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Existence results for impulsive semilinear differential inclusions with nonlinear boundary conditions

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Abstract

In this paper, we discuss the nonlinear boundary problem for first-order impulsive semilinear differential inclusions. We establish existence results by using Martelli's fixed point theorem with upper and lower solutions method. We find that by giving different definitions of lower and upper solutions we can get all existence results. We also present an example.

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1 Introduction

In this paper, we consider the first-order impulsive semilinear differential inclusions with nonlinear boundary conditions

$$x'(t) \in Ax(t) + F\big(t, x(t)\big), \quad t \in J', \tag{1.1}$$

$$\Delta x(t_k) = I_k\big(x(t_k)\big), \quad k = 1, \ldots, m, \tag{1.2}$$

$$g\big(x(0), x(b)\big) = 0, \tag{1.3}$$

where $J' = J \backslash \{t_1, \ldots, t_m\}$, $J = [0, b]$, $b > 0$, $0 < t_1 < t_2 < \cdots < t_m < b$, A is the infinitesimal generator of a strongly continuous semigroup $T(t)$, $t \geq 0$, $F : J \times R \to P(R)$ is a multivalued map, $P(R)$ is the family of all nonempty subsets of R, $\Delta x(t_k) = x(t_k^+) - x(t_k^-)$, $x(t_k^+) = \lim_{\varepsilon \to 0^+} x(t_k + \varepsilon)$, $x(t_k^-) = \lim_{\varepsilon \to 0^+} x(t_k - \varepsilon)$, $I_k \in C(R, R)$ $(k = 1, \ldots, m)$, and $g : R^2 \to R$ is a single-valued map.

The evolving process of dynamics is often subjected to abrupt changes such as shocks, harvesting, and natural disasters. These short-term perturbations are often treated as having acted instantaneously or in the form of impulses. For example, (1.1) subjects to impulse effects (1.2). Impulsive differential equations have been developed in modeling impulsive problems in physics, population dynamics, biotechnology, pharmacokinetics, industrial robotics, and so forth. In the case where the right-hand side of (1.1) has discontinuities and differential inclusions, $F(t, x(t))$ has played an important role in modeling phenomena, especially in scenarios involving automatic control systems.

Variational methods and critical point theory plays a major role in discussing the existence of solutions for boundary problem for impulsive differential inclusions; see [1–9]. There are many other methods such as in [10–12]. In [13], the authors considered periodic boundary conditions $g(x,y) = x - y$, that is, $x(0) = x(b)$. Those results are applicable in some important cases. However, they are not valid for antiperiodic boundary conditions, for example, $x(0) = -x(b)$, which corresponds to $g(x,y) = x + y$. Note that, in this case, g is nondecreasing in the second variable, and hence the results are not applicable. To the author's best knowledge, there is no paper discussing such a boundary problem for impulsive differential inclusions.

Motivated by the works mentioned, the aim of this paper is to study the existence of solutions for nonlinear boundary problem (1.1)–(1.3) by Martelli's fixed point theorem with upper and lower solutions method. The rest of the paper is organized as follows. In Sect. 2, we briefly introduce some notations and necessary preliminaries. In Sect. 3, we prove existence results of solutions for system (1.1)–(1.3), and we give some corollaries in Sect. 4. Finally, in Sect. 5, we present an example to illustrate the main result.

2 Preliminaries

We introduce some notations, definitions, and preliminary facts.

Let X be a Banach space, and let Z be a subset of X. We denote $P(X) = \{Z \subset X \mid Z \neq \emptyset\}$, $P_{cv}(X) = \{Z \subset P(X) \mid Z \text{ is convex}\}$, $P_{cp}(X) = \{Z \subset P(X) \mid Z \text{ is compact}\}$, $P_{cv,cp}(X) = P_{cv}(X) \cap P_{cp}(X)$, and so forth.

Let $L^1(J,R) = \{x : J \to R \| x| : J \to [0,+\infty) \text{ is Lebesgue integrable}\}$. Then $L^1(J,R)$ is a Banach space with norm $\|x\|_{L^1} = \int_0^b |x(t)|\,dt$.

$$\begin{aligned} PC(J,R) = \Bigl\{ & x : J \to R | x(t) \text{ is continuous everywhere except for some } t_k, \\ & \text{at which } x\bigl(t_k^-\bigr), x\bigl(t_k^+\bigr) \text{ exist, and } x\bigl(t_k^-\bigr) = x(t_k), k = 1,\dots,m\Bigr\}, \end{aligned}$$

which is a Banach space with norm $\|x\|_{PC} = \sup\{|x(t)| : t \in J\}$.

Let $L(R) = \{N : R \to R \mid N \text{ is linear bounded}\}$, and for $N \in L(R)$, we define $\|N\|_{L(R)} = \inf\{r > 0 \mid \forall x \in R, |N(x)| < r|x|\}$. Then $(L(R), \|\cdot\|_{L(R)})$ is a Banach space.

By $AC(J,R)$ we denote the space of all absolutely continuous functions $x : J \to R$.

