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Weighted integral inequality and applications in general energy decay estimate for a variable density wave equation with memory

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Abstract

This paper develops a weighted integral inequality to derive decay estimates for the quasilinear viscoelastic wave equation with variable density

$$|u_t|^\rho u_{tt} - \Delta u - \Delta u_{tt} + \int_0^t g(t-s)\Delta u(s)\,ds = 0 \quad \text{in } \Omega \times (0,\infty)$$

with initial conditions and boundary condition, where g is a memory kernel function and ρ is a positive constant. Depending on the properties of convolution kernel g at infinity, we establish a general decay rate of the solution such that the exponential and polynomial decay results in some literature are special cases of this paper, and we improve the integral method used in the literature.

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Keywords: Weighted integral inequality; Viscoelastic equation; Memory kernel; General decay estimate

1 Introduction

Using Lyapunov technique for some perturbed energy, Messaoudiet and Khulaifi [24] studied the following problem:

$$\begin{cases} |u_t|^\rho u_{tt} - \Delta u - \Delta u_{tt} + \int_0^t g(t-s)\Delta u(s)\,ds = 0 & \text{in } \Omega \times (0,\infty), \\ u = 0 \quad \text{on } \partial\Omega \times (0,\infty), & \\ u|_{t=0} = u_0, \qquad u_t|_{t=0} = u_1 & \text{in } \Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$ $(n \geq 1)$ is a bounded domain with smooth boundary $\partial\Omega$ such that the divergence theorem can be applied.

Cavalcanti et al. [3] proved that the finite energy solutions of nonlinear abstract PDE with a memory term exhibit exponential decay rates when strong damping $-\Delta u_t$ is active, this uniform decay is no longer valid (by spectral analysis arguments) for dynamics subjected to frictional damping only, say $g = 0$. Viscoelastic equations with variable density have been studied by many authors, and several stability results have been established in [2, 9, 10, 23, 25, 26]. As we know, all the stability results were obtained by establishing differential inequalities on the functional equivalent to the original energy. Our approach is based on integral inequalities and multiplier techniques. Indeed, instead of using Lyapunov technique for some perturbed energy, we rather concentrate on the original energy, showing that it satisfies a weighted integral inequality which, in turn, yields the final decay estimate. We prove a general decay rate from which the exponential decay and the polynomial decay are only special cases. Due to the assumption on g, the weighted inequality established in this paper improves the integral inequality in [1].

We mention here some related works concerning the energy for the evolution equations. For the nonlinear damped wave equations and Marguerre–von Karman system, some energy decay rate estimates were obtained in [11–15, 19, 20] and the references therein. Li and his coauthors [7, 16–18] studied the blow up phenomena of the solutions for evolution equations. This research laid a good foundation for our further study. For the stability and convergence results of evolution equations, the readers can refer to [6, 21, 22].

The outline of this paper is as follows. In Sect. 2, we present the preliminaries and our important result. In Sect. 3, we construct an energy inequality, prove the main Theorem 2.2 and give applications to various functions $\xi(t)$.

To simplify calculations in our analysis, we introduce the following notations:

$$G(t) = \int_0^t g(s)\,ds \text{ – strength of memory}, \qquad g \circ h = \int_0^t g(t-s)\big\|h(s)-h(t)\big\|^2\,ds,$$

$$g * h = \int_0^t g(t-s)h(s)\,ds, \qquad (u,v) = \int_\Omega uv\,dx, \qquad \mathbb{R}^+ := [0,+\infty),$$

$$\|v\|_{\rho+2} := \|v\|_{L^{\rho+2}(\Omega)}, \qquad \|v\| := \|v\|_{L^2(\Omega)}.$$

2 Preliminaries and main result

In this section we prepare some material needed in the proof of our result and state our main result. Throughout this paper, C denotes a generic positive constant. We impose the following assumptions on ρ and g.

Assumption 2.1 Set $G(t) = \int_0^t g(s)\,ds$. We assume that

1.
$$0 < \rho \le \frac{2}{n-2} \quad \text{if } n \ge 3; \qquad \rho > 0 \quad \text{if } n = 1, 2,$$
   which implies that
$$H_0^1(\Omega) \hookrightarrow L^{2(\rho+1)}(\Omega).$$

2. $g : [0,\infty) \to [0,\infty)$ is a locally absolutely continuous function such that
$$G(\infty) < 1, \qquad g(0) > 0, \qquad g'(t) \le 0, \quad \text{for a.e. } t \ge 0.$$

3. There exists a non-increasing function $\xi \in C^1[0,+\infty)$ with $\int_0^{+\infty} \xi(\tau)\,d\tau = +\infty$ satisfying
$$g'(t) \le -\xi(t)g(t), \qquad \xi(t) > 0, \quad \forall t \ge 0.$$

Theorem 2.1 ([24]) *Suppose that* $(u_0, u_1) \in H_0^1(\Omega) \times H_0^1(\Omega)$. *Under Assumption* 2.1, *there exists a unique solution* u *of* (1.1) *satisfying*

$$u \in L^\infty\bigl(0,\infty; H_0^1(\Omega)\bigr), \qquad u_t \in L^\infty\bigl(0,\infty; H_0^1(\Omega)\bigr), \qquad u_{tt} \in L^\infty\bigl(0,\infty; L^2(\Omega)\bigr),$$

and

$$u(x,t) \to u_0(x) \quad \text{in } H_0^1(\Omega), \qquad u_t(x,t) \to u_1(x) \quad \text{in } H_0^1(\Omega), \text{as } t \to 0^+.$$

Lemma 2.1 ([4, 5, 8]) $H_0^1(\Omega) \hookrightarrow L^r(\Omega)$ *with*

$$r: \begin{cases} 0 < r \le \frac{2n}{n-2}, & n \ge 3, \\ \ge 2, & n = 1, 2, \end{cases}$$

which implies

$$\|\varphi\|_r \le B\|\nabla\varphi\|_2, \quad \forall \varphi \in H_0^1(\Omega).$$

Lemma 2.2 *Let* u *be the global solution of the problem* (1.1), *then for any suitable function* w *one has*

$$\bigl(|u_t|^\rho u_{tt}, w\bigr) = \frac{1}{\rho+1}\frac{d}{dt}\bigl(|u_t|^\rho u_t, w\bigr) - \frac{1}{\rho+1}\bigl(|u_t|^\rho u_t, w_t\bigr).$$

Proof From

$$\begin{aligned} \frac{d}{dt}\bigl(|u_t|^\rho u_t, w\bigr) &= \frac{d}{dt}\bigl(\bigl(u_t^2\bigr)^{\frac{\rho}{2}} u_t, w\bigr) \\ &= \bigl(|u_t|^\rho u_{tt}, w\bigr) + \frac{\rho}{2}\bigl(\bigl(u_t^2\bigr)^{\frac{\rho-2}{2}} 2u_t u_{tt} u_t, w\bigr) + \bigl(|u_t|^\rho u_t, w_t\bigr) \\ &= (\rho+1)\bigl(|u_t|^\rho u_{tt}, w\bigr) + \bigl(|u_t|^\rho u_t, w_t\bigr), \end{aligned}$$

we have

$$\bigl(|u_t|^\rho u_{tt}, w\bigr) = \frac{1}{\rho+1}\frac{d}{dt}\bigl(|u_t|^\rho u_t, w\bigr) - \frac{1}{\rho+1}\bigl(|u_t|^\rho u_t, w_t\bigr).$$ □

