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Dynamics of a predator–prey system with three species

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Abstract

This paper is concerned with the dynamics of a predator–prey system with three species. When the domain is bounded, the global stability of positive steady state is established by contracting rectangles. When the domain is $\mathbb{R}$, we study the traveling wave solutions implying that one predator and one prey invade the habitat of another prey. More precisely, the existence of traveling wave solutions is proved by combining upper and lower solutions with a fixed point theorem, and the asymptotic behavior of traveling wave solutions is obtained by the idea of contracting rectangle. Moreover, we show the nonexistence of traveling wave solutions by applying the theory of the asymptotic spreading.

Keywords: Predator–prey systems; Upper–lower solutions; Contracting rectangles; Minimal wave speed

1 Introduction

In population dynamics, predator–prey systems have been widely studied due to their importance as well as plentiful dynamical behaviors. When the system involves only one predator and one prey, there have been many results for both ordinary differential systems and reaction–diffusion equations. For the topic, we refer to Cantrell and Cosner [4], Fife [13], Ghergu and Radulescu [17], Murray [35, 36], Pao [39], Smith [42], Smoller [43], Ye et al. [50], Zhang and Feng [52], Zhao [54].

When the spatiotemporal dynamics is concerned, since the pioneer work in [14, 25], much attention has been paid to the traveling wave solutions of parabolic equations; we refer to Volpert et al. [44] for some earlier results and a survey paper by Zhao [55] for some recent conclusions. If a system is of predator–prey type with two species, then several methods, including phase analysis, shooting methods, Conley index and fixed point theorem, have been applied to establish the existence of traveling wave solutions; we refer to some important results by Dunbar [9–11], Gardner and Smoller [16], Gardner and Jones [15], Chen et al. [7], Huang et al. [19], Huang and Zou [21], Huang [23], Hsu et al. [18], Li and Li [26], Lin et al. [30], Lin [28, 29], Lin et al. [32], Pan [37], Wang et al. [45], Wang et al. [47], Zhang et al. [51].

In fact, due to the diversity and complexity of ecosystems, the study of the interaction among multi-species has more practical significance. However, when there are more than 2 species, studying the ecosystem becomes more difficult. For example, it is difficult to study the existence of traveling wave solutions by phase analysis. Moreover, even for ordinary differential systems, the investigation of dynamical behavior becomes more difficult due to the deficiency of general Poincare–Bendixson theorem in $\mathbb{R}^n$ with $n \geq 3$. Of course, there are also some conclusions about the dynamics of predator–prey systems with multiple species. For example, Caristi et al. [6] investigated the coexistence of a predator and two species on bounded habitat with Neumann boundary condition, Du and Xu [8], Shang et al. [41], Zhang et al. [53] proved the existence of traveling wave solutions to a reaction–diffusion system with multiple species.

Recently, Huang and Lin [24] considered the following three species reaction-diffusion system (see Cantrell and Cosner [4]):

$$\begin{cases} \frac{\partial u_1(x,t)}{\partial t} = d_1 \Delta u_1(x,t) + u_1(x,t)[b_1 - u_1(x,t) - b_{12}u_2(x,t) - b_{13}u_3(x,t)], \\ \frac{\partial u_2(x,t)}{\partial t} = d_2 \Delta u_2(x,t) + u_2(x,t)[b_2 - b_{21}u_1(x,t) - u_2(x,t) - b_{23}u_3(x,t)], \\ \frac{\partial u_3(x,t)}{\partial t} = d_3 \Delta u_3(x,t) + u_3(x,t)[b_3 + b_{31}u_1(x,t) + b_{32}u_2(x,t) - u_3(x,t)], \end{cases} \tag{1.1}$$

where all parameters are positive, $u_1(x,t), u_2(x,t)$ denote the densities of two competing prey species located at $x \in \Omega \subseteq \mathbb{R}^n$ at time t, u_3 denotes the density of predator feeding on species 1 and species 2 located at $x \in \Omega \subseteq \mathbb{R}^n$ at time t. The authors obtained the minimal wave speed of non-negative traveling wave solutions connecting trivial equilibrium with positive equilibrium, in which the limit behavior is verified by the abstract results in Lin and Ruan [31].

Because all the parameters are positive in [24], this implies that the predator could survive without two preys in the system. In this paper, we suppose that all the parameters, except $b_3 < 0$, are positive and the predator only feeds on species 1 and 2. Under the conditions of (1.1), there are several different dynamical problems in the literature. For example, u_3 invades the habitat in which u_1, u_2 coexist, u_1, u_3 invade the habitat of u_2. In population dynamics, these problems often imply different thresholds in (1.1), and the corresponding control problem is also very important [40].

The purpose of this paper is to discuss the situation when prey u_2 and predator u_3 invade the habitat of prey u_1. We study the global stability of the positive equilibrium with the help of contracting rectangles [42] when the domain is bounded, then the existence as well as noexistence of traveling wave solutions when $x \in \mathbb{R}$. More precisely, we first establish the existence of a nontrivial traveling wave solution by the generalized upper and lower solution. To verify the limit behavior of traveling wave solutions, we use the results in global stability as well as the asymptotic spreading of Fisher equation [1]. Finally, the nonexistence of traveling wave solutions is obtained by constructing auxiliary equations as well as using the theory of asymptotic spreading.

It should be noted that Lin and Ruan [31] applied the idea of contracting rectangles to verify the limit behavior of a traveling wave solution. But in [31] they needed strictly contracting rectangles, a condition which is stronger than the general stability conditions in [42]. In this paper, using the basic idea in [31] and adding necessary discussion, we prove the limit behavior of traveling wave solutions by general contracting rectangles. We believe the technique can be applied to more models.

The rest of this paper is organized as follows. In Sect. 2, we will give some preliminaries. If the domain is bounded, the global stability of the positive equilibrium is proved in Sect. 3. In Sect. 4, by using Schauder's fixed point theorem and constructing upper and lower solutions, we obtain the existence of non-negative traveling wave solutions. Then the asymptotic behavior of traveling wave solutions is also studied in Sect. 5. In Sect. 6, the nonexistence of traveling wave solutions is established by using the relevant conclusions of the asymptotic speed of spreading.

2 Preliminaries

We first introduce some notations. For $u = (u_1, u_2, u_3), v = (v_1, v_2, v_3) \in \mathbb{R}^3$, we write $u \geq v$ provided $u_i \geq v_i$ for $i = 1, 2, 3$. Let X be the following functional space:

$$X = \{u : \mathbb{R} \to \mathbb{R}^3 \text{ is bounded and uniformly continuous}\},$$

which is a Banach space equipped with the standard supremum norm. If $a, b \in \mathbb{R}^3$ with $a \leq b$, then $X_{[a,b]}$ is defined by

$$X_{[a,b]} = \{u \in X : a \leq u(x) \leq b, x \in \mathbb{R}\}.$$

If $u(x) = (u_1(x), u_2(x), u_3(x))$ and $v(x) = (v_1(x), v_2(x), v_3(x)) \in X$, then $u(x) \geq v(x)$ implies that $u(x) \geq v(x)$ for all $x \in \mathbb{R}$; $u(x) > v(x)$ is interpreted as $u(x) \geq v(x)$ but $u(x) > v(x)$ for some $x \in \mathbb{R}$.

By rescaling, (1.1) is equivalent to the following system:

$$\begin{cases} \frac{\partial u_1(x,t)}{\partial t} = d_1 \Delta u_1(x,t) + r_1 u_1(x,t)[1 - u_1(x,t) - a_{12} u_2(x,t) - a_{13} u_3(x,t)], \\ \frac{\partial u_2(x,t)}{\partial t} = d_2 \Delta u_2(x,t) + r_2 u_2(x,t)[1 - a_{21} u_1(x,t) - u_2(x,t) - a_{23} u_3(x,t)], \\ \frac{\partial u_3(x,t)}{\partial t} = d_3 \Delta u_3(x,t) + r_3 u_3(x,t)[-1 + a_{31} u_1(x,t) + a_{32} u_2(x,t) - u_3(x,t)], \end{cases} \tag{2.1}$$

where all the parameters are positive. Clearly, the corresponding kinetic system is

$$\begin{cases} \frac{du_1(t)}{dt} = r_1 u_1(t)[1 - u_1(t) - a_{12} u_2(t) - a_{13} u_3(t)], \\ \frac{du_1(t)}{dt} = r_2 u_2(t)[1 - a_{21} u_1(t) - u_2(t) - a_{23} u_3(t)], \\ \frac{du_1(t)}{dt} = r_3 u_3(t)[-1 + a_{31} u_1(t) + a_{32} u_2(t) - u_3(t)]. \end{cases} \tag{2.2}$$

Assume that

$$\begin{cases} a_{12} + a_{13}(a_{31} + a_{32} - 1) < 1, \\ a_{21} + a_{23}(a_{31} + a_{32} - 1) < 1, \\ a_{32}[1 - a_{21} - a_{23}(a_{31} + a_{32} - 1)] + a_{31}[1 - a_{12} - a_{13}(a_{31} + a_{32} - 1)] > 1, \end{cases} \tag{2.3}$$

then (2.2) has an equilibrium point $E = (1, 0, 0)$ and a unique positive equilibrium point $K = (k_1, k_2, k_3)$ defined by

$$k_1 = \frac{a_{12}a_{23} + a_{13}a_{32} - a_{23}a_{32} + a_{12} - a_{13} - 1}{a_{12}a_{23}a_{31} + a_{13}a_{32}a_{21} + a_{12}a_{21} - a_{13}a_{31} - a_{23}a_{32} - 1},$$

$$k_2 = \frac{a_{21}a_{13} + a_{23}a_{31} - a_{31}a_{13} + a_{21} - a_{23} - 1}{a_{12}a_{23}a_{31} + a_{13}a_{32}a_{21} + a_{12}a_{21} - a_{13}a_{31} - a_{23}a_{32} - 1},$$

$$k_3 = \frac{a_{32}a_{21} + a_{31}a_{12} - a_{12}a_{21} - a_{31} - a_{32} + 1}{a_{12}a_{23}a_{31} + a_{13}a_{32}a_{21} + a_{12}a_{21} - a_{13}a_{31} - a_{23}a_{32} - 1}.$$

In fact, the existence and uniqueness of K are not evident if (2.3) holds. But this will be clear from the stability analysis in Sect. 3. More precisely, we could obtain the existence of equilibria by the limit behavior in Sect. 3; then for any positive equilibria, we shall prove the global stability which implies the uniqueness of the positive steady state. Furthermore, (2.2) also has several semi-trivial steady states, which do not affect our following discussion.

For convenience, we introduce the definition of contracting rectangle (see Smith [42]). Consider the following initial value problem:

$$\begin{cases} x'(t) = f(x(t)), \\ x(0) = \phi, \end{cases} \tag{2.4}$$

where $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))$, $f(x(t)) = (f_1(x(t)), f_2(x(t)), \ldots, f_n(x(t)))$, $\phi = (\phi_1, \phi_2, \ldots, \phi_n)$ and there exists $e \in \mathbb{R}^n$ such that $f_i(e) = 0$.

Denote a one-parameter family of order intervals $\Sigma(s) = [a(s), b(s)]$, $0 \le s \le 1$ such that for $0 \le s_1 \le s_2 \le 1$,

$$a(0) \le a(s_1) \le a(s_2) \le a(1) = e = b(1) \le b(s_2) \le b(s_1) \le b(0),$$

where

$$a(s) = \bigl(a_1(s), a_2(s), \ldots, a_n(s)\bigr), \qquad b(s) = \bigl(b_1(s), b_2(s), \ldots, b_n(s)\bigr),$$

and $a_i(s)$, $b_i(s)$, $i = 1, 2, \ldots, n$, are continuous of $s \in [0, 1]$.