Definition 2.1 Throughout this paper, a multivalued map $F : J \times R \to P(R)$ is said to be L^1-Carathéodory if

(i) $t \to F(t,x)$ is measurable for each $x \in R$,
(ii) $x \to F(t,x)$ is upper semicontinuous on R for almost all $t \in J$,
(iii) for each $\rho > 0$, there exists $\varphi_\rho \in L^1(J,[0,+\infty))$ such that

$$\bigl\|F(t,x)\bigr\|_{P(R)} = \sup\bigl\{|v| : v \in F(t,x)\bigr\} \le \varphi_\rho(t), \quad \forall |x| \le \rho \text{ and a.e. } t \in J.$$

Definition 2.2 Functions $\alpha, \beta \in PC(J,R) \cap AC(J',R)$ are said to be related lower and upper solutions for problem (1.1)–(1.3) if there exist $v_1, v_2 \in L^1(J,R)$ such that

$$v_1(t) \in F\bigl(t,\alpha(t)\bigr), \quad \text{a.e. } t \in J, \tag{2.1}$$

$$\alpha'(t) \le A\alpha(t) + v_1(t), \quad \text{a.e. } t \in J', \tag{2.2}$$

$$\Delta\alpha(t_k) \leq I_k\big(\alpha(t_k)\big), \quad k = 1, \ldots, m,$$
$$g\big(\alpha(0), \beta(b)\big) \leq 0,$$

and

$$v_2(t) \in F\big(t, \beta(t)\big), \quad \text{a.e. } t \in J, \tag{2.3}$$
$$\beta'(t) \geq A\beta(t) + v_2(t), \quad \text{a.e. } t \in J', \tag{2.4}$$
$$\Delta\beta(t_k) \geq I_k\big(\beta(t_k)\big), \quad k = 1, \ldots, m,$$
$$g\big(\beta(0), \alpha(b)\big) \geq 0.$$

Definition 2.3 A function $x \in PC(J,R) \cap AC(J',R)$ is said to be a solution of (1.1)–(1.3) if $g(x(0), x(b)) = 0$, $\Delta x(t_k) = I_k(x(t_k))$, $k = 1, \ldots, m$, and there exists a function $v \in L^1(J,R)$ such that $v(t) \in F(t, x(t))$ a.e. on J, $x'(t) = Ax(t) + v(t)$.

Lemma 2.4 (see [14]) *Let X be a Banach space, let $F : J \times X \to P_{cv,cp}(X)$ be a L^1-Carathéodory multivalued map with*

$$S_{F,x} = \big\{f \in L^1(J,X) \,|\, f(t) \in F\big(t, x(t)\big) \text{ for a.e. } t \in J\big\} \neq \emptyset,$$

and let $\Gamma : L^1(J,X) \to C(J,X)$ be a linear continuous mapping. Then the operator

$$\Gamma \circ S_F : C(J,X) \to P_{cv,cp}\big(C(J,X)\big), \qquad u \mapsto (\Gamma \circ S_F)(x) := \Gamma(S_{F,x})$$

is a closed graph operator in $C(J,X) \times C(J,X)$.

Lemma 2.5 (Martelli's fixed point theorem [15]) *Let X be a Banach space, and let $G : X \to P_{cv,cp}(X)$ be an upper semicontinuous and condensing map. If the set $\Re = \{x \in X : \lambda x \in G(x)$ for some $\lambda > 1\}$ is bounded, then G has a fixed point.*

Remark 2.6
- (i) If a multivalued map F is completely continuous with nonempty compact values, then F is upper semicontinuous if and only if F has a closed graph (i.e., $x_n \to x^*$, $y_n \to y^*$, $y_n \in F(x_n)$ imply $y^* \in F(x^*)$).
- (ii) If a multivalued map F is completely continuous, then F is condensing. For general information, see [16].

Let $J_0 = [0, t_1]$, $J_k = (t_k, t_{k+1}]$, $k = 1, \ldots, m$, $t_{m+1} = b$.

Definition 2.7 (See [17]) A family of functions S is said to be quasiequicontinuous on J if for every $\varepsilon > 0$, there exists $\delta > 0$ such that if $x \in S$, $k = 0, 1, \ldots, m$, then

$$\big\|x(t_1) - x(t_2)\big\| < \varepsilon, \quad \forall t_1, t_2 \in J_k \text{ such that } |t_1 - t_2| < \delta.$$

Lemma 2.8 (Compactness criterion; see [17]) *The set $S \in PC(J, R^n)$ is relatively compact if and only if*
- (i) *S is bounded, that is, $\|x\| < c$ for each $x \in S$ and some $c > 0$,*
- (ii) *S is quasiequicontinuous on J.*

Definition 2.9 Let X be a Banach space. A multivalued map F is said to be completely continuous if $F(U)$ is relatively compact for every bounded subset $U \subseteq X$.

Lemma 2.10 *If x is a solution of the inclusion*

$$\begin{cases} x'(t) + x(t) \in Ax(t) + F(t, x(t)), & t \in J', \\ \Delta x(t_k) = I_k(x(t_k)), & k = 1, \dots, m, \\ x(0) = u_0, \end{cases} \tag{2.5}$$

then it is given by

$$\begin{aligned} x(t) &= T(t)x(0) + \int_0^t T(t-s)\bigl[v(s) - x(s)\bigr]\,ds \\ &\quad + \sum_{0<t_k<t} T(t-t_k)I_k\bigl(x(t_k)\bigr), \quad v \in S_{F,x}, \end{aligned} \tag{2.6}$$

where $u_0 \in R$.