Lemma 2.3 *Let* u *be the global solution of the problem* (1.1), *then*

$$\frac{d}{dt}E(t) = \frac{1}{2}g' \circ \nabla u(t) - \frac{1}{2}g(t)\|\nabla u\|^2,$$

where

$$E(t) = \frac{1}{\rho+2}\|u_t\|_{\rho+2}^{\rho+2} + \frac{1}{2}\bigl(1 - G(t)\bigr)\bigl\|\nabla u(t)\bigr\|^2 + \frac{1}{2}\|\nabla u_t\|^2 + \frac{1}{2}(g \circ \nabla u)(t).$$

Proof Multiplying equation (1.1) by u_t, integrating by parts over Ω and using Lemma 2.2, we obtain the conclusion. □

Our main result is the following decay theorem.

Theorem 2.2 *Let u be the global solution of problem* (1.1) *with Assumption* 2.1. *We define the energy functional as*

$$E(t) = \frac{1}{\rho+2}\|u_t\|_{\rho+2}^{\rho+2} + \frac{1}{2}\bigl(1-G(t)\bigr)\bigl\|\nabla u(t)\bigr\|^2 + \frac{1}{2}\|\nabla u_t\|^2 + \frac{1}{2}(g\circ\nabla u)(t).$$

Then, for some $t_0 > 0$ there exist positive constants C_0 and ω such that

$$E(t) \le C_0 e^{-\omega\int_{t_0}^t \xi(s)\,ds}.$$

3 The proof of main result

In order to derive the desired result of Theorem 2.2 by the integral method, we establish the following weighted integral inequality.

Lemma 3.1 *Let u be the solution of* (1.1) *under Assumption* 2.1, *then*

$$\int_S^T \xi(t)E(t)\,dt \le CE(S)$$

for some constant $C > 0$.

To prove the above inequality, we need the following two lemmas.

Lemma 3.2 *Let u be the solution of* (1.1) *under Assumption* 2.1, *then*

$$\int_S^T \xi(t)E(t)\,dt \le C\int_S^T \xi(t)\|u_t\|^2\,dt + C\int_S^T \xi(t)\|\nabla u_t\|^2\,dt + CE(S).$$

Proof Multiplying by $\xi(t)u(t)$ both sides of equation (1.1), integrating the resulting equation over $\Omega\times[S,T]$ $(0\le S\le T)$, then using the boundary conditions and Lemma 2.2, we have

$$\begin{aligned}
0 &= \int_S^T \xi(t)\left(u, |u_t|^\rho u_{tt} - \Delta u - \Delta u_{tt} + \int_0^t g(t-s)\Delta u(s)\,ds\right)dt \\
&= \int_S^T \xi(t)\bigl(u,|u_t|^\rho u_{tt}\bigr)\,dt + \int_S^T \xi(t)\|\nabla u\|^2\,dt + \int_S^T \xi(t)(\nabla u,\nabla u_{tt})\,dt \\
&\quad - \int_S^T \xi(t)\left(\nabla u, \int_0^t g(t-s)\nabla u(s)\,ds\right)dt \\
&= \int_S^T \xi(t)\bigl(u,|u_t|^\rho u_{tt}\bigr)\,dt + \int_S^T \xi(t)\bigl(1-G(t)\bigr)\|\nabla u\|^2\,dt + \int_S^T \xi(t)(\nabla u,\nabla u_{tt})\,dt \\
&\quad - \int_S^T \xi(t)\left(\nabla u, \int_0^t g(t-s)\bigl(\nabla u(s)-\nabla u(t)\bigr)\,ds\right)dt.
\end{aligned} \tag{3.1}$$

According to the definition of the energy functional $E(t)$, we get

$$\bigl(1-G(t)\bigr)\|\nabla u\|^2 = 2E(t) - \frac{2}{\rho+2}\|u_t\|_{\rho+2}^{\rho+2} - \|\nabla u_t\|^2 - g\circ\nabla u(t). \tag{3.2}$$

Combining (3.1) with (3.2), we deduce that

$$\begin{aligned}\int_S^T \xi(t)E(t)\,dt &= \frac{1}{\rho+2}\int_S^T \xi(t)\|u_t\|_{\rho+2}^{\rho+2}\,dt + \frac{1}{2}\int_S^T \xi(t)\|\nabla u_t\|^2\,dt \\ &\quad + \frac{1}{2}\int_S^T \xi(t)(g\circ\nabla u)(t)\,dt \\ &\quad - \frac{1}{2}\int_S^T \xi(t)\big(u,|u_t|^\rho u_{tt}\big)\,dt - \frac{1}{2}\int_S^T \xi(t)(\nabla u,\nabla u_{tt})\,dt \\ &\quad + \frac{1}{2}\int_S^T \xi(t)\left(\nabla u,\int_0^t g(t-s)\big(\nabla u(s)-\nabla(t)\big)\,ds\right)dt. \end{aligned} \tag{3.3}$$

From Lemma 2.3, we see that

$$\frac{d}{dt}E(t) = \frac{1}{2}g'\circ\nabla u(t) - \frac{1}{2}g(t)\|\nabla u\|^2 \le \frac{1}{2}g'\circ\nabla u(t),$$

that is,

$$-g'\circ\nabla u(t) \le -2E'(t),$$

which, together with Assumption 2.1, implies

$$\int_S^T \xi(t)(g\circ\nabla u)(t)\,dt \le -\int_S^T \big(g'\circ\nabla u\big)(t)\,dt \le -2\int_S^T E'(t)\,dt. \tag{3.4}$$

For the fourth term on the right-hand side of (3.3), integrating by parts and using Lemma 2.2, we have

$$\begin{aligned}-\int_S^T \xi(t)\big(u,|u_t|^\rho u_{tt}\big)\,dt &= -\frac{1}{\rho+1}\big(\xi(t)u,|u_t|^\rho u_t\big)\Big|_S^T + \frac{1}{\rho+1}\int_S^T \big(\big(\xi(t)u\big)_t,|u_t|^\rho u_t\big)\,dt \\ &= -\frac{1}{\rho+1}\big(\xi(t)u,|u_t|^\rho u_t\big)\Big|_S^T + \frac{1}{\rho+1}\int_S^T \big(\xi'(t)u,|u_t|^\rho u_t\big)\,dt \\ &\quad + \frac{1}{\rho+1}\int_S^T \xi(t)\|u_t\|_{\rho+2}^{\rho+2}\,dt. \end{aligned} \tag{3.5}$$