Definition 2.1 $\Sigma(s)$ is said to be a contracting rectangle of (2.4) if for any $s \in [0, 1)$ and $\phi = (\phi_1, \phi_2, \ldots, \phi_n) \in \Sigma(s)$, we have

(1) $f_i(\phi) \ge 0$ whenever $\phi \in \Sigma(s)$ and $\phi_i = a_i(s)$ while $f_i(\phi) \le 0$ whenever $\phi \in \Sigma(s)$ and $\phi_i = b_i(s)$;

(2) for each s, at least one of above $2n$ inequalities is strict.

Using the contracting rectangle, we have the following stability result (see Smith [42, Theorem 5.2.5]).

Lemma 2.2 *Assume that $\Sigma(s)$ is a contracting rectangle of* (2.4). *If $\phi \in \Sigma(0)$, then e is globally stable.*

We also present some results for the Fisher equation. Assume that D, R, M are positive constants, $Z(x) > 0$ is a bounded and continuous function with nonempty support. Consider the initial value problem associated to the Fisher equation

$$\begin{cases} \frac{\partial Z(x,t)}{\partial t} = D\Delta Z(x,t) + RZ(x,t)[1 - MZ(x,t)], \\ Z(x,0) = Z(x), \quad x \in \mathbb{R}. \end{cases} \tag{2.5}$$

From Fife [13], Ye et al. [50] and Aronson and Weinberger [1], (2.5) has the following properties.

Lemma 2.3

(i) *Equation* (2.5) *admits a unique solution* $Z(x,t) \geq 0$ *which is twice differentiable in* $x \in \mathbb{R}$ *and differentiable in* $t > 0$.

(ii) *Assume that* $\overline{Z}(x,t)$, $\underline{Z}(x,t)$ *are continuous and bounded for* $x \in \mathbb{R}$, $t \geq 0$, *twice differentiable in* $x \in \mathbb{R}$ *and differentiable in* $t > 0$. *If they satisfy*

$$\begin{cases} \frac{\partial \overline{Z}(x,t)}{\partial t} \geq D\Delta \overline{Z}(x,t) + R\overline{Z}(x,t)[1 - M\overline{Z}(x,t)], \\ \overline{Z}(x,0) \geq Z(x), \quad x \in \mathbb{R} \end{cases}$$

and

$$\begin{cases} \frac{\partial \underline{Z}(x,t)}{\partial t} \leq D\Delta \underline{Z}(x,t) + R\underline{Z}(x,t)[1 - M\underline{Z}(x,t)], \\ \underline{Z}(x,0) \leq Z(x), \quad x \in \mathbb{R}, \end{cases}$$

then $\overline{Z}(x,t) \geq Z(x,t) \geq \underline{Z}(x,t)$, *where* $Z(x,t)$ *is a solution to* (2.5).

(iii) *If* $Z(x,t)$ *satisfies* (2.5), *and* $Z(x)$ *admits nonempty support, then*

$$\liminf_{t\to\infty} \inf_{|x|<(2\sqrt{DR}-\epsilon)t} Z(x,t) = \limsup_{t\to\infty} \sup_{|x|<(2\sqrt{DR}-\epsilon)t} Z(x,t) = \frac{1}{M}$$

for any $\epsilon \in (0, 2\sqrt{DR})$.

3 Stability of the positive equilibrium

In this section, we shall establish global stability of the positive equilibrium (2.1) on a smooth bounded domain Ω with Neumann boundary condition by using the contracting rectangles, throughout which (2.3) holds. We first consider (2.2) if

$$u_1(0) > 0, \qquad u_2(0) > 0, \qquad u_3(0) > 0.$$

By the quasipositivity, we see that

$$u_1(t) > 0, \qquad u_2(t) > 0, \qquad u_3(t) > 0$$

for all $t > 0$. It is evident that this model is defined for all $t \in (0, \infty)$.

Due to positivity, we see that

$$\frac{du_1(t)}{dt} \leq r_1 u_1(t)\bigl[1 - u_1(t)\bigr]$$

and so

$$\limsup_{t\to\infty} u_1(t) \leq 1.$$

Similarly, we have

$$\limsup_{t\to\infty} u_2(t) \leq 1,$$

which further indicates that

$$\limsup_{t\to\infty} u_3(t) \le a_{31} + a_{32} - 1.$$

Returning to the first equation of (2.2), we obtain

$$\liminf_{t\to\infty} u_1(t) \ge 1 - a_{12} - a_{13}(a_{31} + a_{32} - 1) =: \underline{u}_1.$$

Similarly, we have

$$\liminf_{t\to\infty} u_2(t) \ge 1 - a_{21} - a_{23}(a_{31} + a_{32} - 1) =: \underline{u}_2$$

and

$$\begin{aligned}\liminf_{t\to\infty} u_3(t) &\ge a_{31}\bigl[1 - a_{12} - a_{13}(a_{31} + a_{32} - 1)\bigr] \\ &\quad + a_{32}\bigl[1 - a_{21} - a_{23}(a_{31} + a_{32} - 1)\bigr] - 1 \\ &= a_{31}\underline{u}_1 + a_{32}\underline{u}_2 - 1 =: \underline{u}_3.\end{aligned}$$

Repeating the process, we further have

$$\limsup_{t\to\infty} u_1(t) \le 1 - a_{12}\underline{u}_2 - a_{13}\underline{u}_3 < 1,$$

$$\limsup_{t\to\infty} u_2(t) \le 1 - a_{21}\underline{u}_2 - a_{23}\underline{u}_3 < 1$$

and

$$0 < \limsup_{t\to\infty} u_3(t) < a_{31} + a_{32} - 1.$$

By the theory of dynamical systems [52], we see that (2.2) has at least one positive steady state $K = (k_1, k_2, k_2)$ with

$$0 < \underline{u}_i < k_i < 1, \quad i = 1, 2,$$

and

$$0 < \underline{u}_3 < k_3 < a_{31} + a_{32} - 1.$$

If K is globally stable, then K is the unique positive steady state, which implies the conclusions in Sect. 2. We now prove the global stability by contracting rectangles.

Lemma 3.1 *Assume that* (2.3) *holds. Denote* $\Sigma(s) = [a(s), b(s)]$ *with* $a(s) = (a_1(s), a_2(s), a_3(s))$, $b(s) = (b_1(s), b_2(s), b_3(s))$ *and*

$$a_1(s) = sk_1 + (1 - s)\underline{u}_1, \qquad b_1(s) = sk_1 + (1 - s),$$

$$a_2(s) = sk_2 + (1 - s)\underline{u}_2, \qquad b_2(s) = sk_2 + (1 - s),$$

$$a_3(s) = sk_3 + (1-s)\underline{u}_3, \qquad b_3(s) = sk_3 + (1-s)(a_{31} + a_{32} - 1).$$

Then $\Sigma(s) = [a(s), b(s)], s \in (0,1)$ *is a contracting rectangle of* (2.2). *Moreover*, (k_1, k_2, k_3) *is globally stable*.

Proof We now verify the definition of a contracting rectangle.

(1) (i) If $u_1 = sk_1 + (1-s)\underline{u}_1$, then

$$u_2 \le sk_2 + (1-s), \qquad u_3 \le sk_3 + (1-s)(a_{31} + a_{32} - 1)$$

so that

$$\begin{aligned} &1 - u_1 - a_{12}u_2 - a_{13}u_3 \\ &\quad \ge 1 - [sk_1 + (1-s)\underline{u}_1] - a_{12}[sk_2 + (1-s)] - a_{13}[sk_3 + (1-s)(a_{31} + a_{32} - 1)] \\ &\quad = (1-s) - (1-s)\{1 - [a_{12} + a_{13}(a_{31} + a_{32} - 1)] + [a_{12} + a_{13}(a_{31} + a_{32} - 1)]\} \\ &\quad = 0. \end{aligned}$$

Therefore

$$f_1(u_1, u_2, u_3)|_{u_1 = sk_1 + (1-s)\underline{u}_1} \ge 0.$$

(ii) If $u_1 = sk_1 + (1-s)$, then

$$u_2 \ge sk_2 + (1-s)\underline{u}_2, \qquad u_3 \ge sk_3 + (1-s)\underline{u}_3$$

so that

$$\begin{aligned} &1 - u_1 - a_{12}u_2 - a_{13}u_3 \\ &\quad \le 1 - [sk_1 + (1-s)] - a_{12}[sk_2 + (1-s)\underline{u}_2] - a_{13}[sk_3 + (1-s)\underline{u}_3] \\ &\quad = -(1-s)(a_{12}\underline{u}_2 + a_{13}\underline{u}_3) \\ &\quad < 0. \end{aligned}$$

Therefore

$$f_1(u_1, u_2, u_3)|_{u_1 = sk_1 + (1-s)} < 0.$$

(2) (i) If $u_2 = sk_2 + (1-s)\underline{u}_2$, then

$$u_1 \le sk_1 + (1-s), \qquad u_3 \le sk_3 + (1-s)(a_{31} + a_{32} - 1)$$

so that

$$1 - a_{21}u_1 - u_2 - a_{23}u_3 \ge 0$$

and

$$f_2(u_1,u_2,u_3)|_{u_2=sk_2+(1-s)\underline{u}_2} \geq 0.$$

(ii) If $u_2 = sk_2 + (1-s)$, then

$$u_1 \geq sk_1 + (1-s)\underline{u}_1, \qquad u_3 \geq sk_3 + (1-s)\underline{u}_3$$

so that

$$1 - a_{21}u_1 - u_2 - a_{23}u_3 < 0$$

and

$$f_2(u_1,u_2,u_3)|_{u_2=sk_2+(1-s)} < 0.$$

(3) (i) If $u_3 = sk_3 + (1-s)\underline{u}_3$, then

$$u_1 \geq sk_1 + (1-s)\underline{u}_1, \qquad u_2 \geq sk_2 + (1-s)\underline{u}_2$$

so that

$$-1 + a_{31}u_1 + a_{32}u_2 - u_3 \geq 0$$

and

$$f_3(u_1,u_2,u_3)|_{u_3=sk_3+(1-s)\underline{u}_3} \geq 0.$$

(ii) If $u_3 = sk_3 + (1-s)(a_{31} + a_{32} - 1)$, then

$$u_1 \leq sk_1 + (1-s), \qquad u_2 \leq sk_2 + (1-s)$$

so that

$$-1 + a_{31}u_1 + a_{32}u_2 - u_3 \leq 0$$

and

$$f_3(u_1,u_2,u_3)|_{u_3=sk_3+(1-s)(a_{31}+a_{32}-1)} \leq 0.$$

According to the definition of a contracting rectangle, $\Sigma(s) = [a(s), b(s)]$ is a contracting rectangle of (2.2). Then (k_1,k_2,k_3) is globally stable by Lemma 2.2. The proof is complete. □

The stability result further implies the following properties.

