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Limiting behavior of blow-up solutions for the cubic nonlinear beam equation

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Abstract

In terms of the sharp Gagliardo–Nirenberg–Sobolev inequalities, we find the precisely sharp criteria of blow-up and global existence for the cubic nonlinear beam equations with L^2 critical nonlinearities. And we further study the limiting profile of blow-up solutions.

MSC: 35G20; 35Q40; 35B40

Keywords: Nonlinear beam equation; Blow-up solution; Sharp criteria; Limiting profile

1 Introduction

The nonlinear beam equation is a class of fourth-order partial differential equations appearing in different physical settings (see [26, 27] for a review), and it models the weak interactions of dispersive waves in [1] and the motion of the clamped plate and beams in [20]. In general, the nonlinear beam equation is in the following form:

$$\frac{\partial^2}{\partial t^2}u + \Delta^2 u + mu - |u|^{p-1}u = 0, \quad x \in \mathbb{R}^D, \tag{1.1}$$

where the parameter $m > 0$. $u = u(t,x) : [0,T) \times \mathbb{R}^D \to \mathbb{R}$ and $0 < T \le +\infty$. $\Delta^2 = \Delta\Delta$ and Δ is the Laplacian. $1 < p \le \frac{2D}{(D-4)^+} - 1$, where $\frac{2D}{(D-4)^+} = +\infty$ when $D = 1,2,3,4$; $\frac{2D}{(D-4)^+} = \frac{2D}{D-4}$ when $D \ge 5$. When $m = 0$, the scaling symmetry $\lambda^{\frac{4}{p-1}}u(\lambda^2 t, \lambda x)$ of Eq. (1.1) implies two critical exponents: one is the L^2 critical exponent $p = 1 + \frac{8}{D}$, and the other is the H^2 energy-critical exponent $p = \frac{2D}{D-4} - 1$ (see [12]). Hence, when $p = 3$ and $D = 4,6,8$, we call Eq. (1.1) the L^2 critical, L^2 super-critical, and H^2 energy-critical cubic nonlinear beam equation, respectively. We supplement Eq. (1.1) with the initial data

$$u(0,x) = u_0, \qquad \frac{\partial}{\partial t}u(0,x) = u_1. \tag{1.2}$$

In the last two decades, Eq. (1.1) has been widely studied. The local well-posedness of the Cauchy problem (1.1)–(1.2) for $1 < p < \frac{2D}{(D-4)^+} - 1$ and $p = \frac{2D}{(D-4)^+} - 1$ was established in [17] and in [12], respectively. The stability of traveling waves and standing waves for Eq. (1.1) was obtained in [17]. There are a lot of papers on the asymptotic behavior and scattering properties of global solutions (see [9, 10, 15, 18, 21, 24, 25, 28, 31, 32]). To our knowledge, the only result on the blow-up solutions of Eq. (1.1) is [12], in which authors gave a sufficient condition for the existence of blow-up solutions. This motivates us to study the following properties of Eq. (1.1): How to separate the domains of blow-up and global existence? What is the limiting behavior of blow-up solutions?

In terms of the sharp criteria for the nonlinear Schrödinger equation in [8, 14, 16, 19, 23, 36], we study the sharp criteria for the nonlinear Beam equation. First, for the L^2 critical case: $p = 3$ and $D = 4$, we have the following theorem.

Theorem 1.1 *Let $m = 1$, $p = 3$, and $D = 4$. Let R be a ground state of*

$$\frac{3}{2}\Delta^2 R + \frac{1}{2}R - |R|^2 R = 0, \quad R \in H^2. \tag{1.3}$$

If the initial data $(u_0, u_1) \in H^2 \times L^2$ satisfies

$$E\big((u_0, u_1)\big) < E(R, 0) = \frac{1}{2}\|R\|_2^2, \tag{1.4}$$

then we have the following:

(i) *If $\|u_0\|_2 < \|R\|_2$, then the solution $u(t,x)$ of the Cauchy problem (1.1)–(1.2) exists globally, and $u(t,x)$ satisfies that, for all time t,*

$$\big\|u(t)\big\|_2^2 + \left\|\frac{\partial}{\partial t}u(t)\right\|_2^2 < \|R\|_2^2 \quad \text{and} \quad \big\|\Delta u(t)\big\|_2^2 + \left\|\frac{\partial}{\partial t}u(t)\right\|_2^2 < \|R\|_2^2. \tag{1.5}$$

