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On the Darboux problem involving the distributional Henstock–Kurzweil integral

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Abstract

In this paper, we apply the method associated with the technique of measure of noncompactness and the Darbo fixed point theorem to study the existence of solutions of the Darboux problem involving the distributional Henstock–Kurzweil integral. Meanwhile, an example is provided to illustrate our results.

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Keywords: Darboux problem; Distributional Henstock–Kurzweil integral; Measure of noncompactness; Darbo fixed point theorem

1 Introduction

The Darboux problems have been studied by many authors (see [1–3]). In [2], the authors used a fixed point theorem and some properties of measure of weak noncompactness to prove the existence of pseudo-solutions for the Darboux problem in a Banach space E:

$$\begin{cases} \frac{\partial^2 u}{\partial x \partial y}(x,y) = g(x,y,u(x,y)), & (x,y) \in \Delta, \\ u(x,0) = 0, \\ u(0,y) = 0, \end{cases} \tag{1}$$

where $\Delta = \{(x,y) : 0 \le x \le a_1, 0 \le y \le a_2\}$, g is Pettis-integrable but not necessarily Bochner integrable, and $\frac{\partial^2 u}{\partial x \partial y}$ denotes the second-order mixed pseudo-derivative.

In this paper, instead of the pseudo-derivative, we use the distributional derivative to establish the existence of solutions for the following Darboux problem involving the distributional Henstock–Kurzweil integral (D_{HK}-integral):

$$\begin{cases} \partial_{xy} u(x,y) = g(x,y,u(x,y)) + f(x,y), & (x,y) \in \Delta, \\ u(x,0) - \psi_1(x,u) = h_1(y), \\ u(0,y) - \psi_2(y,u) = h_2(x), \end{cases} \tag{2}$$

where ∂_{xy} denotes the second-order mixed distributional derivative, g is Henstock–Kurzweil integral (HK-integral), ψ_i, h_i are continuous, f is D_{HK}-integrable.

If there exists a continuous function $F \in C_0(\bar{Q})$ on $\overline{Q}$ such that f is the distributional derivative of F (the definition of $C_0(\bar{Q})$ will be introduced in Sect. 2), then the distribution

f is D_{HK}-integrable on Q. In other words, F is the primitive function of f. So we can see from the definition of D_{HK}-integral that the D_{HK}-integral is a kind of integration that is more extensive than the Riemann integral, Lebesgue integral, Henstock–Kurzweil integral and Denjoy integral (see [4–9]).

In order to prove the existence of solutions of the Darboux problem involving the distributional Henstock–Kurzweil integral, the method associated with the technique of measure of noncompactness and the Darbo fixed point theorem will be used.

This paper is organized as follows. In Sect. 2, we recall some fundamental concepts and basic results of the D_{HK}-integral and measure of noncompactness. In Sect. 3, we apply the Darbo fixed point theorem [10, Theorem 2] related to measure of noncompactness to verify the existence of solutions of Eq. (2). In Sect. 4, we give an example to illustrate Theorem 3.3 in this paper.

2 Preliminaries

In this section, we provide preliminary material with respect to the D_{HK}-integral and the measure of noncompactness.

2.1 D_{HK} integral

Let Q be an open rectangle $(a,b) \times (c,d)$ in the plane $\mathbb{R}^2$, and $\mathcal{D}(Q)$ be a subset of $C^\infty(Q)$ such that each $\phi \in \mathcal{D}(Q)$ has a compact support in Q. The continuous linear functional on $\mathcal{D}(Q)$ is called a distribution on Q.

We denote by $\partial = \partial_{xy} = \partial_{yx}$ the mixed distributional derivative, by ∂_1 and ∂_2 the distributional derivatives with respect to x and y, respectively, and by '$\int$' the D_{HK}-integral.

Let

$$C_0(\bar{Q}) = \bigl\{F \in C(\bar{Q}) : F(a,y) = F(x,c) = 0 \text{ for } (x,y) \in \bar{Q}\bigr\},$$

where $\bar{Q}$ is the closure of Q.