Theorem 3.2 *Assume that Ω is a bounded domain with smooth boundary $\partial\Omega$. Consider the following initial boundary value problem*:

$$\begin{cases} \frac{\partial u_1(x,t)}{\partial t} = d_1\Delta u_1(x,t) + r_1u_1(x,t)[1-u_1(x,t)-a_{12}u_2(x,t)-a_{13}u_3(x,t)], \\ \quad x\in\Omega, t>0, \\ \frac{\partial u_2(x,t)}{\partial t} = d_2\Delta u_2(x,t) + r_2u_2(x,t)[1-a_{21}u_1(x,t)-u_2(x,t)-a_{23}u_3(x,t)], \\ \quad x\in\Omega, t>0, \\ \frac{\partial u_3(x,t)}{\partial t} = d_3\Delta u_3(x,t) + r_3u_3(x,t)[-1+a_{31}u_1(x,t)+a_{32}u_2(x,t)-u_3(x,t)], \\ \quad x\in\Omega, t>0, \\ \frac{\partial u_1(x,t)}{\partial \mathbf{n}} = \frac{\partial u_2(x,t)}{\partial \mathbf{n}} = \frac{\partial u_3(x,t)}{\partial \mathbf{n}} = 0, \quad x\in\partial\Omega, t>0, \\ u_i(x,0)=\phi_i(x)>0, \quad i=1,2,3, x\in\Omega, \end{cases} \tag{3.1}$$

where $\mathbf{n}$ *is the outward unit normal vector of* $\partial\Omega$, $\phi_i(x), i=1,2,3$, *are continuous and bounded. If* (2.3) *holds, then*

$$\lim_{t\to\infty} u_i(\cdot,t) = k_i, \quad i=1,2,3.$$

Since the boundary condition is of Neumann type, the proof is similar to that of ODEs with nice properties of Laplacian operator, and we omit the proof here. We end this section by making the following remark.

Remark 3.3 The dynamics of this model has been investigated by other methods, for instance, Cantrell et al. [5] studied it under Dirichlet conditions. We now give the proof in order to study the traveling wave solutions in Sect. 5.

4 Existence of the traveling wave solutions

We now investigate the existence of traveling wave solutions and first give the following definition.

Definition 4.1 A traveling wave solution of (2.1) is a special solution taking the form $u(x,t)=\Phi(x+ct)\in C^2(\mathbb{R},\mathbb{R}^3)$ with

$$u(x,t) = \big(u_1(x,t),u_2(x,t),u_3(x,t)\big), \qquad \Phi(\xi) = \big(\phi_1(\xi),\phi_2(\xi),\phi_3(\xi)\big), \quad \xi = x+ct,$$

in which Φ is the wave profile that propagates through the one-dimension spatial domain $\mathbb{R}$ at the constant wave speed $c>0$.

By definition, $\Phi(\xi)=(\phi_1(\xi),\phi_2(\xi),\phi_3(\xi))$ must satisfy

$$\begin{cases} d_1\phi_1''(\xi) - c\phi_1'(\xi) + r_1\phi_1(\xi)[1-\phi_1(\xi)-a_{12}\phi_2(\xi)-a_{13}\phi_3(\xi)] = 0, \\ d_2\phi_2''(\xi) - c\phi_2'(\xi) + r_2\phi_2(\xi)[1-a_{21}\phi_1(\xi)-\phi_2(\xi)-a_{23}\phi_3(\xi)] = 0, \\ d_3\phi_3''(\xi) - c\phi_3'(\xi) + r_3\phi_3(\xi)[-1+a_{31}\phi_1(\xi)+a_{32}\phi_2(\xi)-\phi_3(\xi)] = 0. \end{cases} \tag{4.1}$$

Since we shall discuss the dynamical behavior that u_2 and u_3 invade the habitat of u_1, $\Phi(\xi)$ will satisfy the following asymptotic boundary conditions:

$$\begin{aligned} &\lim_{\xi\to-\infty}\bigl(\phi_1(\xi),\phi_2(\xi),\phi_3(\xi)\bigr)=(1,0,0),\\ &\lim_{\xi\to+\infty}\bigl(\phi_1(\xi),\phi_2(\xi),\phi_3(\xi)\bigr)=(k_1,k_2,k_3). \end{aligned} \tag{4.2}$$

In population dynamics, positive solutions of (4.1)–(4.2) describe the following biological process: at any fixed location $x\in\mathbb{R}$, there was only one prey a long time ago ($t\to-\infty$ such that $x+ct\to-\infty$), and the predator and two preys will coexist after a long-term species interaction ($t\to+\infty$ such that $x+ct\to+\infty$).

Letting $\Psi(\xi)=(\varphi_1,\varphi_2,\varphi_3)(\xi)=(1-\phi_1,\phi_2,\phi_3)(\xi)$, we obtain

$$\begin{cases} d_1\varphi_1''(\xi)-c\varphi_1'(\xi)+r_1[1-\varphi_1(\xi)][a_{12}\varphi_2(\xi)+a_{13}\varphi_3(\xi)-\varphi_1(\xi)]=0,\\ \quad \xi\in\mathbb{R},\\ d_2\varphi_2''(\xi)-c\varphi_2'(\xi)+r_2\varphi_2(\xi)[1-a_{21}+a_{21}\varphi_1(\xi)-\varphi_2(\xi)-a_{23}\varphi_3(\xi)]=0,\\ \quad \xi\in\mathbb{R},\\ d_3\varphi_3''(\xi)-c\varphi_3'(\xi)+r_3\varphi_3(\xi)[a_{31}-1+a_{32}\varphi_2(\xi)-a_{31}\varphi_1(\xi)-\varphi_3(\xi)]=0,\\ \quad \xi\in\mathbb{R}. \end{cases} \tag{4.3}$$

Due to (4.2), we have

$$\begin{aligned} &\lim_{\xi\to-\infty}\bigl(\varphi_1(\xi),\varphi_2(\xi),\varphi_3(\xi)\bigr)=(0,0,0),\\ &\lim_{\xi\to+\infty}\bigl(\varphi_1(\xi),\varphi_2(\xi),\varphi_3(\xi)\bigr)=(1-k_1,k_2,k_3). \end{aligned} \tag{4.4}$$

Define a constant

$$\beta\ge r_2\bigl[a_{21}+a_{23}(a_{31}+a_{32}-1)\bigr]+r_3(a_{31}+a_{32}) \tag{4.5}$$

such that

$$\beta\varphi_1+r_1[1-\varphi_1][a_{12}\varphi_2+a_{13}\varphi_3-\varphi_1]$$

is nondecreasing with respect to $0\le\varphi_1\le1$, $0\le\varphi_2\le1$, $0\le\varphi_3\le a_{31}+a_{32}-1$,

$$\beta\varphi_2+r_2\varphi_2[1-a_{21}+a_{21}\varphi_1-\varphi_2-a_{23}\varphi_3]$$

is nondecreasing with respect to $0\le\varphi_1\le1$, $0\le\varphi_2\le1$, while it is nonincreasing with respect to $0\le\varphi_3\le a_{31}+a_{32}-1$, and

$$\beta\varphi_3+r_3\varphi_3[a_{31}-1+a_{32}\varphi_2-a_{31}\varphi_1-\varphi_3]$$

is nonincreasing with respect to $0\le\varphi_1\le1$, while it is nondecreasing with respect to $0\le\varphi_2\le1$, $0\le\varphi_3\le a_{31}+a_{32}-1$.

For $\Psi(\xi) = (\varphi_1, \varphi_2, \varphi_3)(\xi) \in X_{[0,M]}$ with $M = (1, 1, a_{31} + a_{32} - 1)$, denote

$$\begin{cases} F_1(\varphi_1, \varphi_2, \varphi_3)(\xi) = \beta\varphi_1(\xi) + r_1[1 - \varphi_1(\xi)][a_{12}\varphi_2(\xi) + a_{13}\varphi_3(\xi) - \varphi_1(\xi)], \\ \quad \xi \in \mathbb{R}, \\ F_2(\varphi_1, \varphi_2, \varphi_3)(\xi) = \beta\varphi_2(\xi) + r_2\varphi_2(\xi)[1 - a_{21} + a_{21}\varphi_1(\xi) - \varphi_2(\xi) - a_{23}\varphi_3(\xi)], \\ \quad \xi \in \mathbb{R}, \\ F_3(\varphi_1, \varphi_2, \varphi_3)(\xi) = \beta\varphi_3(\xi) + r_3\varphi_3(\xi)[a_{31} - 1 + a_{32}\varphi_2(\xi) - a_{31}\varphi_1(\xi) - \varphi_3(\xi)], \\ \quad \xi \in \mathbb{R}. \end{cases}$$

Then (4.3) can be rewritten as

$$d_i\varphi_i''(\xi) - c\varphi_i'(\xi) - \beta\varphi_i(\xi) + F_i(\Psi)(\xi) = 0, \quad i = 1, 2, 3. \tag{4.6}$$

Define constants

$$\lambda_{i1}(c) = \frac{c - \sqrt{c^2 + 4\beta d_i}}{2d_i}, \qquad \lambda_{i2}(c) = \frac{c + \sqrt{c^2 + 4\beta d_i}}{2d_i}, \quad i = 1, 2, 3.$$

Then $\beta > 0$ implies $\lambda_{i1} < 0 < \lambda_{i2}$ and

$$d_i\lambda_{i1}^2 - c\lambda_{i1} - \beta = 0, \qquad d_i\lambda_{i2}^2 - c\lambda_{i2} - \beta = 0, \quad i = 1, 2, 3.$$

For $\Psi(\xi) = (\varphi_1, \varphi_2, \varphi_3)(\xi) \in X_{[0,M]}$, define an operator $P = (P_1, P_2, P_3) : X_{[\mathbf{0},\mathbf{M}]} \to X$ (see Wu and Zou [49]) by

$$P_i(\Psi)(\xi) = \frac{1}{d_i(\lambda_{i2} - \lambda_{i1})}\left[\int_{-\infty}^{\xi} e^{\lambda_{i1}(\xi - s)} + \int_{\xi}^{+\infty} e^{\lambda_{i2}(\xi - s)}\right]F_i(\Psi)(s)\,ds, \tag{4.7}$$

where $i = 1, 2, 3$, $\xi \in \mathbb{R}$. Then a fixed point of operator P is a solution of (4.3) or (4.6). On the other hand, a solution of (4.3) or (4.6) is a fixed point of operator P (see Huang [22]).

In the following, we will establish the existence of a nontrivial positive solution of (4.3) by combining Schauder's fixed point theorem with the method of upper and lower solutions (for quasimonotone systems, we refer to [20, 34, 46, 49]). We now introduce the definition of upper and lower solutions of (4.3).