Proof Let x be a solution of problem (2.5). Then there exists $v \in S_{F,x}$ such that $x'(t) + x(t) = Ax(t) + v(t)$. We put $w(s) = T(t-s)x(s)$. Then

$$\begin{aligned} w'(s) &= -T'(t-s)x(s) + T(t-s)x'(s) \\ &= -AT(t-s)x(s) + T(t-s)x'(s) \\ &= T(t-s)\bigl[x'(s) - Ax(s)\bigr] \\ &= T(t-s)\bigl[v(s) - x(s)\bigr]. \end{aligned} \tag{2.7}$$

If $t < t_1$, then integrating (2.7), we have

$$\begin{aligned} w(t) - w(0) &= \int_0^t w'(s)\,ds = \int_0^t T(t-s)\bigl[v(s) - x(s)\bigr]\,ds, \\ x(t) &= T(t)x(0) + \int_0^t T(t-s)\bigl[v(s) - x(s)\bigr]\,ds. \end{aligned}$$

If $t_k < t$, $k = 1, \dots, m$, then integrating (2.7), we have

$$\int_0^{t_1} w'(s)\,ds + \int_{t_1}^{t_2} w'(s)\,ds + \cdots + \int_{t_k}^t w'(s)\,ds = \int_0^t T(t-s)\bigl[v(s) - x(s)\bigr]\,ds,$$

that is,

$$w\bigl(t_1^-\bigr) - w(0) + w\bigl(t_2^-\bigr) - w\bigl(t_1^+\bigr) + \cdots + w(t) - w\bigl(t_k^+\bigr) = \int_0^t T(t-s)\bigl[v(s) - x(s)\bigr]\,ds,$$

and consequently

$$w(t) = w(0) + \sum_{0<t_k<t} \bigl[w\bigl(t_k^+\bigr) - w\bigl(t_k^-\bigr)\bigr] + \int_0^t T(t-s)\bigl[v(s) - x(s)\bigr]\,ds,$$

$$x(t) = T(t)x(0) + \int_0^t T(t-s)\big[v(s) - x(s)\big]\,ds$$
$$+ \sum_{0<t_k<t} T(t-t_k)I_k\big(x(t_k)\big),$$

which completes the proof. □

3 Main result

Theorem 3.1 *Assume that the following conditions hold.*

(H1) *$F : J \times R \to P_{cv,cp}(R)$ is an L^1-Carathéodory multivalued map.*

(H2) *Functions $\alpha, \beta \in PC(J,R) \cap AC(J',R)$ are related lower and upper solutions of problem* (1.1)–(1.3), *which are given in Definition* 2.2 *and satisfy $\alpha(t) \le \beta(t)$, $t \in J$.*

(H3) *$I_k \in C(R,R)$, $k = 1,\dots,m$.*

(H4) *g is a continuous single-valued map in $(x,y) \in [\alpha(0),\beta(0)] \times [\alpha(b),\beta(b)]$ and nondecreasing in $y \in [\alpha(b),\beta(b)]$.*

(H5) *A is the infinitesimal generator of a linear bounded semigroup $T(t)$, $t \ge 0$, and there exists $M > 0$ such that $\|T(t)\|_{L(R)} \le M$.*

(H6) *For $x(t) < \alpha(t)$, $v \in F(t,\alpha(t))$, $t \in J$, and $Ax(t) + v(t) \ge A\alpha(t) + v_1(t)$, and for $x(t) > \beta(t)$, $v \in F(t,\beta(t))$, $t \in J$, and $Ax(t) + v(t) \le A\beta(t) + v_2(t)$, where $v_1, v_2 \in L^1(J,R)$ satisfy* (2.1)–(2.4).

Then system (1.1)–(1.3) *has at least one solution x such that $\alpha(t) \le x(t) \le \beta(t)$ for all $t \in J$.*

Proof We transform (1.1)–(1.3) into a fixed point problem. Consider the modified problem

$$\begin{cases} x'(t) + x(t) \in Ax(t) + F_1(t,x(t)), & t \in J', \\ \Delta x(t_k) = I_k(\tau(t_k, x(t_k))), & k = 1,\dots,m, \\ x(0) = \tau(0, x(0) - g(\tau(0,x), \tau(b,x))), \end{cases} \tag{3.1}$$

where $F_1(t,x) = F(t,\tau(t,x)) + \tau(t,x)$, and $\tau : C(J,R) \to C(J,R)$ is defined by

$$\tau(t,x) = \begin{cases} \beta(t), & x(t) > \beta(t), \\ x(t), & \alpha(t) \le x(t) \le \beta(t), \\ \alpha(t), & x(t) < \alpha(t). \end{cases}$$

Evidently, if x is a solution of (3.1), $\alpha(t) \le x(t) \le \beta(t)$, and $\alpha(0) \le x(0) - g(\tau(0,x), \tau(T,x)) \le \beta(0)$, then x is a solution of (1.1)–(1.3).