(ii) *If $\|u_0\|_2 > \|R\|_2$, then the solution $u(t,x)$ of the Cauchy problem (1.1)–(1.2) blows up in finite time $0 < T < +\infty$.*

The above sharp criteria of Eq. (1.1) are different from those of the L^2 critical nonlinear Schrödinger equation in [30], where Weinstein proved that for the initial data $\|u_0\|_2 > \|Q\|_2$, the corresponding solution may blow up. However, for Eq. (1.1), in the case $\|u_0\|_2 > \|R\|_2$, the corresponding solution must blow up in a finite time.

Furthermore, we obtain the following limiting profile of blow-up solutions for Eq. (1.1).

Theorem 1.2 *Let $m = 1$, $p = 3$, and $D = 4$. If $(u_0, u_1) \in H^2 \times L^2$ and $u(t,x)$ is the corresponding blow-up solution of the Cauchy problem (1.1)–(1.2) with $\lim_{t\to T}\|u(t)\|_2 = +\infty$ and $\|\Delta u(t)\|_2 = \|\Delta R\|_2$, where $0 < T < +\infty$ is the blow-up time, then there exist $y(t) \in \mathbb{R}^4$, $\gamma(t) \in \mathbb{R}$ such that*

$$u\big(t, \lambda(t)\big(\cdot + y(t)\big)\big) \to R(\cdot) \quad \text{strongly in } H^2 \text{ as } t \to T, \tag{1.6}$$

where $\lambda(t) = (\frac{\|u(t)\|_2}{\|R\|_2})^{\frac{1}{2}}$ and R is a ground state of (1.3).

Throughout this paper, we assume $m = 1$ and $p = 3$ for simplification, and the general cases $m = 1$ and $1 < p < \frac{2D}{(D-4)^+} - 1$ can be obtained by entirely the same way. We extend the sharp criteria argument in [14, 16] for nonlinear Schrödinger equations to that for the nonlinear beam equation (1.1). This is very nontrivial because Eq. (1.1) does not have the conservation of L^2-norm and scaling invariance. The argument in this paper may have a potential application to other nonlinear wave equations without scaling invariance.

In this paper, we abbreviate $L^q(\mathbb{R}^D)$, $\|\cdot\|_{L^q(\mathbb{R}^D)}$, $H^2(\mathbb{R}^D)$, and $\int_{\mathbb{R}^D}\cdot\,dx$ by L^q, $\|\cdot\|_q$, H^2, and $\int\cdot\,dx$. The various positive constants will be denoted by C.

2 Notations and preliminaries

For the Cauchy problem (1.1)–(1.2), the work space is defined by

$$H^2 := \left\{ v \in L^2 \;\Big|\; \int \left(|v|^2 + |\nabla v|^2 + |\Delta v|^2\right) dx < +\infty \right\}.$$

It is easy to check that $(\|v\|_2^2 + \|\Delta v\|_2^2)^{\frac{1}{2}}$ is an equivalent norm of H^2, and this equivalent norm is used to decompose the bounded sequences in H^2 (see [37]). Define two functionals in $H^2 \times L^2$ by

$$E\left(\left(v(t), \frac{\partial}{\partial t}v(t)\right)\right) := \int \left[\frac{1}{2}\left|\frac{\partial}{\partial t}v(t)\right|^2 + \frac{1}{2}\left|\Delta v(t)\right|^2 + \frac{1}{2}\left|v(t)\right|^2 - \frac{1}{4}\left|v(t)\right|^4\right] dx,$$

$$H\left(v(t)\right) := \int \left[\frac{1}{2}\left|v(t)\right|^2 dx - \frac{1}{4}\left|v(t)\right|^4\right] dx.$$

The functionals E and H are well-defined by the Sobolev embedding theorem (see [12]). Moreover, Hebey and Pausader established the local well-posedness of the Cauchy problem (1.1)–(1.2) in the energy space $H^2 \times L^2$ in [12] as follows.