By Young inequality, Lemma 2.1 and the definition of $E(t)$, we have

$$\begin{aligned}\left|-\frac{1}{\rho+1}\big(\xi(t)u,|u_t|^\rho u_t\big)\right| &\le \frac{1}{\rho+1}\left[\xi(t)\left(\frac{1}{\rho+2}\|u\|_{\rho+2}^{\rho+2} + \frac{\rho+1}{\rho+2}\|u_t\|_{\rho+2}^{\rho+2}\right)\right] \\ &\le \xi(t)\left(\frac{1}{(\rho+1)(\rho+2)}\left(\frac{2}{1-G(\infty)}\right)^{\frac{\rho}{2}+1} B^{\rho+2}E^{\frac{\rho}{2}}(0) + 1\right)E(t) \\ &\le k_1\xi(t)E(t), \end{aligned}$$

with some positive constant k_1. Hence,

$$\left|-\frac{1}{\rho+1}\big(\xi(t)u,|u_t|^\rho u_t\big)\Big|_S^T\right| \le 2k_1\xi(0)E(S). \tag{3.6}$$

Similarly,

$$\left|\frac{1}{\rho+1}\int_S^T \bigl(\xi'(t)u, |u_t|^\rho u_t\bigr)\,dt\right| \le k_1\int_S^T \bigl|\xi'(t)\bigr|E(t)\,dt = -k_1\int_S^T \xi'(t)E(t)\,dt. \tag{3.7}$$

For the fifth term on the right-hand side of (3.3), integrating by parts, we have

$$\begin{aligned}
&-\int_S^T \xi(t)(\nabla u, \nabla u_{tt})\,dt \\
&\quad= -\bigl(\xi(t)\nabla u, \nabla u_t\bigr)\Big|_S^T + \int_S^T \bigl(\bigl(\xi(t)\nabla u\bigr)_t, \nabla u_t\bigr)\,dt \\
&\quad\le \xi(t)\left(\frac{1}{2}\|\nabla u\|^2 + \frac{1}{2}\|\nabla u_t\|^2\right)\Big|_S^T + \int_S^T \bigl(\xi'(t)\nabla u, \nabla u_t\bigr)\,dt + \int_S^T \xi(t)\|\nabla u_t\|^2\,dt \\
&\quad\le \left(\frac{1}{1-G(\infty)}+1\right)\left[2\xi(0)E(S) + \int_S^T \bigl|\xi'(t)\bigr|E(t)\,dt\right] + \int_S^T \xi(t)\|\nabla u_t\|^2\,dt \\
&\quad\le 2k_2\xi(0)E(S) - k_2\int_S^T \xi'(t)E(t)\,dt + \int_S^T \xi(t)\|\nabla u_t\|^2\,dt,
\end{aligned} \tag{3.8}$$

with some positive constant k_2.

For the sixth term on the right-hand side of (3.3), we have

$$\begin{aligned}
&\left(\nabla u, \int_0^t g(t-s)\bigl(\nabla u(s) - \nabla u(t)\bigr)\,ds\right) \\
&\quad\le \varepsilon\|\nabla u\|^2 + \frac{1}{4\varepsilon}\int_\Omega\left(\int_0^t g(t-s)\bigl(\nabla u(s) - \nabla u(t)\bigr)\,ds\right)^2 dx \\
&\quad\le \varepsilon\|\nabla u\|^2 + \frac{1}{4\varepsilon}\int_0^t g(s)\,ds\int_\Omega\int_0^t g(t-s)\bigl|\nabla u(s) - \nabla u(t)\bigr|^2\,ds\,dx \\
&\quad\le \frac{2\varepsilon}{1-G(\infty)}E(t) + \frac{G(\infty)}{2\varepsilon}(g\circ\nabla u)(t).
\end{aligned}$$

Combining with (3.4), we obtain

$$\begin{aligned}
&\int_S^T \xi(t)\left(\nabla u(t), \int_0^t g(t-s)\bigl(\nabla u(s) - \nabla u(t)\bigr)\,ds\right) dt \\
&\quad\le \frac{2\varepsilon}{1-G(\infty)}\int_S^T \xi(t)E(t)\,dt - \frac{G(\infty)}{\varepsilon}\int_S^T E'(t)\,dt.
\end{aligned} \tag{3.9}$$

By (3.3)–(3.9), we get

$$\begin{aligned}
\int_S^T \xi(t)E(t)\,dt \le{}& \left(\frac{1}{\rho+2} + \frac{1}{2(\rho+1)}\right)\int_S^T \xi(t)\|u_t\|_{\rho+2}^{\rho+2}\,dt + \frac{3}{2}\int_S^T \xi(t)\|\nabla u_t\|^2\,dt \\
&+ \frac{\varepsilon}{1-G(\infty)}\int_S^T \xi(t)E(t)\,dt - \frac{1}{2}(k_1+k_2)\int_S^T \xi'(t)E(t)\,dt \\
&- \left(1 + \frac{G(\infty)}{2\varepsilon}\right)\int_S^T E'(t)\,dt + (k_1+k_2)\xi(0)E(S).
\end{aligned} \tag{3.10}$$

Integrating by parts (and noting $E'(t) \le 0$), we have

$$\begin{aligned} -\int_S^T \xi'(t)E(t)\,dt &= -\xi(t)E(t)|_S^T + \int_S^T \xi(t)E'(t)\,dt \\ &= -\xi(T)E(T) + \xi(S)E(S) + \int_S^T \xi(t)E'(t)\,dt \\ &\le \xi(0)E(S), \end{aligned} \tag{3.11}$$

and

$$-\int_S^T E'(t)\,dt = E(S) - E(T) \le E(S). \tag{3.12}$$

Owing to (3.10)–(3.12), we get

$$\begin{aligned} &\int_S^T \xi(t)E(t)\,dt \\ &\quad\le \left(\frac{1}{\rho+2} + \frac{1}{2(\rho+1)}\right)\int_S^T \xi(t)\|u_t\|_{\rho+2}^{\rho+2}\,dt + \frac{3}{2}\int_S^T \xi(t)\|\nabla u_t\|^2\,dt \\ &\qquad + \frac{\varepsilon}{1-G(\infty)}\int_S^T \xi(t)E(t)\,dt + \left(1 + \frac{G(\infty)}{2\varepsilon} + \frac{3}{2}(k_1+k_2)\xi(0)\right)E(S). \end{aligned} \tag{3.13}$$

Choosing ε small enough, we obtain from (3.13) that

$$\int_S^T \xi(t)E(t)\,dt \le C\int_S^T \xi(t)\|u_t\|^2\,dt + C\int_S^T \xi(t)\|\nabla u_t\|^2\,dt + CE(S).$$

The proof of Lemma 3.2 is completed. □

Lemma 3.3 *Let u be the solution of* (1.1) *under Assumption* 2.1, *then*

$$\int_S^T \xi(t)\|u_t\|_{\rho+2}^{\rho+2}\,dt + \int_S^T \xi(t)\|\nabla u_t\|^2\,dt \le \varepsilon C\int_S^T \xi(t)E(t)\,dt + C(\varepsilon)E(S).$$