By Lemma 2.10 we have that a solution of (3.1) is a fixed point of the operator $N : PC(J,R) \to P(PC(J,R))$ defined by

$$N(x) = \Bigg\{ h \in PC(J,R) : h(t) = T(t)x(0) + \int_0^t T(t-s)\big[v(s) + \tau(s,x) - x(s)\big]\,ds$$
$$+ \sum_{0<t_k<t} T(t-t_k)I_k\big(\tau\big(t_k, x(t_k)\big)\big), v \in S_{F,\tau(t,x)} \Bigg\},$$

where

$$S_{F,\tau(t,x)} = \{v \in L^1(J,R) : v(t) \in F(t,\tau(t,x)) \text{ for a.e. } t \in J\}.$$

Note that, for each $x \in C(J,R)$, $S_{F,x}$ is nonempty (see [14]), so $S_{F,\tau(t,x)}$ is nonempty.

Next, we will show that N has a fixed point by applying Lemma 2.5. The proof will be given in several steps. We first show that N is a completely continuous multivalued map, upper semicontinuous with convex closed values.

Step 1. $N(x)$ is convex for each $x \in PC(J,R)$.

Indeed, if h_1, h_2 belong to $N(x)$, then there exist $\overline{v}_1, \overline{v}_2 \in S_{F,\tau(t,x)}$ such that

$$\begin{aligned} h_i(t) = {} & T(t)x(0) + \int_0^t T(t-s)\big[\overline{v}_i(s) + \tau(s,x) - x(s)\big]\,ds \\ & + \sum_{0<t_k<t} T(t-t_k)I_k\big(\tau\big(t_k, x(t_k)\big)\big), \quad i = 1,2. \end{aligned}$$

Let $0 \le l \le 1$. Then, for each $t \in J$, we have

$$\begin{aligned} \big[lh_1 + (1-l)h_2\big](t) = {} & T(t)x(0) + \sum_{0<t_k<t} T(t-t_k)I_k\big(\tau\big(t_k, x(t_k)\big)\big) \\ & + \int_0^t T(t-s)\big[l\overline{v}_1(s) + (1-l)\overline{v}_2(s) + \tau(s,x) - x(s)\big]\,ds. \end{aligned}$$

Since $S_{F,\tau(t,x)}$ is convex (because F has convex values in (H1)), then $lh_1 + (1-l)h_2 \in N(x)$, so $N(x)$ is convex.

Step 2. N is completely continuous.

First, we show that N maps bounded sets into bounded sets in $PC(J,R)$. Let q be a positive constant, $B_q = \{x \in PC(J,R) : \|x\|_{PC} < q\}$ be a bounded set, and $x \in B_q$. Then for each $h \in N(x)$, there exists $v \in S_{F,\tau(t,x)}$ such that

$$\begin{aligned} h(t) = {} & T(t)x(0) + \int_0^t T(t-s)\big[v(s) + \tau(s,x) - x(s)\big]\,ds \\ & + \sum_{0<t_k<t} T(t-t_k)I_k\big(\tau\big(t_k, x(t_k)\big)\big). \end{aligned} \tag{3.2}$$

Noting the boundary condition of (3.1) and the definition of τ, we have

$$\alpha(0) \le x(0) \le \beta(0), \tag{3.3}$$

$$\alpha(t) \le \tau(t,x) \le \beta(t). \tag{3.4}$$

Let $\rho_1 = \max(q, \sup_{t\in J}|\alpha(t)|, \sup_{t\in J}|\beta(t)|)$. Then $|\tau(t,x)| \le \rho_1$. By (H1) there exists $\varphi_{\rho_1} \in L^1(J,[0,+\infty))$ such that

$$\sup\{|v| : v \in F(t,\tau(t,x))\} \le \varphi_{\rho_1}(t). \tag{3.5}$$

If $x \in B_q$, then there exist $c_k > 0$, $k = 1, \ldots, m$, such that $|I_k(\tau(t_k, x(t_k)))| \le c_k$, since I_k are continuous in (H3) and (3.4). So, with (3.3), (3.5), and (H5), we have

$$\begin{aligned}
\bigl|h(t)\bigr| &\le \bigl\|T(t)\bigr\|_{L(R)}\bigl|x(0)\bigr| + \int_0^d \bigl\|T(t-s)\bigr\|_{L(R)}\bigl[\bigl|v(s)\bigr| + \bigl|\tau(s,x)\bigr| + \bigl|x(s)\bigr|\bigr]\,ds \\
&\quad + \sum_{0<t_k<t} \bigl\|T(t-t_k)\bigr\|_{L(R)}\bigl|I_k\bigl(\tau\bigl(t_k, x(t_k)\bigr)\bigr)\bigr| \\
&\le M\max\bigl(\bigl|\alpha(0)\bigr|, \bigl|\beta(0)\bigr|\bigr) + M\|\varphi_{\rho_1}\|_{L^1} + Md(\rho_1 + q) \\
&\quad + M\sum_{k=1}^m c_k := K,
\end{aligned}$$

and thus $\|N(x)\|_{PC} \le K$.