Proposition 2.1 *Let $m = 1$, $1 < p < \frac{2D}{(D-4)^+} - 1$, and $(u_0, u_1) \in H^2 \times L^2$. There exists a unique solution $u(t,x)$ of the Cauchy problem (1.1)–(1.2) on the maximal time $[0,T)$ such that $u(t,x) \in C([0,T); H^2 \times L^2)$. Moreover, the following properties hold: either $T = +\infty$ (global existence), or $0 < T < +\infty$ and $\lim_{t\to T}\|u(t,x)\|_{H^2} = +\infty$ (blow-up). Furthermore, for all $t \in [0,T)$,*

$$E\left(\left(u(t), \frac{\partial}{\partial t}u(t)\right)\right) = E\left((u_0, u_1)\right). \tag{2.1}$$

Remark 2.2 Particularly, in the H^2 energy-critical case: $p = \frac{2D}{(D-4)^+} - 1$, Hebey and Pausader's result in [12] implies that the local well-posedness of the Cauchy problem (1.1)–(1.2) also holds in $H^2 \times L^2$. Moreover, for the solution $u(t,x) \in C([0,T); H^2 \times L^2)$, if $0 < T < +\infty$, then $\lim_{t\to T}\|u(t,x)\|_{H^2} = +\infty$, or $\limsup_{t\to T}\|u(t,x)\|_{L_t^q([0,T);L_x^r)} = +\infty$ (blow-up), where

$$(q,r) = \left(\frac{2(D+4)}{D-4}, \frac{2D(D+4)}{(D-4)(D+2)}\right)$$

is the B-admissible.

Now, we introduce the following important sharp inequalities: the sharp generalized Gagliardo–Nirenberg inequality established in [7] and in [37, 38]. Some other sharp Gagliardo–Nirenberg inequalities can be established by the profile arguments (see [2–4, 11, 22, 29, 34, 35]).

Lemma 2.3 *Let R be a ground state of* (1.3) *Then, when $D = 4$, for all $v \in H^2$, we have*

$$\|v\|_4^4 \leq \frac{2}{\|R\|_2^2}\|v\|_2^2\|\Delta v\|_2^2. \tag{2.2}$$

3 Sharp criteria of blow-up and global existence

In terms of the sharp Gagliardo–Nirenberg type inequalities, sharp Sobolev inequality, and some new estimates, we obtain the precisely sharp criteria of blow-up for Eq. (1.1): if $\|u_0\|_2 < \|R\|_2$, then the solution exists globally; if $\|u_0\|_2 > \|R\|_2$, then the solution blows up in finite time, where R is a ground state of (1.3). Now, we give the proof of Theorem 1.1.

Proof (i) Applying the sharp inequality (2.2) to the energy functional E, we deduce that, for all $t \in I$(maximal existence interval),

$$\begin{aligned} 2E\left(\left(u(t), \frac{\partial}{\partial t}u(t)\right)\right) &= \left\|\frac{\partial}{\partial t}u(t)\right\|_2^2 + \left\|u(t)\right\|_2^2 + \left\|\Delta u(t)\right\|_2^2 - \frac{1}{2}\left\|u(t)\right\|_4^4 \\ &\geq \left\|\frac{\partial}{\partial t}u(t)\right\|_2^2 + \left\|u(t)\right\|_2^2 + \left(1 - \frac{\|u(t)\|_2^2}{\|R\|_2^2}\right)\left\|\Delta u(t)\right\|_2^2. \end{aligned} \tag{3.1}$$

By the bootstrap and continuity argument, we claim that if $\|u_0\|_2 < \|R\|_2$, then, for all $t \in I$,

$$\left\|u(t)\right\|_2 < \|R\|_2. \tag{3.2}$$

Indeed, if (3.2) is not true, then there exists $t_1 \in I$ such that $\|u(t_1)\|_2 \geq \|R\|_2$. Since the solution $u(t,x)$ is continuous with respect to t, there exists $0 < t_0 \leq t_1$ such that $\|u(t_0)\|_2 = \|R\|_2$. But from (1.4), (3.1), and the conservation of energy $E((u_0,u_1)) = E((u(t), \frac{\partial}{\partial t}u(t)))$ with $t = t_0$, we get

$$\|R\|_2^2 > 2E\left(\left(u(t_0), \frac{\partial}{\partial t}u(t_0)\right)\right) \geq \left\|\frac{\partial}{\partial t}u(t_0)\right\|_2^2 + \left\|u(t_0)\right\|_2^2 \geq \left\|u(t_0)\right\|_2^2.$$

This contradicts $\|u(t_0)\|_2 = \|R\|_2$. Hence, claim (3.2) holds.