Obviously, $C_0(\bar{Q})$ is a closed subspace of $C(\bar{Q})$ endowed with the norm $\|F\|_\infty = \max\{|F(x,y)| : (x,y) \in \bar{Q}, F(x,y) \in C(\bar{Q})\}$.

We define the distributional Henstock–Kurzweil integral:

$$D_{\mathrm{HK}}(Q) = \bigl\{f \in \mathcal{D}'(Q) | f = \partial F, F \in C_0(\bar{Q})\bigr\}.$$

Definition 2.1 ([11, Lemma 2]) If $f \in \mathcal{D}'(Q)$, then

$$\begin{aligned} &F_1(x,y) + F_1(a,c) - F_1(a,y) - F_1(x,c) \\ &\quad = F_2(x,y) + F_2(a,c) - F_2(a,y) - F_2(x,c) \end{aligned}$$

for all $F_1, F_2 \in C_0(\bar{Q})$, $(x,y) \in \bar{Q}$. Moreover, there exists a unique $F_f \in D_{\mathrm{HK}}(Q)$ such that

$$F_f(a,y) = F_f(x,c) = 0,$$

$\forall x \in [a,b], y \in [c,d]$.

This leads to the next definition.

Definition 2.2 A distribution f is D_{HK}-integrable on Q if $f \in D_{\mathrm{HK}}(Q)$.

The D_{HK}-integral of f on Q is given by $\int_Q f = F_f(b,d)$, where F_f is given by Definition 2.1. We consider the structure of space $D_{\mathrm{HK}}(Q)$. For $f \in D_{\mathrm{HK}}(Q)$, endowed with the norm

$$\|f\| = \sup\left\{\left|\int_{(a,x)\times(c,y)} f\right| : (x,y) \in \bar{Q}\right\}.$$

Lemma 2.3 ([11, Theorem 1]) *The normed space* $(D_{\mathrm{HK}}(Q), \|\cdot\|)$ *is complete, separable and isomorphic to* $(C_0(\bar{Q}), \|\cdot\|_\infty)$.

Before stating a Fubini-type theorem for D_{HK}-integral which will be used later, we introduce some definitions.

Definition 2.4 Let $f \in D_{\mathrm{HK}}(Q)$, $x \in [a,b]$, $y \in [c,d]$. We define

$$\int_a^x f(\xi,\cdot)\,d\xi = \partial_2 F_f(x,\cdot) \quad \text{in } \mathcal{D}'\big((c,d)\big),$$
$$\int_c^y f(\cdot,\eta)\,d\eta = \partial_1 F_f(\cdot,y) \quad \text{in } \mathcal{D}'\big((a,b)\big).$$

It is clear that

$$\int_a^x f(s,\cdot)\,ds \in D_{\mathrm{HK}}\big((c,d)\big), \qquad \int_c^y f(\cdot,t)\,dt \in D_{\mathrm{HK}}\big((a,b)\big),$$

where $D_{\mathrm{HK}}((a,b))$ and $D_{\mathrm{HK}}((c,d))$ are, respectively, the spaces of D_{HK}-integrable distributions on (a,b) and (c,d), i.e.

$$D_{\mathrm{HK}}\big((a,b)\big) = \big\{f \in \mathcal{D}'\big((a,b)\big) | f = \partial F, F \in C_0(\bar{Q})\big\},$$

where $C_0(\bar{Q}) = \{F \in C([a,b]) : F(a) = 0\}$, and f is the distributional derivative of F.

Lemma 2.5 ([11, Theorem 4]) *For all* $f \in D_{\mathrm{HK}}(Q)$, *we have*

$$\int_Q f = \int_a^b \left(\int_c^d f(\cdot,\eta)\,d\eta\right) = \int_c^d \left(\int_a^b f(\xi,\cdot)\,d\xi\right).$$

Lemma 2.6 ([4, Theorem 4.3]) *If the following conditions are satisfied*:

(i) $f_n(x) \to f(x)$ *almost everywhere in* $[a,b]$ *as* $n \to \infty$ *where each* f_n *is Henstock integrable on* $[a,b]$.