Second, we prove that N maps bounded sets into quasiequicontinuous sets of $PC(J,R)$. Let $u_1, u_2 \in J_k$, $k = 0, 1, \ldots, m$, $u_1 < u_2$, $x \in B_q$, and $h \in N(x)$. Then

$$\begin{aligned}
\bigl|h(u_2) - h(u_1)\bigr| &\le \bigl|T(u_2) - T(u_1)\bigr|\max\bigl(\bigl|\alpha(0)\bigr|, \bigl|\beta(0)\bigr|\bigr) \\
&\quad + \int_0^{u_1} \bigl|T(u_2 - s) - T(u_1 - s)\bigr|\bigl(\varphi_{\rho_1}(s) + \rho_1 + q\bigr)\,ds \\
&\quad + \int_{u_1}^{u_2} M\bigl(\varphi_{\rho_1}(s) + \rho_1 + q\bigr)\,ds \\
&\quad + \sum_{0<t_k<u_1} \bigl|T(u_2 - t_k) - T(u_1 - t_k)\bigr|c_k + \sum_{u_1<t_k<u_2} Mc_k.
\end{aligned}$$

As $u_2 \to u_1$, the right-hand side of this inequality tends to zero since $T(t)$ is strongly continuous. This proves that $N(B_q)$ is quasiequicontinuous. By Lemma 2.8, N is completely continuous and therefore a condensing map.

Step 3. N has a closed graph.

Let $x_n \to x^*$, $h_n \in N(x_n)$, and $h_n \to h^*$. We will prove that $h^* \in N(x^*)$. Since $h_n \in N(x_n)$, there exist $v_n \in S_{F,\tau(t,x_n)}$ such that

$$\begin{aligned}
h_n(t) &= T(t)x_n(0) + \int_0^t T(t-s)\bigl[v_n(s) + \tau(s,x_n) - x_n(s)\bigr]\,ds \\
&\quad + \sum_{0<t_k<t} T(t-t_k)I_k\bigl(\tau\bigl(t_k, x_n(t_k)\bigr)\bigr).
\end{aligned}$$

Next, we need prove that there exists $v^* \in S_{F,\tau(t,x^*)}$ such that, for each $t \in J$,

$$\begin{aligned}
h^*(t) &= T(t)x^*(0) + \int_0^t T(t-s)\bigl[v^*(s) + \tau\bigl(s,x^*\bigr) - x^*(s)\bigr]\,ds \\
&\quad + \sum_{0<t_k<t} T(t-t_k)I_k\bigl(\tau\bigl(t_k, x^*(t_k)\bigr)\bigr).
\end{aligned}$$

Since $x_n \to x^*$, $h_n \to h^*$, and $I_k \in C(R,R)$ in (H3), by the definition of τ we have

$$\begin{aligned}
&\bigg\| h_n(t) - T(t)x_n(0) - \int_0^t T(t-s)\big[\tau(s,x_n) - x_n(s)\big]\,ds \\
&\quad - \sum_{0<t_k<t} T(t-t_k)I_k\big(\tau\big(t_k,x_n(t_k)\big)\big) \\
&\quad - \bigg[h^*(t) - T(t)x^*(0) - \int_0^t T(t-s)\big[\tau\big(s,x^*\big) - x^*(s)\big]\,ds \\
&\quad - \sum_{0<t_k<t} T(t-t_k)I_k\big(\tau\big(t_k,x^*(t_k)\big)\big)\bigg]\bigg\|_{PC} \to 0 \qquad (3.6)
\end{aligned}$$

as $n \to \infty$. Consider the linear continuous operator $\Gamma : L^1(J,R) \to C(J,R)$ defined by

$$v \mapsto \Gamma(v)(t) = \int_0^t T(t-s)v(s)\,ds.$$

Note that $S_{F,\tau(t,x)}$ is nonempty, so by Lemma 2.4, $\Gamma \circ S_F$ is a closed graph operator. Moreover,

$$\begin{aligned}
&h_n(t) - T(t)x_n(0) - \int_0^t T(t-s)\big[\tau(s,x_n) - x_n(s)\big]\,ds \\
&\quad - \sum_{0<t_k<t} T(t-t_k)I_k\big(\tau\big(t_k,x_n(t_k)\big)\big) \in \Gamma(S_{F,\tau(t,x_n)}). \qquad (3.7)
\end{aligned}$$

Since $x_n \to x^*$, by (3.6) and (3.7) there exists $v^* \in S_{F,\tau(t,x^*)}$ satisfying

$$\begin{aligned}
&h^*(t) - T(t)x^*(0) - \int_0^t T(t-s)\big[\tau\big(s,x^*\big) - x^*(s)\big]\,ds \\
&\quad - \sum_{0<t_k<t} T(t-t_k)I_k\big(\tau\big(t_k,x^*(t_k)\big)\big) = \int_0^t T(t-s)v^*(s)\,ds.
\end{aligned}$$

As a consequence of Steps 1 to 3, N is a completely continuous multivalued upper semicontinuous map with convex closed values.