Now, we can prove (1.5). By injecting (3.2) into (3.1), we can obtain the first estimate in (1.5) by (1.4) and (3.1). For the second estimate in (1.5), from (3.1), we see that $\forall t \in I$

$$\|R\|_2^2 > 2E\left(\left(u(t), \frac{\partial}{\partial t}u(t)\right)\right) \geq \left\|u(t)\right\|_2^2 + \frac{\|R\|_2^2 - \|u(t)\|_2^2}{\|R\|_2^2}\left\|\Delta u(t)\right\|_2^2,$$

and so, for all $t \in I$,

$$\left(\|R\|_2^2 - \left\|u(t)\right\|_2^2\right)\left(\|R\|_2^2 - \left\|\Delta u(t)\right\|_2^2\right) \geq 0. \tag{3.3}$$

Inject (3.2) into (3.3). We get $\|\Delta u(t)\|_2^2 < \|R\|_2^2$ for all $t \in I$. Moreover, by rewriting (3.1), we see that, for all $t \in I$,

$$\begin{aligned} \|R\|_2^2 > 2E\left(\left(u(t), \frac{\partial}{\partial t}u(t)\right)\right) &\geq \left\|\frac{\partial}{\partial t}u(t)\right\|_2^2 + \left\|\Delta u(t)\right\|_2^2 + \left(1 - \frac{\|\Delta u(t)\|_2^2}{\|R\|_2^2}\right)\left\|u(t)\right\|_2^2 \\ &\geq \left\|\frac{\partial}{\partial t}u(t)\right\|_2^2 + \left\|\Delta u(t)\right\|_2^2. \end{aligned}$$

Then the second estimate in (1.5) is true.

(ii) We claim that if $\|u_0\|_2 > \|R\|_2$, then for all $t \in I$

$$\|u(t)\|_2 > \|R\|_2 \quad \text{and} \quad \|\Delta u(t)\|_2 > \|R\|_2. \tag{3.4}$$

Indeed, if $\|u(t)\|_2 > \|R\|_2$ is not true for all $t \in I$, then there exists $t_2 \in I$ such that $\|u(t_2)\|_2 \leq \|R\|_2$. From the continuity of $\|u(t)\|_2$, there exists $0 < t_3 \leq t_2$ such that $\|u(t_3)\|_2 = \|R\|_2$. Injecting this into (3.1) with $t = t_3$, we get

$$2E\left(\left(u(t_3), \frac{\partial}{\partial t}u(t_3)\right)\right) \geq \|u(t_3)\|_2^2 = \|R\|_2^2,$$

which contradicts $E((u(t), \frac{\partial}{\partial t}u(t))) = E((u_0, u_1)) < \frac{1}{2}\|R\|_2^2$ for all $t \in I$. Then we prove that $\|u(t)\|_2 > \|R\|_2$ for all $t \in I$. Moreover, injecting $\|u(t)\|_2 > \|R\|_2$ for all $t \in I$ into (3.3), we obtain that $\|\Delta u(t)\|_2 > \|R\|_2$ for all $t \in I$. This completes the proof of claim (3.4). Now, let $J(t) := \int |u(t,x)|^2\,dx$. By some basic computations, we deduce that, for all $t \in I$, $J'(t) = 2\int u(t)\frac{\partial}{\partial t}u(t)\,dx$ and

$$\begin{aligned} J''(t) &= 6\left\|\frac{\partial}{\partial t}u(t)\right\|_2^2 + 2\|u(t)\|_2^2 + 2\|\Delta u(t)\|_2^2 - 8E\big((u_0, u_1)\big) \\ &> 6\left\|\frac{\partial}{\partial t}u(t)\right\|_2^2, \end{aligned} \tag{3.5}$$

where the last step employs (1.4) and (3.4). Hence, $J''(t)$ is positive and has a lower bound for all $t \in I$. Notice that $J'(t)^2 \leq 4\|u(t)\|_2^2\|\frac{\partial}{\partial t}u(t)\|_2^2$. Then

$$J(t)J''(t) > 6\left\|\frac{\partial}{\partial t}u(t)\right\|_2^2\|u(t)\|_2^2 > \frac{3}{2}J'(t)^2. \tag{3.6}$$

