n\sum_{n=0}^{\infty}\|z_n\|_{L^1[0,\infty)} \\
&\le \frac{|y_0|}{|\gamma+\lambda|} + \frac{1}{|\gamma+\lambda|}\frac{\sup_{x\in[0,\infty)}s(x)}{\Re\gamma+\lambda+\inf_{x\in[0,\infty)}s(x)}\|z_0\|_{L^1[0,\infty)} \\
&\quad + \frac{1}{\Re\gamma+\lambda+\inf_{x\in[0,\infty)}r(x)}\sum_{n=1}^{\infty}\|y_n\|_{L^1[0,\infty)} \\
&\quad + \frac{1}{\Re\gamma+\lambda+\inf_{x\in[0,\infty)}s(x)}\sum_{n=0}^{\infty}\left(\frac{\lambda}{\Re\gamma+\lambda+\inf_{x\in[0,\infty)}s(x)}\right)^n\sum_{n=0}^{\infty}\|z_n\|_{L^1[0,\infty)} \\
&\le \frac{|y_0|}{\Re\gamma+\lambda} + \frac{1}{\Re\gamma+\lambda}\frac{\sup_{x\in[0,\infty)}s(x)}{\Re\gamma+\lambda+\inf_{x\in[0,\infty)}s(x)}\|z_0\|_{L^1[0,\infty)} \\
&\quad + \frac{1}{\Re\gamma+\lambda+\inf_{x\in[0,\infty)}r(x)}\sum_{n=1}^{\infty}\|y_n\|_{L^1[0,\infty)} + \frac{1}{\Re\gamma+\inf_{x\in[0,\infty)}s(x)}\sum_{n=0}^{\infty}\|z_n\|_{L^1[0,\infty)} \\
&\le \sup\left\{\frac{1}{\Re\gamma+\lambda}+\frac{1}{\Re\gamma+\lambda+\inf_{x\in[0,\infty)}r(x)},\right. \\
&\quad \left.\frac{1}{\Re\gamma+\lambda}\frac{\sup_{x\in[0,\infty)}s(x)}{\Re\gamma+\lambda+\inf_{x\in[0,\infty)}s(x)}+\frac{1}{\Re\gamma+\inf_{x\in[0,\infty)}s(x)}\right\}\|(y,z)\|,
\end{aligned}
$$

which means that the result of this lemma is right.

**Lemma 2.3** *For $\gamma \in \{\gamma \in \mathbb{C} \mid \Re\gamma + \lambda > 0, \Re\gamma + \inf_{x\in[0,\infty)} s(x) > 0\} \subset \rho(A_0)$, we have*

$$
(p,Q) \in \ker(\gamma I - A_m) \quad \Leftrightarrow \quad p_0 = \frac{b_0}{\gamma+\lambda}\int_0^{\infty} s(x)e^{-(\gamma+\lambda)x-\int_0^x s(\xi)\,d\xi}\,dx, \tag{2.31}
$$

$$
p_n(x) = a_n e^{-(\gamma+\lambda)x-\int_0^x r(\xi)\,d\xi}, \quad \forall n \ge 1, \tag{2.32}
$$

$$
Q_n(x) = e^{-(\gamma+\lambda)x-\int_0^{\infty} s(\xi)\,d\xi}\sum_{k=0}^{n}\frac{(\lambda x)^k}{k!}b_{n-k}, \quad \forall n \ge 0, \tag{2.33}
$$

$$
\overrightarrow{a} = (a_1, a_2, a_3, \ldots) \in l^1, \qquad \overrightarrow{b} = (b_0, b_1, b_2, \ldots) \in l^1. \tag{2.34}
$$

*Proof.* If $(p,Q) \in \ker(\gamma I - A_m)$, then $(\gamma I - A_m)(p,Q) = 0$, which is equivalent to

$$
(\gamma+\lambda)p_0 = \int_0^{\infty} Q_0(x)s(x)\,dx, \tag{2.35}
$$

$$
\frac{dp_n(x)}{dx} = -\big[\gamma+\lambda+r(x)\big]p_n(x), \quad \forall n \ge 1, \tag{2.36}
$$

$$
\frac{dQ_0(x)}{dx} = -\big[\gamma+\lambda+s(x)\big]Q_0(x), \tag{2.37}
$$

$$
\frac{dQ_n(x)}{dx} = -\big[\gamma+\lambda+s(x)\big]Q_n(x) + \lambda Q_{n-1}(x), \quad \forall n \ge 1. \tag{2.38}
$$

By solving (2.36)–(2.38) we have

$$p_n(x) = a_n e^{-(\gamma+\lambda)x-\int_0^x r(\xi)\,d\xi}, \quad \forall n \geq 1, \tag{2.39}$$

$$Q_0(x) = b_0 e^{-(\gamma+\lambda)x-\int_0^x s(\xi)\,d\xi}, \tag{2.40}$$

$$\begin{aligned} Q_n(x) &= b_n e^{-(\gamma+\lambda)x-\int_0^x s(\xi)\,d\xi} \\ &\quad + \lambda e^{-(\gamma+\lambda)x-\int_0^x s(\xi)\,d\xi} \int_0^x Q_{n-1}(\tau) e^{(\gamma+\lambda)\tau+\int_0^\tau s(\xi)\,d\xi}\,d\tau, \quad \forall n \geq 1. \end{aligned} \tag{2.41}$$

By inserting (2.40) into (2.35) and using (2.40) and (2.41) repeatedly we deduce

$$p_0 = \frac{b_0}{\gamma+\lambda} \int_0^\infty s(x) e^{-(\gamma+\lambda)x-\int_0^x s(\xi)\,d\xi}\,dx, \tag{2.42}$$

$$Q_n(x) = e^{-(\gamma+\lambda)x-\int_0^x s(\xi)\,d\xi} \sum_{k=0}^n \frac{(\lambda x)^k}{k!} b_{n-k}, \quad \forall n \geq 0. \tag{2.43}$$

Since $(p, Q) \in \ker(\gamma I - A_m)$, by Theorem 4.12 in Adams [19], which implies that $W^{1,1}[0,\infty) \hookrightarrow L^\infty[0,\infty)$, we have

$$\begin{aligned} \sum_{n=1}^\infty |a_n| &= \sum_{n=1}^\infty \bigl|p_n(0)\bigr| \leq \sum_{n=1}^\infty \|p_n\|_{L^\infty[0,\infty)} \\ &\leq \sum_{n=1}^\infty \left\{ \|p_n\|_{L^1[0,\infty)} + \left\| \frac{dp_n}{dx} \right\|_{L^1[0,\infty)} \right\} < \infty, \end{aligned} \tag{2.44}$$

$$\begin{aligned} \sum_{n=0}^\infty |b_n| &= \sum_{n=0}^\infty \bigl|Q_n(0)\bigr| \leq \sum_{n=0}^\infty \|Q_n\|_{L^\infty[0,\infty)} \\ &\leq \sum_{n=0}^\infty \left\{ \|Q_n\|_{L^1[0,\infty)} + \left\| \frac{dQ_n}{dx} \right\|_{L^1[0,\infty)} \right\} < \infty. \end{aligned} \tag{2.45}$$

Relations (2.39) and (2.42)–(2.45) show that (2.31)–(2.34) are true.

Conversely, if (2.31)–(2.34) hold, then by using the formulas

$$\int_0^\infty e^{-cx} x^k\,dx = \frac{k!}{c^{k+1}}, \quad c > 0, k \in \mathbb{N}, \tag{2.46}$$

$$\int_0^\infty r(x) e^{-\int_0^x r(\xi)\,d\xi}\,dx = -e^{-\int_0^x r(\xi)\,d\xi}\Big|_0^\infty = 1, \tag{2.47}$$

$$\int_0^\infty s(x) e^{-\int_0^x s(\xi)\,d\xi}\,dx = -e^{-\int_0^x s(\xi)\,d\xi}\Big|_0^\infty = 1, \tag{2.48}$$

integration by parts, and the Cauchy product, we estimate

$$\begin{aligned} \|p_n\|_{L^1[0,\infty)} &= \int_0^\infty \bigl|a_n e^{-(\gamma+\lambda)x-\int_0^x r(\xi)\,d\xi}\bigr|\,dx \leq |a_n| \int_0^\infty e^{-(\Re\gamma+\lambda)x-\int_0^x r(\xi)\,d\xi}\,dx \\ &\leq |a_n| \int_0^\infty e^{-[\Re\gamma+\lambda+\inf_{x\in[0,\infty)} r(x)]x}\,dx = \frac{1}{\Re\gamma+\lambda+\inf_{x\in[0,\infty)} r(x)} |a_n|, \quad \forall n \geq 1 \\ &\Rightarrow \end{aligned}$$

$$
\begin{aligned}
&|p_0| + \sum_{n=1}^{\infty} \|p_n\|_{L^1[0,\infty)} \\
&\quad \le \frac{|b_0|}{|\gamma+\lambda|} \int_0^{\infty} \left| s(x) e^{-(\gamma+\lambda)x - \int_0^x s(\xi)\,d\xi} \right| dx + \frac{1}{\Re\gamma + \lambda + \inf_{x\in[0,\infty)} r(x)} \sum_{n=1}^{\infty} |a_n| \\
&\quad \le \frac{|b_0|}{|\gamma+\lambda|} \int_0^{\infty} s(x) e^{-\int_0^x s(\xi)\,d\xi}\, dx + \frac{1}{\Re\gamma + \lambda + \inf_{x\in[0,\infty)} r(x)} \sum_{n=1}^{\infty} |a_n| \\
&\quad = \frac{|b_0|}{|\gamma+\lambda|} + \frac{1}{\Re\gamma + \lambda + \inf_{x\in[0,\infty)} r(x)} \sum_{n=1}^{\infty} |a_n| < \infty,
\end{aligned}
\tag{2.49}
$$