Definition 4.2 $\overline{\Psi}(\xi) = (\overline{\varphi}_1, \overline{\varphi}_2, \overline{\varphi}_3)(\xi)$, $\underline{\Psi}(\xi) = (\underline{\varphi}_1, \underline{\varphi}_2, \underline{\varphi}_3)(\xi) \in X_{[0,M]}$ are a pair of upper and lower solutions of (4.3), if $\overline{\Psi}'', \overline{\Psi}', \underline{\Psi}'', \underline{\Psi}'$ are bounded and continuous for each $\xi \in \mathbb{R} \setminus \mathbb{T}$ with $\mathbb{T} = \{T_1, T_2, \ldots, T_m\}$ and they satisfy

$$\begin{cases} d_1\overline{\varphi}_1''(\xi) - c\overline{\varphi}_1'(\xi) + r_1[1 - \overline{\varphi}_1(\xi)][a_{12}\overline{\varphi}_2(\xi) + a_{13}\overline{\varphi}_3(\xi) - \overline{\varphi}_1(\xi)] \le 0, \\ d_2\overline{\varphi}_2''(\xi) - c\overline{\varphi}_2'(\xi) + r_2\overline{\varphi}_2(\xi)[1 - a_{21} + a_{21}\overline{\varphi}_1(\xi) - \overline{\varphi}_2(\xi) - a_{23}\underline{\varphi}_3(\xi)] \le 0, \\ d_3\overline{\varphi}_3''(\xi) - c\overline{\varphi}_3'(\xi) + r_3\overline{\varphi}_3(\xi)[a_{31} - 1 + a_{32}\overline{\varphi}_2(\xi) - a_{31}\underline{\varphi}_1(\xi) - \overline{\varphi}_3(\xi)] \le 0, \\ d_1\underline{\varphi}_1''(\xi) - c\underline{\varphi}_1'(\xi) + r_1[1 - \underline{\varphi}_1(\xi)][a_{12}\underline{\varphi}_2(\xi) + a_{13}\underline{\varphi}_3(\xi) - \underline{\varphi}_1(\xi)] \ge 0, \\ d_2\underline{\varphi}_2''(\xi) - c\underline{\varphi}_2'(\xi) + r_2\underline{\varphi}_2(\xi)[1 - a_{21} + a_{21}\underline{\varphi}_1(\xi) - \underline{\varphi}_2(\xi) - a_{23}\overline{\varphi}_3(\xi)] \ge 0, \\ d_3\underline{\varphi}_3''(\xi) - c\underline{\varphi}_3'(\xi) + r_3\underline{\varphi}_3(\xi)[a_{31} - 1 + a_{32}\underline{\varphi}_2(\xi) - a_{31}\overline{\varphi}_1(\xi) - \underline{\varphi}_3(\xi)] \ge 0. \end{cases} \tag{4.8}$$

Lemma 4.3 *Assume that* (4.3) *has a pair of upper and lower solutions satisfying*
(1) $\underline{\Psi}(\xi) \le \overline{\Psi}(\xi)$, $\xi \in \mathbb{R}$;
(2) $\underline{\Psi}'(\xi_-) \le \overline{\Psi}'(\xi_+)$, $\overline{\Psi}'(\xi_+) \le \overline{\Psi}'(\xi_-)$, $\xi \in \mathbb{T}$, *herein*

$$\underline{\Psi}'(\xi_\pm) = \lim_{t\to\xi_\pm} \underline{\Psi}'(t), \qquad \overline{\Psi}'(\xi_\pm) = \lim_{t\to\xi_\pm} \overline{\Psi}'(t).$$

Then (4.3) *has a positive solution* $\Psi(\xi)$ *such that* $\underline{\Psi}(\xi) \le \Psi(\xi) \le \overline{\Psi}(\xi)$.

Proof We prove this lemma by Schauder's fixed point theorem. Since a similar result has been proved in several earlier papers mentioned above [31], we only give the scheme.

Define

$$B_\mu(\mathbb{R},\mathbb{R}^3) = \Big\{u \in X : \sup_{\xi\in\mathbb{R}}\big\{\|\mathbf{u}(\xi)\|e^{-\mu|\xi|}\big\} < \infty\Big\},$$

$$\big|\mathbf{u}(\xi)\big|_\mu = \sup_{\xi\in\mathbb{R}}\big\{\|\mathbf{u}(\xi)\|e^{-\mu|\xi|}\big\},$$

where

$$\mu \in \big(0, \min\{-\lambda_{11}, -\lambda_{21}, -\lambda_{31}\}\big),$$

then $(B_\mu(\mathbb{R},\mathbb{R}^3), |\cdot|_\mu)$ is a Banach space. Let

$$\Lambda = \big\{\Psi(\xi) \in X_{[\mathbf{0},\mathbf{M}]} : \underline{\Psi}(\xi) \le \Psi(\xi) \le \overline{\Psi}(\xi)\big\}.$$

Obviously, Λ is nonempty and convex. It is also closed and bounded with respect to the decay norm $|\cdot|_\mu$.

We now verify that $P : \Lambda \to \Lambda$. For $\Psi(\xi) = (\varphi_1, \varphi_2, \varphi_3)(\xi) \in \Lambda$ and each fixed $\xi \in \mathbb{R}$, the definition of operator P and the choice of β imply that it suffices to prove that

$$\begin{cases} \underline{\varphi}_1(\xi) \le P_1(\underline{\varphi}_1, \underline{\varphi}_2, \underline{\varphi}_3)(\xi) \le P_1(\overline{\varphi}_1, \overline{\varphi}_2, \overline{\varphi}_3)(\xi) \le \overline{\varphi}_1(\xi), \\ \underline{\varphi}_2(\xi) \le P_2(\underline{\varphi}_1, \underline{\varphi}_2, \overline{\varphi}_3)(\xi) \le P_2(\overline{\varphi}_1, \overline{\varphi}_2, \underline{\varphi}_3)(\xi) \le \overline{\varphi}_2(\xi), \\ \underline{\varphi}_3(\xi) \le P_3(\overline{\varphi}_1, \underline{\varphi}_2, \underline{\varphi}_3)(\xi) \le P_3(\underline{\varphi}_1, \overline{\varphi}_2, \overline{\varphi}_3)(\xi) \le \overline{\varphi}_3(\xi). \end{cases} \tag{4.9}$$

Without loss of generality, we assume that $T_1 < T_2 < \cdots < T_m$ and denote $T_0 = -\infty$, $T_{m+1} = +\infty$. If $\xi \in \mathbb{R}\backslash\mathbb{T}$, namely, $\xi \in (T_k, T_{k+1})$ with some $k \in \{0, 1, \ldots, m\}$, then

$$\begin{aligned} &P_1(\underline{\varphi}_1, \underline{\varphi}_2, \underline{\varphi}_3)(\xi) \\ &\quad= \frac{1}{d_1(\lambda_{12}-\lambda_{11})}\bigg[\int_{-\infty}^{\xi} e^{\lambda_{11}(\xi-s)} + \int_{\xi}^{+\infty} e^{\lambda_{12}(\xi-s)}\bigg]F_1(\underline{\varphi}_1, \underline{\varphi}_2, \underline{\varphi}_3)(s)\,ds \\ &\quad= \frac{1}{d_1(\lambda_{12}-\lambda_{11})}\bigg[\int_{-\infty}^{\xi} e^{\lambda_{11}(\xi-s)} + \int_{\xi}^{+\infty} e^{\lambda_{12}(\xi-s)}\bigg]\big[\beta\underline{\varphi}_1(s) + c\underline{\varphi}_1'(s) - d_1\underline{\varphi}_1''(s)\big]\,ds \\ &\quad\ge \underline{\varphi}_1(\xi) + \frac{1}{\lambda_{12}-\lambda_{11}}\Bigg[\sum_{j=1}^{k} e^{\lambda_{11}(\xi-T_j)}\big(\underline{\varphi}_1'(T_{j+}) - \underline{\varphi}_1'(T_{j-})\big) \end{aligned}$$

$$\left. + \sum_{j=k+1}^{m} e^{\lambda_{12}(\xi - T_j)} \left(\underline{\varphi}_1'(T_{j+}) - \underline{\varphi}_1'(T_{j-}) \right) \right]$$
$$\geq \underline{\varphi}_1(\xi).$$

Because ξ was arbitrary and due to the continuity, we have $P_1(\underline{\varphi}_1, \underline{\varphi}_2, \underline{\varphi}_3)(\xi) \geq \underline{\varphi}_1(\xi)$ in $\mathbb{R}$. In a similar way, we can verity the remainder of (4.9).

Note that the compactness in [24] is independent of the monotonicity, then $P : \Lambda \to \Lambda$ is compact in the sense of the decay norm $|\cdot|_\mu$ by a discussion similar to that in [24].

By Schauder's fixed point theorem, there exists $\Psi(\xi) = (\varphi_1, \varphi_2, \varphi_3)(\xi) \in \Lambda$ which is a positive solution of (4.3) satisfying $\underline{\Psi}(\xi) \leq \Psi(\xi) \leq \overline{\Psi}(\xi)$. The proof is complete. □

Next, we construct the upper and lower solutions of (4.3), and we assume that

$$a_{31} > 1. \tag{4.10}$$

For any fixed

$$c > \max\left\{2\sqrt{d_2 r_2(1 - a_{21})}, 2\sqrt{d_3 r_3(a_{31} - 1)}\right\} := c^*,$$

we define positive constants $\gamma_{21} < \gamma_{22}, \gamma_{31} < \gamma_{32}$ such that

$$d_2\gamma_{21}^2 - c\gamma_{21} + r_2(1 - a_{21}) = d_2\gamma_{22}^2 - c\gamma_{22} + r_2(1 - a_{21}) = 0,$$
$$d_3\gamma_{31}^2 - c\gamma_{31} + r_3(a_{31} - 1) = d_3\gamma_{32}^2 - c\gamma_{32} + r_3(a_{31} - 1) = 0.$$

Further choose $\epsilon > 0$ such that

$$\gamma_{21} + \epsilon < \min\{2\gamma_{21}, \gamma_{22}, \gamma_{21} + \gamma_{31}\},$$

and

$$\gamma_{31} + \epsilon < \min\{2\gamma_{31}, \gamma_{32}, \gamma_{21} + \gamma_{31}\}.$$

Let

$$\gamma_{11} = \min\{\gamma_{21}, \gamma_{31}\}.$$

We now assume that

$$d_1\gamma_{11}^2 - c\gamma_{11} < 0. \tag{4.11}$$

Define $\Gamma = (\gamma_{21}, \gamma_{22}) \cap (\gamma_{31}, \gamma_{32})$, then Γ is nonempty if c is large enough or other parameters satisfy suitable conditions. In particular, when Γ is nonempty, we further assume that there exists $\gamma \in \Gamma$ such that

$$d_1\gamma^2 - c\gamma < 0, \quad \gamma < \gamma_{21} + \gamma_{31}. \tag{4.12}$$

Remark 4.4 Assume that all the parameters in (2.1) are fixed. Then there exists $c' \geq c^*$ such that (4.11)–(4.12) hold.

For any given $c > c^*$, we now fix these constants and define continuous functions as follows:

$$\begin{aligned}
&\overline{\varphi}_1(\xi) = \min\{1, e^{\gamma_{11}\xi} + pe^{\gamma\xi}\}, \qquad \underline{\varphi}_1(\xi) = 0,\\
&\overline{\varphi}_2(\xi) = \min\{1, e^{\gamma_{21}\xi} + pe^{\gamma\xi}\}, \qquad \underline{\varphi}_2(\xi) = \max\{0, e^{\gamma_{21}\xi} - qe^{(\gamma_{21}+\epsilon)\xi}\},\\
&\overline{\varphi}_3(\xi) = \min\{a_{31} + a_{32} - 1, e^{\gamma_{31}\xi} + pa_{32}e^{\gamma\xi}\}, \qquad \underline{\varphi}_3(\xi) = \max\{0, e^{\gamma_{31}\xi} - qe^{(\gamma_{31}+\epsilon)\xi}\},
\end{aligned}$$

in which $p > 1$, $q > 1$ will be clarified in the following lemma.

Lemma 4.5 *Assume that $c > c^*$. Further suppose that Γ is nonempty such that* (4.11)–(4.12) *hold. Then there exist p, q such that $\overline{\Psi}(\xi) = (\overline{\varphi}_1(\xi), \overline{\varphi}_2(\xi), \overline{\varphi}_3(\xi))$ and $\underline{\Psi}(\xi) = (\underline{\varphi}_1(\xi), \underline{\varphi}_2(\xi), \underline{\varphi}_3(\xi))$ are a pair of upper and lower solutions of* (4.3).

