Boundary Value Problems
a SpringerOpen Journal

RESEARCH Open Access

Oscillation theorems for nonlinear fractional difference equations

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Abstract

In this study, we discuss some theorems related to the oscillatory behavior of nonlinear fractional difference equations equipped with well-known fractional Riemann–Liouville difference operator. Then we give an example for the illustration of the results obtained.

Keywords: Fractional difference operator; Riemann–Liouville; Oscillation; Oscillation theory; Fractional difference equations; Oscillation criteria

1 Introduction and preliminaries

Fractional calculus has proved to be valuable tools in describing and solving a large number of problems in various fields of sciences and engineering [1, 2]. Their treatment from the viewpoint of difference equations can additionally open up new perspectives. Thus we decided to study fractional difference equations.

Up to now, many authors have investigated the oscillatory behaviors of solutions of various equations, including differential equations, fractional differential equations, difference equations, fractional difference equations, partial differential equations, fractional partial differential equations and dynamic equations on time scales [3–27]. Motivated by this work, we are concerned with the following equations:

$$\Delta\big(r(t)\Delta^{\alpha}x(t)\big) + q(t)f\big(G(t)\big) = 0, \tag{1}$$

where $t \in \mathbf{N}_{t_0+1-\alpha}$, $G(t) = \sum_{s=t_0}^{t-1+\alpha}(t-s-1)^{(-\alpha)}x(s)$, $q(t)$ and $r(t)$ is positive sequences, $f : \mathbf{R} \to \mathbf{R}$ is continuous function satisfies $xf(x) > 0$ for $x \neq 0$ and Δ^{α} denotes the Riemann–Liouville fractional difference operator of order $0 < \alpha \leq 1$. Throughout the study, we consider

$$R(t) = \sum_{s=t_0}^{t-1}\frac{1}{r(t)}$$

and $\lim_{t\to\infty} R(t) = \infty$.

By a solution of Eq. (1), we mean a real-valued sequence $x(t)$ satisfying Eq. (1) for $t \in \mathbf{N}_{t_0}$. A solution $x(t)$ of Eq. (1) is called oscillatory if for every positive integers $T_0 > t_0$ there exists $t \geq T_0$ such that $x(t)x(t+1) \leq 0$, otherwise it is called non-oscillatory. Equation (1) is called oscillatory if all its solutions are oscillatory.

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Definition 1 ([28]) Let $\nu > 0$. The νth fractional sum f is defined by

$$\Delta^{-\nu}f(t) = \frac{1}{\Gamma(\nu)}\sum_{s=a}^{t-\nu}(t-s-1)^{\nu-1}f(s), \tag{2}$$

where f is defined for $s \equiv a \mod (1)$, $\Delta^{-\nu}f$ is defined for $t \equiv (a+\nu) \mod (1)$ and $t^{(\nu)} = \frac{\Gamma(t+1)}{\Gamma(t-\nu+1)}$. The fractional sum $\Delta^{-\nu}f$ maps functions defined on $\mathbf{N}_a$ to functions defined on $\mathbf{N}_{a+\nu}$, where $\mathbf{N}_t = \{t, t+1, t+2, \dots\}$.

Definition 2 ([28]) Let $\nu > 0$ and $m-1 < \mu < m$, where m denotes a positive integer, $m = \lceil \mu \rceil$. Set $\nu = m - \mu$. The μth fractional difference is defined as

$$\Delta^{\mu}f(t) = \Delta^{m-\nu}f(t) = \Delta^{m}\Delta^{-\nu}f(t), \tag{3}$$

where $\lceil \mu \rceil$ is the ceiling function of μ.

2 Main result

Lemma 1 ([25]) *Let $x(t)$ be a solution of Eq. (1) and let*

$$G(t) = \sum_{s=t_0}^{t-1+\alpha}(t-s-1)^{(-\alpha)}x(s), \tag{4}$$

then

$$\Delta\bigl(G(t)\bigr) = \Gamma(1-\alpha)\Delta^{\alpha}x(t). \tag{5}$$

Theorem 1 *Suppose that*

$$\sum_{s=t_0}^{\infty} q(s) = \infty \tag{6}$$

and

$$\lim_{t\to\infty} \inf f(t) > 0. \tag{7}$$

Then every solution of (1) is oscillatory.