Proof Multiplying by $\xi(t)\int_0^t g(t-s)(u(s)-u(t))\,ds$ both sides of equation (1.1) and then integrating the resulting equation over $\Omega \times [S,T]$ $(0 \le S \le T)$ gives

$$\begin{aligned} &\int_S^T \left(\xi(t)\int_0^t g(t-s)\big(u(s)-u(t)\big)\,ds,\right. \\ &\quad \left.|u_t|^\rho u_{tt} - \Delta u - \Delta u_{tt} + \int_0^t g(t-s)\Delta u(s)\,ds\right)dt = 0. \end{aligned} \tag{3.14}$$

Integrating by parts and using Lemma 2.2, we obtain

$$\begin{aligned} &\int_S^T \left(|u_t|^\rho u_{tt}, \xi(t)\int_0^t g(t-s)\big(u(s)-u(t)\big)\,ds\right)dt \\ &\quad = \frac{1}{\rho+1}\left(|u_t|^\rho u_t, \xi(t)\int_0^t g(t-s)\big(u(s)-u(t)\big)\,ds\right)\Big|_S^T \end{aligned}$$

$$
\begin{aligned}
&- \frac{1}{\rho+1}\int_S^T \biggl(|u_t|^\rho u_t, \xi'(t)\int_0^t g(t-s)\bigl(u(s)-u(t)\bigr)\,ds \biggr)\,dt \\
&- \frac{1}{\rho+1}\int_S^T \biggl(|u_t|^\rho u_t, \xi(t)\int_0^t g'(t-s)\bigl(u(s)-u(t)\bigr)\,ds \biggr)\,dt \\
&+ \frac{1}{\rho+1}\int_S^T \bigl(\xi(t)G(t)\bigr)\bigl\|u_t(t)\bigr\|_{\rho+2}^{\rho+2}\,dt.
\end{aligned}
$$

Moreover, we have

$$
\begin{aligned}
&\int_S^T \biggl(\xi(t)\int_0^t g(t-s)\bigl(u(s)-u(t)\bigr)\,ds, -\Delta u_{tt} \biggr)\,dt \\
&\quad = \int_S^T \biggl(\xi(t)\int_0^t g(t-s)\bigl(\nabla u(s)-\nabla u(t)\bigr)\,ds, \nabla u_{tt} \biggr)\,dt \\
&\quad = \biggl(\nabla u_t, \xi(t)\int_0^t g(t-s)\bigl(\nabla u(s)-\nabla u(t)\bigr)\,ds \biggr)\bigg|_S^T \\
&\qquad - \int_S^T \biggl(\nabla u_t, \xi'(t)\int_0^t g(t-s)\bigl(\nabla u(s)-\nabla u(t)\bigr)\,ds \biggr)\,dt \\
&\qquad - \int_S^T \biggl(\nabla u_t, \xi(t)\int_0^t g'(t-s)\bigl(\nabla u(s)-\nabla u(t)\bigr)\,ds \biggr)\,dt + \int_S^T \xi(t)G(t)\|\nabla u_t\|^2\,dt
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_S^T \biggl(\xi(t)\int_0^t g(t-s)\bigl(u(s)-u(t)\bigr)\,ds, -\Delta u + \int_0^t g(t-s)\Delta u(s)\,ds \biggr)\,dt \\
&\quad = \int_S^T \biggl(\xi(t)\int_0^t g(t-s)\bigl(\nabla u(s)-\nabla u(t)\bigr)\,ds, \nabla u(t)\,dx \biggr)\,dt \\
&\qquad - \int_S^T \biggl(\xi(t)\int_0^t g(t-s)\bigl(\nabla u(s)-\nabla u(t)\bigr)\,ds, \int_0^t g(t-s)\nabla u(s)\,ds \biggr)\,dt \\
&\quad = -\int_S^T \xi(t)\biggl\| \int_0^t g(t-s)\bigl(\nabla u(s)-\nabla u(t)\bigr)\,ds \biggr\|^2\,dt \\
&\qquad + \int_S^T \xi(t)\bigl(1-G(t)\bigr)\biggl(\int_0^t g(t-s)\bigl(\nabla u(s)-\nabla u(t)\bigr)\,ds, \nabla u(t) \biggr)\,dt.
\end{aligned}
$$

Therefore, plugging the above three identities into (3.14), we get

$$
\begin{aligned}
&\frac{1}{\rho+1}\int_S^T \bigl(\xi(t)G(t)\bigr)\bigl\|u_t(t)\bigr\|_{\rho+2}^{\rho+2}\,dt + \int_S^T \xi(t)G(t)\|\nabla u_t\|^2\,dt \\
&\quad = -\frac{1}{\rho+1}\biggl(|u_t|^\rho u_t, \xi(t)\int_0^t g(t-s)\bigl(u(s)-u(t)\bigr)\,ds \biggr)\bigg|_S^T \\
&\qquad + \frac{1}{\rho+1}\int_S^T \biggl(|u_t|^\rho u_t, \xi'(t)\int_0^t g(t-s)\bigl(u(s)-u(t)\bigr)\,ds \biggr)\,dt \\
&\qquad + \frac{1}{\rho+1}\int_S^T \biggl(|u_t|^\rho u_t, \xi(t)\int_0^t g'(t-s)\bigl(u(s)-u(t)\bigr)\,ds \biggr)\,dt \\
&\qquad - \biggl(\nabla u_t, \xi(t)\int_0^t g(t-s)\bigl(\nabla u(s)-\nabla u(t)\bigr)\,ds \biggr)\bigg|_S^T \\
&\qquad + \int_S^T \biggl(\nabla u_t, \xi'(t)\int_0^t g(t-s)\bigl(\nabla u(s)-\nabla u(t)\bigr)\,ds \biggr)
\end{aligned}
$$

$$
\begin{aligned}
&+\int_S^T\Big(\nabla u_t,\xi(t)\int_0^t g'(t-s)\big(\nabla u(s)-\nabla u(t)\big)\,ds\Big)\\
&+\int_S^T\xi(t)\Big\|\int_0^t g(t-s)\big(\nabla u(s)-\nabla u(t)\big)\,ds\Big\|^2\,dt\\
&-\int_S^T\xi(t)\big(1-G(t)\big)\Big(\int_0^t g(t-s)\big(\nabla u(s)-\nabla u(t)\big),\nabla u(t)\,dx\Big)\,dt. \qquad (3.15)
\end{aligned}
$$