Proof It suffices to verify (4.8) one by one.

(1) (i) If $\overline{\varphi}_1(\xi) = 1 < e^{\gamma_{11}\xi} + pe^{\gamma\xi}$, then

$$d_1\overline{\varphi}_1''(\xi) - c\overline{\varphi}_1'(\xi) + r_1[1 - \overline{\varphi}_1(\xi)][a_{12}\overline{\varphi}_2(\xi) + a_{13}\overline{\varphi}_3(\xi) - \overline{\varphi}_1(\xi)] = 0.$$

(ii) If $\overline{\varphi}_1(\xi) = e^{\gamma_{11}\xi} + pe^{\gamma\xi} < 1$, then $\xi < 0$ and

$$\overline{\varphi}_2(\xi) \leq e^{\gamma_{21}\xi} + pe^{\gamma\xi}, \qquad \overline{\varphi}_3(\xi) \leq e^{\gamma_{31}\xi} + pa_{32}e^{\gamma_{31}\xi}$$

so that

$$\begin{aligned}
&a_{12}\overline{\varphi}_2(\xi) + a_{13}\overline{\varphi}_3(\xi) - \overline{\varphi}_1(\xi)\\
&\quad\leq a_{12}(e^{\gamma_{21}\xi} + pe^{\gamma\xi}) + a_{13}(e^{\gamma_{31}\xi} + pa_{32}e^{\gamma\xi}) - (e^{\gamma_{11}\xi} + pe^{\gamma\xi})\\
&\quad\leq (a_{12} + a_{13} - 1)e^{\gamma_{11}\xi} + (a_{12} + a_{13}a_{32} - 1)pe^{\gamma\xi}\\
&\quad< 0
\end{aligned}$$

because $\gamma_{11} = \min\{\gamma_{21}, \gamma_{31}\}$, $a_{12} + a_{13}a_{32} < 1$ and $a_{12} + a_{13} < 1$. Therefore, we have

$$\begin{aligned}
&d_1\overline{\varphi}_1''(\xi) - c\overline{\varphi}_1'(\xi) + r_1[1 - \overline{\varphi}_1(\xi)][a_{12}\overline{\varphi}_2(\xi) + a_{13}\overline{\varphi}_3(\xi) - \overline{\varphi}_1(\xi)]\\
&\quad\leq d_1\overline{\varphi}_1''(\xi) - c\overline{\varphi}_1'(\xi)\\
&\quad= (d_1\gamma_{11}^2 - c\gamma_{11})e^{\gamma_{11}\xi} + (d_1\gamma^2 - c\gamma)pe^{\gamma\xi}\\
&\quad< 0.
\end{aligned}$$

(2) (i) If $\overline{\varphi}_2(\xi) = 1 < e^{\gamma_{21}\xi} + pe^{\gamma\xi}$, then $\overline{\varphi}_1(\xi) \leq 1$ so that

$$\begin{aligned}
&d_2\overline{\varphi}_2''(\xi) - c\overline{\varphi}_2'(\xi) + r_2\overline{\varphi}_2(\xi)[1 - a_{21} + a_{21}\overline{\varphi}_1(\xi) - \overline{\varphi}_2(\xi) - a_{23}\underline{\varphi}_3(\xi)]\\
&\quad\leq d_2\overline{\varphi}_2''(\xi) - c\overline{\varphi}_2'(\xi) + r_2\overline{\varphi}_2(\xi)[1 - a_{21} + a_{21}\overline{\varphi}_1(\xi) - \overline{\varphi}_2(\xi)]
\end{aligned}$$

$$
\begin{aligned}
&\le r_2(1-a_{21}+a_{21}-1)\\
&=0.
\end{aligned}
$$

(ii) If $\overline{\varphi}_2(\xi)=e^{\gamma_{21}\xi}+pe^{\gamma\xi}<1$, then

$$
\begin{aligned}
&d_2\overline{\varphi}_2''(\xi)-c\overline{\varphi}_2'(\xi)+r_2\overline{\varphi}_2(\xi)\bigl[1-a_{21}+a_{21}\overline{\varphi}_1(\xi)-\overline{\varphi}_2(\xi)-a_{23}\underline{\varphi}_3(\xi)\bigr]\\
&\quad\le d_2\overline{\varphi}_2''(\xi)-c\overline{\varphi}_2'(\xi)+r_2\overline{\varphi}_2(\xi)\bigl[1-a_{21}+a_{21}\overline{\varphi}_1(\xi)-\overline{\varphi}_2(\xi)\bigr]\\
&\quad\le d_2\bigl(\gamma_{21}^2e^{\gamma_{21}\xi}+p\gamma^2e^{\gamma\xi}\bigr)-c\bigl(\gamma_{21}e^{\gamma_{21}\xi}+p\gamma e^{\gamma\xi}\bigr)\\
&\qquad+r_2\bigl(e^{\gamma_{21}\xi}+pe^{\gamma\xi}\bigr)\bigl[1-a_{21}+a_{21}\bigl(e^{\gamma_{11}\xi}+pe^{\gamma\xi}\bigr)-\bigl(e^{\gamma_{21}\xi}+pe^{\gamma\xi}\bigr)\bigr].
\end{aligned}
$$

If $\gamma_{11}=\gamma_{21}\le\gamma_{31}$, then

$$
a_{21}\bigl(e^{\gamma_{11}\xi}+pe^{\gamma\xi}\bigr)\le\bigl(e^{\gamma_{21}\xi}+pe^{\gamma\xi}\bigr)
$$

so that

$$
\begin{aligned}
&d_2\overline{\varphi}_2''(\xi)-c\overline{\varphi}_2'(\xi)+r_2\overline{\varphi}_2(\xi)\bigl[1-a_{21}+a_{21}\overline{\varphi}_1(\xi)-\overline{\varphi}_2(\xi)-a_{23}\underline{\varphi}_3(\xi)\bigr]\\
&\quad\le d_2\bigl(\gamma_{21}^2e^{\gamma_{21}\xi}+p\gamma^2e^{\gamma\xi}\bigr)-c\bigl(\gamma_{21}e^{\gamma_{21}\xi}+p\gamma e^{\gamma\xi}\bigr)+r_2\bigl(e^{\gamma_{21}\xi}+pe^{\gamma\xi}\bigr)[1-a_{21}]\\
&\quad\le 0
\end{aligned}
$$

by the definitions of γ_{21} and γ. Otherwise, $\gamma_{11}=\gamma_{31}<\gamma_{21}$ so that

$$
\gamma<\gamma_{11}+\gamma_{21}
$$

and

$$
\begin{aligned}
&d_2\overline{\varphi}_2''(\xi)-c\overline{\varphi}_2'(\xi)+r_2\overline{\varphi}_2(\xi)\bigl[1-a_{21}+a_{21}\overline{\varphi}_1(\xi)-\overline{\varphi}_2(\xi)-a_{23}\underline{\varphi}_3(\xi)\bigr]\\
&\quad\le d_2\bigl(\gamma_{21}^2e^{\gamma_{21}\xi}+p\gamma^2e^{\gamma\xi}\bigr)-c\bigl(\gamma_{21}e^{\gamma_{21}\xi}+p\gamma e^{\gamma\xi}\bigr)\\
&\qquad+r_2\bigl(e^{\gamma_{21}\xi}+pe^{\gamma\xi}\bigr)\bigl[1-a_{21}+a_{21}e^{\gamma_{11}\xi}\bigr]\\
&\quad\le\bigl[d_2\gamma^2-c\gamma+r_2(1-a_{21})\bigr]pe^{\gamma\xi}+r_2a_{21}e^{(\gamma_{11}+\gamma_{21})\xi}+pr_2a_{21}e^{(\gamma_{11}+\gamma)\xi}\\
&\quad\le 0
\end{aligned}
$$

provided that

$$
\bigl[d_2\gamma^2-c\gamma+r_2(1-a_{21})\bigr]pe^{\gamma\xi}+2r_2a_{21}e^{(\gamma_{11}+\gamma_{21})\xi}\le 0 \tag{4.13}
$$

and

$$
\bigl[d_2\gamma^2-c\gamma+r_2(1-a_{21})\bigr]+2r_2a_{21}e^{\gamma_{11}\xi}\le 0. \tag{4.14}
$$

Clearly, (4.13) is true if

$$
p>1-\frac{2r_2a_{21}}{d_2\gamma^2-c\gamma+r_2(1-a_{21})}:=p_1(>1)
$$

and (4.14) is true if

$$p > \left(\frac{2r_2a_{21}}{-(d_2\gamma^2 - c\gamma + r_2(1-a_{21}))}\right)^{\frac{\gamma}{\gamma_{11}}} + 1 := p_2 > 1.$$

(3) (i) If $\overline{\varphi}_{31}(\xi) = a_{31} + a_{32} - 1 < e^{\gamma_{31}\xi} + pa_{32}e^{\gamma\xi}$, then $\overline{\varphi}_2(\xi) \le 1$ so that

$$\begin{aligned}
&d_3\overline{\varphi}_3''(\xi) - c\overline{\varphi}_3'(\xi) + r_3\overline{\varphi}_3(\xi)\big[a_{31} - 1 + a_{32}\overline{\varphi}_2(\xi) - a_{31}\underline{\varphi}_1(\xi) - \overline{\varphi}_3(\xi)\big] \\
&\quad\le d_3\overline{\varphi}_3''(\xi) - c\overline{\varphi}_3'(\xi) + r_3\overline{\varphi}_3(\xi)\big[a_{31} - 1 + a_{32} - \overline{\varphi}_3(\xi)\big] \\
&\quad= 0.
\end{aligned}$$

(ii) If $\overline{\varphi}_3(\xi) = e^{\gamma_{31}\xi} + pa_{32}e^{\gamma\xi} < a_{31} + a_{32} - 1$, then $\overline{\varphi}_2(\xi) \le e^{\gamma_{21}\xi} + pe^{\gamma\xi}$ so that

$$\begin{aligned}
&d_3\overline{\varphi}_3''(\xi) - c\overline{\varphi}_3'(\xi) + r_3\overline{\varphi}_3(\xi)\big[a_{31} - 1 + a_{32}\overline{\varphi}_2(\xi) - a_{31}\underline{\varphi}_1(\xi) - \overline{\varphi}_3(\xi)\big] \\
&\quad\le d_3\overline{\varphi}_3''(\xi) - c\overline{\varphi}_3'(\xi) + r_3\overline{\varphi}_3(\xi)\big[a_{31} - 1 + a_{32}\overline{\varphi}_2(\xi) - \overline{\varphi}_3(\xi)\big] \\
&\quad\le \big[d_3\gamma_3^2 - c\gamma_{31} + r_3(a_{31} - 1)\big]e^{\gamma_{31}\xi} + \big[d_3\gamma^2 - c\gamma + r_3(a_{31} - 1)\big]pa_{32}e^{\gamma\xi} \\
&\qquad + r_3\big(e^{\gamma_{31}\xi} + pa_{32}e^{\gamma\xi}\big)\big[a_{32}\big(e^{\gamma_{21}\xi} + pe^{\gamma\xi}\big) - \big(e^{\gamma_{31}\xi} + pa_{32}e^{\gamma\xi}\big)\big] \\
&\quad\le \big[d_3\gamma^2 - c\gamma + r_3(a_{31} - 1)\big]pa_{32}e^{\gamma\xi} + r_3a_{32}e^{(\gamma_{21}+\gamma_{31})\xi} + r_3{a_{32}}^2pe^{(\gamma_{21}+\gamma)\xi} \\
&\quad\le 0
\end{aligned}$$

provided that

$$\big[d_3\gamma^2 - c\gamma + r_3(a_{31} - 1)\big]pe^{\gamma\xi} + 2r_3e^{(\gamma_{21}+\gamma_{31})\xi} \le 0 \tag{4.15}$$

and

$$\big[d_3\gamma^2 - c\gamma + r_3(a_{31} - 1)\big] + 2r_3a_{32}e^{\gamma_{21}\xi} \le 0. \tag{4.16}$$