Proof Suppose to the contrary that $x(t)$ is a non-oscillatory solution of Eq. (1). Then, without loss of generality, we assume that $x(t)$ is eventually positive solution of (1) on $[t_0, \infty)$, then $G(t) > 0$ on $[t_0, \infty)$. From (1), we have

$$\Delta\bigl(r(t)\Delta^{\alpha}x(t)\bigr) = -q(t)f\bigl(G(t)\bigr) < 0. \tag{8}$$

That is, $r(t)\Delta^{\alpha}x(t)$ is an eventually nonincreasing sequence on $[t_0, \infty)$. We claim that $r(t)\Delta^{\alpha}x(t) > 0$ on $[t_1, \infty)$, where t_1 is sufficiently large. Otherwise, assume that there exists a $t_2 > t_1$ such that $r(t)\Delta^{\alpha}x(t) < 0$ on $[t_2, \infty)$. Then we have

$$r(t)\Delta^{\alpha}x(t) \le r(t_2)\Delta^{\alpha}x(t_2) = c < 0,$$

or

$$\Delta^\alpha x(t) \le \frac{c}{r(t)}.$$

Then we get

$$\Delta\bigl(G(t)\bigr) \le \Gamma(1-\alpha)\frac{c}{r(t)}. \tag{9}$$

Summing both sides of (9) from t_2 to $t-1$, we have

$$G(t) \le G(t_2) + \Gamma(1-\alpha)\sum_{s=t_2}^{t-1}\frac{c}{r(s)}.$$

Letting $t \to \infty$, we get $\lim_{t\to\infty} G(t) = -\infty$, which contradicts the fact that $G(t) > 0$ on $[t_0,\infty)$. Hence we obtain $r(t)\Delta^\alpha x(t) > 0$ on $[t_1,\infty)$. Hence we obtain $\Delta^\alpha x(t) > 0$ and $\Delta(r(t)\Delta^\alpha x(t)) < 0$ on $[t_1,\infty)$. Now we consider that

$$\lim_{t\to\infty} G(t) = k.$$

Then $k > 0$ is finite or infinite.

Case 1 $k > 0$ is finite.

Since f is a continuous function, we get

$$\lim_{t\to\infty} f\bigl(G(t)\bigr) = f(k) > 0.$$

This implies we have for sufficiently large $t_3 > t_2$ and $t \ge t_3$

$$f\bigl(G(t)\bigr) > \frac{1}{2}f(k). \tag{10}$$

Substituting (10) in (8), we get

$$\Delta\bigl(r(t)\Delta^\alpha x(t)\bigr) \le -q(t)\frac{1}{2}f(k)$$

or

$$\Delta\bigl(r(t)\Delta^\alpha x(t)\bigr) + \frac{1}{2}f(k)q(t) \le 0. \tag{11}$$

Then summing both sides of the last inequality from t_3 to $t-1$, we have

$$r(t)\Delta^\alpha x(t) - r(t_3)\Delta^\alpha x(t_3) + \frac{1}{2}f(k)\sum_{s=t_3}^{t-1} q(t) \le 0.$$

Hence we obtain for $t \ge t_3$

$$\frac{1}{2}f(k)\sum_{s=t_3}^{t-1} q(t) \le r(t_3)\Delta^\alpha x(t_3),$$

which contradicts with (6).

Case 2 $k = \infty$.
From the condition (7), we have

$$\lim_{t\to\infty} \inf f\big(G(t)\big) > 0.$$

Then we can choose a positive constant c and sufficiently large $t_4 > t_3$ such that for $t \geq t_4$

$$f\big(G(t)\big) > c. \tag{12}$$

Substituting (12) in (8),

$$\Delta\big(r(t)\Delta^{\alpha}x(t)\big) \leq -cq(t).$$

This implies

$$\Delta\big(r(t)\Delta^{\alpha}x(t)\big) + cq(t) \leq 0.$$

Summing the last inequality from t_4 to $t-1$, we get

$$r(t)\Delta^{\alpha}x(t) - r(t_4)\Delta^{\alpha}x(t_4) + c\sum_{s=t_4}^{\infty} q(s) \leq 0.$$

Thus

$$c\sum_{s=t_4}^{\infty} q(s) \leq r(t_4)\Delta^{\alpha}x(t_4),$$

which contradicts (6). Then the proof is complete. □

Theorem 2 *Assume that*

$$\sum_{s=t_0}^{\infty} R(s)q(s) = \infty. \tag{13}$$

Then every bounded solution of (1) *is oscillatory.*

Proof Proceeding as in the proof of Theorem 1 with the assumption that $x(t)$ is a bounded non-oscillatory solution of (1), from (11), we have for $t \geq t_0$ where t_0 is sufficiently large

$$R(t)\Delta\big(r(t)\Delta^{\alpha}x(t)\big) + \frac{1}{2}f(k)R(t)q(t) \leq 0. \tag{14}$$