Using Cauchy and Hölder inequalities as well as Lemma 2.1, we have

$$
\begin{aligned}
&\Big|\Big(|u_t|^\rho u_t,\int_0^t g(t-s)\big(u(s)-u(t)\big)\,ds\Big)\Big|\\
&\le\frac{\rho+1}{\rho+2}\|u_t\|_{\rho+2}^{\rho+2}+\frac{1}{\rho+2}\int_\Omega\Big(\int_0^t g(t-s)\big|u(s)-u(t)\big|\,ds\Big)^{\rho+2}dx\\
&=\frac{\rho+1}{\rho+2}\|u_t\|_{\rho+2}^{\rho+2}+\frac{1}{\rho+2}\int_\Omega\Big(\int_0^t g^{\frac{\rho+1}{\rho+2}}(t-s)g^{\frac{1}{\rho+2}}(t-s)\big|u(s)-u(t)\big|\,ds\Big)^{\rho+2}dx\\
&\le\frac{\rho+1}{\rho+2}\|u_t\|_{\rho+2}^{\rho+2}+\frac{1}{\rho+2}\Big(\int_0^t g(t-s)\,ds\Big)^{\rho+1}\int_\Omega\Big(\int_0^t g(t-s)\big|u(s)-u(t)\big|^{\rho+2}\,ds\Big)\,dx\\
&\le\frac{\rho+1}{\rho+2}\|u_t\|_{\rho+2}^{\rho+2}+\frac{1}{\rho+2}G^{\rho+1}(t)\int_0^t g(t-s)\big\|u(s)-u(t)\big\|_{\rho+2}^{\rho+2}\,ds\\
&\le\frac{\rho+1}{\rho+2}\|u_t\|_{\rho+2}^{\rho+2}+\frac{1}{\rho+2}G^{\rho+1}(t)B^{\rho+2}\int_0^t g(t-s)\big\|\nabla u(s)-\nabla u(t)\big\|^{\rho+2}\,ds\\
&\le\frac{\rho+1}{\rho+2}\|u_t\|_{\rho+2}^{\rho+2}+\frac{1}{\rho+2}G^{\rho+1}(t)\\
&\quad\times B^{\rho+2}\int_0^t g(t-s)\big(\big\|\nabla u(s)\big\|+\big\|\nabla u(t)\big\|\big)^\rho\big\|\nabla u(s)-\nabla u(t)\big\|^2\,ds\\
&\le\frac{\rho+1}{\rho+2}\|u_t\|_{\rho+2}^{\rho+2}+\frac{1}{\rho+2}G^{\rho+1}(t)\\
&\quad\times B^{\rho+2}\int_0^t g(t-s)\Big[2\Big(\frac{2E(0)}{1-G(\infty)}\Big)^{\frac12}\Big]^\rho\big\|\nabla u(s)-\nabla u(t)\big\|^2\,ds\\
&=\frac{\rho+1}{\rho+2}\|u_t\|_{\rho+2}^{\rho+2}+\frac{2^\rho}{\rho+2}G^{\rho+1}(t)B^{\rho+2}\Big(\frac{2E(0)}{1-G(\infty)}\Big)^{\frac{\rho}{2}}\int_0^t g(t-s)\big\|\nabla u(s)-\nabla u(t)\big\|^2\,ds\\
&\le(\rho+1)E(t)+\frac{2^\rho}{\rho+2}G^{\rho+1}(t)B^{\rho+2}\Big(\frac{2E(0)}{1-G(\infty)}\Big)^{\frac{\rho}{2}}g\circ\nabla u\\
&\le\Big((\rho+1)+\frac{2^{\rho+1}}{\rho+2}G^{\rho+1}(t)B^{\rho+2}\Big(\frac{2E(0)}{1-G(\infty)}\Big)^{\frac{\rho}{2}}\Big)E(t), \qquad (3.16)
\end{aligned}
$$

which implies that

$$
\begin{aligned}
&-\frac{1}{\rho+1}\Big(|u_t|^\rho u_t,\xi(t)\int_0^t g(t-s)\big(u(s)-u(t)\big)\,ds\Big)\Big|_S^T\\
&\le\frac{2\xi(0)}{\rho+1}\Big((\rho+1)+\frac{2^{\rho+1}}{\rho+2}G^{\rho+1}(t)B^{\rho+2}\Big(\frac{2E(0)}{1-G(\infty)}\Big)^{\frac{\rho}{2}}\Big)E(S)\\
&\le\frac{2\xi(0)}{\rho+1}\Big((\rho+1)+\frac{2^{\rho+1}}{\rho+2}G^{\rho+1}(t)B^{\rho+2}\Big(\frac{2E(0)}{1-G(\infty)}\Big)^{\frac{\rho}{2}}\Big)E(S). \qquad (3.17)
\end{aligned}
$$

In order to estimate the second term on the right-hand side of (3.15), we apply (3.16) and (3.11) to get

$$\begin{aligned}
&\frac{1}{\rho+1}\int_S^T\left(|u_t|^\rho u_t,\xi'(t)\int_0^t g(t-s)\big(u(s)-u(t)\big)\,ds\right)dt\\
&\quad\le\frac{1}{\rho+1}\left((\rho+1)+\frac{2^{\rho+1}}{\rho+2}G^{\rho+1}(t)B^{\rho+2}\left(\frac{2E(0)}{1-G(\infty)}\right)^{\frac{\rho}{2}}\right)\int_S^T\big|\xi'(t)\big|E(t)\,dt\\
&\quad=-\frac{1}{\rho+1}\left((\rho+1)+\frac{2^{\rho+1}}{\rho+2}G^{\rho+1}(t)B^{\rho+2}\left(\frac{2E(0)}{1-G(\infty)}\right)^{\frac{\rho}{2}}\right)\int_S^T\xi'(t)E(t)\,dt\\
&\quad\le\frac{\xi(0)}{\rho+1}\left((\rho+1)+\frac{2^{\rho+1}}{\rho+2}G^{\rho+1}(t)B^{\rho+2}\left(\frac{2E(0)}{1-G(\infty)}\right)^{\frac{\rho}{2}}\right)E(S).
\end{aligned}\tag{3.18}$$

In addition, for any $\delta>0$, using Young inequality, we have

$$\begin{aligned}
&\int_S^T\left(|u_t|^\rho u_t,\xi(t)\int_0^t g'(t-s)\big(u(s)-u(t)\big)\,ds\right)dt\\
&\quad=\int_S^T\left(\xi^{\frac{\rho+1}{\rho+2}}(t)|u_t|^\rho u_t,\xi^{\frac{1}{\rho+2}}(t)\int_0^t g'(t-s)\big(u(s)-u(t)\big)\,ds\right)dt\\
&\quad\le\delta\int_S^T\xi(t)\|u_t\|_{\rho+2}^{\rho+2}\,dt+C(\delta)\int_S^T\xi(t)\left[\int_\Omega\left(\int_0^t\big|g'(t-s)\big|\big|u(s)-u(t)\big|\,ds\right)^{\rho+2}dx\right]dt.
\end{aligned}$$