(ii) $g(x) \le f_n(x) \le h(x)$ *for almost all* $x \in [a,b]$ *and all* n. *If* g *and* h *are also Henstock integrable on* $[a,b]$,

then f *is Henstock integrable on* $[a,b]$ *and*

$$\int_a^b f_n \to \int_a^b f \quad \textit{as } n \to \infty.$$

2.2 Measure of noncompactness

In this subsection, we recall the definition and some basic properties concerning measure of noncompactness [10]. Let $(E, \|\cdot\|)$ be a real Banach space with zero element 0, $B(x,r)$ be the closed ball in E centered at x and of radius r, and B_r be the ball $B(0,r)$. Denote by $\overline{X}, \overline{\text{conv}}X$ the closure and the closed convex hull of a nonempty subset X of E, respectively. Finally, denote by $\mathfrak{m}_E$ and $\mathfrak{n}_E$ the family of all nonempty and bounded subsets of E and the subfamilies of all relatively compact subsets, respectively.

Definition 2.7 ([12]) Let (E,d) be a metric space and X a bounded subset of E. The Hausdorff measure of noncompactness (μ-measure or ball measure of noncompactness) of the set X, denoted by $\mu(X)$ is defined to be the infimum of the set of all reals $\varepsilon > 0$ such that X can be covered by a finite number of balls of sets with diameters $< \varepsilon$, that is,

$$\mu(X) = \inf\{\varepsilon > 0 : X \text{ has a finite } \varepsilon\text{-net in } E\}. \tag{3}$$

The function μ is called the Hausdorff measure of noncompactness.

Definition 2.8 ([10]) A mapping $\mu : \mathfrak{m}_E \to \mathbb{R}_+$ is said to be a measure of noncompactness in E if it satisfies the following conditions:

(1) The family $\ker\mu = \{X \in \mathfrak{m}_E : \mu(X) = 0\}$ is nonempty and $\ker\mu \subseteq \mathfrak{n}_E$.
(2) if $X \subseteq Y \Rightarrow \mu(X) \leq \mu(Y)$.
(3) $\mu(\overline{\text{conv}}X) = \mu(X) = \mu(\overline{X})$.
(4) $\mu(\lambda X + (1-\lambda)Y) \leq \lambda\mu(X) + (1-\lambda)\mu(Y)$ for $\lambda \in [0,1)$.
(5) If (X_n) is a sequence of closed sets from $\mathfrak{m}_E$ such that $X_{n+1} \subseteq X_n (n \geq 1)$ and if $\lim_{n\to\infty} \mu(X_n) = 0$, then the intersection set $X_\infty = \bigcap_{n=1}^{\infty} X_n$ is nonempty.

In what follows, we will work in the space $C(\Delta)$ consisting of all real valued continuous functions on Δ. The space $C(\Delta)$ is equipped with the supremum norm

$$\|u\| = \sup\bigl\{\bigl|u(x,y)\bigr| : (x,y) \in \Delta\bigr\}.$$

For each $u \in C(\Delta)$, we define

$$\omega(u,\varepsilon) = \sup\bigl\{\bigl|u(x_2,y_2) - u(x_1,y_1)\bigr| : |x_2 - x_1| \leq \varepsilon, |y_2 - y_1| \leq \varepsilon\bigr\}.$$

Obviously, $\omega(u,\varepsilon) \to 0$, as $\varepsilon \to 0$, since u is uniformly continuous on Δ. Moreover, if this limit relation holds uniformly for u running over some bounded set $X \subset C(\Delta)$, then X is equicontinuous, and vice versa. Therefore, we have the following.

Lemma 2.9 ([13, Theorem 2.2]) *On the space $C(\Delta)$, the measure of noncompactness (3) is equivalent to*

$$\mu(X) = \lim_{\varepsilon\to 0} \sup_{u\in X} \omega(u,\varepsilon) \tag{4}$$

for all bounded sets $X \subset C(\Delta)$.

Lemma 2.10 ([10, Theorem 2, Darbo]) *Let Ω be a nonempty, bounded, closed and convex subset of a Banach space E and let $T : \Omega \to \Omega$ be a continuous mapping. Assume that there exists a constant $k \in [0,1)$ such that*

$$\mu\bigl(T(X)\bigr) \le k\mu(X)$$

for any $X \subset \Omega$. Then T has a fixed point.