Additionally we have

$$R(t)\Delta\big(r(t)\Delta^{\alpha}x(t)\big) \geq \Delta\big(R(t)r(t)\Delta^{\alpha}x(t)\big) - \big(\Delta R(t)\big)r(t)\Delta^{\alpha}x(t). \tag{15}$$

From (14) and (15),

$$\Delta\big(R(t)r(t)\Delta^{\alpha}x(t)\big) - \frac{\Delta G(t)}{\Gamma(1-\alpha)} + \frac{1}{2}f(k)R(t)q(t) \leq 0$$

and summing both sides of the last inequality from t_0 to $t-1$, we get

$$\begin{aligned} & R(t)r(t)\Delta^{\alpha}x(t) - R(t_0)r(t_0)\Delta^{\alpha}x(t_0) - \frac{1}{\Gamma(1-\alpha)}\big(G(t)-G(t_0)\big) \\ & \quad + \frac{1}{2}f(k)\sum_{s=t_0}^{t-1} R(s)q(s) \le 0. \end{aligned}$$

That is,

$$\frac{1}{2}f(k)\sum_{s=t_0}^{t-1} R(s)q(s) \le \frac{G(t)-G(t_0)}{\Gamma(1-\alpha)} + R(t_0)r(t_0)\Delta^{\alpha}x(t_0).$$

Since $x(t)$ is bounded, we can choose a positive constant c such that

$$\sum_{s=t_0}^{t-1} R(s)q(s) \le c,$$

which is a contradiction to the assumption of the theorem. □

Theorem 3 *Assume that* (6), f *is non-decreasing and there is a non-negative constant* M *such that*

$$\limsup_{t\to 0} \frac{t}{f(t)} = M \tag{16}$$

and there exists a $r_1(t)$ *positive subsequence of* $r(t)$ *such that* $r_1(t) \le 1$. *Then the fractional difference* $\Delta^{\alpha}x(t)$ *of every solution* $x(t)$ *of* (1) *oscillates.*

Proof Suppose to the contrary that Eq. (1) has a solution $x(t)$ such that its fractional difference $\Delta^{\alpha}x(t)$ is non-oscillatory. Firstly we assume that $\Delta^{\alpha}x(t)$ is eventually negative. Then there exists a positive integer t_0 such that $\Delta^{\alpha}x(t) < 0$ and $G(t)$ is decreasing on $[t_0,\infty)$. This implies that $x(t)$ is also non-oscillatory. Then we consider the following function for $t \ge t_1 \ge t_0$:

$$\omega(t) = \frac{r(t)\Delta^{\alpha}x(t)}{f(G(t))}.$$

Thus

$$\begin{aligned} \Delta\omega(t) &= \frac{\Delta(r(t)\Delta^{\alpha}x(t))f(G(t)) - r(t)\Delta^{\alpha}x(t)\Delta f(G(t))}{f(G(t))f(G(t+1))} \\ &= \frac{\{r(t+1)\Delta^{\alpha}x(t+1) - r(t)\Delta^{\alpha}x(t)\}f(G(t))}{f(G(t))f(G(t+1))} \\ &\quad - \frac{r(t)\Delta^{\alpha}x(t)\{f(G(t+1) - f(G(t)))\}}{f(G(t))f(G(t+1))} \\ &= \frac{r(t+1)\Delta^{\alpha}x(t+1)f(G(t)) - r(t)\Delta^{\alpha}x(t)f(G(t+1))}{f(G(t))f(G(t+1))} \\ &= \frac{r(t+1)\Delta^{\alpha}x(t+1)}{f(G(t+1))} - \frac{r(t)\Delta^{\alpha}x(t)}{f(G(t))} \end{aligned}$$

$$\begin{aligned}
&= \frac{r(t+1)\Delta^{\alpha}x(t+1)}{f(G(t+1))} - \frac{\{r(t+1)\Delta^{\alpha}x(t+1) - \Delta(r(t)\Delta^{\alpha}x(t))\}}{f(G(t))} \\
&= \frac{\Delta(r(t)\Delta^{\alpha}x(t))}{f(G(t))} + \frac{r(t+1)\Delta^{\alpha}x(t+1)\{f(G(t)) - f(G(t+1))\}}{f(G(t))f(G(t+1))} \\
&\leq \frac{\Delta(r(t)\Delta^{\alpha}x(t))}{f(G(t))} = -q(t).
\end{aligned}$$