$$
\begin{aligned}
&\|Q_n\|_{L^1[0,\infty)} \\
&\quad = \int_0^{\infty} \left| e^{-(\gamma+\lambda)x - \int_0^x s(\xi)\,d\xi} \sum_{k=0}^{n} \frac{(\lambda x)^k}{k!} b_{n-k} \right| dx \\
&\quad \le \sum_{k=0}^{n} \frac{\lambda^k}{k!} |b_{n-k}| \int_0^{\infty} x^k e^{-[\Re\gamma + \lambda + \inf_{x\in[0,\infty)} s(x)]x}\, dx \\
&\quad = \sum_{k=0}^{n} \frac{\lambda^k}{k!} |b_{n-k}| \frac{k!}{[\Re\gamma + \lambda + \inf_{x\in[0,\infty)} s(x)]^{k+1}} \\
&\quad = \frac{1}{\Re\gamma + \lambda + \inf_{x\in[0,\infty)} s(x)} \sum_{k=0}^{n} \left( \frac{\lambda}{\Re\gamma + \lambda + \inf_{x\in[0,\infty)} s(x)} \right)^k |b_{n-k}| \\
&\quad \Rightarrow \\
&\sum_{n=0}^{\infty} \|Q_n\|_{L^1[0,\infty)} \\
&\quad \le \sum_{n=0}^{\infty} \frac{1}{\Re\gamma + \lambda + \inf_{x\in[0,\infty)} s(x)} \sum_{k=0}^{n} \left( \frac{\lambda}{\Re\gamma + \lambda + \inf_{x\in[0,\infty)} s(x)} \right)^k |b_{n-k}| \\
&\quad = \frac{1}{\Re\gamma + \lambda + \inf_{x\in[0,\infty)} s(x)} \sum_{k=0}^{\infty} \left( \frac{\lambda}{\Re\gamma + \lambda + \inf_{x\in[0,\infty)} s(x)} \right)^k \sum_{n=0}^{\infty} |b_n| \\
&\quad = \frac{1}{\Re\gamma + \inf_{x\in[0,\infty)} s(x)} \sum_{n=0}^{\infty} |b_n| < \infty.
\end{aligned}
\tag{2.50}
$$

By direct calculation it is not difficult to verify (2.35)–(2.38). In addition, (2.36)–(2.38) imply

$$
\sum_{n=1}^{\infty} \left\| \frac{dp_n}{dx} \right\|_{L^1[0,\infty)} \le \Bigl[ |\gamma| + \lambda + \sup_{x\in[0,\infty)} r(x) \Bigr] \sum_{n=1}^{\infty} \|p_n\|_{L^1[0,\infty)} < \infty, \tag{2.51}
$$

$$
\sum_{n=0}^{\infty} \left\| \frac{dQ_n}{dx} \right\|_{L^1[0,\infty)} \le \Bigl[ |\gamma| + 2\lambda + \sup_{x\in[0,\infty)} s(x) \Bigr] \sum_{n=0}^{\infty} \|Q_n\|_{L^1[0,\infty)} < \infty. \tag{2.52}
$$

Relations (2.49)–(2.52) mean that $(p,Q) \in D(A_m)$ and $(\gamma I - A_m)(p,Q) = 0$.

It is not difficult to see that $\Psi$ is surjective. Moreover,

$$
\Psi|_{\ker(\gamma I - A_m)} : \ker(\gamma I - A_m) \to \partial X
$$

is invertible for $\gamma \in \rho(A_0)$. For $\gamma \in \rho(A_0)$ we define the Dirichlet operator as

$$D_\gamma := (\Psi|_{\ker(\gamma I - A_m)})^{-1} : \partial X \to \ker(\gamma I - A_m).$$

Lemma 2.3 gives an explicit form of $D_\gamma$ for $\gamma \in \rho(A_0)$:

$$D_\gamma(\vec{a}, \vec{b}) = \left( \begin{pmatrix} 0 & 0 & 0 & \cdots \\ \varepsilon & 0 & 0 & \cdots \\ 0 & \varepsilon & 0 & \cdots \\ 0 & 0 & \varepsilon & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ \vdots \end{pmatrix} + \begin{pmatrix} \frac{1}{\gamma+\lambda}\phi\delta_0 & 0 & \cdots \\ 0 & 0 & \cdots \\ 0 & 0 & \cdots \\ 0 & 0 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \\ \vdots \end{pmatrix}, \right.$$
$$\left. \begin{pmatrix} \delta_0 & 0 & 0 & 0 & \cdots \\ \delta_1 & \delta_0 & 0 & 0 & \cdots \\ \delta_2 & \delta_1 & \delta_0 & 0 & \cdots \\ \delta_3 & \delta_2 & \delta_1 & \delta_0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \\ \vdots \end{pmatrix} \right), \tag{2.53}$$

where

$$\varepsilon = e^{-(\gamma+\lambda)x - \int_0^x r(\xi)\,d\xi}, \qquad \delta_k = \frac{(\lambda x)^k}{k!} e^{-(\gamma+\lambda)x - \int_0^x s(\xi)\,d\xi}, \quad k \geq 0.$$

From (2.53) and the definition of $\Phi$ it is easy to determine the expression of $\Phi D_\gamma$ by

$$\Phi D_\gamma(a,b) = \left( \begin{pmatrix} \phi\delta_1 & \phi\delta_0 & 0 & 0 & \cdots \\ \phi\delta_2 & \phi\delta_1 & \phi\delta_0 & 0 & \cdots \\ \phi\delta_3 & \phi\delta_2 & \phi\delta_1 & \phi\delta_0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} b_0 \\ b_1 \\ b_2 \\ \vdots \end{pmatrix}, \right.$$
$$\left. \begin{pmatrix} \psi\varepsilon & 0 & 0 & \cdots \\ \lambda H\varepsilon & \psi\varepsilon & 0 & \cdots \\ 0 & \lambda H\varepsilon & \psi\varepsilon & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \end{pmatrix} + \begin{pmatrix} \frac{\lambda}{\gamma+\lambda}\phi\delta_0 & 0 & \cdots \\ 0 & 0 & \cdots \\ 0 & 0 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} b_0 \\ b_1 \\ b_2 \\ \vdots \end{pmatrix} \right).$$

Haji and Radl [18] gave the following result, through which we deduce the resolvent set of $A$ on the imaginary axis.

**Lemma 2.4** *If $\gamma \in \rho(A_0)$ and $1 \notin \sigma(\Phi D_\gamma)$, then*

$$\gamma \in \sigma(A) \quad \Leftrightarrow \quad 1 \in \sigma(\Phi D_\gamma).$$

By using Lemma 2.4 and p. 297 of Nagel [20] we derive the following result.

**Lemma 2.5** *Let $r(x), s(x) : [0,\infty) \to [0,\infty)$ be measurable,*

$$0 < \inf_{x\in[0,\infty)} r(x) \leq \sup_{x\in[0,\infty)} r(x) < \infty, \quad \textit{and} \quad 0 < \inf_{x\in[0,\infty)} s(x) \leq \sup_{x\in[0,\infty)} s(x) < \infty.$$

*Then all points on the imaginary axis except zero belong to the resolvent set of $A$.*

*Proof.* Let $\gamma = im$, $m \in \mathbb{R}\backslash\{0\}$, $\overrightarrow{a} = (a_1, a_2, \ldots) \in l^1$, and $\overrightarrow{b} = (b_0, b_1, b_2, \ldots) \in l^1$. The Riemann–Lebesgue lemma

$$\lim_{m\to\infty}\int_0^\infty f(x)e^{imx}\,dx = 0, \quad f \in L^1[0,\infty), f(x) > 0,$$

implies that there exist $\mathcal{M} > 0$ and $\theta_1 \in (0,1)$ such that

$$\left|\int_0^\infty f(x)e^{-imx}\,dx\right| < \theta_1\int_0^\infty f(x)\,dx, \quad |m| > \mathcal{M}. \tag{2.54}$$

Replacing $f(x)$ in (2.54) with $f(x) = e^{-\lambda x - \int_0^x r(\xi)\,d\xi}$ and $f(x) = e^{-\lambda x - \int_0^x s(\xi)\,d\xi}$ and using (2.47)–(2.48), we have

$$\begin{aligned}
&\left|\int_0^\infty e^{-(im+\lambda)x - \int_0^x r(\xi)\,d\xi}\,dx\right| < \theta_2\int_0^\infty e^{-\lambda x - \int_0^x r(\xi)\,d\xi}\,dx, \quad \theta_2 \in (0,1),\\
&\left|\int_0^\infty \frac{(\lambda x)^n}{n!}e^{-(im+\lambda)x - \int_0^x s(\xi)\,d\xi}\,dx\right| < \theta_2\int_0^\infty \frac{(\lambda x)^n}{n!}e^{-\lambda x - \int_0^x s(\xi)\,d\xi}\,dx, \quad \forall n \geq 0, \theta_2 \in (0,1),
\end{aligned}$$

recalling that a convergent positive number series still converges to the original limit if the orders of its terms are changed we derive, for $|m| > \mathcal{M}$ and $(\overrightarrow{a}, \overrightarrow{b}) \neq (\overrightarrow{0}, \overrightarrow{0})$,

$$\begin{aligned}
&\left\|\Phi D_\gamma(\overrightarrow{a}, \overrightarrow{b})\right\|\\
&\quad= |\phi\delta_1 b_0 + \phi\delta_0 b_1| + |\phi\delta_2 b_0 + \phi\delta_1 b_1 + \phi\delta_0 b_2|\\
&\qquad + |\phi\delta_3 b_0 + \phi\delta_2 b_1 + \phi\delta_1 b_2 + \phi\delta_0 b_3|\\
&\qquad + |\phi\delta_4 b_0 + \phi\delta_3 b_1 + \phi\delta_2 b_2 + \phi\delta_1 b_3 + \phi\delta_0 b_4| + \cdots\\
&\qquad + \left|\psi\varepsilon a_1 + \frac{\lambda}{\gamma+\lambda}\phi\delta_0 b_0\right| + |\lambda H\varepsilon a_1 + \psi\varepsilon a_2|\\
&\qquad + |\lambda H\varepsilon a_2 + \psi\varepsilon a_3| + |\lambda H\varepsilon a_3 + \psi\varepsilon a_4| + \cdots\\
&\quad\leq \sum_{n=1}^\infty |\phi\delta_n||b_0| + \sum_{k=1}^\infty |b_k|\sum_{n=0}^\infty |\phi\delta_n|\\
&\qquad + \left|\frac{\lambda}{\gamma+\lambda}\right||\phi\delta_0||b_0| + \bigl(|\psi\varepsilon| + \lambda|H\varepsilon|\bigr)\sum_{n=1}^\infty |a_n|\\
&\quad< |b_0|\sum_{n=1}^\infty |\phi\delta_n| + \sum_{k=1}^\infty |b_k|\sum_{n=0}^\infty |\phi\delta_n|\\
&\qquad + |b_0||\phi\delta_0| + \bigl(|\psi\varepsilon| + \lambda|H\varepsilon|\bigr)\sum_{n=1}^\infty |a_n|\\
&\quad= \sum_{k=0}^\infty |b_k|\sum_{n=0}^\infty |\phi\delta_n| + \bigl(|\psi\varepsilon| + \lambda|H\varepsilon|\bigr)\sum_{n=1}^\infty |a_n|\\
&\quad= \sum_{n=0}^\infty \left|\int_0^\infty s(x)\frac{(\lambda x)^n}{n!}e^{-(im+\lambda)x - \int_0^x s(\xi)\,d\xi}\,dx\right|\sum_{k=0}^\infty |b_k|
\end{aligned}$$