Step 4. The set $\Re = \{x \in PC(J,R) : \lambda x \in N(x) \text{ for some } \lambda > 1\}$ is bounded.

Let $x \in \Re$. Then $\lambda x \in N(x)$ for some $\lambda > 1$. Thus, for each $t \in J$,

$$\begin{aligned}
x(t) = \lambda^{-1}\bigg[&T(t)x(0) + \int_0^t T(t-s)\big[v(s) + \tau(s,x) - x(s)\big]\,ds \\
&+ \sum_{0<t_k<t} T(t-t_k)I_k\big(\tau\big(t_k,x(t_k)\big)\big)\bigg]
\end{aligned}$$

for some $v \in S_{F,\tau(t,x)}$. Let $\rho_2 = \max(\sup_{t\in J}|\alpha(t)|, \sup_{t\in J}|\beta(t)|)$. From (3.4) it follows that $|\tau(t,x)| \le \rho_2$. By (H1) there exists $\varphi_{\rho_2} \in L^1(J,[0,+\infty))$ such that

$$\sup\big\{|v| : v \in F\big(t,\tau(t,x)\big)\big\} \le \varphi_{\rho_2}(t). \qquad (3.8)$$

Since $I_k \in C(R,R)$ in (H3) and (3.4), there exist $c'_k > 0$, $k = 1,\dots,m$, such that $|I_k(\tau(t_k, x(t_k)))| \le c'_k$. So, by (3.3) and (3.8), for each $t \in J$, we have

$$\begin{aligned} |x(t)| &\le M\left[|x(0)| + \int_0^t \left[|v(s)| + |\tau(s,x)| + |x(s)|\right] ds + \sum_{0<t_k<t} \left|I_k\left(\tau\left(t_k, x(t_k)\right)\right)\right|\right] \\ &\le M\left[\max\left(|\alpha(0)|, |\beta(0)|\right) + \|\varphi_{\rho_2}\|_{L^1} + b\rho_2 + \int_0^t |x(s)|\, ds + \sum_{k=1}^m c'_k\right]. \end{aligned}$$

Set

$$K_0 = M\left[\max\left(|\alpha(0)|, |\beta(0)|\right) + \|\varphi_{\rho_2}\|_{L^1} + b\rho_2 + \sum_{k=1}^m c'_k\right].$$

Using Gronwall's lemma (see [18], p. 36), for each $t \in J$, we have

$$|x(t)| \le K_0 e^{Mt}.$$

So,

$$\|x\|_{PC} \le K_0 e^{Mb}.$$

This shows that the set $\Re$ is bounded. As a consequence of Lemma 2.5, we deduce that N has a fixed point, which is a solution of problem (3.1).

Step 5. The solution x of (3.1) satisfies

$$\alpha(t) \le x(t) \le \beta(t), \quad t \in J, \tag{3.9}$$

and

$$\alpha(0) \le x(0) - g\left(\tau(0,x), \tau(b,x)\right) \le \beta(0). \tag{3.10}$$

We first prove (3.9). Let x be a solution of (3.1). We prove that $x(t) \le \beta(t)$ for all $t \in J$. Suppose that $x - \beta$ attains a positive maximum on J at s_0. As (3.3), we consider the only possible case $s_0 \in (0, T]$. Then there exists $s_1 \in (0, s_0)$, $s_1 \ne t_k$ ($k = 1, 2, \dots, m$), such that

$$0 < x(t) - \beta(t) \le x(s_0) - \beta(s_0), \quad t \in [s_1, s_0].$$

So, $\tau(t,x) = \beta(t)$ for $t \in [s_1, s_0]$, and there exists $v \in F(t, \beta(t))$ such that

$$\begin{aligned} \beta(s_0) - \beta(s_1) &\le x(s_0) - x(s_1) \\ &= \int_{s_1}^{s_0} \left[Ax(s) + v(s) + \beta(s) - x(s)\right] ds \\ &< \int_{s_1}^{s_0} \left[Ax(s) + v(s)\right] ds. \end{aligned}$$

By (H6) and Definition 2.2 we have

$$\begin{aligned}\beta(s_0)-\beta(s_1) &\leq \int_{s_1}^{s_0}\bigl[Ax(s)+v(s)\bigr]\,ds \leq \int_{s_1}^{s_0}\bigl[A\beta(s)+v_2(s)\bigr]\,ds \\ &\leq \int_{s_1}^{s_0}\beta'(s)\,ds = \beta(s_0)-\beta(s_1).\end{aligned}$$

This is a contradiction. Consequently, $x(t) \leq \beta(t)$ for all $t \in J$.

Similarly, we can prove that $\alpha(t) \leq x(t)$ on J. This shows that (3.9) holds.