3 Main results

In this section, we shall prove the existence of solutions of Eq. (2).

Firstly, we give the following assumptions:

(D_1) The functions $\psi_i : [0,a_i] \times \mathbb{R} \to \mathbb{R}$ are continuous, and there exist nonnegative constants $c_i > 0 (i = 1,2)$ such that, for each $u, v \in C(\Delta)$,

$$\begin{aligned} &\bigl|\psi_1(x,u) - \psi_1(x,v)\bigr| \le c_1\bigl(|u-v|\bigr), \\ &\bigl|\psi_2(y,u) - \psi_2(y,v)\bigr| \le c_2\bigl(|u-v|\bigr), \quad c_1 + c_2 < 1. \end{aligned}$$

Let $M_1 = \sup\{|\psi_1(x,0)| : x \in [0,a_1]\}, M_2 = \sup\{|\psi_2(y,0)| : y \in [0,a_2]\}$.

(D_2) The functions $h_1 : [0,a_2] \to \mathbb{R}, h_2 : [0,a_1] \to \mathbb{R}$ are continuous, and $h_1(0) = h_2(0)$, let $M_3 = \sup\{|h_1(y)| : y \in [0,a_2]\}, M_4 = \sup\{|h_2(x)| : x \in [0,a_1]\}$.

(D_3) The function f is D_{HK}-integrable on Δ, and $M_5 = \sup_{(x,y)\in\Delta}\{|\int_0^x \int_0^y f(s,t)\,dt\,ds|\}$.

(D_4) The function $g : \Delta \times \mathbb{R} \to \mathbb{R}$ is HK-integrable, for each $(x,y) \in \Delta$, $z \mapsto g(x,y,z)$ is continuous, and there exists HK-integral function $g_-, g_+ : \Delta \to \mathbb{R}$ such that

$$g_1(\cdot,\cdot) \le g(\cdot,\cdot,z) \le g_+(\cdot,\cdot),$$

and $M_6 = \sup_{(x,y)\in\Delta}\{|\int_0^x \int_0^y g_-(s,t)\,dt\,ds| + |\int_0^x \int_0^y g_+(s,t)\,dt\,ds|\}$.

(D_5) There exists $r > 0$ such that

$$M_1 + M_2 + 2M_3 + M_4 + M_5 + M_6 + (c_1 + c_2)r \le r.$$

Theorem 3.1 *Under the assumptions (D_1)–(D_5), Eq. (2) is equivalent to the integral equation*

$$\begin{aligned} u(x,y) = {}& \psi_1(x,u) + h_1(y) + \psi_2(y,u) + h_2(x) - h_1(0) + \int_0^x \int_0^y f(s,t)\,dt\,ds \\ & + \int_0^x \int_0^y g\bigl(s,t,u(s,t)\bigr)\,dt\,ds, \quad (x,y) \in \Delta. \end{aligned} \tag{5}$$

Proof For all $(x,y) \in \Delta$, by the properties of the distributional derivative, we have

$$\int_0^y \partial_{st}u(s,t)\,dt = \int_0^y g\bigl(s,t,u(s,t)\bigr)\,dt + \int_0^y f(s,t)\,dt. \tag{6}$$

Since

$$\partial_s u(s,y) - \partial_s u(s,0) = \int_0^y g\bigl(s,t,u(s,t)\bigr)\,dt + \int_0^y f(s,t)\,dt, \tag{7}$$

then

$$\int_0^x \partial_s u(s,y)\,ds = \int_0^x \partial_s u(s,0)\,ds + \int_0^x \int_0^y g\big(s,t,u(s,t)\big)\,dt\,ds + \int_0^x \int_0^y f(s,t)\,dt\,ds. \tag{8}$$

We obtain

$$\begin{aligned} u(x,y) &= u(x,0) + u(0,y) - u(0,0) + \int_0^x \int_0^y g\big(s,t,u(s,t)\big)\,dt\,ds + \int_0^x \int_0^y f(s,t)\,dt\,ds \\ &= \psi_1(x,u) + h_1(y) + \psi_2(y,u) + h_2(x) - h_1(0) + \int_0^x \int_0^y f(s,t)\,dt\,ds \\ &\quad + \int_0^x \int_0^y g\big(s,t,u(s,t)\big)\,dt\,ds. \end{aligned} \tag{9}$$