$$
\begin{aligned}
&+ \left( \left| \int_0^\infty r(x) e^{-(im+\lambda)x - \int_0^x r(\xi)\,d\xi}\,dx \right| \right. \\
&\left. + \lambda \left| \int_0^\infty e^{-(im+\lambda)x - \int_0^x r(\xi)\,d\xi}\,dx \right| \right) \sum_{n=1}^\infty |a_n| \\
< {}& \sum_{n=0}^\infty \left( \theta_2 \int_0^\infty s(x) \frac{(\lambda x)^n}{n!} e^{-\lambda x - \int_0^x s(\xi)\,d\xi}\,dx \right) \sum_{k=0}^\infty |b_k| \\
&+ \left[ \theta_2 \int_0^\infty r(x) e^{-\lambda x - \int_0^x r(\xi)\,d\xi}\,dx + \theta_2 \lambda \int_0^\infty e^{-\lambda x - \int_0^x r(\xi)\,d\xi}\,dx \right] \sum_{n=1}^\infty |a_n| \\
= {}& \theta_2 \sum_{k=0}^\infty |b_k| \int_0^\infty s(x) \sum_{n=0}^\infty \frac{(\lambda x)^n}{n!} e^{-\lambda x - \int_0^x s(\xi)\,d\xi}\,dx \\
&+ \theta_2 \sum_{n=1}^\infty |a_n| \int_0^\infty \bigl[\lambda + r(x)\bigr] e^{-\int_0^x [\lambda + r(\xi)]\,d\xi}\,dx \\
= {}& \theta_2 \left( \sum_{n=1}^\infty |a_n| + \sum_{k=0}^\infty |b_k| \int_0^\infty s(x) e^{\lambda x} e^{-\lambda x - \int_0^x s(\xi)\,d\xi}\,dx \right) \\
= {}& \theta_2 \left( \sum_{n=1}^\infty |a_n| + \sum_{k=0}^\infty |b_k| \int_0^\infty s(x) e^{-\int_0^x s(\xi)\,d\xi}\,dx \right) \\
= {}& \theta_2 \left( \sum_{n=1}^\infty |a_n| + \sum_{k=0}^\infty |b_k| \right) \\
= {}& \theta_2 \bigl\| (\vec{a}, \vec{b}) \bigr\| \\
\Rightarrow {}&
\end{aligned}
$$

$$
\|\Phi D_\gamma\| = \sup_{\|(\vec{a}, \vec{b})\| \neq 0} \frac{\|\Phi D_\gamma(\vec{a}, \vec{b})\|}{\|(\vec{a}, \vec{b})\|} \le \theta_2 < 1. \tag{2.55}
$$

This means that when $|m| > \mathcal{M}$, the spectral radius $r(\Phi D_\gamma) \le \|\Phi D_\gamma\| \le \theta_2 < 1$, which implies $1 \notin \sigma(\Phi D_\gamma)$ for $|m| > \mathcal{M}$, and therefore by Lemma 2.4 we know that $\gamma = im \notin \sigma(A)$ for $|m| > \mathcal{M}$, that is,

$$
\bigl\{ im \mid |m| > \mathcal{M} \bigr\} \subset \rho(A), \qquad \bigl\{ im \mid |m| \le \mathcal{M} \bigr\} \subset \sigma(A) \cap i\mathbb{R}. \tag{2.56}
$$

On the other hand, since $T(t)$ is positive uniformly bounded by Theorem 1.1, by Corollary 2.3 in Nagel [20], p. 297, we know that $\sigma(A) \cap i\mathbb{R}$ is imaginary additively cyclic, which states that $im \in \sigma(A) \cap i\mathbb{R} \Rightarrow imk \in \sigma(A) \cap i\mathbb{R}$ for all integers $k$, from which, together with (2.56) and Lemma 2.1, we conclude that $\sigma(A) \cap i\mathbb{R} = \{0\}$, that is, all points on the imaginary axis except zero belong to the resolvent set of $A$.

According to Zhang and Gupur [9] and Jiang and Gupur [10], $X^*$, the dual space of $X$, is as follows:

$$
X^* = \left\{ (p^*, Q^*) \left| \begin{array}{l} p^* \in \mathbb{R} \times L^\infty[0,\infty) \times L^\infty[0,\infty) \times \cdots, \\ Q^* \in L^\infty[0,\infty) \times L^\infty[0,\infty) \times \cdots, \\ \|(p^*, Q^*)\| = \max\{|p_0^*|, \\ \sup_{n \ge 1} \|p_n^*\|_{L^\infty[0,\infty)}, \sup_{n \ge 0} \|Q_n^*\|_{L^\infty[0,\infty)}\} < \infty \end{array} \right. \right\}.
$$

It is obvious that $X^*$ is a Banach space. Zhang and Gupur [9] gave the following expression of $A^*$, the adjoint operator of $A$:

$$A^*(p^*,Q^*) = (L+N+R)(p^*,Q^*), \quad (p^*,Q^*) \in D(L),$$

where

$$L(p^*,Q^*) = \left( \begin{pmatrix} -\lambda & 0 & 0 & \cdots \\ 0 & \frac{d}{dx} - [\lambda + r(x)] & 0 & \cdots \\ 0 & 0 & \frac{d}{dx} - [\lambda + r(x)] & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} p_0^* \\ p_1^*(x) \\ p_2^*(x) \\ \vdots \end{pmatrix}, \right.$$
$$\left. \begin{pmatrix} \frac{d}{dx} - [\lambda + s(x)] & 0 & 0 & \cdots \\ 0 & \frac{d}{dx} - [\lambda + s(x)] & 0 & \cdots \\ 0 & 0 & \frac{d}{dx} - [\lambda + s(x)] & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} Q_0^*(x) \\ Q_1^*(x) \\ Q_2^*(x) \\ \vdots \end{pmatrix} \right),$$

$$D(L) = \left\{ (p^*,Q^*) \,\middle|\, \begin{array}{l} \frac{dp_i^*(x)}{dx}\ (i \geq 1), \frac{dQ_n^*(x)}{dx}\ (n \geq 0) \text{ exist and} \\ p_i^*(\infty) = K_0\ (i \geq 1), Q_n^*(\infty) = K_0\ (n \geq 0) \end{array} \right\},$$

where $K_0$ is a constant which is irrelevant to $i$, $n$ in $D(L)$.

$$N(p^*,Q^*) = \left( \begin{pmatrix} \lambda & 0 & 0 & \cdots \\ 0 & \lambda & 0 & \cdots \\ 0 & 0 & \lambda & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} Q_0^*(0) \\ Q_1^*(0) \\ Q_2^*(0) \\ \vdots \end{pmatrix}, \begin{pmatrix} 0 & \lambda & 0 & \cdots \\ 0 & 0 & \lambda & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} Q_0^*(x) \\ Q_1^*(x) \\ Q_2^*(x) \\ \vdots \end{pmatrix} \right),$$

$$R(p^*,Q^*) = \left( \begin{pmatrix} 0 & 0 & 0 & \cdots \\ r(x) & 0 & 0 & \cdots \\ 0 & r(x) & 0 & \cdots \\ 0 & 0 & r(x) & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} Q_0^*(0) \\ Q_1^*(0) \\ Q_2^*(0) \\ Q_3^*(0) \\ \vdots \end{pmatrix}, \right.$$
$$\left. \begin{pmatrix} s(x) & 0 & 0 & 0 & \cdots \\ 0 & s(x) & 0 & 0 & \cdots \\ 0 & 0 & s(x) & 0 & \cdots \\ 0 & 0 & 0 & s(x) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} p_0^* \\ p_1^*(0) \\ p_2^*(0) \\ p_3^*(0) \\ \vdots \end{pmatrix} \right).$$

Since $T(t)$ is uniformly bounded, by Lemma 2.3 in Arendt and Batty [21] and Lemma 2.1 we know that 0 is an eigenvalue of $A^*$. Furthermore, replacing $\alpha$ and $\beta$ in [9] with $r(x)$ and $s(x)$, respectively, we deduce the following result.

**Lemma 2.6** *If*

$$\lambda \int_0^\infty e^{-\lambda x - \int_0^x r(\xi)\,d\xi}\,dx + \int_0^\infty \lambda x s(x) e^{-\int_0^x s(\xi)\,d\xi}\,dx < 1,$$

*then* 0 *is an eigenvalue of* $A^*$ *with geometric multiplicity one.*

Since by combining Lemmas 2.1 and 2.6 and Lemma 5 in Gupur [22] we know that 0 is an eigenvalue of $A^*$ with algebraic multiplicity one, Theorem 1.1, Lemma 2.1, Lemma 2.5, and Lemma 2.6 satisfy the conditions of Theorem 14 in Gupur, Li, and Zhu [11] (Theorem 1.96 in Gupur [12]). Thus, we obtain the desired result in this section, which is the direct result of Theorem 14 in Gupur, Li, and Zhu [11].