There exists $t_0 > 0$ such that $J'(t) > 0$ for all $t > t_0$. Thus, for all $t > t_0$, we get $\frac{J''(t)}{J'(t)} > \frac{3}{2}\frac{J'(t)}{J(t)}$, which implies that there exists $K > 0$ such that $J'(t) > KJ(t)^{\frac{3}{2}}$. Due to $\frac{3}{2} > 1$, we deduce that, for all $t > t_0$,

$$J(t) > \left(\frac{2\sqrt{J(t_0)}}{2 - K\sqrt{J(t_0)}(t - t_0)}\right)^2.$$

Then there exists $0 < T < +\infty$ such that $\lim_{t\to T}\|u(t)\|_2^2 = \lim_{t\to T} J(t) = +\infty$. The solution $u(t,x)$ blows up in finite time $0 < T < +\infty$. □

4 Limiting profile of blow-up solutions

In this section, we assume that the ground state of (1.3) is unique up to translations in space and dilations, which is also denoted by R. A similar assumption has been used for the ground state of the classical second-order nonlinear Schrödinger equation in [30]. First, we show the variational characteristics of R.

Lemma 4.1 *Let $D = 4$. If $v \in H^2$ satisfies $\|\Delta v\|_2 = \|\Delta R\|_2$ and $H(v) = 0$, then $v(x)$ is of the following form*:

$$v(x) = R(\lambda x + x_0) \quad \textit{for some } \lambda > 0, x_0 \in \mathbb{R}^4, \tag{4.1}$$

where R is a ground state of (1.3).

Proof According to the hypothesis $H(v) = 0$, we see that $\int |v|^2\,dx = \frac{1}{2}\int |v|^4\,dx$. Hence, inject this into the following functional:

$$I(v) := \frac{(\int |v|^2\,dx)(\int |\Delta v|^2\,dx)}{\int |v|^4\,dx} = \frac{1}{2}\|\Delta v\|_2^2 = \frac{1}{2}\|\Delta R\|_2^2 = \frac{1}{2}\|R\|_2^2 = I_*,$$

which implies that v is a minimizer of $I(v)$. According to Zhu, Zhang, and Yang's result in [37], we can deduce that $v(x)$ is of the form $v(x) = R(\lambda x + x_0)$ by the uniqueness of R. □

The main tool to study the limit of the blow-up solutions for Eq. (1.1) is the profile decomposition established by Zhu, Zhang, and Yang in [37]. This argument has been applied to study the stability of standing waves (see [5, 6, 33]).

Proposition 4.2 *Let $D = 4$ and $\{v_n\}_{n=1}^{+\infty}$ be a bounded sequence in H^2. Then there exist a subsequence of $\{v_n\}_{n=1}^{+\infty}$(still denoted $\{v_n\}_{n=1}^{+\infty}$), a family $\{x_n^j\}_{j=1}^{+\infty}$ of sequences in $\mathbb{R}^4$, and a sequence $\{V^j\}_{j=1}^{+\infty}$ in H^2 such that*

(i) *for every $k \neq j$, $|x_n^k - x_n^j| \to +\infty$ as $n \to +\infty$;*

(ii) *for every $l \geq 1$ and every $x \in \mathbb{R}^4$, $v_n(x) = \sum_{j=1}^{l} V^j(x - x_n^j) + v_n^l(x)$ with $\lim_{l\to+\infty}\limsup_{n\to+\infty}\|v_n^l\|_q = 0$ for every $q \in (2, +\infty)$.*

Moreover, as $n \to +\infty$, we have

$$\|v_n\|_2^2 = \sum_{j=1}^{l}\|V^j\|_2^2 + \|v_n^l\|_2^2 + o(1),$$

$$\|\Delta v_n\|_2^2 = \sum_{j=1}^{l}\|\Delta V^j\|_2^2 + \|\Delta v_n^l\|_2^2 + o(1),$$

$$\left\|\sum_{j=1}^{l} V^j(x - x_n^j)\right\|_4^4 = \sum_{j=1}^{l}\|V^j(x - x_n^j)\|_4^4 + o(1), \tag{4.2}$$

where $o(1) := o_n(1) \to 0$ as $n \to +\infty$.