Finally, we prove that the solution x of (3.1) satisfies (3.10). Suppose that

$$\alpha(0) > x(0) - g\bigl(\tau(0,x),\tau(b,x)\bigr). \tag{3.11}$$

Then by the boundary condition of (3.1) and the definition of τ we have

$$x(0) = \alpha(0). \tag{3.12}$$

By (3.9) and the definition of τ we have

$$\tau(0,x) = x(0), \qquad \tau(b,x) = x(b). \tag{3.13}$$

From (3.11) to (3.13) we get

$$g\bigl(\alpha(0),x(b)\bigr) = g\bigl(\tau(0,x),\tau(b,x)\bigr) > 0.$$

Since g is nondecreasing in the second variable in (H4) and $x(b) \leq \beta(b)$, we have

$$g\bigl(\alpha(0),\beta(b)\bigr) \geq g\bigl(\alpha(0),x(b)\bigr) > 0,$$

which contradicts $g(\alpha(0),\beta(b)) \leq 0$ in Definition 2.2. So, we have

$$\alpha(0) \leq x(0) - g\bigl(\tau(0,x),\tau(b,x)\bigr). \tag{3.14}$$

Analogously, we can prove that

$$x(0) - g\bigl(\tau(0,x),\tau(b,x)\bigr) \leq \beta(0). \tag{3.15}$$

Inequalities (3.14) and (3.15) show that (3.10) holds.

According to Steps 1 to 5, the solution x of (3.1) is also a solution of (1.1)–(1.3). The proof is complete. □

Remark 3.2 If $g(x(0),x(b)) = x(0) + x(b)$ in (1.1)–(1.3), that is, $x(0) = -x(b)$, which satisfies (H4), then (1.1)–(1.3) become an antiperiodic boundary value problem for semilinear impulsive differential inclusions.

4 Corollary

Definition 4.1 Functions $\alpha, \beta \in PC(J,R) \cap AC(J',R)$ are said to be lower and upper solutions for problem (1.1)–(1.3) if there exist $v_1, v_2 \in L^1(J,R)$ such that

$$\begin{cases} v_1(t) \in F(t,\alpha(t)), \quad \text{a.e. } t \in J, \\ \alpha'(t) \leq A\alpha(t) + v_1(t), \quad \text{a.e. } t \in J', \\ \Delta\alpha(t_k) \leq I_k(\alpha(t_k)), \quad k = 1,\dots,m, \\ g(\alpha(0),\alpha(b)) \leq 0, \end{cases}$$

and

$$\begin{cases} v_2(t) \in F(t,\beta(t)), \quad \text{a.e. } t \in J, \\ \beta'(t) \geq A\beta(t) + v_2(t), \quad \text{a.e. } t \in J', \\ \Delta\beta(t_k) \geq I_k(\beta(t_k)), \quad k = 1,\dots,m, \\ g(\beta(0),\beta(b)) \geq 0. \end{cases}$$

Corollary 4.2 *Assume* (H1), (H3), (H5), (H6), *and the following conditions hold.*

(H7) *Functions $\alpha, \beta \in PC(J,R) \cap AC(J',R)$ are lower and upper solutions of problem* (1.1)–(1.3) *given in Definition* 4.1 *and satisfying $\alpha(t) \leq \beta(t)$, $t \in J$.*

(H8) *g is a continuous single-valued map in $(x,y) \in [\alpha(0),\beta(0)] \times [\alpha(b),\beta(b)]$ and nonincreasing in $y \in [\alpha(b),\beta(b)]$.*

Then system (1.1)–(1.3) *has at least one solution x such that $\alpha(t) \leq x(t) \leq \beta(t)$ for all $t \in J$.*

The proof is similar to that of Theorem 3.1, and we omit it.

Remark 4.3 If $g(x(0),x(b)) = x(0) - x(b)$ in (1.1)–(1.3), that is, $x(0) = x(b)$, which satisfies (H8), then (1.1)–(1.3) become a periodic boundary value problem for impulsive semilinear differential inclusions.

Definition 4.4 Functions $\alpha, \beta \in PC(J,R) \cap AC(J',R)$ are said to be lower and upper solutions for problem (1.1)–(1.3) if there exist $v_1, v_2 \in L^1(J,R)$ such that

$$\begin{cases} v_1(t) \in F(t,\alpha(t)), \quad \text{a.e. } t \in J, \\ \alpha'(t) \leq A\alpha(t) + v_1(t), \quad \text{a.e. } t \in J', \\ \Delta\alpha(t_k) \leq I_k(\alpha(t_k)), \quad k = 1,\dots,m, \\ g(\alpha(0),\alpha(b)) \geq 0, \end{cases}$$

and

$$\begin{cases} v_2(t) \in F(t,\beta(t)), \quad \text{a.e. } t \in J, \\ \beta'(t) \geq A\beta(t) + v_2(t), \quad \text{a.e. } t \in J', \\ \Delta\beta(t_k) \geq I_k(\beta(t_k)), \quad k = 1,\dots,m, \\ g(\beta(0),\beta(b)) \leq 0. \end{cases}$$

Corollary 4.5 *Assume that* (H1), (H3), (H4), (H5), (H6), *and the following condition hold.*

(H9) *Functions $\alpha, \beta \in PC(J,R) \cap AC(J',R)$ are lower and upper solutions of problem* (1.1)–(1.3) *given in Definition* 4.4 *and satisfying $\alpha(t) \leq \beta(t)$, $t \in J$.*

Then system (1.1)–(1.3) *has at least one solution x such that $\alpha(t) \leq x(t) \leq \beta(t)$ for all $t \in J$.*

Proof We transform (1.1)–(1.3) into a fixed point problem. Consider the modified problem

$$\begin{cases} x'(t) + x(t) \in Ax(t) + F_1(t, x(t)), & t \in J', \\ \Delta x(t_k) = I_k(\tau(t_k, x(t_k))), & k = 1, \ldots, m, \\ x(0) = \tau(0, x(0) + g(\tau(0,x), \tau(b,x))), \end{cases} \tag{4.1}$$

where F_1 and τ are defined in (3.1).