**Theorem 2.1** *Let* $r(x), s(x) : [0,\infty) \to [0,\infty)$ *be measurable,*

$$0 < \inf_{x\in[0,\infty)} r(x) \le \sup_{x\in[0,\infty)} r(x) < \infty \quad \textit{and} \quad 0 < \inf_{x\in[0,\infty)} s(x) \le \sup_{x\in[0,\infty)} s(x) < \infty.$$

*If*

$$\lambda \int_0^\infty e^{-\lambda x - \int_0^x r(\xi)\,d\xi}\,dx + \int_0^\infty \lambda x s(x) e^{-\int_0^x s(\xi)\,d\xi}\,dx < 1,$$

*then the time-dependent solution of system* (1.9) *converges strongly to its steady-state solution, that is,*

$$\lim_{t\to\infty} \left\| (p,Q)(\cdot,t) - \left\langle (p^*,Q^*), \left(p(0),Q(0)\right)\right\rangle (p,Q)(\cdot) \right\| = 0,$$

*where* $(p^*, Q^*)$ *and* $(p, Q)$ *are the eigenvectors in Lemmas* 2.6 *and* 2.1, *respectively.*

*Remark* 2.1 Gomez-Corral [4] obtained that the Markov process is ergodic if and only if $\lambda\tilde{\beta}_1 < \tilde{\alpha}(\lambda)$. This condition is quite different from the conditions in Theorem 2.1. Hence, it is worth studying the relation between the ergodicity of the Markov process and the conditions in Theorem 2.1. This is our future research topic.

## 3 Point spectrum of $A$ when $r(x) = \alpha$, $s(x) = \beta$

When $r(x) = \alpha$, $s(x) = \beta$, we have the following result.

**Theorem 3.1** *If* $\alpha, \beta, \lambda > 0$ *and* $\frac{\lambda(\lambda+\alpha)}{\alpha\beta} < 1$, *then all points in the set*

$$\left\{ \gamma \in \mathbb{C} \,\middle|\, \begin{aligned} &\Re\gamma + \lambda + \alpha > 0, \Re\gamma + \beta > 0, \\ &|[(\gamma+\lambda+\alpha)(\gamma+\lambda+\beta) - \lambda\beta] \\ &\quad \pm \sqrt{[(\gamma+\lambda+\alpha)(\gamma+\lambda+\beta)-\lambda\beta]^2 - 4\alpha\beta\lambda(\gamma+\lambda+\alpha)}| < 2\alpha\beta \end{aligned} \right\} \cup \{0\}$$

*are eigenvalues of* $A$ *with geometric multiplicity* 1. *Especially, the interval*

$$\left( \frac{-(2\lambda+\alpha+\beta) + \sqrt{(\alpha+\beta)^2 + 4\lambda\beta}}{2}, 0 \right]$$

*belongs to the point spectrum of* $A$.

*Proof.* Consider the equation $A(p,Q) = \gamma(p,Q)$, that is,

$$(\gamma+\lambda)p_0 = \beta \int_0^\infty Q_0(x)\,dx, \tag{3.1}$$

$$\frac{dp_n(x)}{dx} = -(\gamma+\lambda+\alpha)p_n(x), \quad \forall n \ge 1, \tag{3.2}$$

$$\frac{dQ_0(x)}{dx} = -(\gamma+\lambda+\beta)Q_0(x), \tag{3.3}$$

$$\frac{dQ_{n(x)}}{dx} = -(\gamma+\lambda+\beta)Q_n(x) + \lambda Q_{n-1}(x), \quad \forall n \geq 1, \tag{3.4}$$

$$p_n(0) = \beta \int_0^\infty Q_n(x)\,dx, \quad \forall n \geq 1, \tag{3.5}$$

$$Q_0(0) = \lambda p_0 + \alpha \int_0^\infty p_1(x)\,dx, \tag{3.6}$$

$$Q_n(0) = \lambda \int_0^\infty p_n(x)\,dx + \alpha \int_0^\infty p_{n+1}(x)\,dx, \quad \forall n \geq 1. \tag{3.7}$$

By solving (3.2)–(3.4) we have

$$p_n(x) = a_n e^{-(\gamma+\lambda+\alpha)x}, \quad \forall n \geq 1, \tag{3.8}$$

$$Q_0(x) = b_0 e^{-(\gamma+\lambda+\beta)x}, \tag{3.9}$$

$$Q_n(x) = b_n e^{-(\gamma+\lambda+\beta)x} + \lambda e^{-(\gamma+\lambda+\beta)x} \int_0^x e^{(\gamma+\lambda+\beta)\tau} Q_{n-1}(\tau)\,d\tau, \quad \forall n \geq 1. \tag{3.10}$$

By combining (3.9) with (3.10) we deduce

$$Q_n(x) = e^{-(\gamma+\lambda+\beta)x} \sum_{j=0}^{n} b_{n-j} \frac{(\lambda x)^j}{j!}, \quad \forall n \geq 0. \tag{3.11}$$

By (3.5)–(3.10), using the Fubini theorem, we calculate, for $\Re\gamma + \lambda + \alpha > 0$ and $\Re\gamma + \beta > 0$,

$$\begin{aligned} b_{n+1} = Q_{n+1}(0) &= \lambda \int_0^\infty p_{n+1}(x)\,dx + \alpha \int_0^\infty p_{n+2}(x)\,dx \\ &= \lambda \int_0^\infty a_{n+1} e^{-(\gamma+\lambda+\alpha)x}\,dx + \alpha \int_0^\infty a_{n+2} e^{-(\gamma+\lambda+\alpha)x}\,dx \\ &= \frac{\lambda}{\gamma+\lambda+\alpha} a_{n+1} + \frac{\alpha}{\gamma+\lambda+\alpha} a_{n+2}, \quad \forall n \geq 1, \end{aligned} \tag{3.12}$$

$$\begin{aligned} a_{n+1} = p_{n+1}(0) &= \beta \int_0^\infty Q_{n+1}(x)\,dx \\ &= \beta \int_0^\infty \left[ b_{n+1} e^{-(\gamma+\lambda+\beta)x} + \lambda e^{-(\gamma+\lambda+\beta)x} \int_0^x e^{(\gamma+\lambda+\beta)\tau} Q_n(\tau)\,d\tau \right] dx \\ &= \frac{\beta}{\gamma+\lambda+\beta} b_{n+1} + \lambda\beta \int_0^\infty e^{(\gamma+\lambda+\beta)\tau} Q_n(\tau) \int_\tau^\infty e^{-(\gamma+\lambda+\beta)x}\,dx\,d\tau \\ &= \frac{\beta}{\gamma+\lambda+\beta} b_{n+1} + \frac{\lambda\beta}{\gamma+\lambda+\beta} \int_0^\infty Q_n(\tau)\,d\tau \\ &= \frac{\beta}{\gamma+\lambda+\beta} b_{n+1} + \frac{\lambda}{\gamma+\lambda+\beta} p_n(0) \\ &= \frac{\beta}{\gamma+\lambda+\beta} b_{n+1} + \frac{\lambda}{\gamma+\lambda+\beta} a_n, \quad \forall n \geq 1. \end{aligned} \tag{3.13}$$

By inserting (3.12) into (3.13) and rearranging we immediately get

$$a_{n+2} = \frac{(\gamma+\lambda+\alpha)(\gamma+\lambda+\beta) - \lambda\beta}{\alpha\beta} a_{n+1} - \frac{\lambda(\gamma+\lambda+\alpha)}{\alpha\beta} a_n, \quad \forall n \geq 1. \tag{3.14}$$

If we set

$$\nu + h = \frac{(\gamma+\lambda+\alpha)(\gamma+\lambda+\beta)-\lambda\beta}{\alpha\beta}, \qquad \nu h = \frac{\lambda(\gamma+\lambda+\alpha)}{\alpha\beta}, \tag{3.15}$$

then it is easy to calculate that

$$\begin{aligned} \nu, h = \frac{1}{2\alpha\beta}\Big\{&\big[(\gamma+\lambda+\alpha)(\gamma+\lambda+\beta)-\lambda\beta\big] \\ &\pm\sqrt{\big[(\gamma+\lambda+\alpha)(\gamma+\lambda+\beta)-\lambda\beta\big]^2 - 4\alpha\beta\lambda(\gamma+\lambda+\alpha)}\Big\}. \end{aligned} \tag{3.16}$$

By comparing (3.14) with (3.15) we find

$$\begin{aligned} &a_{n+2} = (\nu+h)a_{n+1} - \nu h a_n \\ &\quad \Rightarrow \\ &a_{n+2} - \nu a_{n+1} = h(a_{n+1} - \nu a_n) = h^2(a_n - \nu a_{n-1}) = h^3(a_{n-1} - \nu a_{n-2}) \\ &\qquad\qquad = \cdots = h^n(a_2 - \nu a_1), \quad n \geq 1 \\ &\qquad\qquad \Rightarrow \\ &a_{n+2} - \nu a_{n+1} = h^n(a_2 - \nu a_1), \\ &\nu a_{n+1} - \nu^2 a_n = \nu h^{n-1}(a_2 - \nu a_1), \\ &\nu^2 a_n - \nu^3 a_{n-1} = \nu^2 h^{n-2}(a_2 - \nu a_1), \\ &\qquad\qquad \cdots \\ &\nu^{n-3}a_5 - \nu^{n-2}a_4 = \nu^{n-3}h^3(a_2 - \nu a_1), \\ &\nu^{n-2}a_4 - \nu^{n-1}a_3 = \nu^{n-2}h^2(a_2 - \nu a_1), \\ &\nu^{n-1}a_3 - \nu^n a_2 = \nu^{n-1}h(a_2 - \nu a_1). \end{aligned}$$

By adding up all these equations together we have

$$\begin{aligned} a_{n+2} - \nu^n a_2 &= \big(h^n + \nu h^{n-1} + \nu^2 h^{n-2} + \cdots + \nu^{n-3}h^3 + \nu^{n-2}h^2 + \nu^{n-1}h\big) \\ &\quad \times (a_2 - \nu a_1) \\ &= \begin{cases} n\nu^n a_2 - n\nu^{n+1}a_1 & \text{if } \nu = h, \\ (\frac{\nu^{n+1}-h^{n+1}}{\nu-h} - \nu^n)(a_2 - \nu a_1) & \text{if } \nu \neq h. \end{cases} \end{aligned}$$