Remark 4.3 By using the inequality

$$\left|\left|\sum_{j=1}^{l} a_j\right|^{p+1} - \sum_{j=1}^{l}|a_j|^{p+1}\right| \leq C\sum_{j\neq k}|a_j||a_k|^p$$

for $p > 1$, we can prove that the mixed terms in $\|\sum_{j=1}^{l} V^j(x - x_n^j)\|_{p+1}^{p+1}$ vanish. Thus, (4.2) is true.

In terms of Hmidi and Keraani's argument in [13] and Zhu, Zhang, and Yang's argument in [37], we prove the following refined compactness result by Proposition 4.2.

Lemma 4.4 *Let $D = 4$ and $\{v_n\}_{n=1}^{+\infty}$ be a bounded sequence in H^2 such that*

$$\limsup_{n\to+\infty}\|v_n\|_2 \leq M \quad \textit{and} \quad \limsup_{n\to+\infty}\|v_n\|_4 \geq N > 0. \tag{4.3}$$

Then there exists a sequence $\{x_n\}_{n=1}^{+\infty}$ *of* $\mathbb{R}^4$ *such that up to a subsequence*

$$v_n(x+x_n) \rightharpoonup V(x) \quad \textit{weakly in } H^2 \tag{4.4}$$

with $\|\Delta V\|_2^2 \geq \frac{\|\Delta R\|_2^2 N^4}{2M^2}$, *and R is a ground state of* (1.3).

Proof The proof of Lemma 4.4 is fully similar to that of Theorem 1.1 in [37]. The key is injecting the decomposition: $v_n(x) = \sum_{j=1}^{l} V^j(x-x_n^j) + v_n^l(x)$ into (4.3). Indeed, from (4.2), we deduce that as $n \to +\infty$, $l \to +\infty$,

$$N^4 \leq \limsup_{n\to+\infty} \left(\left\| \sum_{j=1}^{l} V^j\left(x-x_n^j\right) \right\|_4 + \left\| v_n^l(x) \right\|_4 \right)^4 \leq \sum_{j=1}^{+\infty} \left\| V^j \right\|_4^4.$$

The left proof is similar to the proof of Theorem 1.1 in [37], and is omitted. □

Applying Lemma 4.4, we obtain the following limiting profile of blow-up solutions for Eq. (1.1) in Theorem 1.2: for the blow-up solution of the Cauchy problem (1.1)–(1.2) satisfying $\lim_{t\to T} \|u(t)\|_2 = +\infty$ and $\|\Delta u(t)\|_2 = \|\Delta R\|_2$, where $0 < T < +\infty$ is the blow-up time, we prove that $u(t,x)$ remains close to the ground state R in H^2 up to scaling and translation in the nonradial case, where R is the ground state of (1.3).

Proof of Theorem 1.2 By the assumptions, for any $t_n \to T$ as $n \to +\infty$, take

$$\lambda_n^2 := \frac{\|u(t_n,x)\|_2}{\|R\|_2} \to +\infty \quad \text{as } n \to +\infty, \tag{4.5}$$

and $U_n = u(t_n, \lambda_n x)$. We note that

$$\begin{cases} \|U_n\|_2 = \frac{1}{\lambda_n^2}\|u(t_n)\|_2 = \|R\|_2, \\ \|\Delta U_n\|_2 = \|\Delta u(t_n)\|_2 = \|\Delta R\|_2. \end{cases} \tag{4.6}$$