For (4.15), since $\gamma < \gamma_{21} + \gamma_{31}$, we have

$$\begin{aligned}
&\big[d_3\gamma^2 - c\gamma + r_3(a_{31} - 1)\big]pe^{\gamma\xi} + 2r_3e^{(\gamma_{21}+\gamma_{31})\xi} \\
&\quad\le e^{\gamma\xi}\big\{\big[d_3\gamma^2 - c\gamma + r_3(a_{31} - 1)\big]p + 2r_3\big\},
\end{aligned}$$

which is true if

$$p \ge \frac{-2r_3}{d_3\gamma^2 - c\gamma + r_3(a_{31} - 1)} + 1 := p_3.$$

Since $e^{\gamma_{31}\xi} + pa_{32}e^{\gamma\xi} < a_{31} + a_{32} - 1$, then

$$pe^{\gamma\xi} < \frac{a_{31} + a_{32} - 1}{a_{32}}$$

so that

$$\xi < \frac{1}{\gamma}\ln\frac{a_{31} + a_{32} - 1}{pa_{32}}$$

and

$$e^{\gamma_{21}\xi} < \left(\frac{a_{31} + a_{32} - 1}{pa_{32}}\right)^{\frac{\gamma_{21}}{\gamma}}.$$

Clearly, (4.16) holds if

$$\left(\frac{a_{31} + a_{32} - 1}{pa_{32}}\right)^{\frac{\gamma_{21}}{\gamma}} < \frac{d_3\gamma^2 - c\gamma + r_3(a_{31} - 1)}{-2r_3a_{32}},$$

which is true provided that

$$p > \frac{a_{31} + a_{32} - 1}{a_{32}}\left[\left(\frac{d_3\gamma^2 - c\gamma + r_3(a_{31} - 1)}{-2r_3a_{32}}\right)^{\frac{\gamma}{\gamma_{21}+\gamma_{31}}} + 1\right] := p_4.$$

(4) If $\underline{\varphi}_1(\xi) = 0$, then

$$\begin{aligned}
&d_1\underline{\varphi}_1''(\xi) - c\underline{\varphi}_1'(\xi) + r_1\big[1 - \underline{\varphi}_1(\xi)\big]\big[a_{12}\underline{\varphi}_2(\xi) + a_{13}\underline{\varphi}_3(\xi) - \underline{\varphi}_1(\xi)\big] \\
&\quad = r_1\big[a_{12}\underline{\varphi}_2(\xi) + a_{13}\underline{\varphi}_3(\xi)\big] \\
&\quad \geq 0.
\end{aligned}$$

(5) (i) If $\underline{\varphi}_2(\xi) = 0 > e^{\gamma_{21}\xi} - qe^{(\gamma_{21}+\epsilon)\xi}$, then

$$d_2\underline{\varphi}_2''(\xi) - c\underline{\varphi}_2'(\xi) + r_2\underline{\varphi}_2(\xi)\big[1 - a_{21} + a_{21}\underline{\varphi}_1(\xi) - \underline{\varphi}_2(\xi) - a_{23}\overline{\varphi}_3(\xi)\big] = 0.$$

(ii) If $\underline{\varphi}_2(\xi) = e^{\gamma_{21}\xi} - qe^{(\gamma_{21}+\epsilon)\xi} > 0$, then

$$\underline{\varphi}_2(\xi) < e^{\gamma_{21}\xi}, \qquad \overline{\varphi}_3(\xi) \leq e^{\gamma_{31}\xi} + pa_{32}e^{\gamma\xi}$$

so that

$$\begin{aligned}
&d_2\underline{\varphi}_2''(\xi) - c\underline{\varphi}_2'(\xi) + r_2\underline{\varphi}_2(\xi)\big[1 - a_{21} + a_{21}\underline{\varphi}_1(\xi) - \underline{\varphi}_2(\xi) - a_{23}\overline{\varphi}_3(\xi)\big] \\
&\quad \geq d_2\underline{\varphi}_2''(\xi) - c\underline{\varphi}_2'(\xi) + r_2\underline{\varphi}_2(\xi)\big[1 - a_{21} - \underline{\varphi}_2(\xi) - a_{23}\overline{\varphi}_3(\xi)\big] \\
&\quad \geq d_2\big[\gamma_2^2e^{\gamma_{21}\xi} - q(\gamma_{21} + \epsilon)^2e^{(\gamma_{21}+\epsilon)\xi}\big] - c\big[\gamma_{21}e^{\gamma_{21}\xi} - q(\gamma_{21} + \epsilon)e^{(\gamma_{21}+\epsilon)\xi}\big] \\
&\qquad + r_2\big(e^{\gamma_{21}\xi} - qe^{(\gamma_{21}+\epsilon)\xi}\big)\big[1 - a_{21} - e^{\gamma_{21}\xi} - a_{23}\big(e^{\gamma_{31}\xi} + pa_{32}e^{\gamma\xi}\big)\big] \\
&\quad \geq -\big[d_2(\gamma_{21} + \epsilon)^2 - c(\gamma_{21} + \epsilon) + r_2(1 - a_{21})\big]qe^{(\gamma_{21}+\epsilon)\xi} \\
&\qquad - r_2a_{23}\big[e^{(\gamma_{21}+\gamma_{31})\xi} + pa_{32}e^{(\gamma+\gamma_{21})\xi}\big] - r_2e^{2\gamma_{21}\xi} \\
&\quad \geq -\big[d_2(\gamma_{21} + \epsilon)^2 - c(\gamma_{21} + \epsilon) + r_2(1 - a_{21})\big]qe^{(\gamma_{21}+\epsilon)\xi} \\
&\qquad - r_2e^{(\gamma_{21}+\epsilon)\xi}\big[1 + a_{23}(1 + a_{32}p)\big].
\end{aligned}$$

Let

$$q \geq q_1 = 1 + \frac{-r_2[1 + a_{23}(1 + a_{32}p)]}{d_2(\gamma_{21} + \epsilon)^2 - c(\gamma_{21} + \epsilon) + r_2(1 - a_{21})},$$

then

$$d_2\underline{\varphi}_2''(\xi) - c\underline{\varphi}_2'(\xi) + r_2\underline{\varphi}_2(\xi)\bigl[1 - a_{21} + a_{21}\underline{\varphi}_1(\xi) - \underline{\varphi}_2(\xi) - a_{23}\overline{\varphi}_3(\xi)\bigr] \geq 0.$$

(6) (i) If $\underline{\varphi}_3(\xi) = 0 > e^{\gamma_{31}\xi} - qe^{(\gamma_{31}+\epsilon)\xi}$, then

$$d_3\underline{\varphi}_3''(\xi) - c\underline{\varphi}_3'(\xi) + r_3\underline{\varphi}_3(\xi)\bigl[a_{31} - 1 + a_{32}\underline{\varphi}_2(\xi) - a_{31}\overline{\varphi}_1(\xi) - \underline{\varphi}_3(\xi)\bigr] = 0.$$

(ii) If $\underline{\varphi}_3(\xi) = e^{\gamma_{31}\xi} - qe^{(\gamma_{31}+\epsilon)\xi} > 0$, then

$$\overline{\varphi}_1(\xi) \leq e^{\gamma_{11}\xi} + pe^{\gamma\xi}, \qquad \underline{\varphi}_3(\xi) < e^{\gamma_{31}\xi}$$

so that

$$\begin{aligned}
&d_3\underline{\varphi}_3''(\xi) - c\underline{\varphi}_3'(\xi) + r_3\underline{\varphi}_3(\xi)\bigl[a_{31} - 1 + a_{32}\underline{\varphi}_2(\xi) - a_{31}\overline{\varphi}_1(\xi) - \underline{\varphi}_3(\xi)\bigr] \\
&\quad\geq d_3\underline{\varphi}_3''(\xi) - c\underline{\varphi}_3'(\xi) + r_3\underline{\varphi}_3(\xi)\bigl[a_{31} - 1 - a_{31}\overline{\varphi}_1(\xi) - \underline{\varphi}_3(\xi)\bigr] \\
&\quad\geq d_3\bigl[\gamma_{31}^2 e^{\gamma_{31}\xi} - q(\gamma_{31}+\epsilon)^2 e^{(\gamma_{31}+\epsilon)\xi}\bigr] - c\bigl[\gamma_{31}e^{\gamma_{31}\xi} - q(\gamma_{31}+\epsilon)e^{(\gamma_{31}+\epsilon)\xi}\bigr] \\
&\qquad+ r_3\bigl(e^{\gamma_{31}\xi} - qe^{(\gamma_{31}+\epsilon)\xi}\bigr)\bigl[a_{31} - 1 - a_{31}\bigl(e^{\gamma_{11}\xi} + pe^{\gamma\xi}\bigr) - e^{\gamma_{31}\xi}\bigr] \\
&\quad\geq -\bigl[d_3(\gamma_{31}+\epsilon)^2 - c(\gamma_{31}+\epsilon) + r_3(a_{31}-1)\bigr]qe^{(\gamma_{31}+\epsilon)\xi} \\
&\qquad- r_3a_{31}\bigl[e^{(\gamma_{31}+\gamma_{11})\xi} + pe^{(\gamma_{31}+\gamma)\xi}\bigr] - r_3e^{2\gamma_{31}\xi} \\
&\quad\geq -\bigl[d_3(\gamma_{31}+\epsilon)^2 - c(\gamma_{31}+\epsilon) + r_3(a_{31}-1)\bigr]qe^{(\gamma_{31}+\epsilon)\xi} \\
&\qquad- r_3e^{(\gamma_{31}+\epsilon)\xi}\bigl[1 + a_{31}(1+p)\bigr].
\end{aligned}$$

Let

$$q \geq q_2 = 1 + \frac{-r_3[1 + a_{31}(1+p)]}{d_3(\gamma_{31}+\epsilon)^2 - c(\gamma_{31}+\epsilon) + r_3(a_{31}-1)},$$

then

$$d_3\underline{\varphi}_3''(\xi) - c\underline{\varphi}_3'(\xi) + r_3\underline{\varphi}_3(\xi)\bigl[a_{31} - 1 + a_{32}\underline{\varphi}_2(\xi) - a_{31}\overline{\varphi}_1(\xi) - \underline{\varphi}_3(\xi)\bigr] \geq 0.$$

By what we have done, we first fix $p = p_1 + p_2 + p_3 + p_4$, then let $q = q_1 + q_2$, which completes the proof. □

Summarizing the above, we have the following conclusions.

Theorem 4.6 *Assume that* (4.10) *holds. If* $c > c^*$ *is such that* (4.11)–(4.12) *are true, then* (4.3) *has a nonconstant positive solution.*

About the traveling wave solution, we also give the following remark.