On the other hand, it is not difficult to see that Eq. (2) holds by taking the derivative of both sides of (9). This completes the proof. □

To simplify, we define an operator F on $C(\Delta)$ by

$$\begin{aligned} Fu(x,y) &= \psi_1(x,u) + h_1(y) + \psi_2(y,u) + h_2(x) - h_1(0) + \int_0^x \int_0^y f(s,t)\,dt\,ds \\ &\quad + \int_0^x \int_0^y g\big(s,t,u(s,t)\big)\,dt\,ds, \quad (x,y) \in \Delta. \end{aligned} \tag{10}$$

Then we have the following statement.

Theorem 3.2 *Under the assumptions* (D_1)*–*(D_5)*, the operator* F *given in* (10) *has at least one fixed point in the space* $C(\Delta)$.

Proof (i) For any $u \in C(\Delta)$, with $\|u\| \leq r$,

$$\begin{aligned} \big|Fu(x,y)\big| &\leq \big|\psi_1(x,u)\big| + \big|h_1(y)\big| + \big|\psi_2(y,u)\big| + \big|h_2(x)\big| + \big|h_1(0)\big| + \left|\int_0^x \int_0^y f(s,t)\,dt\,ds\right| \\ &\quad + \left|\int_0^x \int_0^y g\big(s,t,u(s,t)\big)\,dt\,ds\right| \\ &\leq \big|\psi_1(x,u) - \psi_1(x,0)\big| + \big|\psi_1(x,0)\big| + M_3 + \big|\psi_2(y,u) - \psi_2(y,0)\big| + \big|\psi_2(y,0)\big| \\ &\quad + M_4 + M_3 + M_5 + \left|\int_0^x \int_0^y g_-(s,t)\,dt\,ds\right| + \left|\int_0^x \int_0^y g_+(s,t)\,dt\,ds\right| \\ &\leq M_1 + M_2 + 2M_3 + M_4 + M_5 + M_6 + (c_1 + c_2)\|u\| \\ &\leq r. \end{aligned}$$

This implies that F maps the space B_r into B_r, where $B_r = \{u \in C(\Delta) : \|u\| \leq r\}$, r is a constant appearing in assumption (D_5).

(ii) We prove that the operator F is continuous on B_r. For arbitrary $u \in B_r$ and $\varepsilon > 0$, now let $u_n \in B_r$ with $\|u_n - u\| < \varepsilon$, then we have

$$\big|Fu_n(x,y) - Fu(x,y)\big|$$

$$\begin{aligned}
&\le \left|\psi_1(x,u_n)-\psi_1(x,u)\right| + \left|\psi_2(y,u_n)-\psi_2(y,u)\right| \\
&\quad + \left|\int_0^x\int_0^y g\big(s,t,u_n(s,t)\big)\,dt\,ds - \int_0^x\int_0^y g\big(s,t,u(s,t)\big)\,dt\,ds\right|. \qquad (11)
\end{aligned}$$

Now by the uniform continuity of the function $\psi_i(i=1,2)$ on the set $[0,a_i]\times[-r,r]$, we infer that

$$\lim_{n\to\infty}\psi_1(x,u_n)=\psi_1(x,u),\ \lim_{n\to\infty}\psi_2(y,u_n)=\psi_2(y,u).$$

According to (D_4) and Lemma 2.6, we have

$$\lim_{n\to\infty}\int_0^x\int_0^y g\big(s,t,u_n(s,t)\big)\,dt\,ds=\int_0^x\int_0^y g\big(s,t,u(s,t)\big)\,dt\,ds.$$

From estimate (11), for each $(x,y)\in\Delta$, we get

$$\left|Fu_n(x,y)-Fu(x,y)\right|\le\varepsilon.$$

Hence, we conclude that the operator F is continuous on B_r.