Using Hölder inequality and Lemmas 2.1 and 2.3, we have

$$\begin{aligned}
&\int_S^T\xi(t)\left[\int_\Omega\left(\int_0^t|g'|^{\frac{\rho+1}{\rho+2}}(t-s)|g'|^{\frac{1}{\rho+2}}(t-s)\big|u(s)-u(t)\big|\,ds\right)^{\rho+2}dx\right]dt\\
&\quad\le\int_S^T\xi(t)\left(\int_0^t-g'(t-s)\,dt\right)^{\rho+1}\left[\int_\Omega\left(\int_0^t\big|g'(t-s)\big|\big|u(s)-u(t)\big|^{\rho+2}\,ds\right)dx\right]dt\\
&\quad\le g^{\rho+1}(0)\xi(0)\int_S^T\left(\int_0^t\big|g'(t-s)\big|\big\|u(s)-u(t)\big\|_{\rho+2}^{\rho+2}\,ds\right)dt\\
&\quad\le g^{\rho+1}(0)\xi(0)B^{\rho+2}\int_S^T\left(\int_0^t\big|g'(t-s)\big|\big\|\nabla u(s)-\nabla u(t)\big\|_{\rho+2}^{\rho+2}\,ds\right)dt\\
&\quad\le 2^\rho g^{\rho+1}(0)B^{\rho+2}\left(\frac{2E(0)}{1-G(\infty)}\right)^{\frac{\rho}{2}}\xi(0)\int_S^T\left(\int_0^t\big|g'(t-s)\big|\big\|\nabla u(s)-\nabla u(t)\big\|^2\,ds\right)dt\\
&\quad=-2^\rho g^{\rho+1}(0)B^{\rho+2}\left(\frac{2E(0)}{1-G(\infty)}\right)^{\frac{\rho}{2}}\xi(0)\int_S^T\left(\int_0^t g'(t-s)\big\|\nabla u(s)-\nabla u(t)\big\|^2\,ds\right)dt\\
&\quad\le-2^{\rho+1}g^{\rho+1}(0)B^{\rho+2}\left(\frac{2E(0)}{1-G(\infty)}\right)^{\frac{\rho}{2}}\xi(0)\int_S^T E'(t)\,dt\\
&\quad\le 2^{\rho+1}g^{\rho+1}(0)B^{\rho+2}\left(\frac{2E(0)}{1-G(\infty)}\right)^{\frac{\rho}{2}}\xi(0)E(S).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\int_S^T\left(|u_t|^\rho u_t,\xi(t)\int_0^t g'(t-s)\big(u(s)-u(t)\big)\,ds\right)dt\\
&\quad\le\delta\int_S^T\xi(t)\|u_t\|_{\rho+2}^{\rho+2}\,dt+2^{\rho+1}C(\delta)g^{\rho+1}(0)B^{\rho+2}\left(\frac{2E(0)}{1-G(\infty)}\right)^{\frac{\rho}{2}}\xi(0)E(S).
\end{aligned}\tag{3.19}$$

Since

$$\begin{aligned}
&\left|\left(\nabla u_t, \xi(t)\int_0^t g(t-s)\big(\nabla u(s)-\nabla u(t)\big)\,ds\right)\right| \\
&\quad\le \frac{1}{2}\xi(t)\|\nabla u_t\|^2 + \frac{1}{2}\xi(t)\int_\Omega\left(\int_0^t g(t-s)\big|\nabla u(s)-\nabla u(t)\big|\,ds\right)^2 dx \\
&\quad= \frac{1}{2}\xi(t)\|\nabla u_t\|^2 + \frac{1}{2}\xi(t)G(t)\int_0^t g(t-s)\big\|\nabla u(s)-\nabla u(t)\big\|^2\,ds\,dx \\
&\quad\le \xi(0)E(t) + \frac{1}{2}\xi(t)G(t)g\circ\nabla u \le 2\xi(0)E(t),
\end{aligned} \tag{3.20}$$

one obtains

$$-\left(\nabla u_t, \xi(t)\int_0^t g(t-s)\big(\nabla u(s)-\nabla u(t)\big)\,ds\right)\Big|_S^T \le 4\xi(0)E(S). \tag{3.21}$$

Using (3.20) and (3.11), we have

$$\int_S^T\left(\nabla u_t, \xi'(t)\int_0^t g(t-s)\big(\nabla u(s)-\nabla u(t)\big)\,ds\right) \le 2\int_S^T\big|\xi'(t)\big|E(t) \le 2\xi(0)E(S). \tag{3.22}$$

Similarly,

$$\begin{aligned}
&\int_S^T\left(\nabla u_t, \xi(t)\int_0^t g'(t-s)\big(\nabla u(s)-\nabla u(t)\big)\,ds\right)dt \\
&\quad\le \frac{\delta}{2}\int_S^T \xi(t)\|\nabla u_t\|^2\,dt \\
&\qquad+ \frac{1}{2\delta}\int_S^T \xi(t)\left(-\int_0^t g'(t-s)\,ds\right)\cdot\left(\int_0^t\big|g'(t-s)\big|\big\|\nabla u(s)-\nabla u(t)\big\|^2\,ds\right)dt \\
&\quad= \frac{\delta}{2}\int_S^T \xi(t)\|\nabla u_t\|^2\,dt - \frac{1}{2\delta}\int_S^T\xi(t)\int_0^t g'(s)\,ds\int_0^t\big|g'(t-s)\big|\big\|\nabla u(s)-\nabla u(t)\big\|^2\,ds\,dt \\
&\quad\le \frac{\delta}{2}\int_S^T \xi(t)\|\nabla u_t\|^2\,dt + \frac{g(0)}{2\delta}\int_S^T\xi(t)\int_0^t\big|g'(t-s)\big|\big\|\nabla u(s)-\nabla u(t)\big\|^2\,ds\,dt \\
&\quad\le \frac{\delta}{2}\int_S^T \xi(t)\|\nabla u_t\|^2\,dt - \frac{g(0)}{\delta}\int_S^T\xi(t)E'(t)\,dt \\
&\quad\le \frac{\delta}{2}\int_S^T \xi(t)\|\nabla u_t\|^2\,dt + \frac{g(0)\xi(0)}{\delta}E(S).
\end{aligned} \tag{3.23}$$

Now, since $g(t)\ge 0$ and $G(\infty)<1$, we get

$$\begin{aligned}
&\int_S^T \xi(t)\left\|\int_0^t g(t-s)\big(\nabla u(s)-\nabla u(t)\big)\,ds\right\|^2 dt \\
&\quad\le \int_S^T \xi(t)\left(\int_0^t g(s)\,ds\right)\left(\int_0^t g(t-s)\big\|\nabla u(s)-\nabla u(t)\big\|^2\,ds\right)dt \\
&\quad\le G(\infty)\int_S^T \xi(t)\left(\int_0^t g(t-s)\big\|\nabla u(s)-\nabla u(t)\big\|^2\,ds\right)dt
\end{aligned}$$

$$
\begin{aligned}
&\le G(\infty)\int_S^T\left(\int_0^t \frac{\xi(t)}{\xi(t-s)}\xi(t-s)g(t-s)\left\|\nabla u(s)-\nabla u(t)\right\|^2\,ds\right)dt\\
&\le G(\infty)\int_S^T\left(\int_0^t \xi(t-s)g(t-s)\left\|\nabla u(s)-\nabla u(t)\right\|^2\,ds\right)dt\\
&\le -G(\infty)\int_S^T\left(\int_0^t g'(t-s)\left\|\nabla u(s)-\nabla u(t)\right\|^2\,ds\right)dt\\
&\le -2G(\infty)\int_S^T E'(t)\,dt \le 2G(\infty)E(S)\le 2E(S)
\end{aligned}
\tag{3.24}
$$