If $\nu = h$, then

$$a_{n+2} - \nu^n a_2 = n\nu^n a_2 - n\nu^{n+1}a_1 \quad \Rightarrow \quad a_{n+2} = (n+1)\nu^n a_2 - n\nu^{n+1}a_1, \quad \forall n \geq 1. \tag{3.17}$$

This and (3.12) give

$$b_{n+2} = \frac{\lambda}{\gamma+\lambda+\alpha}a_{n+2} + \frac{\alpha}{\gamma+\lambda+\alpha}a_{n+3}$$

$$\begin{aligned}
&= \frac{\lambda}{\gamma+\lambda+\alpha}\big[(n+1)\nu^n a_2 - n\nu^{n+1}a_1\big] \\
&\quad + \frac{\alpha}{\gamma+\lambda+\alpha}\big[(n+2)\nu^{n+1}a_2 - (n+1)\nu^{n+2}a_1\big].
\end{aligned} \tag{3.18}$$

From this, together with (3.8), (3.11), (3.17), (2.46), and the Cauchy product, we estimate for $\Re\gamma+\lambda+\alpha>0$, $\Re\gamma+\beta>0$:

$$\begin{aligned}
\|p\| &= |p_0| + \sum_{n=1}^{\infty}\|p_n\|_{L^1[0,\infty)} \\
&= |p_0| + \sum_{n=1}^{\infty}\int_0^{\infty}\big|p_n(x)\big|\,dx \\
&= |p_0| + \sum_{n=1}^{\infty}\int_0^{\infty}\big|a_n e^{-(\gamma+\lambda+\alpha)x}\big|\,dx = |p_0| + \frac{1}{\Re\gamma+\lambda+\alpha}\sum_{n=1}^{\infty}|a_n| \\
&= |p_0| + \frac{1}{\Re\gamma+\lambda+\alpha}\left(|a_1|+|a_2|+\sum_{n=1}^{\infty}|a_{n+2}|\right) \\
&= |p_0| + \frac{1}{\Re\gamma+\lambda+\alpha}\left(|a_1|+|a_2|+\sum_{n=1}^{\infty}\big|(n+1)\nu^n a_2 - n\nu^{n+1}a_1\big|\right) \\
&\le |p_0| + \frac{1}{\Re\gamma+\lambda+\alpha}\left(|a_1|+|a_2|+\sum_{n=1}^{\infty}(n+1)|\nu|^n|a_2| + \sum_{n=1}^{\infty}n|\nu|^{n+1}|a_1|\right) \\
&= |p_0| + \frac{|a_1|+|a_2|}{\Re\gamma+\lambda+\alpha} + \frac{1}{\Re\gamma+\lambda+\alpha}|a_2|\sum_{n=1}^{\infty}(n+1)|\nu|^n \\
&\quad + \frac{1}{\Re\gamma+\lambda+\alpha}|a_1|\sum_{n=1}^{\infty}n|\nu|^{n+1},
\end{aligned} \tag{3.19}$$

$$\begin{aligned}
\|Q_n\|_{L^1[0,\infty)} &= \int_0^{\infty}\big|Q_n(x)\big|\,dx = \int_0^{\infty}\left|e^{-(\gamma+\lambda+\beta)x}\sum_{j=0}^{n}\frac{(\lambda x)^j}{j!}b_{n-j}\right|dx \\
&\le \sum_{j=0}^{n}\frac{\lambda^j}{j!}|b_{n-j}|\int_0^{\infty}x^j e^{-(\Re\gamma+\lambda+\beta)x}\,dx \\
&= \sum_{j=0}^{n}\frac{\lambda^j}{j!}|b_{n-j}|\frac{j!}{(\Re\gamma+\lambda+\beta)^{j+1}} = \sum_{j=0}^{n}\frac{\lambda^j}{(\Re\gamma+\lambda+\beta)^{j+1}}|b_{n-j}| \\
&\Rightarrow
\end{aligned}$$

$$\begin{aligned}
\|Q\| &= \sum_{n=0}^{\infty}\|Q_n\|_{L^1[0,\infty)} \le \sum_{n=0}^{\infty}\sum_{j=0}^{n}\frac{\lambda^j}{(\Re\gamma+\lambda+\beta)^{j+1}}|b_{n-j}| \\
&= \sum_{n=0}^{\infty}|b_n|\sum_{j=0}^{\infty}\frac{\lambda^j}{(\Re\gamma+\lambda+\beta)^{j+1}} = \frac{1}{\Re\gamma+\beta}\sum_{n=0}^{\infty}|b_n| \\
&= \frac{1}{\Re\gamma+\beta}\left(|b_0|+|b_1|+|b_2|+\sum_{n=1}^{\infty}|b_{n+2}|\right)
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{|b_0|+|b_1|+|b_2|}{\Re\gamma+\beta} + \frac{1}{\Re\gamma+\beta}\sum_{n=1}^{\infty}\left|\frac{\lambda}{\gamma+\lambda+\alpha}\left[(n+1)\nu^n a_2 - n\nu^{n+1}a_1\right]\right| \\
&\quad + \frac{1}{\Re\gamma+\beta}\sum_{n=1}^{\infty}\left|\frac{\alpha}{\gamma+\lambda+\alpha}\left[(n+2)\nu^{n+1}a_2 - (n+1)\nu^{n+2}a_1\right]\right| \\
&\leq \frac{|b_0|+|b_1|+|b_2|}{\Re\gamma+\beta} + \frac{\lambda}{\Re\gamma+\beta}\left|\frac{a_2}{\gamma+\lambda+\alpha}\right|\sum_{n=1}^{\infty}(n+1)|\nu|^n \\
&\quad + \frac{\lambda}{\Re\gamma+\beta}\left|\frac{a_1}{\gamma+\lambda+\alpha}\right|\sum_{n=1}^{\infty}n|\nu|^{n+1} + \frac{\alpha}{\Re\gamma+\beta}\left|\frac{a_2}{\gamma+\lambda+\alpha}\right|\sum_{n=1}^{\infty}(n+2)|\nu|^{n+1} \\
&\quad + \frac{\alpha}{\Re\gamma+\beta}\left|\frac{a_1}{\gamma+\lambda+\alpha}\right|\sum_{n=1}^{\infty}(n+1)|\nu|^{n+2}. \qquad (3.20)
\end{aligned}$$

If $\nu \neq h$, then

$$\begin{aligned}
&a_{n+2} - \nu^n a_2 = \left(\frac{\nu^{n+1}-h^{n+1}}{\nu-h} - \nu^n\right)(a_2 - \nu a_1) \\
&\qquad \Rightarrow \\
&a_{n+2} = \nu^{n+1}a_1 + \frac{\nu^{n+1}-h^{n+1}}{\nu-h}(a_2-\nu a_1), \quad \forall n \geq 1. \qquad (3.21)
\end{aligned}$$

From this and (3.12) we determine

$$\begin{aligned}
b_{n+2} &= \frac{\lambda}{\gamma+\lambda+\alpha}a_{n+2} + \frac{\alpha}{\gamma+\lambda+\alpha}a_{n+3} \\
&= \frac{\lambda}{\gamma+\lambda+\alpha}\left(\nu^{n+1}a_1 + \frac{\nu^{n+1}-h^{n+1}}{\nu-h}(a_2-\nu a_1)\right) \\
&\quad + \frac{\alpha}{\gamma+\lambda+\alpha}\left(\nu^{n+2}a_1 + \frac{\nu^{n+2}-h^{n+2}}{\nu-h}(a_2-\nu a_1)\right), \quad \forall n \geq 1. \qquad (3.22)
\end{aligned}$$

From this, together with (3.8), (3.11), (3.21),(3.22), as in (3.19) and (3.20), we estimate for $\Re\gamma+\lambda+\alpha > 0$ and $\Re\gamma+\beta > 0$:

$$\begin{aligned}
\|p\| &\leq |p_0| + \frac{|a_1|+|a_2|}{\Re\gamma+\lambda+\alpha} + \frac{1}{\Re\gamma+\lambda+\alpha} \\
&\quad \times \left[|a_1|\sum_{n=1}^{\infty}|\nu|^{n+1} + \left|\frac{a_2-\nu a_1}{\nu-h}\right|\left(\sum_{n=1}^{\infty}|\nu|^{n+1} + \sum_{n=1}^{\infty}|h|^{n+1}\right)\right], \qquad (3.23) \\
\|Q\| &\leq \frac{|b_0|+|b_1|+|b_2|}{\Re\gamma+\beta} + \frac{\lambda}{\Re\gamma+\beta}\left|\frac{a_1}{\gamma+\lambda+\alpha}\right|\sum_{n=1}^{\infty}|\nu|^{n+1} \\
&\quad + \frac{\lambda}{\Re\gamma+\beta}\left|\frac{a_2-\nu a_1}{(\gamma+\lambda+\alpha)(\nu-h)}\right|\left(\sum_{n=1}^{\infty}|\nu|^{n+1} + \sum_{n=1}^{\infty}|h|^{n+1}\right) \\
&\quad + \frac{\alpha}{\Re\gamma+\beta}\left|\frac{a_1}{\gamma+\lambda+\alpha}\right|\sum_{n=1}^{\infty}|\nu|^{n+2} \\
&\quad + \frac{\alpha}{\Re\gamma+\beta}\left|\frac{a_2-\nu a_1}{(\gamma+\lambda+\alpha)(\nu-h)}\right|\left(\sum_{n=1}^{\infty}|\nu|^{n+2} + \sum_{n=1}^{\infty}|h|^{n+2}\right). \qquad (3.24)
\end{aligned}$$