(iii) Let us take an arbitrary nonempty subset V of the ball B_r. Fix $\varepsilon>0$, choose arbitrarily $(x_1,y_1),(x_2,y_2)\in\Delta$ such that $|x_2-x_1|\le\varepsilon$, $|y_2-y_1|\le\varepsilon$. Then for arbitrary $u\in V$, we get

$$\begin{aligned}
&\left|Fu(x_2,y_2)-Fu(x_1,y_1)\right| \\
&\quad\le \left|\psi_1\big(x_2,u(x_2,y_2)\big)-\psi_1\big(x_1,u(x_1,y_1)\big)\right| + \left|h_1(y_2)-h_1(y_1)\right| \\
&\qquad + \left|\psi_2\big(y_2,u(x_2,y_2)\big)-\psi_2\big(y_1,u(x_1,y_1)\big)\right| + \left|h_2(x_2)-h_2(x_1)\right| \\
&\qquad + \left|\int_0^{x_2}\int_0^{y_2} f(s,t)\,dt\,ds - \int_0^{x_1}\int_0^{y_1} f(s,t)\,dt\,ds\right| \\
&\qquad + \left|\int_0^{x_2}\int_0^{y_2} g\big(s,t,u(s,t)\big)\,dt\,ds - \int_0^{x_1}\int_0^{y_1} g\big(s,t,u(s,t)\big)\,dt\,ds\right| \\
&\quad\le \left|\psi_1\big(x_2,u(x_2,y_2)\big)-\psi_1\big(x_1,u(x_2,y_2)\big)\right| + \left|\psi_1\big(x_1,u(x_2,y_2)\big)-\psi_1\big(x_1,u(x_1,y_1)\big)\right| \\
&\qquad + \omega(h_1,\varepsilon) + \left|\psi_2\big(y_2,u(x_2,y_2)\big)-\psi_2\big(y_1,u(x_2,y_2)\big)\right| \\
&\qquad + \left|\psi_2\big(y_1,u(x_2,y_2)\big)-\psi_2\big(y_1,u(x_1,y_1)\big)\right| + \omega(h_2,\varepsilon) \\
&\qquad + \left|\int_0^{x_1}\int_{y_1}^{y_2} f(s,t)\,dt\,ds + \int_{x_1}^{x_2}\int_0^{y_2} f(s,t)\,dt\,ds\right| \\
&\qquad + \left|\int_0^{x_1}\int_{y_1}^{y_2} g\big(s,t,u(s,t)\big)\,dt\,ds + \int_{x_1}^{x_2}\int_0^{y_2} g\big(s,t,u(s,t)\big)\,dt\,ds\right| \\
&\quad\le \omega(\psi_1,\varepsilon) + c_1\big(\left|u(x_2,y_2)-u(x_1,y_1)\right|\big) + \omega(h_1,\varepsilon) + \omega(\psi_2,\varepsilon) \\
&\qquad + c_2\big(\left|u(x_2,y_2)-u(x_1,y_1)\right|\big) + \omega(h_2,\varepsilon) \\
&\qquad + \left|\int_0^{x_1}\int_{y_1}^{y_2} f(s,t)\,dt\,ds + \int_{x_1}^{x_2}\int_0^{y_2} f(s,t)\,dt\,ds\right| \\
&\qquad + \left|\int_0^{x_1}\int_{y_1}^{y_2} g\big(s,t,u(s,t)\big)\,dt\,ds + \int_{x_1}^{x_2}\int_0^{y_2} g\big(s,t,u(s,t)\big)\,dt\,ds\right|, \qquad (12)
\end{aligned}$$

where

$$\begin{aligned}
&\omega(\psi_1,\varepsilon)=\sup\bigl\{\bigl|\psi_1(x_2,u)-\psi_1(x_1,u)\bigr|:u\in[-r,r],|x_2-x_1|\le\varepsilon\bigr\},\\
&\omega(h_1,\varepsilon)=\sup\bigl\{\bigl|h_1(y_2)-h_1(y_1)\bigr|:|y_2-y_1|\le\varepsilon\bigr\},\\
&\omega(\psi_2,\varepsilon)=\sup\bigl\{\bigl|\psi_2(y_2,u)-\psi_2(y_1,u)\bigr|:u\in[-r,r],|y_2-y_1|\le\varepsilon\bigr\},\\
&\omega(h_2,\varepsilon)=\sup\bigl\{\bigl|h_2(x_2)-h_2(x_1)\bigr|:|x_2-x_1|\le\varepsilon\bigr\}.
\end{aligned}$$