and

$$
\begin{aligned}
&-\int_S^T \xi(t)\big(1-G(t)\big)\left(\int_0^t g(t-s)\big(\nabla u(s)-\nabla u(t)\big),\nabla u(t)\right)dt\\
&\le \varepsilon\int_S^T \xi(t)\|\nabla u\|^2\,dt+\frac{G(\infty)}{4\varepsilon}\int_S^T \xi(t)\left(\int_0^t g(t-s)\left\|\nabla u(s)-\nabla u(t)\right\|^2\,ds\right)dt\\
&= \varepsilon\int_S^T \xi(t)\|\nabla u\|^2\,dt\\
&\quad+\frac{G(\infty)}{4\varepsilon}\int_S^T\left(\int_0^t \frac{\xi(t)}{\xi(t-s)}\xi(t-s)g(t-s)\left\|\nabla u(s)-\nabla u(t)\right\|^2\,ds\right)dt\\
&\le \varepsilon\int_S^T \xi(t)\|\nabla u\|^2\,dt+\frac{G(\infty)}{4\varepsilon}\int_S^T\left(\int_0^t \xi(t-s)g(t-s)\left\|\nabla u(s)-\nabla u(t)\right\|^2\,ds\right)dt\\
&\le \varepsilon\int_S^T \xi(t)\|\nabla u\|^2\,dt-\frac{G(\infty)}{4\varepsilon}\int_S^T\left(\int_0^t g'(t-s)\left\|\nabla u(s)-\nabla u(t)\right\|^2\,ds\right)dt\\
&\le \varepsilon\int_S^T \xi(t)\|\nabla u\|^2\,dt-\frac{G(\infty)}{2\varepsilon}\int_S^T E'(t)\,dt\\
&\le \varepsilon\int_S^T \xi(t)\|\nabla u\|^2\,dt+\frac{1}{2\varepsilon}E(S)\le \varepsilon\int_S^T \xi(t)E(t)\,dt+\frac{1}{2\varepsilon}E(S).
\end{aligned}
\tag{3.25}
$$

Combining (3.15)–(3.25), we obtain

$$
\begin{aligned}
&\frac{1}{\rho+1}\int_S^T\big(\xi(t)G(t)\big)\|u_t\|_{\rho+2}^{\rho+2}\,dt+\int_S^T \xi(t)G(t)\|\nabla u_t\|^2\,dt\\
&\le \delta\int_S^T \xi(t)\|u_t\|_{\rho+2}^{\rho+2}\,dt+\frac{\delta}{2}\int_S^T \xi(t)\|\nabla u_t\|^2\,dt+\varepsilon\int_S^T \xi(t)E(t)\,dt\\
&\quad+\left[C_1+\frac{g(0)\xi(0)}{\delta}+\frac{1}{2\varepsilon}\right]E(S).
\end{aligned}
\tag{3.26}
$$

Since g is continuous and $g(0)>0$, for any $t_0>0$, we have

$$
G(t)=\int_0^t g(s)\,ds\ge\int_0^{t_0} g(s)\,dx>0,\quad \forall t\ge t_0.
$$

Now, if we fix $\delta>0$ small enough such that

$$
\delta<\int_0^{t_0} g(s)\,ds,
$$

then by (3.26), for $T > S \geq t_0$, we have

$$\int_S^T \xi(t)\|u_t\|_{\rho+2}^{\rho+2}\,dt + \int_S^T \xi(t)\|\nabla u_t\|^2\,dt \leq \varepsilon C\int_S^T \xi(t)E(t)\,dt + C(\varepsilon)E(S). \qquad \square$$

Proof of Lemma 3.1 Plugging the estimate of Lemma 3.3 into the inequality of Lemma 3.2, and fixing ε small enough, we obtain

$$\int_S^T \xi(t)E(t)\,dt \leq CE(S)$$

for some constant $C > 0$. Letting $T \to +\infty$, we have

$$\int_t^{+\infty} \psi'(s)E(s)\,ds \leq cE(t), \quad \forall t \geq t_0, \tag{3.27}$$

with

$$\psi(t) := \int_{t_0}^t \xi(s)\,ds. \qquad \square$$

Proof of Theorem 2.2 From Assumption 2.1 and Lemma 2.3, we know that $E(t)$ is a non-increasing function and $\psi : [t_0, +\infty) \to \mathbb{R}^+$ is a strictly increasing C^2 function such that $\psi(t_0) = 0$ and $\lim_{t\to+\infty}\psi(t) = +\infty$. Firstly, we define a new function $f : [t_0, \infty) \to \mathbb{R}^+$ as follows:

$$f(\tau) = E\big(\psi^{-1}(\tau)\big),$$

then f is a non-increasing function such that

$$\begin{aligned}\int_{\psi(S)}^{\psi(T)} f(\tau)\,d\tau &= \int_{\psi(S)}^{\psi(T)} E\big(\psi^{-1}(\tau)\big)\,d\tau = \int_S^T E(t)\psi'(t)\,dt \\ &\leq \int_S^{+\infty} E(t)\psi'(t)\,dt \leq cE(S) = cf\big(\psi(S)\big), \quad \forall t_0 \leq S < T < \infty.\end{aligned}$$

Set $t = \psi(S)$. Since $\lim_{T\to+\infty}\psi(T) = +\infty$, we get

$$\int_t^{+\infty} f(\tau)\,d\tau \leq cf(t), \quad \forall t \geq t_0. \tag{3.28}$$

Next, we define the following function:

$$h(x) = e^{\frac{1}{c}x}\int_x^{+\infty} f(s)\,ds, \tag{3.29}$$

where $c > 0$ is a constant. Noting (3.28), we obtain

$$h'(x) = \frac{1}{c}e^{\frac{1}{c}x}\left(\int_x^{+\infty} f(s)\,ds - cf(x)\right) \leq 0. \tag{3.30}$$

Integrating (3.30) over $[t_0, t]$ and noting (3.28), we have

$$h(t) \le h(t_0) = e^{\frac{1}{c}t_0} \int_{t_0}^{+\infty} f(s)\,ds \le c_1 f(t_0). \tag{3.31}$$

Furthermore, using (3.29) and (3.31), we obtain

$$\int_t^{+\infty} f(s)\,ds \le c_1 f(t_0) e^{-\frac{1}{c}t}. \tag{3.32}$$

On the other hand, noting that f is a positive non-increasing function, it is easy to see that

$$\int_t^{+\infty} f(s)\,ds \ge \int_t^{t+c} f(s)\,ds \ge cf(t+c). \tag{3.33}$$

Combining (3.32) with (3.33), we get

$$f(t+c) \le c_2 f(t_0) e^{-\frac{1}{c}t}. \tag{3.34}$$

Letting $s = t + c$ in (3.34), we have

$$f(s) \le c_2 f(t_0) e^{1-\frac{1}{c}s},$$

that is,

$$E\bigl(\psi^{-1}(s)\bigr) \le c_2 f(t_0) e^{1-\frac{1}{c}s}. \tag{3.35}$$

Moreover, letting $t = \psi^{-1}(s)$ in (3.35), we get

$$E(t) \le c_2 f(t_0) e^{1-\frac{1}{c}\psi(t)},$$

that is,

$$E(t) \le C_0 e^{-\omega\psi(t)} = C_0 e^{-\omega \int_{t_0}^t \xi(s)\,ds},$$

for some constants C_0 and $\omega = \frac{1}{c} > 0$.