From (3.1), (3.5)–(3.8), and (3.11) by direct calculation we determine

$$b_0 = \frac{(\gamma+\lambda+\beta)(\gamma+\lambda)}{\beta} p_0, \tag{3.25}$$

$$a_1 = \frac{(\gamma+\lambda+\alpha)[(\gamma+\lambda)^2+\gamma\beta]}{\alpha\beta} p_0, \tag{3.26}$$

$$b_1 = \frac{(\gamma+\lambda+\alpha)(\gamma+\lambda+\beta)[(\gamma+\lambda)^2+\gamma\beta]-\lambda\alpha\beta(\gamma+\lambda)}{\alpha\beta^2} p_0, \tag{3.27}$$

$$\begin{aligned} a_2 &= \frac{\gamma+\lambda+\alpha}{(\alpha\beta)^2} \\ &\quad \times \bigl\{\bigl[(\gamma+\lambda)^2+\gamma\beta\bigr]\bigl[(\gamma+\lambda+\alpha)(\gamma+\lambda+\beta)-\lambda\beta\bigr]-\lambda\alpha\beta(\gamma+\lambda)\bigr\}p_0, \end{aligned} \tag{3.28}$$

$$\begin{aligned} b_2 &= \frac{(\gamma+\lambda+\alpha)(\gamma+\lambda+\beta)}{\alpha^2\beta^3} \\ &\quad \times \bigl\{\bigl[(\gamma+\lambda)^2+\lambda\beta\bigr]\bigl[(\gamma+\lambda+\alpha)(\gamma+\lambda+\beta)-\lambda\beta\bigr]-\lambda\alpha\beta(\gamma+\lambda)\bigr\}p_0 \\ &\quad + \frac{\lambda}{\gamma+\lambda+\beta}\bigl\{(\gamma+\lambda+\alpha)(\gamma+\lambda+\beta)\bigl[(\gamma+\lambda)^2+\gamma\beta\bigr]-\lambda\alpha\beta(\gamma+\lambda)\bigr\}p_0 \\ &\quad + \frac{\lambda^2(\gamma+\lambda)}{\gamma+\lambda+\beta} p_0. \end{aligned} \tag{3.29}$$

For simplicity, we introduce the notation

$$\Lambda := \left\{ \gamma \in \mathbb{C} \,\middle|\, \begin{array}{l} \Re\gamma+\lambda+\alpha>0, \Re\gamma+\beta>0, \\ |[(\gamma+\lambda+\alpha)(\gamma+\lambda+\beta)-\lambda\beta] \\ \quad \pm\sqrt{[(\gamma+\lambda+\alpha)(\gamma+\lambda+\beta)-\lambda\beta]^2-4\alpha\beta\lambda(\gamma+\lambda+\alpha)}| < 2\alpha\beta \end{array} \right\}.$$

From (3.16) it is easy to see that

$$\gamma \in \Lambda \quad \Leftrightarrow \quad \Re\gamma+\lambda+\alpha>0, \qquad \Re\gamma+\beta>0, \qquad |\nu|<1, \qquad |h|<1.$$

From this, together with (3.19), (3.20), and (3.23)–(3.29), we have, for $\gamma \in \Lambda$,

$$\bigl\|(p,Q)\bigr\| = \|p\| + \|Q\| < \infty,$$

which shows that all $\gamma \in \Lambda$ are eigenvalues of $A$. Moreover, from (3.17), (3.18), (3.21), (3.22), (3.25)∼(3.29), (3.8), and (3.11) we know that their geometric multiplicity is one.

In the following, we discuss the case $\gamma \in \mathbb{R}$. Since Theorem 1.1 implies that all $\gamma \in (0,\infty)$ belong to the resolvent set of $A$, the spectral set of $A$ belongs to $(-\infty,0]$. This includes the following three cases:

(1) $[(\gamma+\lambda+\alpha)(\gamma+\lambda+\beta)-\lambda\beta]^2-4\alpha\beta\lambda(\gamma+\lambda+\alpha)>0 \Leftrightarrow |(\gamma+\lambda+\alpha)(\gamma+\lambda+\beta)-\lambda\beta| > \sqrt{4\alpha\beta\lambda(\gamma+\lambda+\alpha)}$, which implies, for $\gamma+\lambda+\alpha>0$,

$$\begin{aligned} &\left(\gamma - \frac{-(2\lambda+\alpha+\beta)+\sqrt{(\alpha-\beta)^2+4\lambda\beta}}{2}\right)\left(\gamma - \frac{-(2\lambda+\alpha+\beta)-\sqrt{(\alpha-\beta)^2+4\lambda\beta}}{2}\right) \\ &\quad = (\gamma+\lambda+\alpha)(\gamma+\lambda+\beta)-\lambda\beta > \sqrt{4\alpha\beta\lambda(\gamma+\lambda+\alpha)} > 0 \end{aligned}$$

$$\Rightarrow \quad \gamma > \frac{-(2\lambda+\alpha+\beta)+\sqrt{(\alpha-\beta)^2+4\lambda\beta}}{2} \quad \text{or}$$
$$\gamma < \frac{-(2\lambda+\alpha+\beta)-\sqrt{(\alpha-\beta)^2+4\lambda\beta}}{2}.$$

If $\alpha \geq \beta$, then $4\lambda\beta > 0 \Rightarrow \sqrt{(\alpha-\beta)^2+4\lambda\beta} > \alpha-\beta \Rightarrow \beta+\sqrt{(\alpha-\beta)^2+4\lambda\beta} > \alpha$. If $\alpha < \beta$, then $\beta+\sqrt{(\alpha-\beta)^2+4\lambda\beta} > \alpha$. So, for all $\alpha > 0$, $\beta > 0$, $\lambda > 0$

$$\begin{aligned}
&\beta+\sqrt{(\alpha-\beta)^2+4\lambda\beta} > \alpha\\
&\quad\Leftrightarrow\quad -(2\lambda+\alpha+\beta)-\sqrt{(\alpha-\beta)^2+4\lambda\beta} < -2(\lambda+\alpha)\\
&\quad\Leftrightarrow\quad \frac{-(2\lambda+\alpha+\beta)-\sqrt{(\alpha-\beta)^2+4\lambda\beta}}{2} < -(\lambda+\alpha).
\end{aligned}$$

This, together with

$$\gamma < \frac{-(2\lambda+\alpha+\beta)-\sqrt{(\alpha-\beta)^2+4\lambda\beta}}{2},$$

imply that $\gamma < -(\lambda+\alpha)$, which contradicts to the condition $\gamma+\lambda+\alpha > 0$. Hence,

$$\gamma > \frac{-(2\lambda+\alpha+\beta)+\sqrt{(\alpha-\beta)^2+4\lambda\beta}}{2}. \tag{3.30}$$

It is easy to see that

$$\begin{aligned}
&\alpha\beta+\lambda(\lambda+\alpha) > 0\\
&\quad\Leftrightarrow\quad 4\alpha\beta+4\lambda^2+4\lambda\alpha+4\lambda\beta > 4\lambda\beta\\
&\quad\Leftrightarrow\quad (\alpha+\beta)^2+4\lambda^2+4\lambda(\alpha+\beta) > (\alpha-\beta)^2+4\lambda\beta\\
&\quad\Leftrightarrow\quad (2\lambda+\alpha+\beta)^2 > (\alpha-\beta)^2+4\lambda\beta\\
&\quad\Leftrightarrow\quad 2\lambda+\alpha+\beta > \sqrt{(\alpha-\beta)^2+4\lambda\beta}\\
&\quad\Leftrightarrow\quad -(2\lambda+\alpha+\beta)+\sqrt{(\alpha-\beta)^2+4\lambda\beta} < 0.
\end{aligned} \tag{3.31}$$

If $\beta \geq \alpha$, then

$$\begin{aligned}
&\beta \geq \alpha\\
&\quad\Rightarrow\quad \beta-\alpha < \sqrt{(\alpha-\beta)^2+4\lambda\beta}\\
&\quad\Rightarrow\quad -2(\lambda+\alpha) < -(2\lambda+\alpha+\beta)+\sqrt{(\alpha-\beta)^2+4\lambda\beta}\\
&\quad\Rightarrow\quad -(\lambda+\alpha) < \frac{-(2\lambda+\alpha+\beta)+\sqrt{(\alpha-\beta)^2+4\lambda\beta}}{2}.
\end{aligned}$$

If $\beta < \alpha$, then

$$\begin{aligned}
&\beta < \alpha\\
&\quad\Rightarrow\quad \alpha(\alpha-\beta)+2\lambda\beta+\alpha\sqrt{(\alpha-\beta)^2+4\lambda\beta} > 0
\end{aligned}$$

$$
\begin{aligned}
&\Rightarrow \quad 2\alpha^2 - 2\alpha\beta + 4\lambda\beta + 2\alpha\sqrt{(\alpha-\beta)^2 + 4\lambda\beta} > 0 \\
&\Rightarrow \quad \beta^2 < \alpha^2 + (\alpha-\beta)^2 + 4\lambda\beta + 2\alpha\sqrt{(\alpha-\beta)^2 + 4\lambda\beta} = \Bigl[\alpha + \sqrt{(\alpha-\beta)^2 + 4\lambda\beta}\Bigr]^2 \\
&\Rightarrow \quad \beta < \alpha + \sqrt{(\alpha-\beta)^2 + 4\lambda\beta} \\
&\Rightarrow \quad -(\lambda+\alpha) < \frac{-(2\lambda+\alpha+\beta) + \sqrt{(\alpha-\beta)^2 + 4\lambda\beta}}{2}.
\end{aligned}
$$

The above two inequalities give, for all $\beta > 0$, $\alpha > 0$, $\lambda > 0$,

$$
-(\lambda+\alpha) < \frac{-(2\lambda+\alpha+\beta) + \sqrt{(\alpha-\beta)^2 + 4\lambda\beta}}{2}. \tag{3.32}
$$