By (D_1)–(D_2), $\psi_i(i=1,2), h_1, h_2$ are uniformly continuous on $[0,a_i],[0,a_2],[0,a_1]$ respectively, so

$$\omega(\psi_i,\varepsilon)\to 0,\qquad \omega(h_i,\varepsilon)\to 0\quad \text{as } \varepsilon\to 0.$$

Further, by the condition (D_4), we have the following inequalities:

$$\begin{aligned}
&\left|\int_0^{x_1}\int_{y_1}^{y_2}g\bigl(s,t,u(s,t)\bigr)\,dt\,ds+\int_{x_1}^{x_2}\int_0^{y_2}g\bigl(s,t,u(s,t)\bigr)\,dt\,ds\right|\\
&\quad\le\left|\int_0^{x_1}\int_{y_1}^{y_2}g_-(s,t)\,dt\,ds\right|+\left|\int_0^{x_1}\int_{y_1}^{y_2}g_+(s,t)\,dt\,ds\right|\\
&\qquad+\left|\int_{x_1}^{x_2}\int_0^{y_2}g_-(s,t)\,dt\,ds\right|+\left|\int_{x_1}^{x_2}\int_0^{y_2}g_+(s,t)\,dt\,ds\right|.
\end{aligned}$$

Since g_-,g_+ are HK-integrable, the primitives of g_-,g_+ are continuous and so are uniformly continuous on Δ.

Moreover, the fact that $f\in D_{\mathrm{HK}}$ also implies that the primitive of f is uniformly continuous on Δ.

From (12), we get

$$\bigl|Fu(x_2,y_2)-Fu(x_1,y_1)\bigr|\le c_1\omega(u,\varepsilon)+c_2\omega(u,\varepsilon),\quad \text{as } \varepsilon\to 0. \tag{13}$$

Since u is an arbitrary element of V in (13), we obtain

$$\omega(FV,\varepsilon)\le c_1\omega(u,\varepsilon)+c_2\omega(u,\varepsilon).$$

Hence,

$$\lim_{\varepsilon\to 0}\sup\omega(FV,\varepsilon)\le(c_1+c_2)\lim_{\varepsilon\to 0}\sup\omega(u,\varepsilon). \tag{14}$$

It follows from (14) and Lemma 2.9 that

$$\mu(FV))\le(c_1+c_2)\mu(V). \tag{15}$$

According to Lemma 2.10, F has at least one fixed point in the space B_r. The proof is therefore complete. □

According to Theorem 3.2 and (10) the definition of the operator F, we have:

Theorem 3.3 *Under the assumptions* (D_1)–(D_5), *Eq.* (2) *has at least one solution in the space* $C(\Delta)$.

4 Application

Example 4.1 Consider the following Darboux problem:

$$\begin{cases} \frac{\partial^2 u}{\partial x\,\partial y}(x,y) = x^2e^{-y}\sin\frac{u(x,y)}{2} + yR'(x) + xR'(y), \quad (x,y)\in[0,1]\times[0,1], \\ u(x,0) - (\frac{x}{4(1+x^2)} + \frac{u(x,y)}{4+x^2}) = \frac{1}{4+y^2}, \\ u(0,y) - \frac{1}{2}\arctan[\frac{u(x,y)}{8+\sqrt{y}} + \frac{2y}{1+y^2}] = \frac{1}{2}\arctan x + \frac{1}{4}, \end{cases} \tag{16}$$

where $R'(x)$ is the distributional derivative of the Riemann function $R(x) = \Sigma_{n=1}^{\infty}\frac{\sin n^2\pi x}{n^2}$.