Remark 4.7 By direct calculations in P, we see that $\varphi_i'(\xi), \varphi_i''(\xi), i = 1, 2, 3$, are uniformly bounded.

5 Asymptotic behavior of traveling wave solutions

In this section, we consider the asymptotic behavior of traveling wave solutions of (4.3) by using the idea of contracting rectangles.

Theorem 5.1 *Assume that* (2.3) *holds. If* $\Psi(\xi) = (\varphi_1(\xi), \varphi_2(\xi), \varphi_3(\xi))$ *is a positive solution of* (4.3) *given in Theorem* 4.6, *then* (4.4) *is true.*

Proof According to Theorem 4.6, we have $\lim_{\xi\to-\infty} \Psi(\xi) = \mathbf{0}$. Now we verify

$$\lim_{\xi\to+\infty} \bigl(\varphi_1(\xi), \varphi_2(\xi), \varphi_3(\xi)\bigr) = (1 - k_1, k_2, k_3),$$

which is equivalent to

$$\lim_{\xi\to+\infty} \bigl(\phi_1(\xi), \phi_2(\xi), \phi_3(\xi)\bigr) = (k_1, k_2, k_3). \tag{5.1}$$

By Theorem 4.6, $u_1(x,t) = \phi_1(\xi)$ satisfies

$$\begin{cases} \frac{\partial u_1(x,t)}{\partial t} = d_1 \Delta u_1(x,t) + r_1 u_1(x,t)[1 - u_1(x,t) - a_{12}u_2(x,t) - a_{13}u_3(x,t)], \\ u_1(x,0) = \phi_1(x) > 0, \end{cases}$$

and so

$$\begin{cases} \frac{\partial u_1(x,t)}{\partial t} \ge d_1 \Delta u_1(x,t) + r_1 u_1(x,t)[\underline{u}_1 - u_1(x,t)], \\ u_1(x,0) = \phi_1(x) > 0 \end{cases}$$

for all $x \in \mathbb{R}, t > 0$. Then Lemma 2.3 indicates that

$$\liminf_{t\to\infty} u_1(0,t) \ge \underline{u}_1 > 0,$$

which implies that

$$\liminf_{\xi\to\infty} \phi_1(\xi) \ge \underline{u}_1 > 0.$$

Similarly, we can verify that

$$\liminf_{\xi\to+\infty} \phi_2(\xi) \ge \underline{u}_2 > 0, \qquad \limsup_{\xi\to+\infty} \phi_3(\xi) \le a_{32} + a_{31} - 1.$$

We now verify that

$$\limsup_{\xi\to\infty} \phi_1(\xi) < 1.$$

By what we have done, we only need to show that $\limsup_{\xi\to\infty} \phi_1(\xi) = 1$ is impossible. If $\limsup_{\xi\to\infty} \phi_1(\xi) = 1$, then there exists $\{\xi_m\}_{m=1}^{\infty}$ such that

$$\xi_m \to \infty, \quad \phi_1(\xi_m) \to 1, \quad m \to \infty,$$

and Remark 4.7 indicates that

$$\lim_{m\to\infty}\bigl[d_1\phi_1''(\xi_m) - c\phi_1'(\xi_m)\bigr] \le 0.$$

Moreover, if m is large, then $2\phi_2(\xi_m) \ge \underline{u}_2$ and

$$r_1\phi_1(\xi_m)\bigl[1 - \phi_1(\xi_m) - a_{12}\phi_2(\xi_m) - a_{13}\phi_3(\xi_m)\bigr] < 0$$

and so

$$d_1\phi_1''(\xi_m) - c\phi_1'(\xi_m) + r_1\phi_1(\xi_m)\bigl[1 - \phi_1(\xi_m) - a_{12}\phi_2(\xi_m) - a_{13}\phi_3(\xi_m)\bigr] < 0,$$

which indicates a contradiction.

By a similar discussion, we can prove that

$$a_i(0) < \liminf_{\xi\to\infty}\phi_i(\xi) \le \limsup_{\xi\to\infty}\phi_i(\xi) < b_i(0), \quad i = 1,2,3,$$

where $a_i(0), b_i(0)$ are defined by Lemma 3.1.

If (5.1) does not hold, then there exists some $s_0 \in (0,1)$ such that

$$a_i(s_0) \le \liminf_{\xi\to\infty}\phi_i(\xi) \le \limsup_{\xi\to\infty}\phi_i(\xi) \le b_i(s_0), \quad i = 1,2,3, \tag{5.2}$$

and at least one equality holds.

If $b_1(s_0) = \limsup_{\xi\to\infty}\phi_1(\xi)$, then there exists $\{\xi_m\}_{m=1}^{\infty}$ such that

$$\xi_m \to \infty, \quad \phi_1(\xi_m) \to b_1(s_0), \quad m \to \infty,$$

and

$$\limsup_{m\to\infty}\bigl[d_1\phi_1''(\xi_m) - c\phi_1'(\xi_m)\bigr] \le 0.$$

Moreover, we have

$$\begin{aligned}&\limsup_{m\to\infty}\bigl\{r_1\phi_1(\xi_m)\bigl[1 - \phi_1(\xi_m) - a_{12}\phi_2(\xi_m) - a_{13}\phi_3(\xi_m)\bigr]\bigr\}\\ &\quad < -r_1b_1(s_0)(1-s_0)(a_{12}\underline{u}_2 + a_{13}\underline{u}_3) < 0\end{aligned}$$

by Lemma 3.1, which indicates that

$$d_1\phi_1''(\xi_m) - c\phi_1'(\xi_m) + r_1\phi_1(\xi_m)\bigl[1 - \phi_1(\xi_m) - a_{12}\phi_2(\xi_m) - a_{13}\phi_3(\xi_m)\bigr] < 0$$

if m is large enough. Thus, $b_1(s_0) > \limsup_{\xi\to\infty}\phi_1(\xi)$.

If $\limsup_{\xi\to\infty}\phi_3(\xi) = b_3(s_0)$, then there exists $\{\xi_m\}_{m=1}^{\infty}$ such that

$$\xi_m \to \infty, \quad \phi_3(\xi_m) \to b_3(s_0), \quad m \to \infty,$$

and

$$\limsup_{m\to\infty}\left[d_1\phi_1''(\xi_m)-c\phi_1'(\xi_m)\right]\le 0.$$

Moreover, by Sect. 3 and $b_1(s_0) > \limsup_{\xi\to\infty}\phi_1(\xi)$, we have

$$\begin{aligned}&\limsup_{m\to\infty}\left\{r_3\phi_3(\xi)\left[-1+a_{31}\phi_1(\xi)+a_{32}\phi_2(\xi)-\phi_3(\xi)\right]\right\}\\&\quad< r_3b_3(s_0)\left[-1+a_{31}b_1(s_0)+a_{32}b_2(s_0)-b_3(s_0)\right]=0,\end{aligned}$$

and so

$$d_3\phi_3''(\xi_m)-c\phi_3'(\xi_m)+r_3\phi_3(\xi_m)\left[-1+a_{31}\phi_1(\xi_m)+a_{32}\phi_2(\xi_m)-\phi_3(\xi_m)\right]<0$$

if m is large enough. Thus, $\limsup_{\xi\to\infty}\phi_3(\xi) < b_3(s_0)$. It should be noted that in Sect. 3, we cannot obtain a strict inequality for $b_3(s)$, but we can obtain strict a inequality in the above inequality since $b_1(s_0) > \limsup_{\xi\to\infty}\phi_1(\xi)$.

By similar discussions, we find that every inequality in (5.2) is strict. A contradiction occurs, which implies (5.1). The proof is complete. □

Remark 5.2 Different from Lin and Ruan [31], we did not use a strictly contracting rectangle.

6 Minimal wave speed

In this section, we shall prove that (4.3)–(4.4) has no positive solution if $c < c^*$, which implies that c^* is the minimal wave speed. The method is similar to that in Lin [29] and Lin and Ruan [31].

Theorem 6.1 *If $c < c^*$, then (4.3)–(4.4) has no positive solution.*

Proof If the statement is false, then there exists some $c' < c^*$ such that (4.3)–(4.4) with $c = c'$ has a positive solution $\Psi(\xi) = (\varphi_1(\xi), \varphi_2(\xi), \varphi_3(\xi))$ which satisfies (4.4). We now discuss two cases: $c^* = 2\sqrt{d_2r_2(1-a_{21})}$ and $c^* = 2\sqrt{d_3r_3(a_{31}-1)}$.

If $c^* = 2\sqrt{d_2r_2(1-a_{21})}$, then there exists $\epsilon_1 > 0$ such that

$$2\sqrt{d_2r_2(1-a_{21}-2a_{23}\epsilon_1)}>c'.$$

By the asymptotic boundary condition $\lim_{\xi\to-\infty}\varphi_3(\xi) = 0$ and the strict positivity of solution to (4.3), there exists $\xi_1 \in \mathbb{R}$ such that

$$a_{23}\varphi_3(\xi)<a_{23}\epsilon_1,\quad \xi<\xi_1.$$

When $\xi \ge \xi_1$, the positivity and limit behavior of $\varphi_2(\xi)$ indicate that

$$\inf_{\xi\ge\xi_1}\varphi_2(\xi)>0.$$

Let

$$L = \frac{a_{23}(a_{31} + a_{32} - 1)}{\inf_{\xi \geq \xi_1} \varphi_2(\xi)},$$

then $a_{23}\varphi_3(\xi) \leq L\varphi_2(\xi)$ and

$$1 - a_{21} + a_{21}\varphi_1(\xi) - \varphi_2(\xi) - a_{23}\varphi_3(\xi) > 1 - a_{21} - (L+1)\varphi_2(\xi)$$

for $\xi \geq \xi_1$.

Therefore, we have

$$\begin{cases} \frac{\partial w_2(x,t)}{\partial t} \geq d_1 \Delta w_2(x,t) + r_2 w_2(x,t)(1 - a_{21} - a_{23}\epsilon_1 - (L+1)w_2(x,t)), \\ w_2(x,0) = \varphi_2(x) > 0 \end{cases}$$

for $x \in \mathbb{R}, t > 0$.

Namely, $\varphi_2(x + c't)$ is the upper solution of

$$\begin{cases} \frac{\partial w_2(x,t)}{\partial t} = d_2 \Delta w_2(x,t) + r_2 w_2(x,t)(1 - a_{21} - a_{23}\epsilon_1 - (L+1)w_2(x,t)), \\ w_2(x,0) = \varphi_2(x). \end{cases}$$

By the theory of asymptotic spreading (Lemma 2.3), we see that

$$\liminf_{t\to\infty} \inf_{|x|\leq c_1 t} w_2(x,t) \geq \frac{1 - a_{21} - a_{23}\epsilon_1}{L+1} > 0$$

with $c_1 = 2\sqrt{d_2 r_2(1 - a_{21} - 2a_{23}\epsilon_1)}$. Letting $-x = c_1 t$, one gets

$$\liminf_{t\to\infty} w_2(-c_1 t, t) \geq \frac{1 - a_{21} - a_{23}\epsilon_1}{L+1} > 0.$$

At the same time, we have

$$\xi = x + c't = \left(c' - c_1\right)t \to -\infty, \quad t \to \infty$$

so that

$$\lim_{\xi\to-\infty} \varphi_2(\xi) = \lim_{t\to\infty} w_2(-c_1 t, t) = 0,$$

which gives a contradiction.

Similarly, we can obtain a contradiction if $c^* = 2\sqrt{d_3 r_3(a_{31} - 1)}$.