The conditions $\frac{\lambda(\lambda+\alpha)}{\alpha\beta} < 1$ and $\beta > 0$, $\alpha > 0$, $\lambda > 0$ imply

$$
\begin{aligned}
&\frac{\lambda(\lambda+\alpha)}{\alpha\beta} < 1 \\
&\quad \Rightarrow \quad \frac{\lambda^2}{\alpha\beta} + \frac{\lambda}{\beta} < 1 \\
&\quad \Rightarrow \quad \lambda < \beta \\
&\quad \Rightarrow \quad 4\lambda(\lambda+\alpha) < 4\beta(\beta+\alpha) < 2\beta(\beta+\alpha) + 2\beta\sqrt{(\alpha+\beta)^2 + 4\lambda\beta} + 4\lambda\beta \\
&\quad \Rightarrow \quad (2\lambda+\alpha)^2 = 4\lambda(\lambda+\alpha) + \alpha^2 < \alpha^2 + 2\beta(\beta+\alpha) + 2\beta\sqrt{(\alpha+\beta)^2 + 4\lambda\beta} + 4\lambda\beta \\
&\qquad\qquad = \beta^2 + (\alpha+\beta)^2 + 4\lambda\beta + 2\beta\sqrt{(\alpha+\beta)^2 + 4\lambda\beta} \\
&\qquad\qquad = \Bigl[\beta + \sqrt{(\alpha+\beta)^2 + 4\lambda\beta}\Bigr]^2 \\
&\quad \Rightarrow \quad 2\lambda + \alpha < \beta + \sqrt{(\alpha+\beta)^2 + 4\lambda\beta} \\
&\quad \Rightarrow \quad -\beta < \frac{-(2\lambda+\alpha+\beta) + \sqrt{(\alpha+\beta)^2 + 4\lambda\beta}}{2}.
\end{aligned} \tag{3.33}
$$

In addition, for $\gamma < 0$, $\gamma + \lambda + \alpha > 0$, we have

$$
\frac{\lambda(\lambda+\alpha)}{\alpha\beta} < 1 \quad \Rightarrow \quad \alpha\beta > \lambda(\lambda+\alpha) > \lambda(\gamma+\lambda+\alpha) > (\gamma+\lambda)(\gamma+\lambda+\alpha).
$$

By this we verify that, for $\gamma + \lambda + \alpha > 0$, $\gamma + \beta > 0$, and $\gamma < 0$,

$$
\begin{aligned}
0 &< \bigl[(\gamma+\lambda+\alpha)(\gamma+\lambda+\beta) - \lambda\beta\bigr] \\
&\quad - \sqrt{\bigl[(\gamma+\lambda+\alpha)(\gamma+\lambda+\beta) - \lambda\beta\bigr]^2 - 4\alpha\beta\lambda(\gamma+\lambda+\alpha)} \\
&< \bigl[(\gamma+\lambda+\alpha)(\gamma+\lambda+\beta) - \lambda\beta\bigr] \\
&\quad + \sqrt{\bigl[(\gamma+\lambda+\alpha)(\gamma+\lambda+\beta) - \lambda\beta\bigr]^2 - 4\alpha\beta\lambda(\gamma+\lambda+\alpha)} \\
&< \bigl[(\gamma+\lambda+\alpha)(\gamma+\lambda+\beta) - (\gamma+\lambda)\beta\bigr] \\
&\quad + \sqrt{\bigl[(\gamma+\lambda+\alpha)(\gamma+\lambda+\beta) - (\gamma+\lambda)\beta\bigr]^2 - 4\alpha\beta(\gamma+\lambda)(\gamma+\lambda+\alpha)} \\
&= \bigl[(\gamma+\lambda+\alpha)(\gamma+\lambda) + \alpha\beta\bigr]
\end{aligned}
$$

$$
\begin{aligned}
&+\sqrt{\big[(\gamma+\lambda+\alpha)(\gamma+\lambda)+\alpha\beta\big]^2-4\alpha\beta(\gamma+\lambda)(\gamma+\lambda+\alpha)}\\
&=\big[(\gamma+\lambda+\alpha)(\gamma+\lambda)+\alpha\beta\big]+\sqrt{\big[(\gamma+\lambda+\alpha)(\gamma+\lambda)-\alpha\beta\big]^2}\\
&=\big[(\gamma+\lambda+\alpha)(\gamma+\lambda)+\alpha\beta\big]-(\gamma+\lambda+\alpha)(\gamma+\lambda)+\alpha\beta=2\alpha\beta\\
&\Rightarrow\quad 0<|\nu|<1,\qquad 0<|h|<1\quad\Rightarrow\quad \gamma\in\Lambda.
\end{aligned}
\tag{3.34}
$$

Relations (3.30), (3.31), (3.32), (3.33), (3.34), and

$$\frac{-(2\lambda+\alpha+\beta)+\sqrt{(\alpha-\beta)^2+4\lambda\beta}}{2}<\frac{-(2\lambda+\alpha+\beta)+\sqrt{(\alpha+\beta)^2+4\lambda\beta}}{2}<0$$

imply that all points in $(\frac{-(2\lambda+\alpha+\beta)+\sqrt{(\alpha+\beta)^2+4\lambda\beta}}{2},0)$ are eigenvalues of $A$, which includes the main result in Lv and Gupur [13] that $\frac{-(2\lambda+\alpha+\beta)+\sqrt{(\alpha+\beta)^2+4\lambda\beta}}{4}$ is an eigenvalue of $A$ for $\frac{\lambda(\lambda+\alpha)}{\alpha\beta}<\frac{1}{4}$, the main result in Ismayil and Gupur [14] that all points in $(\frac{-(2\lambda+\alpha+\beta)+\sqrt{(\alpha+\beta)^2+4\lambda\beta}}{4},0)$ are eigenvalues of $A$ for $\frac{\lambda(\lambda+\alpha)}{\alpha\beta}<\frac{1}{4}$, and the main result in Gupur [15] that all points in $(\frac{-(2\lambda+\alpha+\beta)+\sqrt{(\alpha+\beta)^2+4\lambda\beta}}{4},0)$ are eigenvalues of $A$ for $\frac{\lambda(\lambda+\alpha)}{\alpha\beta}=\frac{1}{4}$.

In the case $\gamma=0$, it is just the result of Lemma 2.1, so we omit the proof.

Although it is hard to determine explicitly the roots of the equation $[(\gamma+\lambda+\alpha)(\gamma+\lambda+\beta)-\lambda\beta]^2-4\alpha\beta\lambda(\gamma+\lambda+\alpha)=0$, without loss of generality, we assume that all negative real roots of the equation are $\omega_i$ ($i=1,2,3,4$) satisfying $\omega_1\le\omega_2\le\omega_3\le\omega_4$ (in fact, we discovered numerically that the equation has two real roots). Then from our discussion we know that all points in the interval $(\max\{-(\lambda+\alpha),-\beta,\omega_4\},0]$ are eigenvalues of $A$ with geometric multiplicity one.

(2) $[(\gamma+\lambda+\alpha)(\gamma+\lambda+\beta)-\lambda\beta]^2-4\alpha\beta\lambda(\gamma+\lambda+\alpha)=0$. From this, together with $\frac{\lambda(\lambda+\alpha)}{\alpha\beta}<1$, $\gamma+\lambda+\alpha>0$, $\gamma+\beta>0$, and $\gamma<0$, we have

$$
\begin{aligned}
&\big[(\gamma+\lambda+\alpha)(\gamma+\lambda+\beta)-\lambda\beta\big]^2-4\alpha\beta\lambda(\gamma+\lambda+\alpha)=0\\
&\quad\Rightarrow\quad \big|(\gamma+\lambda+\alpha)(\gamma+\lambda+\beta)-\lambda\beta\big|=2\sqrt{\alpha\beta\lambda(\gamma+\lambda+\alpha)}\\
&\quad\Rightarrow\quad 0<|\nu|=|h|=\left|\frac{1}{2\alpha\beta}\big[(\gamma+\lambda+\alpha)(\gamma+\lambda+\beta)-\lambda\beta\big]\right|\\
&\qquad\qquad=\frac{1}{2\alpha\beta}\cdot 2\sqrt{\alpha\beta\lambda(\gamma+\lambda+\alpha)}\\
&\qquad\qquad=\frac{1}{\alpha\beta}\sqrt{\alpha\beta\lambda(\gamma+\lambda+\alpha)}\\
&\qquad\qquad<\frac{1}{\alpha\beta}\sqrt{\alpha\beta\lambda(\lambda+\alpha)}\\
&\qquad\qquad<\frac{1}{\alpha\beta}\sqrt{(\alpha\beta)^2}=1\quad\Rightarrow\quad\gamma\in\Lambda.
\end{aligned}
\tag{3.35}
$$

Without loss of generality, we assume that all negative real roots of the equation are $\omega_1\le\omega_2\le\omega_3\le\omega_4$. Then (3.35) implies that $\max\{-(\lambda+\alpha),-\beta,\omega_4\}$ is an eigenvalue of $A$ with geometric multiplicity one.

(3) $[(\gamma+\lambda+\alpha)(\gamma+\lambda+\beta)-\lambda\beta]^2-4\alpha\beta\lambda(\gamma+\lambda+\alpha)<0$, from which, together with $\frac{\lambda(\lambda+\alpha)}{\alpha\beta}<1$, $\gamma+\lambda+\alpha>0$, $\gamma+\beta>0$, and $\gamma<0$, we deduce

$$\begin{aligned}
0 &< |\nu| \\
&= \Biggl|\frac{1}{2\alpha\beta}\Bigl\{\bigl[(\gamma+\lambda+\alpha)(\gamma+\lambda+\beta)-\lambda\beta\bigr] \\
&\quad - \sqrt{\bigl[(\gamma+\lambda+\alpha)(\gamma+\lambda+\beta)-\lambda\beta\bigr]^2-4\alpha\beta\lambda(\gamma+\lambda+\alpha)}\Bigr\}\Biggr| \\
&= \Biggl|\frac{1}{2\alpha\beta}\Bigl\{\bigl[(\gamma+\lambda+\alpha)(\gamma+\lambda+\beta)-\lambda\beta\bigr] \\
&\quad - i\sqrt{4\alpha\beta\lambda(\gamma+\lambda+\alpha)-\bigl[(\gamma+\lambda+\alpha)(\gamma+\lambda+\beta)-\lambda\beta\bigr]^2}\Bigr\}\Biggr| \\
&= \frac{1}{2\alpha\beta}\sqrt{4\alpha\beta\lambda(\gamma+\lambda+\alpha)} \\
&< \frac{1}{2\alpha\beta}\sqrt{4\alpha\beta\lambda(\lambda+\alpha)} \\
&< \frac{1}{2\alpha\beta}\sqrt{4(\alpha\beta)^2}=1,
\end{aligned} \tag{3.36}$$

$$\begin{aligned}
0 < |h| &= \Biggl|\frac{1}{2\alpha\beta}\Bigl\{\bigl[(\gamma+\lambda+\alpha)(\gamma+\lambda+\beta)-\lambda\beta\bigr] \\
&\quad + \sqrt{\bigl[(\gamma+\lambda+\alpha)(\gamma+\lambda+\beta)-\lambda\beta\bigr]^2-4\alpha\beta\lambda(\gamma+\lambda+\alpha)}\Bigr\}\Biggr| \\
&= \Biggl|\frac{1}{2\alpha\beta}\Bigl\{\bigl[(\gamma+\lambda+\alpha)(\gamma+\lambda+\beta)-\lambda\beta\bigr] \\
&\quad + i\sqrt{4\alpha\beta\lambda(\gamma+\lambda+\alpha)-\bigl[(\gamma+\lambda+\alpha)(\gamma+\lambda+\beta)-\lambda\beta\bigr]^2}\Bigr\}\Biggr| \\
&= \frac{1}{2\alpha\beta}\sqrt{4\alpha\beta\lambda(\gamma+\lambda+\alpha)} \\
&< \frac{1}{2\alpha\beta}\sqrt{4\alpha\beta\lambda(\lambda+\alpha)} \\
&< \frac{1}{2\alpha\beta}\sqrt{4(\alpha\beta)^2}=1.
\end{aligned} \tag{3.37}$$