With the following choices, it is evident that Eq. (16) is a special case of Eq. (2) with

$$\begin{aligned} &g\bigl(x,y,u(x,y)\bigr) = x^2e^{-y}\sin\frac{u(x,y)}{2}, \\ &f(x,y) = yR'(x) + xR'(y), \\ &\psi_1(x,u) = \frac{x}{4(1+x^2)} + \frac{u(x,y)}{4+x^2}, \\ &\psi_2(y,u) = \frac{1}{2}\arctan\biggl[\frac{u(x,y)}{8+\sqrt{y}} + \frac{2y}{1+y^2}\biggr], \\ &h_1(y) = \frac{1}{4+y^2}, \\ &h_2(x) = \frac{1}{2}\arctan x + \frac{1}{4}, \\ &\Delta = [0,1]\times[0,1]. \end{aligned}$$

Now we show that all the conditions of Theorem 3.2 are satisfied for Eq. (16).

(i) Obviously, $\psi_i (i = 1,2)$ are continuous, and suppose that $(x,y)\in[0,1]\times[0,1]$ and $u,v\in C(\Delta)$. Then we can get the following estimate:

$$\begin{aligned} &\bigl|\psi_1(x,u) - \psi_1(x,v)\bigr| \le \frac{1}{4+x^2}|u-v| \le \frac{1}{4}|u-v|, \quad \text{so } c_1 = \frac{1}{4}, \\ &\bigl|\psi_2(y,u) - \psi_2(y,v)\bigr| \le \frac{1}{2}\cdot\frac{1}{8+\sqrt{y}}|u-v| \le \frac{1}{16}|u-v|, \quad \text{so } c_2 = \frac{1}{16}, \\ &M_1 = \sup\bigl\{\bigl|\psi_1(x,0)\bigr| : x\in[0,1]\bigr\} = \frac{1}{8}, \\ &M_2 = \sup\bigl\{\bigl|\psi_2(y,0)\bigr| : y\in[0,1]\bigr\} = \frac{\pi}{8}. \end{aligned}$$

(ii) Clearly, the functions h_i are continuous, $h_1(0) = h_2(0) = \frac{1}{4}$, and

$$M_3 = \sup\bigl\{\bigl|h_1(y)\bigr| : y\in[0,1]\bigr\} = \frac{1}{4}, \qquad M_4 = \sup\bigl\{\bigl|h_2(x)\bigr| : x\in[0,1]\bigr\} = \frac{\pi}{8} + \frac{1}{4}.$$

(iii) The function f is D_{HK}-integrable on Δ, and

$$M_5 = \sup_{(x,y)\in\Delta}\left|\int_0^x\int_0^y f(s,t)\,dt\,ds\right| < 2.$$

(iv) The function g is continuous, and

$$-x^2e^{-y} \preceq g\bigl(x,y,u(x,y)\bigr) \preceq x^2e^{-y},$$

if we put $g_-(x,y) = -x^2e^{-y}, g_+(x,y) = x^2e^{-y}$, then we have

$$M_6 = \sup_{(x,y)\in\Delta}\left\{\left|\int_0^x\int_0^y g_-(s,t)\,dt\,ds\right| + \left|\int_0^x\int_0^y g_+(s,t)\,dt\,ds\right|\right\} = \frac{2}{3} - \frac{2}{3e}.$$

(v) It is easy to check that for each number $r \geq 5$, we have the following inequality:

$$M_1 + M_2 + 2M_3 + M_4 + M_5 + M_6 + (c_1 + c_2)r < r.$$

Consequently, all the conditions of Theorem 3.2 are satisfied and Eq. (16) has at least one solution in the space $C(\Delta)$.

Remark 4.2 It is well known that the function $R(x)$ given by Riemann is continuous but pointwise differentiable nowhere on $[0,1]$ (see, e.g.,[14]), then the distributional derivative $R'(x)$ in Eq. (16) is neither HK nor Lebesgue integrable. Hence, this example is not covered by any result using HK or Lebesgue integral. Thus, Theorem 3.3 is more extensive.

5 Conclusions

In this research, by using the method associated with the technique of measure of noncompactness and the Darbo fixed points theorem, we studied the existence of solutions for the Darboux problem involving the distributional Henstock–Kurzweil integral, and we obtained the existence of at least one solution for the Darboux problem we considered.

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