Therefore, $\{U_n\}_{n=1}^{+\infty}$ is a uniformly bounded sequence in H^2 and $\{U_n\}_{n=1}^{+\infty}$ has a weakly convergent subsequence $\{U_n\}_{n=1}^{+\infty}$ (still denoted by $\{U_n\}_{n=1}^{+\infty}$). And for the subsequence $\{U_n\}_{n=1}^{+\infty}$, we deduce that

$$\begin{aligned} H(U_n) &= \frac{1}{\lambda_n^4}\left(E\big((u_0,u_1)\big) - \frac{1}{2}\left\| \frac{\partial}{\partial t} u(t_n) \right\|_2^2 - \frac{1}{2}\left\| \Delta u(t_n) \right\|_2^2 \right) \\ &\leq \frac{1}{\lambda_n^4} E\big((u_0,u_1)\big) \to 0 \quad \text{as } n \to +\infty. \end{aligned} \tag{4.7}$$

Meanwhile, from (2.2), we see that $\|\Delta U_n\|_2 = \|\Delta R\|_2$ implies

$$H(U_n) \geq \frac{1}{2}\left(1 - \frac{\|\Delta U_n\|_2^2}{\|\Delta R\|_2^2} \right) \|U_n\|_2^2 \geq 0. \tag{4.8}$$

Then we get $\lim_{n\to+\infty} H(U_n) = 0$ and $\lim_{n\to+\infty} \|U_n\|_4^4 = 2\|R\|_2^2$. By applying Lemma 4.4 to the sequence $\{U_n\}_{n=1}^{+\infty}$ (here, we take $M^2 = \|R\|_2^2$, $N^4 = 2\|R\|_2^2$), there exist $\{y_n\}_{n=1}^{+\infty} \subset \mathbb{R}^4$ and

$U(x) \in H^2$ such that

$$U_n(x+y_n) \rightharpoonup U(x) \quad \text{weakly in } H^2 \text{ as } n \to +\infty \tag{4.9}$$

with $\|\Delta U\|_2 \geq \|\Delta R\|_2$. But by the lower semi-continuity of norm, we get $\|\Delta U\|_2 \leq \liminf_{n\to+\infty} \|\Delta U_n(x+y_n)\|_2 = \|\Delta R\|_2$, and so we get $\|\Delta U\|_2 = \|\Delta R\|_2 = \|\Delta U_n(x+y_n)\|_2$. And from the Brézis–Lieb lemma, we get

$$\lim_{n\to+\infty} \bigl\| \Delta\bigl(U_n(x+y_n) - U(x)\bigr) \bigr\|_2 = 0. \tag{4.10}$$

Applying (2.2) to $U_n(x+y_n) - U$, there exists $C > 0$ such that

$$\bigl\|U_n(x+y_n) - U\bigr\|_4^4 \leq C \bigl\|U_n(x+y_n) - U\bigr\|_2^2 \bigl\|\Delta\bigl(U_n(x+y_n) - U\bigr)\bigr\|_2^2.$$

Inject (4.6) and $\|U_n(x+y_n) - U(x)\|_2 \leq 2\|R\|_2$ into the above estimate.

$$U_n(x+y_n) \to U(x) \quad \text{strongly in } L^4 \text{ as } n \to +\infty. \tag{4.11}$$

Then we have proved that $\lim_{n\to+\infty} H(U_n(x+y_n)) = H(U) = 0$, which implies that $\|U\|_2^2 = \|R\|_2^2$, and so, from the Brézis–Lieb lemma, (4.6), and (4.9), we see that

$$U_n(x+y_n) \to U(x) \quad \text{strongly in } H^2 \text{ as } n \to +\infty.$$

Now, collecting the properties of $U = U(x)$, we see that

$$U \in H^2, \qquad \|U\|_2 = \|R\|_2, \qquad \|\Delta U\|_2 = \|\Delta R\|_2 \quad \text{and} \quad H(U) = 0.$$

Applying the variational characteristic of the ground state (see Lemma 4.1), there exist $\lambda_0 > 0$ and $x_0 \in \mathbb{R}^4$ such that $U(x) = U(\lambda_0 x + x_0)$, and so we can obtain (1.6) by redefining $\lambda(t)$ and $x(t)$. □

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Abbreviations
Not applicable.

Availability of data and materials
Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

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The authors declare that they have no competing interests.

Authors' contributions
PZ and LL have the same contribution to this work. All authors read and approved the final manuscript.

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