The proof of Theorem 2.2 is completed. □

Remark 3.1 From Theorem 2.2, if we choose different $\xi(t)$, we can get different decay results. Choosing $\xi(t) \equiv a$, we get the exponential decay result

$$E(t) \le Ce^{-\omega t}, \quad \forall t \ge t_0.$$

Now consider $\xi(t) = \frac{1}{(1+t)^\gamma}$ $(0 < \gamma \le 1)$. If $\gamma = 1$, we get the polynomial decay result

$$E(t) \le C(1+t)^{-\omega}, \quad \omega > 0, \forall t \ge t_0.$$

If $0 < \gamma < 1$, we get a decay result of the form

$$E(t) \le Ce^{-\frac{\omega}{1-\gamma}(1+t)^{1-\gamma}}, \quad \omega > 0, \forall t \ge t_0.$$

4 Conclusions

In this paper, we present a weighted integral inequality to derive decay estimates for a quasilinear viscoelastic wave equation with variable density and memory. Due to the assumption on the memory kernel function, the weighted inequality established in this paper improves the integral inequality in [1]. We establish a general decay rate of the solution such that the exponential and polynomial decay results are special cases of this paper.

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References

1. Alabau-Boussouira, F., Cannarsa, P., Sforza, D.: Decay estimates for second order evolution euations with memory. J. Funct. Anal. **254**(5), 1342–1372 (2008)
2. Cavalcanti, M.M., Domingos Cavalcanti, V.N., Ferreira, J.: Existence and uniform decay for a non-linear viscoelastic equation with strong damping. Math. Methods Appl. Sci. **24**, 1043–1053 (2001)
3. Cavalcanti, M.M., Domingos Cavalcanti, V.N., Lasiecka, I., Webler, C.M.: Intrinsic decay rates for the energy of a nonlinear viscoelastic equation modeling the vibrations of thin rods with variable density. Adv. Nonlinear Anal. **6**(2), 121–145 (2017)
4. Di Nezza, E., Palatucci, G., Valdinoci, E.: Hitchhiker's guide to the fractional Sobolev spaces. Bull. Sci. Math. **136**(5), 521–573 (2012)
5. Evans, L.C.: Partial Differential Equations. Grad. Stud. Math., vol. 19. Am. Math. Soc., Providence (1998)
6. Feng, Y.H., Liu, C.M.: Stability of steady-state solutions to Navier–Stokes–Poisson systems. J. Math. Anal. Appl. **462**, 1679–1694 (2018)
7. Gao, Q., Li, F., Wang, Y.: Blow-up of the solution for higher-order Kirchhoff-type equations with nonlinear dissipation. Cent. Eur. J. Math. **9**(3), 686–698 (2011)
8. Guariglia, E.: Harmonic Sierpinski gasket and applications. Entropy **20**(9), 714 (2018)
9. Guesmia, A., Messaoudi, S.A., Webler, C.M.: Well-posedness and optimal decay rates for the viscoelastic Kirchhoff equation. Bol. Soc. Parana. Mat. **35**(3), 203–224 (2017)
10. Han, X., Wang, M.: General decay of energy for a viscoelastic equation with nonlinear damping. Math. Methods Appl. Sci. **32**(3), 346–358 (2009)
11. Li, F.: Global existence and uniqueness of weak solution for nonlinear viscoelastic full Marguerre–von Karman shallow shell equations. Acta Math. Sin. Engl. Ser. **25**(12), 2133–2156 (2009)
12. Li, F.: Limit behavior of the solution to nonlinear viscoelastic Marguerre–von Karman shallow shells system. J. Differ. Equ. **249**, 1241–1257 (2010)
13. Li, F., Bai, Y.: Uniform rates of decay for nonlinear viscoelastic Marguerre–von Karman shallow shell system. J. Math. Anal. Appl. **351**(2), 522–535 (2009)
14. Li, F., Bao, Y.: Uniform stability of the solution for a memory-type elasticity system with nonhomogeneous boundary control condition. J. Dyn. Control Syst. **23**, 301–315 (2017)
15. Li, F., Du, G.: General energy decay for a degenerate viscoelastic Petrovsky-type plate equation with boundary feedback. J. Appl. Anal. Comput. **8**, 390–401 (2018)
16. Li, F., Gao, Q.: Blow-up of solution for a nonlinear Petrovsky type equation with memory. Appl. Math. Comput. **274**, 383–392 (2016)
17. Li, F., Li, J.: Global existence and blow-up phenomena for nonlinear divergence form parabolic equations with inhomogeneous Neumann boundary conditions. J. Math. Anal. Appl. **385**, 1005–1014 (2012)

18. Li, F., Li, J.: Global existence and blow-up phenomena for p-Laplacian heat equation with inhomogeneous Neumann boundary conditions. Bound. Value Probl. **2014**, 219 (2014)
19. Li, F., Zhao, C.: Uniform energy decay rates for nonlinear viscoelastic wave equation with nonlocal boundary damping. Nonlinear Anal. **74**, 3468–3477 (2011)
20. Li, F., Zhao, Z., Chen, Y.: Global existence uniqueness and decay estimates for nonlinear viscoelastic wave equation with boundary dissipation. Nonlinear Anal., Real World Appl. **12**, 1770–1784 (2011)
21. Liu, C.M., Peng, Y.J.: Stability of periodic steady-state solutions to a non-isentropic Euler–Maxwell system. Z. Angew. Math. Phys. **68**, 105 (2017)
22. Liu, C.M., Peng, Y.J.: Convergence of a non-isentropic Euler–Poisson system for all time. J. Math. Pures Appl. **119**(9), 255–279 (2018)
23. Liu, W.: General decay and blow-up of solution for a quasilinear viscoelastic problem with nonlinear source. Nonlinear Anal. **73**(6), 1890–1904 (2010)
24. Messaoudi, S.A., Khulaifi, W.: General and optimal decay for a quasilinear viscoelastic equation. Appl. Math. Lett. **66**, 16–22 (2017)
25. Messaoudi, S.A., Mustafa, M.I.: A general stability result for a quasilinear wave equation with memory. Nonlinear Anal., Real World Appl. **14**(4), 1854–1864 (2013)
26. Messaoudi, S.A., Tatar, N.: Exponential and polynomial decay for a quasilinear viscoelastic equation. Nonlinear Anal. **68**(4), 785–793 (2008)