Relations (3.36) and (3.37) imply that $\gamma\in\varLambda$. Without loss of generality, we assume that all negative real roots of the equation are $\omega_i$ $(i=1,2,3,4)$ satisfying $\omega_1\le\omega_2\le\omega_3\le\omega_4$, then all points in $(\max\{-(\lambda+\alpha),-\beta,\omega_3\},\max\{-(\lambda+\alpha),-\beta,\omega_4\})$ are eigenvalues of $A$ with geometric multiplicity one.

By summarizing our discussion we conclude that all points in

$$\left\{\gamma\in\mathbb{C}\;\middle|\;\begin{array}{l}\Re\gamma+\lambda+\alpha>0,\Re\gamma+\beta>0,\\ |[(\gamma+\lambda+\alpha)(\gamma+\lambda+\beta)-\lambda\beta]\\ \quad\pm\sqrt{[(\gamma+\lambda+\alpha)(\gamma+\lambda+\beta)-\lambda\beta]^2-4\alpha\beta\lambda(\gamma+\lambda+\alpha)}|<2\alpha\beta\end{array}\right\}\cup\{0\}$$

are eigenvalues of $A$ with geometric multiplicity 1. In particular, all points in $(\max\{-(\lambda+\alpha), -\beta, \omega_3\}, 0]$ are eigenvalues of $A$ with geometric multiplicity one, and the interval

$$\left(\frac{-(2\lambda+\alpha+\beta)+\sqrt{(\alpha+\beta)^2+4\lambda\beta}}{2}, 0\right]$$

belongs to the point spectrum of $A$.

*Remark* 3.1 From (3.17), (3.18), (3.21), and (3.22) it is easy to see that if $|\nu| > 1$ and $|h| < 1$ or $|\nu| < 1$ and $|h| > 1$, then $\|(p, Q)\| = \|p\| + \|Q\| = \infty$, that is, there are no eigenvalues in

$$\left\{\gamma \in \mathbb{C} \;\middle|\; \begin{array}{l} |[(\gamma+\lambda+\alpha)(\gamma+\lambda+\beta)-\lambda\beta] \\ \quad + \sqrt{[(\gamma+\lambda+\alpha)(\gamma+\lambda+\beta)-\lambda\beta]^2-4\alpha\beta\lambda(\gamma+\lambda+\alpha)}| > 2\alpha\beta, \\ |[(\gamma+\lambda+\alpha)(\gamma+\lambda+\beta)-\lambda\beta] \\ \quad - \sqrt{[(\gamma+\lambda+\alpha)(\gamma+\lambda+\beta)-\lambda\beta]^2-4\alpha\beta\lambda(\gamma+\lambda+\alpha)}| < 2\alpha\beta \\ \text{or } |[(\gamma+\lambda+\alpha)(\gamma+\lambda+\beta)-\lambda\beta] \\ \quad + \sqrt{[(\gamma+\lambda+\alpha)(\gamma+\lambda+\beta)-\lambda\beta]^2-4\alpha\beta\lambda(\gamma+\lambda+\alpha)}| < 2\alpha\beta, \\ |[(\gamma+\lambda+\alpha)(\gamma+\lambda+\beta)-\lambda\beta] \\ \quad - \sqrt{[(\gamma+\lambda+\alpha)(\gamma+\lambda+\beta)-\lambda\beta]^2-4\alpha\beta\lambda(\gamma+\lambda+\alpha)}| > 2\alpha\beta \end{array}\right\}.$$

## 4 Conclusion and discussion

Let $\sigma_p(T(t))$ and $\sigma_p(A)$ be the point spectra of $T(t)$ and $A$, respectively. From Theorem 1.1, Theorem 3.1, and the spectral mapping theorem for the point spectrum ([23], p. 277)

$$\sigma_p\big(T(t)\big) = e^{t\sigma_p(A)} \cup \{0\}$$

we know that $T(t)$ has uncountably many eigenvalues, and therefore it is not compact and even not eventually compact ([23], p. 330).

Corollary 2.11 in Engel and Nagel [23], p. 258, states that if $T(t)$ is a $C_0$-semigroup on the Banach space $X$ with generator $A$, then

I. $\omega_0 = \max\{\omega_{\mathrm{ess}}, \tilde{s}(A)\}$, where $\omega_0$ is the growth bound of $T(t)$, $\omega_{\mathrm{ess}}$ is the essential growth bound of $T(t)$, and $\tilde{s}(A)$ is the spectral bound of $A$.

II. $\sigma(A) \cap \{\gamma \in \mathbb{C} \mid \operatorname{Re}\gamma \geq \omega\}$ is finite for each $\omega > \omega_{\mathrm{ess}}$. Here $\sigma(A)$ is the spectrum of $A$.

Theorem 1.1, Lemma 2.5, and Theorem 3.1 imply that $\omega_0 = 0$ and $\tilde{s}(A) = 0$. These, together with items I and II above, yield $\omega_{\mathrm{ess}} = 0$. From this and Proposition 3.5 in [23], p. 332, we conclude that $T(t)$ is not quasi-compact. Hence, queueing models are essentially different from the population equation [24] and the reliability models that are described by a finite number of partial differential equations with integral boundary conditions [12].

Let $(p, Q)^{(0)}(x)$ be an eigenvector with respect to 0 in Lemma 2.1, and let $(p, Q)^{(\varepsilon)}(x)$ be eigenvectors with respect to $\frac{-(2\lambda+\alpha+\beta)+\sqrt{(\alpha+\beta)^2+4\lambda\beta}}{2}\varepsilon$ for $\varepsilon \in (0, 1)$ in Theorem 3.1. Then, using $A(p, Q)^{(0)}(x) = 0$ and

$$A(p, Q)^{(\varepsilon)}(x) = \frac{-(2\lambda+\alpha+\beta)+\sqrt{(\alpha+\beta)^2+4\lambda\beta}}{2}\varepsilon(p, Q)^{(\varepsilon)}(x),$$

we have, for all $t \geq 0$ and $\varepsilon \in (0, 1)$,

$$T(t)\big[(p, Q)^{(0)}(x) + A(p, Q)^{(\varepsilon)}(x)\big]$$

$$\begin{aligned}
&= T(t)(p,Q)^{(0)}(x) + T(t)A(p,Q)^{(\varepsilon)}(x) \\
&= (p,Q)^{(0)}(x) + T(t)\left[\frac{-(2\lambda+\alpha+\beta)+\sqrt{(\alpha+\beta)^2+4\lambda\beta}}{2}\varepsilon(p,Q)^{(\varepsilon)}(x)\right] \\
&= (p,Q)^{(0)}(x) + \frac{-(2\lambda+\alpha+\beta)+\sqrt{(\alpha+\beta)^2+4\lambda\beta}}{2}\varepsilon T(t)(p,Q)^{(\varepsilon)}(x) \\
&= (p,Q)^{(0)}(x) \\
&\quad + \frac{-(2\lambda+\alpha+\beta)+\sqrt{(\alpha+\beta)^2+4\lambda\beta}}{2}\varepsilon e^{\frac{-(2\lambda+\alpha+\beta)+\sqrt{(\alpha+\beta)^2+4\lambda\beta}}{2}\varepsilon t}(p,Q)^{(\varepsilon)}(x) \\
&\Rightarrow \\
&\left\| T(t)\left[(p,Q)^{(0)}(\cdot) + A(p,Q)^{(\varepsilon)}(\cdot)\right] - (p,Q)^{(0)}(\cdot)\right\| \\
&= \left|\frac{-(2\lambda+\alpha+\beta)+\sqrt{(\alpha+\beta)^2+4\lambda\beta}}{2}\right|\varepsilon e^{\frac{-(2\lambda+\alpha+\beta)+\sqrt{(\alpha+\beta)^2+4\lambda\beta}}{2}\varepsilon t}\left\|(p,Q)^{(\varepsilon)}\right\|.
\end{aligned}$$

This means that there are no positive constants $\overline{M}_0 > 0$ and $\overline{\varpi} > 0$ such that

$$\begin{aligned}
&\left\| T(t)\left[(p,Q)^{(0)}(\cdot) + A(p,Q)(\cdot)\right] - (p,Q)^{(0)}(\cdot)\right\| \\
&\leq \overline{M}_0 e^{-\overline{\varpi}t}\left\|(p,Q)\right\|, \quad \forall t \geq 0, \forall (p,Q) \in D(A),
\end{aligned}$$

that is, it is impossible that the time-dependent solution of system (1.9) exponentially converges to its steady-state solution. In other words, the convergence result given in Theorem 2.1 is optimal.

Until now, we have not described the essential spectrum of $A$ for $r(x) = \alpha$ and $s(x) = \beta$. We have not found an efficient way to describe the spectrum of $A$ when $r(x)$ and $s(x)$ are nonconstant. All of them are our next research topic.

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**Authors' contributions**
NY completed the main study and drafted the paper. GG checked the proofs and verified the calculations. Both authors read and approved the final manuscript.

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