Thus, for any $c < c^*$, (4.3)–(4.4) has no positive solution. The proof is complete. □

7 Conclusion and discussion

For a parabolic system, if there exists a constant c_0 such that $c \geq c_0$ ($c > c_0$) implies that the system has a desired traveling wave solution while $c < c_0$ ($c \leq c_0$) implies the nonexistence of desired traveling wave solution, then c_0 is the so-called minimal wave speed. In population dynamics, the minimal wave speed is an important threshold [36].

In this paper, we obtain the existence of traveling wave solutions if $c > c^*$ and nonexistence of traveling wave solutions if $c < c^*$. In a weaker sense, we formulate the minimal wave speed. However, we cannot directly confirm the existence or nonexistence of traveling wave solutions with $c = c^*$ by the method in this paper.

Besides the minimal wave speed, spreading speed is also an important threshold, which may equal to the minimal wave speed of traveling wave solutions. For monotone systems, some important results have been established, see Fang et al. [12], Liang and Zhao [27], Lui [33], Weinberger et al. [48]. For predator–prey systems of two species, Lin [29] and Pan [38] proved a similar result, also see Bianca et al. [2, 3]. However, for the predator–prey system with three species, the question of estimating the asymptotic spreading of each species remains open.

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References

1. Aronson, D.G., Weinberger, H.F.: Nonlinear diffusion in population genetics, combustion, and nerve pulse propagation. In: Goldstein, J.A. (ed.) Partial Differential Equations and Related Topics. Lecture Notes in Math., vol. 446, pp. 5–49. Springer, Berlin (1975)
2. Bianca, C., Pappalardo, F., Motta, S., Ragusa, M.A.: Persistence analysis in a Kolmogorov-type model for cancer-immune system competition. AIP Conf. Proc. **1558**, 1797–1800 (2013)
3. Bianca, C., Pennisi, M., Motta, S., Ragusa, M.A.: Immune system network and cancer vaccine. AIP Conf. Proc. **1389**, 945–948 (2011)
4. Cantrell, R.S., Cosner, C.: Spatial Ecology Via Reaction–Diffusion Equations. Wiley, Chichester (2003)
5. Cantrell, R.S., Cosner, C., Hutson, V.: Permanence in some diffusive Lotka–Volterra models for three interacting species. Dyn. Syst. Appl. **2**, 505–530 (1993)
6. Caristi, G., Rybakowski, K.P., Wessolek, T.: Persistence and spatial patterns in a one-predator–two-prey Lotka–Volterra model with diffusion. Ann. Mat. Pura Appl. **161**, 345–377 (1992)
7. Chen, Y.Y., Guo, J.S., Yao, C.H.: Traveling wave solutions for a continuous and discrete diffusive predator–prey model. J. Math. Anal. Appl. **445**, 212–239 (2014)
8. Du, Z., Xu, D.: Traveling wave solution for a reaction-diffusion competitive–cooperative system with delays. Bound. Value Probl. **46**, 14 (2016)
9. Dunbar, S.R.: Travelling wave solutions of diffusive Lotka–Volterra equations. J. Math. Biol. **17**, 11–32 (1983)
10. Dunbar, S.R.: Traveling wave solutions of diffusive Lotka–Volterra equations: a heteroclinic connection in $\mathbb{R}^4$. Transl. Am. Math. Soc. **286**, 557–594 (1984)
11. Dunbar, S.R.: Traveling waves in diffusive predator-prey equations: periodic orbits and point-to-periodic heteroclinic orbits. SIAM J. Appl. Math. **46**, 1057–1078 (1986)

12. Fang, J., Yu, X., Zhao, X.: Traveling waves and spreading speeds for time-space periodic monotone systems. J. Funct. Anal. **272**, 4222–4262 (2017)
13. Fife, P.C.: Mathematical Aspects of Reacting and Diffusing Systems. Springer, Berlin (1979)
14. Fisher, R.A.: The wave of advance of advantageous genes. Ann. Eugen. **7**, 355–369 (1937)
15. Gardner, R., Jones, C.K.R.T.: Stability of travelling wave solutions of diffusive predator–prey systems. Transl. Am. Math. Soc. **327**, 465–524 (1991)
16. Gardner, R., Smoller, J.: The existence of periodic travelling waves for singularly perturbed predator–prey equations via the Conley index. J. Differ. Equ. **47**, 133–161 (1983)
17. Ghergu, M., Radulescu, V.D.: Nonlinear PDEs. Mathematical Models in Biology, Chemistry and Population Genetics. Springer Monographs in Mathematics. Springer, Heidelberg (2012)
18. Hsu, C.H., Yang, C.R., Yang, T.H., Yang, T.Z.: Existence of traveling wave solutions for diffusive predator-prey type systems. J. Differ. Equ. **252**, 3040–3075 (2012)
19. Huang, J., Lu, G., Ruan, S.: Existence of traveling wave solutions in a diffusive predator–prey model. J. Math. Biol. **46**, 132–152 (2003)
20. Huang, J., Zou, X.: Existence of traveling wavefronts of delayed reaction diffusion systems without monotonicity. Discrete Contin. Dyn. Syst. **9**, 925–936 (2003)
21. Huang, J., Zou, X.: Traveling wave solutions in delayed reaction diffusion systems with partial monotonicity. Acta Math. Appl. Sin. **22**, 243–256 (2006)
22. Huang, W.: Problem on minimum wave speed for a Lotka–Volterra reaction-diffusion competition model. J. Dyn. Differ. Equ. **22**, 285–297 (2010)
23. Huang, W.: Traveling wave solutions for a class of predator–prey systems. J. Dyn. Differ. Equ. **24**, 633–644 (2012)
24. Huang, Y.L., Lin, G.: Traveling wave solutions in a diffusive system with two preys and one predator. J. Math. Anal. Appl. **418**, 163–184 (2014)
25. Kolmogorov, A.N., Petrovsky, I.G., Piskunov, N.S.: Study of the diffusion equation with growth of the quantity of matter and its application to a biological problem. Byul. Mosk. Gos. Univ. Ser. A: Mat. Mekh **1**, 1–26 (1937)
26. Li, K., Li, X.: Travelling wave solutions in diffusive and competition–cooperation systems with delays. IMA J. Appl. Math. **74**, 604–621 (2009)
27. Liang, X., Zhao, X.Q.: Asymptotic speeds of spread and traveling waves for monotone semiflows with applications. Commun. Pure Appl. Math. **60**, 1–40 (2007)
28. Lin, G.: Spreading speeds of a Lotka–Volterra predator–prey system: the role of the predator. Nonlinear Anal. **74**, 2448–2461 (2011)
29. Lin, G.: Invasion traveling wave solutions of a predator–prey system. Nonlinear Anal. **96**, 47–58 (2014)
30. Lin, G., Li, W.T., Ma, M.: Travelling wave solutions in delayed reaction diffusion systems with applications to multi-species models. Discrete Contin. Dyn. Syst., Ser. B **13**, 393–414 (2010)
31. Lin, G., Ruan, S.: Traveling wave solutions for delayed reaction-diffusion systems and applications to diffusive Lotka–Volterra competition models with distributed delays. J. Dyn. Differ. Equ. **26**, 583–605 (2014)
32. Lin, X., Weng, P., Wu, C.: Traveling wave solutions for a predator–prey system with sigmoidal response function. J. Dyn. Differ. Equ. **23**, 903–921 (2011)
33. Lui, R.: Biological growth and spread modeled by systems of recursions. I. Mathematical theory. Math. Biosci. **93**, 269–295 (1989)
34. Ma, S.: Traveling wavefronts for delayed reaction–diffusion systems via a fixed point theorem. J. Differ. Equ. **171**, 294–314 (2001)
35. Murray, J.D.: Mathematical Biology, I. An Introduction, 3rd edn. Interdisciplinary Applied Mathematics, vol. 17. Springer, New York (2002)
36. Murray, J.D.: Mathematical Biology, II. Spatial Models and Biomedical Applications, 3rd edn. Interdisciplinary Applied Mathematics, vol. 18. Springer, New York (2003)
37. Pan, S.: Convergence and traveling wave solutions for a predator–prey system with distributed delays. Mediterr. J. Math. **14**, Article ID 103 (2017)
38. Pan, S.: Invasion speed of a predator–prey system. Appl. Math. Lett. **74**, 46–51 (2017)
39. Pao, C.V.: Nonlinear Parabolic and Elliptic Equations. Plenum Press, New York (1992)
40. Papageorgiou, N.S., Rădulescu, V.D., Repovš, D.D.: Sensitivity analysis for optimal control problems governed by nonlinear evolution inclusions. Adv. Nonlinear Anal. **6**, 199–235 (2017)
41. Shang, X., Du, Z., Lin, X.: Traveling wave solutions of *n*-dimensional delayed reaction–diffusion systems and application to four-dimensional predator–prey systems. Math. Methods Appl. Sci. **39**, 1607–1620 (2016)
42. Smith, H.L.: Monotone Dynamical System: An Introduction to the Theory of Competitive and Cooperative System. AMS, Providence (1995)
43. Smoller, J.: Shock Waves and Reaction–Diffusion Equations. Springer, New York (1983)
44. Volpert, A.I., Volpert, V.A.: Traveling-wave solutions of parabolic systems with discontinuous nonlinear terms. Nonlinear Anal. **49**, 113–139 (2002)
45. Wang, X.S., Wang, H.Y., Wu, J.: Traveling waves of diffusive predator–prey systems: disease outbreak propagation. Discrete Contin. Dyn. Syst. **32**, 3303–3324 (2012)
46. Wang, Z., Li, W.T., Ruan, S.: Traveling wave fronts of reaction–diffusion systems with spatio-temporal delays. J. Differ. Equ. **222**, 185–232 (2006)
47. Wang, Z.C., Zhang, L., Zhao, X.Q.: Time periodic traveling waves for a periodic and diffusive SIR epidemic model. J. Dyn. Differ. Equ. **30**, 379–403 (2018)
48. Weinberger, H.F., Lewis, M.A., Li, B.: Analysis of linear determinacy for spread in cooperative models. J. Math. Biol. **45**, 183–218 (2002)
49. Wu, J., Zou, X.: Traveling wave fronts of reaction–diffusion systems with delay. J. Dyn. Differ. Equ. **13**, 651–687 (2001)
50. Ye, Q., Li, Z., Wang, M., Wu, Y.: Introduction to Reaction–Diffusion Equations. Science Press, Beijing (2011)
51. Zhang, G., Li, W.T., Lin, G.: Traveling waves in delayed predator–prey systems with nonlocal diffusion and stage structure. Math. Comput. Model. **49**, 1021–1029 (2009)
52. Zhang, J.Y., Feng, B.Y.: Geometrical Theory and Bifurcation Problems of Ordinary Differential Equations. Peking University Press, Beijing (2000)

53. Zhang, Z., Yang, T.-H., Wang, W.: Traveling wave solutions of Lotka–Volterra type two predators-one prey model. Math. Methods Appl. Sci. **39**, 5395–5408 (2016)
54. Zhao, X.Q.: Dynamical Systems in Population Biology. Springer, New York (2003)
55. Zhao, X.Q.: Spatial dynamics of some evolution systems in biology. In: Du, Y., Ishii, H., Lin, W.Y. (eds.) Recent Progress on Reaction–Diffusion Systems and Viscosity Solutions, pp. 332–363. World Scientific, Singapore (2009)