That is,

$$\Delta\omega(t) \leq -q(t).$$

Summing both sides of last inequality from t_1 to $t-1$, we have

$$\omega(t) - \omega(t_1) \leq -\sum_{s=t_1}^{t-1} q(s).$$

That is,

$$\lim_{t\to\infty} \omega(t) = -\infty, \tag{17}$$

thus $f(G(t)) > 0$ and hence

$$G(t) > 0. \tag{18}$$

From (17), for sufficiently large $t_2 > t_1$ and $t \geq t_2$

$$\omega(t) \leq -(M+1).$$

Then we obtain

$$\omega(t) = \frac{r(t)\Delta^{\alpha}x(t)}{f(G(t))} \leq -(M+1)$$

or

$$r(t)\Delta^{\alpha}x(t) + (M+1)f\bigl(G(t)\bigr) \leq 0. \tag{19}$$

Set $\lim_{t\to\infty} G(t) = L$. Then $L \geq 0$. We claim that $L = 0$. If $L > 0$, then $\lim_{t\to\infty} f(G(t)) = f(L) > 0$ by the continuity of f. For sufficiently large $t_3 > t_2$ and $t \geq t_3$ we get

$$f\bigl(G(t)\bigr) > \frac{1}{2}f(L).$$

Substituting the last inequality in (19), we have

$$\Delta^{\alpha}x(t) + \frac{(M+1)f(L)}{2r(t)} \leq 0;$$

in other words

$$\Delta G(t) + \frac{\Gamma(1-\alpha)(M+1)f(L)}{2r(t)} \leq 0.$$

Summing both sides of the last inequality from t_3 to $t-1$, we obtain

$$G(t) - G(t_3) + \frac{1}{2}\Gamma(1-\alpha)(M+1)f(L)\sum_{s=t_3}^{t-1}\frac{1}{r(s)} \le 0.$$

This implies $\lim_{t\to\infty} G(t) = -\infty$, which contradicts (18). Thus $\lim_{t\to\infty} G(t) = 0$. By the assumption (16), we get

$$\lim_{t\to\infty} \sup \frac{G(t)}{f(G(t))} \le M.$$

Then for sufficiently large $t_4 > t_3$ and $t \ge t_4$ we obtain

$$\frac{G(t)}{f(G(t))} < M + 1.$$

That is,

$$G(t) < (M+1)f\bigl(G(t)\bigr).$$

From (19), we obtain

$$r(t)\Delta^\alpha x(t) + G(t) < 0.$$

Then we consider subsequence $r_1(t)$,

$$\Delta^\alpha x(t) + G(t) \le r_1(t)\Delta^\alpha x(t) + G(t) < 0,$$

and we have

$$\frac{\Delta G(t)}{\Gamma(1-\alpha)} + G(t) < 0$$

or

$$0 < G(t+1) - G(t) + \Gamma(1-\alpha)G(t) < 0,$$

which contradicts (18). The case that $\Delta^\alpha x(t)$ is eventually positive can be proved in a similar manner and so the proof is complete. □

3 Applications

Example Consider the following fractional difference equation:

$$\Delta\bigl(t\Delta^\alpha x(t)\bigr) + t^7\left(\sum_{s=t_0}^{t-1+\alpha}(t-s-1)^{(-\alpha)}x(s)\right)\exp\left(\sum_{s=t_0}^{t-1+\alpha}(t-s-1)^{(-\alpha)}x(s)\right) = 0. \tag{20}$$

This corresponds to Eq. (1) with $\alpha \in (0,1]$, $r(t) = t$, $q(t) = t^7$ and $f(x) = xe^x$. Then we have $\lim_{t\to\infty} R(t) = \infty$, $xf(x) > 0$,

$$\sum_{s=t_0}^{\infty} q(s) = \sum_{s=t_0}^{\infty} t^7 = \infty,$$

and

$$\lim_{t\to\infty}\inf f(t) > 0.$$

Thus, (20) is oscillatory from Theorem 1.

Acknowledgements
The author is grateful to the scholars who provided the literature sources.

Availability of data and materials
Not applicable.

Competing interests
The author declares that they have no competing interests.

Authors' contributions
HA contributed to the work totally, and he read and approved the final version of the manuscript.

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Received: 21 October 2018 Accepted: 15 November 2018 Published online: 23 November 2018

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