The rest of the proof of Corollary 4.5 is similar to the proof of Theorem 3.1, and we omit it. □

Definition 4.6 Functions $\alpha, \beta \in PC(J,R) \cap AC(J',R)$ are said to be related lower and upper solutions for problem (1.1)–(1.3) if there exist $v_1, v_2 \in L^1(J,R)$ such that

$$\begin{cases} v_1(t) \in F(t, \alpha(t)), & \text{a.e. } t \in J, \\ \alpha'(t) \leq A\alpha(t) + v_1(t), & \text{a.e. } t \in J', \\ \Delta\alpha(t_k) \leq I_k(\alpha(t_k)), & k = 1, \ldots, m, \\ g(\alpha(0), \beta(b)) \geq 0, \end{cases}$$

and

$$\begin{cases} v_2(t) \in F(t, \beta(t)), & \text{a.e. } t \in J, \\ \beta'(t) \geq A\beta(t) + v_2(t), & \text{a.e. } t \in J', \\ \Delta\beta(t_k) \geq I_k(\beta(t_k)), & k = 1, \ldots, m, \\ g(\beta(0), \alpha(b)) \leq 0. \end{cases}$$

Corollary 4.7 *Assume that* (H1), (H3), (H5), (H6), (H8), *and the following condition hold.*

(H10) *Functions $\alpha, \beta \in PC(J,R) \cap AC(J',R)$ are related lower and upper solutions of problem* (1.1)–(1.3) *given in Definition* 4.6 *and satisfying $\alpha(t) \leq \beta(t)$, $t \in J$.*

Then system (1.1)–(1.3) *has at least one solution x such that $\alpha(t) \leq x(t) \leq \beta(t)$ for all $t \in J$.*

The proof is similar to that of Corollary 4.5, and we omit it.

5 An example

In this section, as an application of our main result, we present an example. We consider the following partial differential equation:

$$\begin{cases} \frac{\partial x(t,\theta)}{\partial t} \in \frac{\partial^2 x(t,\theta)}{\partial \theta^2} + F_1(t, x(t,\theta)), & \theta \in [0,\pi], t \in J', \\ x(t,0) = x(t,\pi) = 0, & t \in [0,b], \\ x(t_k^+, \theta) - x(t_k^-, \theta) = x(t_k, \theta), & k = 1, \ldots, m, \\ x(0,\theta) + x(b,\theta) = 0, \end{cases} \tag{5.1}$$

where $J' = [0,b]\backslash\{t_1,\ldots,t_m\}$. Let $X = L^2([0,\pi],R)$. Define $F : [0,b] \times X \to P(X)$, $I_k : X \to X$, and $g : X \times X \to X$ by

$$F(t,x)(\theta) = F_1\bigl(t,x(t,\theta)\bigr), \quad \theta \in [0,\pi],$$
$$I_k\bigl(x(t_k)\bigr)(\theta) = x(t_k,\theta), \quad \theta \in [0,\pi],$$
$$g\bigl(x(u),y(v)\bigr)(\theta) = x(u,\theta) + y(v,\theta), \quad u,v \in [0,b], \theta \in [0,\pi].$$

Obviously, I_k and g satisfy conditions (H3) and (H4) in Theorem 3.1, respectively.

Define $A : X \to X$ by $Ax = x''$ with

$$D(A) = \bigl\{x \in X, x, x' \text{ are absolutely continuous}, x'' \in X, x(0) = x(\pi) = 0\bigr\}.$$

It is well known that A is the infinitesimal generator of a strongly continuous cosine function $(C(t))_{t\in R}$ on X. The operator A has a discrete spectrum, and the eigenvalues are $-n^2$, $n \in N$, with corresponding eigenvectors $z_n(\theta) = (2/\pi)^{1/2}\sin(n\theta)$. Furthermore, the set $\{z_n : n \in N\}$ is an orthonormal basis of X, and the following properties hold.

(a) For $x \in D(A)$, $Ax = -\sum_{n=1}^{\infty} n^2\langle x, z_n\rangle z_n$.

(b) For $x \in X$, $C(t)x = \sum_{n=1}^{\infty}\cos(nt)\langle x, z_n\rangle z_n$.

Consequently, $\|C(t)\| \le 1$ for all $t \in R$, that is, condition (H5) in Theorem 3.1 is satisfied. More about the cosine family can be found in [19, 20]. Hence the partial differential inclusions (5.1) can be rewritten in an abstract form as system (1.1)–(1.3).

If we assume that conditions (H1), (H2), and (H6) in Theorem 3.1 hold, then system (5.1) has at least one mild solution.

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